UNIT 5B: Hohmann-Style Maneuvers
• Thought Problem: How does one transfer from a 500 km altitude circular Earth orbit to a 10000 km altitude circular orbit in one burn?
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- It’s not possible!
- We need at least two impulsive maneuvers to connect orbits that are not intersecting.
- Transfer orbit must intersect both initial orbit and target orbit.

Orbits relevant to thought problem.
Thought Problem: How does one transfer from a 500 km altitude circular Earth orbit to a 10,000 km altitude circular orbit in one burn?

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- Transfer orbit must intersect both initial orbit and target orbit.

Initial Orbit → burn → Transfer Orbit → burn → Final Orbit
Trajectory Maneuvers

Initial Orbit → burn → Transfer Orbit → burn → Final Orbit

Note: Video is not time-accurate – parameterized by angle.
Transfer Vs. Single Maneuver

- Clearly, in some cases a transfer strategy is always needed.
- Also clear should be the fact that some orbital changes can be conducted with one impulse.
- The general problem facing a trajectory designer is to find the cheapest option.
- Some cases have a known optimal solution.
- In general, we have to apply various “strategies” in search of the best approach.

Possibility of either 1 or 2 impulse transfer.
Transfer Strategies

• We know applying a $\Delta V$ at a point of high speed (periapsis) results in a larger change of energy than the same $\Delta V$ at a point of low speed (apoapsis).

• But, it still matters that the orbit is periodic. **Some transfers cannot be accomplished in a single burn.**

• Often, as in the previous diagram, we need one burn to create a transfer orbit intersecting the target orbit, and one burn at the point of intersection to match the velocity of the target orbit.

• Example: LEO $\rightarrow$ GTO $\rightarrow$ GEO
The most simple 2 impulse transfer strategy is the **Hohmann transfer**.

A Hohmann-style transfer is any approach in which all impulses are applied at points where the position vector is perpendicular to velocity vector (periapsis/apoapsis).

This is the case for the points where burns occur on the initial orbit, the transfer orbit, and the target orbit.
Hohmann Transfer

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• Thus, all burns are solely raising/lowering apoapsis/periapsis!
**Hohmann Transfer**

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*Diagram of the Hohmann Transfer Orbits*

Therefore, all 3 orbits must share an apse line!
Hohmann Transfer

• This can be shown to be the **optimal 2-impulse transfer between circular orbits.**

• All points on the circles would have a perpendicular velocity.

• In this case, the burn points must become the periapse and apoapse for the transfer orbit.
Hohmann Transfer

- Consider a transfer from inner circle to outer circle:
  
  Do we need to speed up or slow down at point A?
Hohmann Transfer

- Consider a transfer from inner circle to outer circle:
  - Do we need to speed up or slow down at point A?
  - SPEED UP!
• Consider a transfer from inner circle to outer circle:
  
  Do we need to speed up or slow down at point A?
  
  SPEED UP!
  
  Do we need to speed up or slow down at point B?
• Consider a transfer from inner circle to outer circle:
  Do we need to speed up or slow down at point A? 
  **SPEED UP!**
  Do we need to speed up or slow down at point B? 
  **SPEED UP!**
Hohmann Transfer

• Consider a transfer from outer circle to inner circle:
  Do we need to speed up or slow down at point A?
Hohmann Transfer

• Consider a transfer from outer circle to inner circle:
  
  Do we need to speed up or slow down at point A?
  
  SLOW DOWN!
Hohmann Transfer

- Consider a transfer from outer circle to inner circle:
  - Do we need to speed up or slow down at point A?
    - SLOW DOWN!
  - Do we need to speed up or slow down at point B?
Hohmann Transfer

- Consider a transfer from outer circle to inner circle:
  - Do we need to speed up or slow down at point A?
    - SLOW DOWN!
  - Do we need to speed up or slow down at point B?
    - SLOW DOWN!

For this case, is the $\Delta V$ negative?

