A new empirical approach to lepton and quark masses

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A lepton ratio with a small dimensionless residual $k_e = m_e m_\tau^2/m_\mu^3$ and mass scaling factor $\alpha_f = 27m_e/m_\tau$ are used to construct empirical formulas for charged leptons, left-handed neutrinos and quarks. The predicted masses are in excellent agreement with known experimental values and constraints.

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I. INTRODUCTION

One outstanding problem in modern physics is the seemingly unrelated particle masses that exist as free parameters in the standard model. Numerous attempts have been made over the years to derive a systematic mathematical relationship between these masses, most famously by Yoshio Koide with his charged lepton formula [1, 2]

$$K = \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} \simeq \frac{2}{3}. \tag{1}$$

While this approximation works for charged leptons, and appears to work for some combinations of quarks with running masses, it does not work for neutrinos or attempt to relate all fundamental fermions in a consistent manner. It does however hint at the possibility of finding other more fundamental relationships. Indeed, Koide and others [1–24] have used it as inspiration for extended models to cover other fermions.

While these extended Koide models are interesting we present an alternative structure for charged leptons that is easily applied to other sectors. we find these formulas appear to relate all fundamental lepton and quark pole (physical) masses. This allows precise predictions of neutrino and quark pole masses. If correct this would significantly reduce the number of free parameters in the standard model (extended for neutrinos with mass).

II. CHARGED LEPTONS

We begin by considering the small dimensionless residual value in a charged lepton structure proposed by Terazawa [25]

$$k_e = \frac{m_e m_\tau^2}{m_\mu^3} \simeq 1.37. \tag{2}$$

As with the Koide formula most of the mass differences cancel despite the input masses having a range of over 3500:1. If this is not just a coincidence we can attempt to find factors that cancel out with this formula to learn more about the underlying structure. We can then attempt to apply those factors to search for similar structures in other sectors.

Mac Gregor proposed using different exponents of the QED coupling constant to explain fermion lifetimes and mass relationships using the zero energy scale $\alpha_{\mathrm{QED}}(0)$ [26, 27]. While we have not found convincing relationships with $\alpha_{\mathrm{QED}}(0)$ we do find

$$\alpha_f = 27\frac{m_e}{m_\tau} \tag{3}$$

interesting with a value $\alpha_f^{-1} \simeq 128.786$. This is close to the effective QED coupling constant $\alpha_{\mathrm{QED}}^{-1}(M_Z^2) = 128.944$ [28]. Despite the similarity in value to an effective QED coupling constant we are assuming that α_f is a separate parameter related only to mass generation and not electric charge. With (2) and (3) we can derive lepton formulas

$$m_\mu = 9\frac{m_e}{k_e^{\frac{1}{3}}\alpha_f^{\frac{2}{3}}}, \tag{4}$$

$$m_\tau = 27\frac{m_e}{\alpha_f}. \tag{5}$$

With our choice of definitions for k_e and α_f we have discovered an interesting and surprisingly simple charged lepton mass structure that could be used to look for similar patterns in other sectors.

III. NEUTRINO SECTOR

We will now use α_f and a sector structure formula similar to (2) to look for formulas for left handed neutrinos and predict mass state values for them. While neutrino mass states have not been directly measured yet numerous neutrino oscillation experiments have established increasingly refined limits on neutrino squared mass differences [29]. Cosmological models also put constraints on the sum of the three mass states [30]. By looking for equations of similar form to (4) and (5) and testing against the experimental neutrino constraints we can identify candidate neutrino mass formulas. Given

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that the experimental bounds on the neutrino mass states have them much closer together than the charged leptons we assume the same exponent of α_f would need to appear in each formula. By also using the same integer coefficients as the charged lepton formulas we find a surprisingly close match to the experimental squared mass differences when setting the neutrino residual $k_\nu \simeq 6.64$. This results in left-handed neutrino sector formulas

$$k_\nu = \frac{m_1 m_3^2}{m_2^3} \simeq 6.64. \tag{6}$$

$$m_1 = \alpha_f^4 m_e \simeq 1.86 \times 10^{-3} \text{ eV}/c^2, \tag{7}$$

$$m_2 = 9\frac{\alpha_f^4}{k_\nu^{\frac{1}{3}}} m_e \simeq 8.90 \times 10^{-3} \text{ eV}/c^2, \tag{8}$$

