Unit 3 Practice Problems

1. A satellite launch resulted in the following observations: Perigee altitude of 350 km, and orbit period of 162.5 minutes. **Determine:**
   a) The eccentricity of the resulting orbit,
   b) The apogee altitude, and
   c) The speeds at perigee and apogee.

\[
\begin{align*}
M &= 3.986 \times 10^5 \text{ km}^3/\text{s}^2 \quad R_E = 6378 \text{ km} \\
\varepsilon_p &= 350 \text{ km} \quad T = 162.5 \text{ min} = 9750 \text{ s} \\
R_p &= a(1 - e) \Rightarrow e = 1 - \frac{R_p}{a} = 0.31794 \Rightarrow e \approx 0.318 \\
r_a &= \sqrt{\frac{GM}{2\pi}} = 18,100 \text{ km} \quad z_a = R_a - R_e = 6622.4 \text{ km} \Rightarrow z \approx 6.62 \times 10^3 \text{ km} \\
h &= \sqrt{\frac{GMa(1-e^2)}{2}} = 59,450.9 \text{ km}^2/\text{s} \\
v_p &= h/r_p = 8.83636 \text{ km/s} \quad v_a = h/r_a = 4.57301 \text{ km/s} \\
\Rightarrow v_p \approx 8.84 \text{ km/s} \quad v_a \approx 4.57 \text{ km/s} \\
\end{align*}
\]

2. A satellite is injected into orbit at an altitude of 487 km, with a total vehicle speed of 9.6 km/s. If the altitude is increasing at a rate of 932 m/s, **determine the altitude at apogee** of the orbit.

\[
\begin{align*}
M &= 3.986 \times 10^5 \text{ km}^3/\text{s}^2 \quad R_E = 6378 \text{ km} \quad z = 487 \text{ km} \quad v = 9.6 \text{ km/s} \\
V_r &= 0.932 \text{ km/s} \quad r = R_e + z = 6865 \text{ km} \\
\epsilon &= \frac{V^2}{2} - \frac{M}{r} = -11,98244 \text{ km}^2/\text{s} \quad \epsilon = \frac{-M}{2a} \Rightarrow a = \frac{-M}{2\epsilon} = 16632.4 \text{ km} \\
V^2 &= V_r^2 + V_z^2 \Rightarrow v_z = \sqrt{v^2 - v_r^2} = 9.55465 \text{ km/s} \Rightarrow h = rv_z = 65,592.7 \text{ km/s} \\
\frac{h}{\rho} &= a(1 - \epsilon^2) \Rightarrow e = \left[1 - \frac{h^2}{\rho a}\right]^{1/2} = 0.597299 \Rightarrow e \approx 0.592 \\
\end{align*}
\]
3. For a certain satellite, the velocity magnitude and radial distance from the Earth are reported as 8.385 km/s and 16,387.5 km at \( \theta = 90^\circ \). **Determine:**

a) The **eccentricity** of the satellite

b) The **angle between the apse line and asymptote**, \( \beta \)

c) The **hyperbolic excess velocity**

\[ \mu = 3.986 \times 10^5 \text{ km}^3/\text{s}^2 \]

\[ V = 8.385 \text{ km/s}, \quad r = 16,387.5 \text{ km}, \quad \theta = 90^\circ \]

\[ \cos \theta = 0 \Rightarrow r = \frac{h^2}{\mu} = \frac{h}{M} \Rightarrow h = \sqrt{\mu r} = 80821.1 \frac{\text{km}^2}{\text{s}} \]

\[ \varepsilon = \frac{v^2}{2} = \frac{M}{2a} \Rightarrow \varepsilon = 10.8307 \text{ km}^2/\text{s}^2 \]

\[ a = \frac{-M}{2 \varepsilon} = -19,401.4 \text{ km}, \quad \frac{h^2}{M} = a (1 - \varepsilon^2) \Rightarrow \varepsilon = \left[ 1 - \frac{h^2}{Ma} \right]^{1/2} = 1.37498 \]

\[ \Rightarrow \varepsilon \approx 1.375 \]