NO! It’s the magnitude that is important! Both transfers use the same $\Delta V$. 
Hohmann Transfer

Relative Time-Accurate Simulation of Hohmann Transfer
**Hohmann Transfer**

- **Characteristics of Hohmann transfer:**
- Since $\Delta V$ is applied tangent to initial and final orbit, these points must match our periapse and apoapse exactly.
- Determines parameters of transfer orbit:

$$
\begin{align*}
 r_p &= r_1 \\
r_a &= r_2 \\
a_T &= \frac{1}{2} (r_a + r_p) \\
e_T &= \frac{r_a - r_p}{r_a + r_p} \\
h_T &= \sqrt{\mu a_T (1 - e_T^2)}
\end{align*}
$$
Hohmann Transfer

• Matching up velocities:

\[ v_{c1} = \sqrt{\frac{\mu}{r_1}} \quad v_{c2} = \sqrt{\frac{\mu}{r_2}} \]

\[ v_p = \frac{h_T}{r_p} \quad v_a = \frac{h_T}{r_a} \]

• Because \( v_p \parallel v_{c1} \quad v_a \parallel v_{c2} \)

\[ \Delta v_1 = \left| v_p - v_{c1} \right| = v_p - v_{c1} \]

\[ \Delta v_2 = \left| v_a - v_{c2} \right| = v_{c2} - v_a \]

\[ \Delta v_T = \Delta v_1 + \Delta v_2 \quad \Delta t_H = \frac{1}{2} T = \frac{\pi}{\sqrt{\mu}} a_T^{3/2} \]

Calculating \( \Delta V \) as a simple magnitude difference

ONLY works for this specialized type of transfer!!!
Example: Consider a Hohmann transfer from a circular 300 km altitude Earth orbit to GEO. Determine:

A) Total required $\Delta V$.
B) Elapsed transfer time.
C) The answers to (A) and (B) for the reverse transfer from GEO to LEO.
Elliptical Transfer (Hohmann)

- To be a Hohmann-type maneuver, burns must occur when $\mathbf{v} \perp \mathbf{r}$ (apoapse or periapse).
- Orbits must therefore share an apse line.
- When either starting orbit or final orbit is an ellipse (or both), we have two options for transfer.
Elliptical Transfer (Hohmann)

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• Orbits must therefore share an apse line.
• When either starting orbit or final orbit is an ellipse (or both), we have two options for transfer.

No way to know a priori which option is cheaper in terms of $\Delta V$, so we have to test both!
Define:
• P1, innermost periapse point.
• P2, outer periapse point.
• A1, apoapse corresponding to P1.
• A2, apoapse corresponding to P2.

Four clear cases:
1) Periapsis points aligned:
   A) Orbits not crossing
   B) Orbits crossing
2) Periapsis points not aligned:
   A) Orbits not crossing
   B) Orbits crossing
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2) Periapsis points not aligned:
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   B) Orbits crossing
Clearly, the optimal result will depend on the relative geometry of the two orbits.

Some common rules of thumb:

1) changing a point from being a periapsis point on one orbit to an apoapsis point on a new orbit (and vice versa) can be expensive.

2) If we can keep periapse and apoapse on the same sides of the focus, that tends to be the more efficient option.

3) Initiating a transfer at periapse (when starting inside) tends to be cheaper.
Always True (strict inequality for orbits not intersecting on apse line and noncircular)

\[ P_1 < P_2 \quad P_1 < A_1 \quad P_2 < A_2 \]

Case 1A

\[ A_1 < A_2 \]

Option 1:

\[ P_1 \rightarrow P_T \quad A_T \rightarrow A_2 \]

Option 2: Exact order depends on if \( A_1 < P_2 \)

\[ A_1 \rightarrow (P_T, A_T) \quad (A_T, P_T) \rightarrow P_2 \]

Option 1 is probably best (possibly all three rules).
Example 1: A launch vehicle releases a satellite with an upper stage into a 250 km orbit. A transfer into a 500x30,000 km orbit is desired. Determine the minimum ΔV required.

Option 1: Raise apogee to 30,000 km altitude from 250 km. Then, raise perigee to 500 km from 250 km.

Option 2: Raise apogee to 500 km from 250 km. Apogee point becomes perigee as apogee is raised to 30,000 km.

