UNIT 2: The Two Body Problem
What is the Two Body Problem?

The **Two Body Problem (2BP)** is the study of two objects under their mutual gravitation

- Derived using the motion of planets around the Sun
- Is this an accurate model for what satellites experience?
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<table>
<thead>
<tr>
<th>Source</th>
<th>Acceleration (m/s²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Earth Spherical Gravity</td>
<td>8.676</td>
</tr>
<tr>
<td>Oblateness of Earth</td>
<td>~0.0088</td>
</tr>
<tr>
<td>Solar Gravity</td>
<td>~0.0059</td>
</tr>
<tr>
<td>Lunar Gravity</td>
<td>~2.9× 10⁻⁵</td>
</tr>
<tr>
<td>Other Planets</td>
<td>&lt; 3.0× 10⁻⁷</td>
</tr>
</tbody>
</table>

The next largest acceleration on a satellite in LEO is about 1000x less in magnitude, so the 2BP is a pretty good model for how satellites behave!
From ancient times, humans have studied celestial objects, mapped them, named them, and tracked changes.

• The Ancient Babylonians and Greeks proposed mathematical models for the timing of celestial events (such as eclipses) and the relative positions of objects in space.

• Arabic and Persian scholars mapped many stars to naked eye precision in the first millennium CE.

• The Catholic church helped to spur European interest in astronomy beginning in about 1200 CE.
  • They wanted to be able to pin down dates for important times in Scripture (such as Easter).
  • It’s important to note that many scholars in Europe were either beholden to the Church for funds and support, or answered to a King who was beholden to the Church.
After the motions of the Sun and the Moon, ancient astronomers dedicated much of their thought to understanding the motions of the planets (which noticeably move against the background stars)

- In fact “planet” comes from the Ancient Greek terms for “wanderers,” or the “wandering stars”
- When Copernicus revealed his heliocentric model in 1543, it was controversial (particularly with the Church), but well evidenced from observations – extending Aristarchus’ ideas from the 3rd century BCE
- Galileo’s telescope observations of the moons of Jupiter and the phases of Venus supported this heliocentric model for planetary motion
Before Tycho Brahe died in 1601, he had been busy collecting data on Mars’ orbit to support his own geocentric solar system model.

Brahe was an aristocrat who also happened to be a mechanical genius.

He was dedicated and meticulous in his collection of astronomical observations, some of which were 5x more accurate than measurements by others, but was jealous with his data and shared little of it.

He was impressed with Johannes Kepler and hired him as an assistant shortly before he died (about 1600).

Kepler inherited Brahe’s detailed data under disputed circumstances. It was even suggested that Kepler had poisoned his mentor! This was not conclusively ruled out until 2010.
A little bit of history

• Though sources disagree on how Kepler retained possession of Brahe’s measurements, it was probably a good thing for science overall.
• Kepler was a brilliant pure mathematician who was poor and sickly for most of his life.
• He was patient, but unskilled in mechanical tasks such as astronomy measurements.
• Kepler likely would not have been successful without Brahe’s detailed, accurate data, and Brahe would not have been able to bring the same mathematical resources to bear on the problem.
• Additionally, Brahe was trying to prove the wrong thing, so he might not have made the same intuitive leaps.
As a result of Brahe’s data on Mars’ orbit, Kepler discovered relationships that proved conclusively the Copernican worldview, and resulting in his publication of what are now known as **Kepler’s Laws**:

1. Orbit of each planet is an ellipse, with the Sun located at one focus (1609).
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3. The square of a planet’s orbital period is proportional to the cube of its mean distance from the Sun (1619).
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It should be noted that Kepler’s Laws were based solely on observation of the motion of the planets. There were no explanations for why these should be the case!
We owe much of the remaining explanation to Sir Isaac Newton, born Christmas Day, 1642.

- Co-invented calculus, discovered inverse square law for gravity, proved many disputed scientific ideas of the day.
- Demonstrated that the Earth should be oblate – the evidence that later proved it is one of the reasons mathematics won out over philosophy as the domain for physics. Take that Descartes!
- What was Newton’s greatest accomplishment?
Newton developed his Theory of Gravitation, including the inverse square law from Unit 1, in 1666.

Due to discrepancies with his calculations of the Moon’s orbit, he did not publish for 20 years.

One day, over drinks, Edmund Halley and Hooke discussed possibility of gravity acting like magnetism \((1/r^2)\)– they made a bet that it would prove Kepler’s Laws.