$$m_3 = 27\alpha_f^4 m_e \simeq 5.02 \times 10^{-2} \text{ eV}/c^2. \tag{9}$$

These proposed masses give squared mass differences of

$$\Delta m_{21}^2 \simeq 7.56 \times 10^{-5} \text{ eV}^2/c^4, \tag{10}$$
$$\Delta m_{31}^2 \simeq 2.51 \times 10^{-3} \text{ eV}^2/c^4, \tag{11}$$
$$\Delta m_{32}^2 \simeq 2.44 \times 10^{-3} \text{ eV}^2/c^4, \tag{12}$$

and the sum of all three mass states

$$\Sigma m_\nu = 6.09 \simeq \times 10^{-2} \text{ eV}/c^2. \tag{13}$$

These predicted values are in excellent agreement with recent global analysis of oscillation experiments [29, 31] for normal mass ordering ($m_{\nu_1} < m_{\nu_2} < m_{\nu_3}$) and the sum is just below cosmological model limits [30]. A comparison of our predicted values to experiments is shown in table I. Remarkably, these equations have the identical integer coefficients (1, 9, 27) as our charged lepton formulas and also satisfy a left handed neutrino sector structure equation similar to (2) for charged leptons. Right handed sterile neutrinos, if the exist do not yet have reliable constraints established from which we could search for similar relationships.

At this point we must ask why the sector residual factors k_e and k_ν only appear as terms in the second generation and if their values are related? While we do not yet have an answer to the first question, the second may be answered by looking at the sum of the lepton residuals

$$k_\ell = k_e + k_\nu \simeq 8. \tag{14}$$

If their sum is exactly 8 we can express the residuals with an angle $\theta_{k_{e\nu}}$ such that

$$k_\nu = 8\cos^2\theta_{k_{e\nu}}, \tag{15}$$

$$k_e = 8\sin^2\theta_{k_{e\nu}}, \tag{16}$$

with $\theta_{k_{e\nu}} \simeq 24.42^\text{o}$. This is smaller than the weak mixing angle $\theta_W \simeq 28.17^\text{o}$ by approximately $\frac{\pi}{48}$ radians. If the relationship is exactly $\theta_W = \theta_{k_{e\nu}} + \frac{\pi}{48}$ then it would be possible to calculate $\sin^2\theta_W = 0.222928(26)$ and $m_W = 80.3834(32)$ MeV, an order of magnitude more precise than previously known.

IV. QUARK SECTOR

We will now attempt to apply the techniques we used relating masses in the lepton sector to quarks. The quark sector residual formulas similar to (2) and (6) are

$$k_d = \frac{m_d m_b^2}{m_s^3}, \tag{17}$$

$$k_u = \frac{m_u m_{\text{top}}^2}{{m_c}^3}. \tag{18}$$

While most attempts to find relationships between leptons and quarks use renormalized running masses we have surprisingly found the best overall results using pole masses for the heavy quarks. The deconfined nature of the top quark has enabled It's mass to be measured through decay products in particle collision experiments. The best fit we have found to the global average m_{top} pole mass using α_f and simple coefficients is

$$m_{\text{top}} = \frac{m_e}{2\pi\alpha_f^3} \simeq 173.72 \text{ GeV/c}^2. \tag{19}$$

Several different analysis modes exists on data from the ongoing top quark experiments at the LHC with a comprehensive summary in [32]. Our predicted pole mass is an excellent match for recent analysis including the 2016 PDG pole mass average of $m_{\text{top}} = 173.5 \pm 1.1$ GeV/c^2 [31]. We also have a particularly close match to the center value of $M_t = 173.7 \pm 1.5 \pm 1.4^{+1.0}_{-0.5}$ reported in [33].

For the bottom and charm quarks we target the pole masses derived from perturbative QCD $\overline{\text{MS}}$ mass analysis from collider experiments. The best formulas we have found using α_f and relatively simple coefficients for the bottom and charm quarks are

$$m_b = \frac{m_e}{\pi^{\frac{1}{2}}\alpha_f^2} \simeq 4.78 \text{ GeV/c}^2, \tag{20}$$

$$m_c = 2\frac{m_e}{k_u^{\frac{1}{3}}\alpha_f^2} \simeq 1.69 \text{ GeV}/c^2. \tag{21}$$

$$k_u = 16 \tag{22}$$

These predicted pole masses are in excellent agreement with recent pole mass evaluations of $m_b = 4.78 \pm 0.06$ GeV and $m_c = 1.67 \pm 0.07$ GeV [31].