Based on previous principles, my guess is option 1 is cheaper.
Example 2: Find the minimum $\Delta V$ required to transfer from orbit 1 to orbit 4 as shown in the figure (all numbers are altitudes above the Earth in km – not drawn to scale). Based on previous principles, my guess is orbit 2, since it begins at a higher energy point (periapse).
When considering non-crossing elliptical orbits with periapsis aligned on the same side of the body, the optimal Hohmann-style transfer intersects the periapsis of the inner orbit.
Always True (strict inequality for orbits not intersecting on apse line and noncircular)

\[ P_1 < P_2 \quad P_1 < A_1 \quad P_2 < A_2 \]

Case 1B

\[ A_2 < A_1 \]

Option 1:

\[ P_1 \rightarrow P_T \quad A_T \rightarrow A_2 \]

Option 2: Exact order is determined in both cases.

\[ A_1 \rightarrow A_T \quad P_T \rightarrow P_2 \]

Option 2 is probably best (largest changes occur at periapse of inner orbit – higher energy).
Elliptical Transfer (Hohmann)

- Example 3: A spacecraft is currently in an Earth orbit with semimajor axis $a_1 = 10,000$ km and eccentricity $e_1 = 0.3$. The vehicle must be transferred to an orbit with $a_2 = 9,200$ km and $e_2 = 0.08$. Perigee is to be maintained on the same side of the body as the current orbit. Calculate the required $\Delta V$ for both Hohmann-style transfer options and determine the minimum cost.
Elliptical Transfer (Hohmann)

When considering crossing orbits with periapsis aligned on the same side of the body, the optimal Hohmann-style transfer intersects the apoapsis of the inner orbit.
• Always True (strict inequality for orbits not intersecting on apse line and noncircular)

\[ P_1 < P_2 \quad P_1 < A_1 \quad P_2 < A_2 \]

• Case 2A

\[ A_1 < P_2 \]

• Option 1:

\[ P_1 \rightarrow P_T \quad A_T \rightarrow P_2 \]

• Option 2: Exact order is determined in both cases.

\[ A_1 \rightarrow P_T \quad A_T \rightarrow A_2 \]

Option 2 is probably best (change from A to P occurs closer – higher energy).
• Example 4: A spacecraft around Mars is in an orbit with a periapsis altitude of 300 km, and a period of 4 hours. It must transfer into an orbit with a period of 12 hours and an eccentricity of 0.3. The apse line is to be rotated by $\pi$ radians (180°), so that periapsis occurs on the opposite side of Mars. Determine the minimum required $\Delta V$ for this transfer using a Hohmann-style maneuver.
Option 2 may be cheaper typically only if $e_1 \lesssim 0.65$ and if $e_2 > e_1$. Otherwise, Option 1 is favorable.

For initial orbits with low eccentricity, a large portion of cases favor Option 2. In fact, if the inner orbit is circular it is always more efficient to intersect the outer orbit at apoapsis.
• Always True (strict inequality for orbits not intersecting on apse line and noncircular)

\[ P_1 < P_2 \quad P_1 < A_1 \quad P_2 < A_2 \]

• Case 2A

\[ A_1 > P_2 \]

• Option 1:

\[ P_1 \rightarrow P_T \quad A_T \rightarrow P_2 \]

• Option 2: Exact order depends on if \( A_1 < A_2 \)

\[ A_1 \rightarrow (P_T, A_T) \quad (A_T, P_T) \rightarrow A_2 \]

Really a toss-up, might lean towards Option 1.
Example 6: Consider the initial and final orbits from Example 4 ($a_1 = 10,000\ km, \ e_1 = 0.3, \ a_2 = 9,200\ km, \ e_2 = 0.08$), but perigee is to switch sides of the body. Therefore, a rotation of the apse line by $\pi$ radians ($180^\circ$) must occur.
For crossing elliptical orbits that are not aligned with periapsis on the same side of the body, the optimal Hohmann-style transfer connects the apoapsis points of the two orbits.
Bi-Elliptic (Hohmann) Transfer

• The Hohmann transfer is known to be the optimal 2-impulse transfer between circular orbits, and is possibly the 2-impulse optimum for coplanar, coaxial ellipses.

• *Can we use more than 2 impulses?*
Bi-Elliptic (Hohmann) Transfer

• The Hohmann transfer is known to be the optimal 2-impulse transfer between circular orbits, and is possibly the 2-impulse optimum for coplanar, coaxial ellipses.

• Can we use more than 2 impulses?
  – YES

• If we do, is the Hohmann transfer still the optimum?
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Can we use more than 2 impulses?

- **YES**

If we do, is the Hohmann transfer still the optimum?

- **IT DEPENDS!**

One potential 3-impulse transfer is the **Bielliptic Maneuver**, or **Hoelker-Silber transfer**.
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• One potential 3-impulse transfer is the Bielliptic Maneuver, or Hoelker-Silber transfer.