Neither of them could figure out the math needed to win the bet.
Theory of Gravitation

• (Not telling him about the bet) Halley asked Newton to weigh in, who claimed to already have a proof.
• Halley convinced Newton that this might be important and that he should publish it.
• Newton published *Philosophiae Naturalis Principia Mathematica* (aka *Principia*) in 1687.
• Contained Newton’s 3 Laws, Theory of Gravitation, hydrostatics, compressible fluids, harmonic motion, theory of optics, etc.
• Assume 2 spherically symmetric masses (gravity treated as point mass toward CG)
• One mass, \( m_1 \), is treated as the “primary mass” for the purposes of relative motion.
• Equation of motion derived in terms of relative position vector \( \mathbf{r} = \mathbf{R}_2 - \mathbf{R}_1 \)
Recap Derivation

• 2BP Equation of Motion – Nonlinear Differential Equation

\[ \ddot{r} + \frac{\mu}{r^3} \mathbf{r} = 0 \]
Recap Derivation

- 2BP Equation of Motion – Nonlinear Differential Equation
  \[ \ddot{r} + \frac{\mu}{r^3} r = 0 \]

- Conservation of Energy (Consequence of EOM)
  \[ \varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \text{constant} \]
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- Conservation of Angular Momentum (EOM)
  \[ \mathbf{h} = \mathbf{r} \times \mathbf{v} = r \mathbf{v}_\perp = \text{constant} \]
Recap Derivation

- **2BP Equation of Motion** – Nonlinear Differential Equation
  \[ \ddot{r} + \frac{\mu}{r^3} r = 0 \]

- **Conservation of Energy** (Consequence of EOM)
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- **Conservation of Angular Momentum** (EOM)
  \[ \mathbf{h} = \mathbf{r} \times \mathbf{v} = r v_{\perp} = \text{constant} \]

- **Solution is the Orbit Equation** (aka Trajectory Equation)
  \[ r = \frac{h^2/\mu}{1 + e \cos \theta} \]
• Equation of a conic section in polar coordinates, measured from one of the foci:

\[ r = \frac{p}{1 + e \cos \theta} \]

• where \( p \) is the conic section focal parameter, \( e \) is the **eccentricity**, and \( \theta \) is an angle measured from the closest approach, known as **true anomaly**.

• \( p \) measures the semi-latus rectum for an ellipse \((e < 1)\)

• In modern parlance, we call \( p \) the **orbit parameter**, or the **semiparameter**
Orbit Equation

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This math formula holds for circles \((e = 0)\), ellipses \((e < 1)\), parabolas \((e = 1)\), and hyperbolas \((e > 1)\)
Orbit Equation

- Equation of a conic section in polar coordinates, measured from one of the foci:
  \[ r = \frac{p}{1 + e \cos \theta} \]
  \[ p = \frac{h^2}{\mu} \]

- where \( p \) is the conic section focal parameter, \( e \) is the **eccentricity**, and \( \theta \) is an angle measured from the closest approach, known as **true anomaly**.
- \( p \) measures the semi-latus rectum for an ellipse (\( e < 1 \))
- In modern parlance, we call \( p \) the **orbit parameter**, or the **semiparameter**

Assuming \( e < 1 \) for the planets, the match between this formula and the trajectory equation proves Kepler’s 1\(^{st}\) Law must be true!
Consequences of our discussion:

1. Trajectory as a function of time **must** be a conic section with eccentricity $e$ and $p = h^2 / \mu$.

2. Body One, the more massive gravity source in the case of a satellite or planet, must lie at the focus of the conic section.

3. Specific Mechanical Energy, $\varepsilon$, is conserved.

4. $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ must be perpendicular to the plane of motion at every instant of time.

5. $\mathbf{h}$ fixed means that this plane must be fixed. The **orbital plane** in which the conic section lies is fixed in inertial space.
Consequences of our discussion (Continued):

6. Angular momentum conserved means that the flight path angle, \( \gamma = \tan^{-1}\left(\frac{v_\perp}{v_r}\right) \), and therefore the trajectory are determined at all points. This means that the position vector and velocity vector are well defined at each point in the orbit given initial data.

7. Orbits (in the case of an ellipse) are periodic. We will return back to the same position/velocity in one orbital period. Thus, position matters when choosing an orbit. As of Engine Cutoff (ECO), our path is ballistic and determined for all future time by the initial conditions of orbital insertion.
Examples

• 1: A body has a speed of 6 km/s at a distance of 10,000 km in its Earth orbit. Determine the energy of the orbit and the speed of the body at a distance of 7,000 km.

• 2: The orbiting body in Example 1 has an initial flight path angle $\gamma_1 = 15.75^\circ$. Determine the flight path angle, the perpendicular component of velocity, and the radial component of velocity at point 2 on the orbit.
Examples

• 3: An Earth orbit has an angular momentum of 59,500 km²/s and an eccentricity of 0.3. Determine the parameter of the orbit and the radial distance when the true anomaly is 120°.