For the light quarks perturbative QCD does not provide reliable pole masses so we are left to extrapolate formulas from the heavy quarks. Having proposed formulas for m_{top} and m_c we can extract from (18)

$$m_u = \frac{\pi^2}{2} m_e \simeq 2.52\ \text{MeV}/c^2.$$

While this value is close to the lattice theory derived $\overline{\text{MS}}$ mass commonly quoted $m_u \simeq 2.2\ \text{MeV}/c^2$, we caution against comparing our predicted light quark pole masses to renormalized light quark masses directly. For the down quark we assume a similar structure to our up quark formula and a ratio $m_u/m_d \simeq 0.5$. With that criteria our proposed formulas for the down and strange quarks are

$$m_d = \pi^2 m_e \simeq 5.04\ \text{MeV/c}^2, \tag{23}$$

$$m_s = \frac{\pi^{\frac{1}{3}} m_e}{k_d^{\frac{1}{3}} \alpha_f^{\frac{4}{3}}} \simeq 96.6\ \text{MeV}/c^2, \tag{24}$$

$$k_d = 128. \tag{25}$$

These are also close to the lattice theory $\overline{\text{MS}}$ masses though as mentioned before it is not appropriate to compare them directly. The quark sector residuals can be expressed in terms of an angle $\theta_{k_{ud}} \simeq 19.47^{\text{o}}$ such that

$$k_d = 144 \cos^2\theta_{k_{ud}}, \tag{26}$$

$$k_u = 144 \sin^2\theta_{k_{ud}}. \tag{27}$$

Looking at different combinations of the lepton and quark residuals we find that

$$k_\ell = \frac{k_d}{k_u} = 8, \tag{28}$$

$$\frac{k_u}{k_\ell} = 2. \tag{29}$$

V. COMBINED VIEW

While the preceding formulas in terms of m_e are convenient for calculating precision predicted values it is more interesting to view them together in terms of the Higgs vacuum expectation value $v \simeq 246.22$ GeV [31]. With the top quark Yukawa coupling $y_t = \sqrt{2} m_{\text{top}}/v \simeq 0.99779$ we can rewrite all of our proposed formulas in terms of y_t and v:

$$\begin{aligned}
m_{\nu_1} &= \left(2\pi\alpha_f^7\right)\frac{y_t v}{\sqrt{2}}, & m_e &= \left(2\pi\alpha_f^3\right)\frac{y_t v}{\sqrt{2}}, & m_d &= \left(2\pi^3\alpha_f^3\right)\frac{y_t v}{\sqrt{2}}, & m_u &= \left(\pi^3\alpha_f^3\right)\frac{y_t v}{\sqrt{2}}, \\
m_{\nu_2} &= 9\left(\frac{2\pi\alpha_f^7}{k_\nu^{\frac{1}{3}}}\right)\frac{y_t v}{\sqrt{2}}, & m_\mu &= 9\left(\frac{2\pi\alpha_f^{\frac{7}{3}}}{k_e^{\frac{1}{3}}}\right)\frac{y_t v}{\sqrt{2}}, & m_s &= \left(\frac{2\pi^{\frac{4}{3}}\alpha_f^{\frac{5}{3}}}{k_d^{\frac{1}{3}}}\right)\frac{y_t v}{\sqrt{2}}, & m_c &= \left(\frac{\pi\alpha_f}{k_u^{\frac{1}{3}}}\right)\frac{y_t v}{\sqrt{2}}, \\
m_{\nu_3} &= 27\left(2\pi\alpha_f^7\right)\frac{y_t v}{\sqrt{2}}, & m_\tau &= 27\left(2\pi\alpha_f^2\right)\frac{y_t v}{\sqrt{2}}, & m_b &= \left(2\pi^{\frac{1}{2}}\alpha_f\right)\frac{y_t v}{\sqrt{2}}, & m_{\text{top}} &= \frac{y_t v}{\sqrt{2}}.
\end{aligned} \tag{30}$$

There are many interesting patterns in these expressions, only some of which are enforced by the sector residual equations (2), (6) , (17) and (18). Despite these patterns there does not appear to be any relationship between them and electric charge or other properties besides generation, sector and mass. The unexpected appearance of the sector residuals on only the second generation is puzzling and may indicate these are just approximations of more exact formulas yet to be found. The top quark Higgs Yukawa coupling appearing in every formula also may be an approximation of slightly different values for each fermion. In table I we provide a summary of the predicted masses and parameters derived from these equations.