Diagram of Bielliptic Maneuver Between Circular Orbits

Initial Orbit → burn → Transfer Orbit 1
The Hohmann transfer is known to be the optimal 2-impulse transfer between circular orbits, and is possibly the 2-impulse optimum for coplanar, coaxial ellipses.

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Diagram of Bielliptic Maneuver Between Circular Orbits

Initial Orbit → burn → Transfer Orbit 1  
→ burn → Transfer Orbit 2
The Hohmann transfer is known to be the optimal 2-impulse transfer between circular orbits, and is possibly the 2-impulse optimum for coplanar, coaxial ellipses.

Can we use more than 2 impulses?  
– YES

If we do, is the Hohmann transfer still the optimum?  
– IT DEPENDS!

One potential 3-impulse transfer is the Bielliptic Maneuver, or Hoelker-Silber transfer.
Bi-Elliptic (Hohmann) Transfer

Initial Orbit → burn → Transfer Orbit 1 → burn → Transfer Orbit 2 → burn → Final Orbit

Note: Video is not time-accurate – parameterized by angle.
It is fairly easy to show that Hohmann transfer is cheaper if

\[ r_2 < r_3 \]

Apoapsis of initial ellipse is chosen to be very large so that the relative velocity needed to raise the other apse is low.

\[ r_1 < r_3 < r_2 \]

The transfer path is the two elliptical arcs connecting the 3 impulses, which is the source of the term “bi-elliptic”
Bi-Elliptic (Hohmann) Transfer

- For first half elliptical-arc:

  \[ r_{p1} = r_1 \quad r_{a1} = r_2 \]

  \[ a_{T1} = \frac{1}{2} (r_{p1} + r_{a1}) \quad e_{T1} = \frac{r_{a1} - r_{p1}}{r_{a1} + r_{p1}} \]

  \[ h_{T1} = \sqrt{\mu a_{T1} (1 - e_{T1}^2)} \]

  \[ v_{p1} = h_{T1} / r_{p1} \quad v_{a1} = h_{T1} / r_{a1} \]

- For second half elliptical-arc:

  \[ r_{p2} = r_3 \quad r_{a2} = r_2 \]

  \[ a_{T2} = \frac{1}{2} (r_{p2} + r_{a2}) \quad e_{T2} = \frac{r_{a2} - r_{p2}}{r_{a2} + r_{p2}} \]

  \[ h_{T2} = \sqrt{\mu a_{T2} (1 - e_{T2}^2)} \]

  \[ v_{p2} = h_{T2} / r_{p2} \quad v_{a2} = h_{T2} / r_{a2} \]

Also

\[ v_{c1} = \sqrt{\mu / r_1} \quad v_{c3} = \sqrt{\mu / r_3} \]
Bi-Elliptic (Hohmann) Transfer

- First Intersection:
  \[ \Delta V_1 = |v_{p1} - v_{c1}| = |v_{p1} - v_{c1}| \]
- Second Intersection:
  \[ \Delta V_2 = |v_{a2} - v_{a1}| = |v_{a2} - v_{a1}| \]
- Third Intersection:
  \[ \Delta V_3 = |v_{c3} - v_{p2}| = |v_{c3} - v_{p2}| \]
- Total Cost:
  \[ \Delta V_T = \Delta V_1 + \Delta V_2 + \Delta V_3 \]

Transfer Time:
\[ \Delta t_B = \frac{1}{2} T_1 + \frac{1}{2} T_2 = \frac{\pi}{\sqrt{\mu}} a_{T1}^{3/2} + \frac{\pi}{\sqrt{\mu}} a_{T2}^{3/2} \]
Example: Compare a Bielliptic transfer to a Hohmann transfer from a 6650 km radius Earth orbit to a 175,000 km radius Earth orbit. Use an intermediate apogee radius of 350,000 km.

a) Determine the ΔV savings.

b) Determine the added flight time to conduct the transfer.
Relative Time-accurate simulation of Bi-elliptic transfer, except inner circular orbit (slowed 30x)
Suppose we have a transfer from two circular orbits of radii $r_1$ and $r_3$

- One option is classical Hohmann maneuver
- Another option is a Bielliptic maneuver with intermediate radius $r_2$

If $\rho = r_3/r_1$ and $\sigma = r_2/r_1$:

$$\Delta V_H = v_{c1} \left[ \rho^{-1/2} + (1 - \rho^{-1}) \sqrt{\frac{2\rho}{1 + \rho}} - 1 \right]$$