• 4: An Earth orbit has an eccentricity of 0.310. At a true anomaly of 100°, the radial distance from Earth is 200,000 km. Determine the angular momentum of the orbit.
Angular Momentum

FIGURE 2.8 The path of $m_2$ around $m_1$ lies in a plane whose normal is defined by $\mathbf{h}$. 

\[ \hat{\mathbf{h}} = \frac{\mathbf{h}}{h} \]
Velocity Components

\[ v_\perp = r \dot{\theta} \]

\[ h = r v_\perp = r^2 \dot{\theta} \]
Velocity Components

\[ v_\perp = r \dot{\theta} \]

\[ h = rv_\perp = r^2 \dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{h}{r^2} \]

Only connection between geometric orbital parameters \((h, e, a, \theta)\) and time!
Angular Rate gives us the information needed to prove Kepler’s 2\textsuperscript{nd} Law
\[ \dot{\theta} = \frac{h}{r^2} \]

\[
dA = \frac{1}{2} (\text{base})(\text{height})
\]

\[
dA = \frac{1}{2} (r)(v_\perp dt)
\]

\[
dA = \frac{1}{2} (r)(r\dot{\theta} dt)
\]
\[ \dot{\theta} = \frac{h}{r^2} \]

\[ dA = \frac{1}{2} (base)(height) \]

\[ dA = \frac{1}{2} (r)(v_\perp dt) \]

\[ dA = \frac{1}{2} (r)(r \dot{\theta} dt) \]

\[ \Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \left( \frac{h}{r^2} \right) = \frac{h}{2} \]
\[ \dot{\theta} = \frac{h}{r^2} \]

\[ dA = \frac{1}{2} \text{(base)} \times \text{(height)} \]

\[ dA = \frac{1}{2} (r)(v_\perp \, dt) \]

\[ dA = \frac{1}{2} (r)(r \dot{\theta} \, dt) \]

\[ \Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \left( \frac{h}{r^2} \right) = \frac{h}{2} = \text{constant} \]

Proves Kepler’s 2nd Law!
Velocity Components

\[ h = r v_\perp \]

\[ \Rightarrow v_\perp = \frac{h}{r} = h \left[ \frac{1 + e \cos \theta}{h^2/\mu} \right] \]
Velocity Components

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\[ = \frac{\mu}{h} (1 + e \cos \theta) \]
\[ h = rv_\perp \]

\[ \Rightarrow v_\perp = \frac{h}{r} = h \left[ \frac{1 + e \cos \theta}{h^2 / \mu} \right] \]

\[ = \frac{\mu}{h} (1 + e \cos \theta) \]

\[ \Rightarrow v_\perp = \frac{\mu}{h} (1 + e \cos \theta) \]
\[ v_r = \frac{dr}{dt} = \frac{d}{dt} \left[ \frac{h^2/\mu}{1+e \cos \theta} \right] \]

\[ = \frac{h^2}{\mu} \frac{d}{dt} (1 + e \cos \theta)^{-1} \]
Velocity Components

\[ v_r = \frac{dr}{dt} = \frac{d}{dt} \left[ \frac{h^2/\mu}{1+e \cos \theta} \right] \]

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\[ = \frac{h^2 e}{\mu} \left( \frac{h}{r^2} \right) \frac{\sin \theta}{(1 + e \cos \theta)^2} \]
\[
\nu_r = \frac{dr}{dt} = \frac{d}{dt} \left[ \frac{h^2}{\mu} \right] \\
= \frac{h^2}{\mu} \frac{d}{dt} \left( 1 + e \cos \theta \right)^{-1} \\
= \frac{h^2 e}{\mu} \left( \frac{h}{r^2} \right) \frac{\sin \theta}{(1 + e \cos \theta)^2} \\
= \frac{h^3 e \sin \theta}{\mu (1 + e \cos \theta)^2} \left[ \frac{(1 + e \cos \theta)^2}{h^4 / \mu^2} \right]
\]
Velocity Components

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\nu_r = \frac{dr}{dt} = \frac{d}{dt} \left[ \frac{h^2/\mu}{1+e \cos \theta} \right]
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\[\Rightarrow \nu_r = \frac{\mu}{h} e \sin \theta\]
Velocity Components Summary:

\[ v_\perp = \frac{\mu}{h} (1 + e \cos \theta) \]

\[ v_r = \frac{\mu}{h} e \sin \theta \]
• 5: An Earth orbit has an energy of -20.82 km²/s². At one point on the orbit, the radial distance is 8000 km and this distance is increasing at a rate of 712 m/s. Determine the eccentricity of this orbit and the current true anomaly.
• Overall Orbital Velocity:

• Know from energy equation that it should be large as we are close to body and small as we are far away.

\[
\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \text{constant}
\]

• Matches our experience with non-orbital ballistic trajectories.

• Also needed to match Kepler’s 2\textsuperscript{nd} Law.

• Velocity Component calculations are consistent with this expectation:

\[
v = \sqrt{v_{\perp}^2 + v_r^2} = \frac{\mu}{h} \sqrt{1 + 2e \cos \theta + e^2}
\]
\[ \varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \text{constant} \]