\[ \beta = \cos^{-1} \left( \frac{1}{\varepsilon} \right) = 0.75644 \text{ rad (43.341°)} \Rightarrow \beta \approx 0.756 \text{ rad (43.3°)} \]

\[ V_\infty = \sqrt{\frac{\mu}{a}} = 4.65418 \text{ km/s} \Rightarrow V_\infty \approx 4.65 \text{ km/s} \]
4. A lunar satellite is in a 200 x 7,000 km orbit. **Determine the new perilune and apolune altitudes** if a $\Delta V$ of 100 m/s is applied along the velocity direction at:

a) Perilune;

b) Apolune;

\[
\mu = 4.903 \text{ km}^3/\text{s}^2 \quad R_E = 1737 \text{ km}
\]

\[
R_p = R_E + 200 \text{ km} = 1937 \text{ km} \quad R_a = R_E + 7000 \text{ km} = 8737 \text{ km}
\]

\[
q = \frac{1}{2} \left( R_p + R_a \right) = 5337 \text{ km} \quad e = \frac{R_a - R_p}{R_a + R_p} = 0.63706
\]

\[
h = \sqrt{\mu a \left( 1 - e^2 \right)} = 3943.0 \text{ km/s}^2
\]

\[
V_p = \frac{h}{v_p} = 2.03563 \text{ km/s} \quad V_a = \frac{h}{v_a} = 0.45130 \text{ km/s}
\]

a) **NEW ORBIT** : \( r = R_p \quad V = V_p + 0.1 \text{ km/s} = 2.13563 \text{ km/s} \)

\( V \) is still > \( V_c \) so this point is still **PERILUNE ON THE NEW ORBIT** because \( r \perp V \) still, \( \Rightarrow V_r = 0 \).

\[
\epsilon = \frac{V^2}{2} - \frac{\mu}{r_p} = -0.25078 \text{ km}^2/\text{s}^2 \quad \epsilon = \frac{\mu}{2a} \implies a = \frac{\mu}{2\epsilon} = 9775.6 \text{ km}
\]

\[
V_p = a(1 - e) \implies e = 1 - \frac{V_p}{a} = 0.80185 \quad V_a = a(1 + e) = 17614.2 \text{ km}
\]

\[
Z_a = \left| \frac{v_a - v_E}{v_a} \right| = 15877.2 \text{ km} \quad \Rightarrow \left| Z_p \approx 200 \text{ km} \right| \left| Z_a \approx 15900 \text{ km} \right|
\]

b) **NEW ORBIT** : \( r = R_a \quad V = V_a + 0.1 \text{ km/s} = 0.55130 \text{ km/s} < V_c \text{ (0.749 km/s)} \)

**SO THIS IS STILL APOLUNE** because \( r \perp V \) still.

\[
\epsilon = \frac{V^2}{2} - \frac{\mu}{r_a} = -0.4821 \text{ km}^2/\text{s}^2 \implies a = \frac{\mu}{2\epsilon} = 5990.8 \text{ km}
\]

\[
\epsilon = \frac{v_a^2}{2} - \frac{\mu}{r_a} \implies v_a = a(1 + e) \implies e = \frac{v_a}{a} - 1 = 0.45840
\]

\[
R_p = a(1 - e) = 3244.6 \text{ km} \quad \Rightarrow Z_p = R_p - R_E = 1507.6 \text{ km}
\]

\[\Rightarrow Z_p \approx 1510 \text{ km} \quad Z_a \approx 7000 \text{ km} \]
5. A satellite is in an orbit around Earth with a perigee radius of 8500 km and an apogee radius of 32,250 km. A $\Delta V$ of 700 m/s is applied at a true anomaly of 45°. **Determine the new semimajor axis, eccentricity, angular momentum, and mechanical energy** if the $\Delta V$ is applied along a vector:

a) 10° above the local horizontal, and
b) in the current velocity direction.

\[ \gamma_\theta = \alpha \left( \frac{r_p + r_a}{r_p r_a} \right) = 20.375 \text{ km} \]
\[ e = \frac{r_a - r_p}{r_p + r_a} = 0.58282 \]
\[ M = 3.986 	imes 10^5 \text{ km}^3 \s^{-2} \]
\[ h = \sqrt{M a (1 - e^2)} = 7.3230.9 \text{ km} \s^{-1} \]
\[ r = \frac{h^2}{M} \left( 1 + e \cos \theta \right) = 9.527.53 \text{ km} \]