**Derivations shown in book!**

$$\Delta V_{BE} = v_{c1} \left[ (1 - \sigma^{-1}) \sqrt{\frac{2\sigma}{1 + \sigma}} + (\rho^{-1} + \sigma^{-1}) \sqrt{\frac{2\rho\sigma}{(\rho + \sigma)}} - (1 + \rho^{-1/2}) \right]$$

**Comparison of $\Delta V_T/\Delta V_H$ for $\sigma < \rho$**
Can show $\Delta V_{BE}$ is a decreasing function of $\sigma$: for constant $\rho$, $\Delta V_{BE} \downarrow$ as $\sigma \uparrow$

The minimum cost is for $\sigma \to \infty$:

$$\Delta V_{\text{min}} = (\sqrt{2} - 1)(1 + \rho^{-1/2})v_{c1}$$

This represents the **biparabolic maneuver**:

$$\Delta V_1 = |v_{\text{esc},1} - v_{c1}| = (\sqrt{2} - 1)v_{c1}$$
$$\Delta V_2 = 0$$
$$\Delta V_3 = |v_{c3} - v_{\text{esc},2}| = (\sqrt{2} - 1)v_{c3} = (\sqrt{2} - 1)\rho^{-1/2}v_{c1}$$

Comparison of $\Delta V_T/\Delta V_H$ for $\sigma > \rho$
• For large enough $\sigma$, the Bi-elliptic maneuver can outperform the Hohmann maneuver as shown.

Critical Value of $\sigma$ for which $\Delta V_{BE} = \Delta V_H$
The first value of $\rho$ for which the best possible Bielliptic maneuver can outperform the Hohmann maneuver is at $\Delta V_{BP} = \Delta V_H$:

$$\rho^3 - (7 + 4\sqrt{2})\rho^2 + (3 + 4\sqrt{2})\rho - 1 = 0 \Rightarrow \rho \approx 11.93876$$

Above this value, if $\sigma$ is chosen over the critical value, Bielliptic is cheaper.
The value of $\rho$ for which any Bielliptic maneuver $\sigma > \rho$ can outperform the Hohmann maneuver is at the maximum cost of the Hohmann maneuver:

$$(d/d\rho)\Delta V_H = 0:
\rho^3 - 15\rho^2 - 9\rho - 1 = 0
\Rightarrow \rho \approx 15.58176$$

Above this value, for any choice of $\sigma$, Bielliptic is cheaper.
• $\rho < 11.93876$: Hohmann transfer is the optimal maneuver;

• $11.93876 < \rho < 15.58176$: A bielliptic transfer can be cheaper than the Hohmann if the intermediate apoapsis is further away than the critical value.

• $\rho > 15.58176$: Any bielliptic transfer is cheaper than the equivalent Hohmann transfer.

Comparison of $\Delta V_T / \Delta V_H$ for $\sigma > \rho$
In Unit 3C, we addressed the possibility of adding more $\Delta V$ than necessary to escape. By adding $\Delta V$ beyond the parabolic speed, we add to $v_\infty$.

Two ways to transfer to hyperbolic orbit:

- We transfer onto a hyperbola with a Hohmann-style transfer, which means the hyperbola must share a periapsis with a point on the apse line of our initial orbit, or
- We transfer onto a hyperbola at any other point, or the apse lines for the orbits are not aligned.
In the first case (Hohmann-style), the periapsis distance would be known since it occurs on the apse line of the initial orbit.

Need one more parameter: we assume $v_\infty$ is known.

\[ v_\infty = \sqrt{-\frac{\mu}{a}} \Rightarrow a = -\frac{\mu}{v_\infty^2} \]

\[ r_p = a(1 - e) \Rightarrow e = 1 - \frac{r_p}{a} \]

\[ e = 1 + \frac{r_p v_\infty^2}{\mu} \]

\[ v_p = \sqrt{v_{esc}^2 + v_\infty^2} = \sqrt{\frac{2\mu}{r_p} + v_\infty^2} \]
Example: A spacecraft departing Earth is in a 185 x 35,786 km altitude Geostationary Transfer Orbit (GTO). The spacecraft is to depart Earth directly from this orbit and achieve a $C_3$ of 16.672 km$^2$/s$^2$. Determine the required $\Delta V$ and the eccentricity of the departure orbit.