Unit 2 Practice Problems

1. A satellite is injected into orbit at an altitude of 487 km, with a total vehicle speed of 9.6 km/s. If the altitude is increasing at a rate of 932 m/s, determine
   a) The specific mechanical energy of the orbit;
   b) The eccentricity of the orbit;

\[
\begin{align*}
\mu &= 3.986 \times 10^5 \text{ km}^3/\text{s}^2 \\
R_E &= 6378 \text{ km} \\
\varepsilon &= 487 \text{ km} \\
V &= 9.6 \text{ km/s}
\end{align*}
\]

\[
\begin{align*}
V_r &= 0.932 \text{ km/s} \\
r &= R_E + \varepsilon = 6865 \text{ km}
\end{align*}
\]

\[\alpha = \frac{\varepsilon^2}{2} - \frac{\mu}{r} = -11.98 \text{ km}^3/\text{s}^2 \quad \Rightarrow \quad \varepsilon \approx -11.98 \text{ km}^3/\text{s}^2\]

\[\begin{align*}
\varepsilon &= \frac{-\mu}{2a} \Rightarrow a = \frac{-\mu}{2\varepsilon} = 16632.4 \text{ km}
\end{align*}\]

\[\begin{align*}
\varepsilon &= V_r^2 + V_\perp^2 \Rightarrow V_\perp = \sqrt{\varepsilon - V_r^2} = 9.55465 \text{ km/s} \\
\ell &= rV_\perp = 65.5927 \text{ km/s}
\end{align*}\]

\[\begin{align*}
\frac{\beta}{\mu} &= a(1 - \varepsilon^2) \Rightarrow \varepsilon = \left[1 - \frac{\beta^2}{\mu a}\right]^{1/2} = 0.59249 \quad \Rightarrow \quad \varepsilon \approx 0.592
\end{align*}\]

2. Consider an object at an altitude of 600 km above the surface of the Earth. Assume the Earth to be a sphere of radius 6378 km.
   a) If the object followed a path that maintained a constant altitude, determine the radial acceleration required to maintain a tangential speed of \(v\). (HINT: what is the acceleration on a path of constant radius of curvature, \(r\), and constant speed, \(v\)?)
   b) What speed, \(v\), is required such that this radial acceleration exactly balances the gravitational acceleration at this altitude?

\[\begin{align*}
a &= \frac{v^2}{r}
\end{align*}\]

\[\begin{align*}
\alpha &= \frac{GM_E}{r^2} \Rightarrow \frac{v^2}{r} &= \frac{GM_E}{r^2} \Rightarrow v^2 &= \frac{GM_E}{r} \Rightarrow v = \sqrt{\frac{GM_E}{r}}
\end{align*}\]

\[\begin{align*}
\Rightarrow v &= \sqrt{\frac{M_E}{r}} \Rightarrow v = \sqrt{\frac{3.986 \times 10^5 \text{ km}^3/\text{s}^2}{\left(6378 \text{ km} + 600 \text{ km}\right)}} = 7.5579 \text{ km/s}
\end{align*}\]

\[\Rightarrow v \approx 7.56 \text{ km/s}\]
3. The position and velocity of an Earth satellite at a given instant are

\[ \mathbf{r} = 2.5\mathbf{I} + 2.5\mathbf{J} + 3.75\mathbf{K} \text{ (Distance Units)} \]
\[ \mathbf{v} = -0.5\mathbf{I} + 0.25\mathbf{J} + 0.5\mathbf{K} \text{ (Distance Units per Time Unit)} \]

Where \( \mathbf{IJK} \) is a nonrotating geocentric coordinate system. Find the specific angular momentum relative to the origin of \( \mathbf{IJK} \) and the specific total mechanical energy of the satellite. Assume that \( \mu = 1 \) (DU\(^3\)/TU\(^2\))

\[
\mathbf{L} = \mathbf{r} \times \mathbf{p} = \begin{vmatrix} \mathbf{I} & \mathbf{J} & \mathbf{K} \\ 2.5 & 2.5 & 3.75 \\ -0.5 & 0.25 & 0.5 \end{vmatrix} \text{ DU/TU} = 0.3125\mathbf{I} - 3.125\mathbf{J} + 1.875\mathbf{K} \text{ DU}\(^2\)/TU
\]

\[ \mathbf{v} = \mathbf{L}/m = 0.3125\mathbf{I} - 3.125\mathbf{J} + 1.875\mathbf{K} \text{ DU}/TU
\]

\[ E = \frac{v^2}{2} - \frac{\mu}{r} = \frac{(0.75 \text{ DU}/\text{m})^2}{2} - \frac{(1 \text{ DU}^3/\text{m}^2 \text{ DU}/\text{m})}{(5.1539 \text{ DU})} = 8.72 \times 10^{-2} \text{ DU}^2/\text{m}^2
\]

4. Two particles of identical mass \( m \) are acted on only by the gravitational force of one upon the other. If the distance \( d \) between the particles is constant, show that the angular velocity of the line joining them is \( \omega = \sqrt{2Gm/d^3} \).
5. After engine cut-off (ECO) a spacecraft has the following position and velocity vectors:

\[
\mathbf{r} = [-6131.1 \quad 4309.2 \quad 303.8]^T \text{ km} \\
\mathbf{v} = [-7.6156 \quad -5.5785 \quad 0.9036]^T \text{ km/s}
\]

a) **Determine the specific angular momentum, \( h \), and the specific mechanical energy, \( \epsilon \), of the resulting orbit.**
   - Using the definition of the specific angular momentum vector, \( \mathbf{h} \), as the cross product of the state vectors, we find its magnitude \( h = |\mathbf{h}| \). We also compute the magnitudes of \( \mathbf{r} \) and \( \mathbf{v} \) to get \( r \) and \( v \), respectively. The energy equation (2.8) can then be used to compute specific mechanical energy, \( \epsilon \).
   Answer: \( h = 67,329 \text{ km}^2/\text{s} \), \( \epsilon = -8.18 \text{ km}^2/\text{s}^2 \)

b) **Determine the radial velocity, \( v_r \), and the perpendicular velocity, \( v_\perp \), components.**
   - The radial velocity is the dot product of the two state vectors divided by the distance \( r \).
   The perpendicular component can then be found using the overall velocity magnitude.
   Answer: \( v_r = 3.06 \text{ km/s} \), \( v_\perp = 8.98 \text{ km/s} \)

c) **Determine the eccentricity, \( e \), and the true anomaly, \( \theta \), at this point in the orbit.**
   - Several strategies would work. One is to use the same approach as Example 2.5.
   Answer: \( \theta = 0.785 \text{ rad} (45.0^\circ) \), \( e = 0.7303 \)

6. For a certain Earth satellite, the observed velocity and radius at a true anomaly of \( \theta = \pi/3 \) radians are 5.610 km/s and 14,602 km, respectively. Find the eccentricity of the orbit for this satellite.
   Answer: \( e = 0.2329 \)