VI. CONCLUSION

In conclusion, we have presented a novel way to view fermion mass relationships with α_f showing potential as a scaling factor relating the widely differing magnitudes of different masses across all sectors. While the predicted masses are all within experimental bounds many of those bounds are relatively wide. Lower uncertainty of the reference masses would help greatly in validating and further developing this model. The most exciting area of experimental confirmation would be refinement of the neutrino oscillation parameters and especially measurement of the neutrino mass states directly. The KATRIN experiment [35] will search for neutrino mass signatures in de-

	Parameter	THIS WORK	Reference Value	Ref.
Model Inputs	m_e		0.5109989461(31) MeV	CODATA 2014 [34]
	m_μ		105.6583745(24) MeV	CODATA 2014 [34]
	m_τ		1776.86 ± 0.12 MeV	PDG 2017 [31]
	$k_\ell = k_e + k_v$	8 (exact)		
	k_u	16 (exact)		
	k_d	128 (exact)		
Model Parameters	α_f	0.00776481(52)		
	α_f^{-1}	128.7862(87)		
	k_e	1.36777(18)		
	$k_\nu = 8 - k_e$	6.63223(18)		
	$\sin^2\theta_{k_{e\nu}}$	0.170972(23)		
Neutrinos	m_1	$1.85756(50) \times 10^{-3}$ eV		
	m_2	$8.8981(23) \times 10^{-3}$ eV		
	m_3	$5.0154(14) \times 10^{-2}$ eV		
	Δm^2_{21}	$7.5626(39) \times 10^{-5}$ eV^2	$7.53 \pm 0.18 \times 10^{-5}$ eV^2	PDG 2017 [31]
	Δm^2_{31}	$2.5120(14) \times 10^{-3}$ eV^2	$2.524^{+0.038}_{-0.041} \times 10^{-3}$ eV^2	Esteban 2016 [29]
	Δm^2_{32}	$2.4363(13) \times 10^{-3}$ eV^2	$2.45 \pm 0.05 \times 10^{-3}$ eV^2	PDG 2017 [31]
	$\sum m_\nu$	$6.0910(16) \times 10^{-2}$ eV	$< 9.68 \times 10^{-2}$ eV	Giusarma 2016 [30]
Down-type	m_d	5.043357447(31) MeV	$4.7^{+0.5}_{-0.4}$ MeV ($\overline{\text{MS}}$)	PDG 2017 [31]
	m_s	96.5812(87) MeV	96^{+8}_{-4} MeV ($\overline{\text{MS}}$)	PDG 2017 [31]
	m_b	4.78172(65) GeV	4.78 ± 0.06 GeV	PDG 2017 [31]
Up-type	m_u	2.521678724(15) MeV	$2.2^{+0.6}_{-0.4}$ MeV ($\overline{\text{MS}}$)	PDG 2017 [31]
	m_c	1.68173(23) GeV	1.67 ± 0.07 GeV	PDG 2017 [31]
	m_{top}	173.719(35) GeV	173.5 ± 1.1 GeV	PDG 2017 [31]
	y_t	0.99779(20)		
Ratios	m_d/m_e	π^2 (exact)		
	m_u/m_e	$\pi^2/2$ (exact)		
	m_u/m_d	0.5 (exact)	$0.38 - 0.58$	PDG 2017 [31]
	$\overline{m} = (m_u + m_d)/2$	3.782518086(23) MeV	$3.5^{+0.7}_{-0.3}$ MeV	PDG 2017 [31]
	m_s/m_d	19.1502(17)	17-22	PDG 2017 [31]
	$m_s/\overline{m}$	25.5336(23)	27.3 ± 0.7	PDG 2017 [31]
Electroweak	$\sin^2\theta_W$ (if $\theta_W = \theta_{k_{e\nu}} + \frac{\pi}{48}$)	0.222928(26)	0.22290(33)	PDG 2017 [31]
	m_W/m_Z (if $\theta_W = \theta_{k_{e\nu}} + \frac{\pi}{48}$)	0.881517(14)	0.88153(17)	PDG 2017 [31]
	m_W (if $\theta_W = \theta_{k_{e\nu}} + \frac{\pi}{48}$)	80.3834(32) GeV	80.385 ± 0.015 GeV	PDG 2017 [31]

Table I. Predicted parameters, pole masses and derived values.

cay from tritium β-decay but has a minimum sensitivity threshold of 0.2 eV, 100 times higher than our predicted m_1 and 25 times higher than our predicted m_3. Further refinement of the heavy quark experimental masses is another important confirmation opportunity.

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