**\[ \gamma = e \left( \frac{h}{2M} \right) = 9.527.53 \text{ km} \]**

\[ \Rightarrow V = \left[ \left( \frac{M - \Delta M}{2\pi} \right) \gamma \right] = 8.00688 \text{ km} \s^{-1} \]
\[ h = r V \cos \gamma \Rightarrow \gamma = \cos^{-1} \left( \frac{h}{r V} \right) = 16.869° \]
\[ = 0.2839 \text{ rad} \]

**b) $\Delta V$ in the current velocity direction.**

\[ \Rightarrow V_+ = V \cos \gamma = 7.686.24 \text{ km} \s^{-1} \]
\[ V_r = V \sin \gamma = 2.24318 \text{ km} \s^{-1} \]

\[ V_{x z} = V_1 + \Delta V \cos 10° = 8.37561 \text{ km} \s^{-1} \]
\[ V_{y z} = V_r + \Delta V \sin 10° = 2.36473 \text{ km} \s^{-1} \]

\[ \eta_z = V V_{z z} = 7.9798.8 \text{ km}^2 \s^{-1} \]
\[ \xi_z = \frac{V z^2 - M}{r} = -3.96530 \text{ km}^2 \s^{-1} \]
\[ a_z = \frac{-M}{2 \xi_z} = 50.241.0 \text{ km} \]

\[ \theta_z = \tan^{-1} \left( \frac{\xi_z}{\eta_z} \right) = \tan^{-1} \left( \frac{V z^2 h_z}{V z^2 \eta_z} \right) = \tan^{-1} \left( \frac{V r^2 h_z}{4M h_z - 1} \right) = \tan^{-1} \left( \frac{V r^2}{V r^2 h_z - \frac{M}{h_z}} \right) = 0.61040 \text{ rad} (35.0°) \]

\[ e_x = \frac{r \sin \theta}{\cos \theta} = \frac{V r^2 h_z}{M \sin \theta \partial_z} = 0.8256 \]

\[ \Rightarrow \eta_z \simeq 50.300 \text{ km} \quad \xi_z \simeq 0.820 \]
\[ h_z \simeq 7.9800 \text{ km} \s^{-1} \quad \eta_z \simeq -3.97 \text{ km}^2 \s^{-1} \]
6. **Determine the altitudes** of the following circular orbits:
   a) **An Earth orbit with a period of 2 sidereal days**;
      - The period is converted to seconds using the definition of a sidereal day. The period formula (3.5) is inverted to find the radius, \( r \). The altitude is computed by subtracting the equatorial radius of Earth, 6378.1 km.
      Answer: 60,553.3 km

   b) **A lunar orbit with a speed of 430 m/s**;
      - The speed is converted to km/s and the circular velocity (3.4) is used to compute the radius. The altitude is computed subtracting the lunar radius, 1737.4 km.
      Answer: 24,779.6 km

   c) **A Mars orbit with a period of 7 hours**.
      - The period is converted to seconds. The period formula is inverted to obtain radius, the altitude is computed subtracting the Mars radius, 3396.2 km.
      Answer: 5,435.7 km

7. **Determine the periods** of the following circular orbits:
   a) **An Earth orbit with an energy of -21.43 km²/s²**;
      - The energy is a direct function of orbital radius, so (3.6) can be solved for radius. This radius can then be applied in the period formula (3.5).
      Answer: 8,925.6 s or 2 hr 28' 45.6''

   b) **A lunar orbit with an altitude of 110 km**;
      - The altitude can be added to the lunar radius of 1737.4 km, and the period computed.
      Answer: 7,125.1 s or 1 hr 58’ 45.1’’

   c) **A Jupiter orbit with an angular momentum of 9.516E6 km²/s**;
      - The angular momentum is related to radius through (3.1), from which the period follows.
      Answer: 337,211.9 s or 3 days 21 hr 40’ 11.9’’

   d) **A Pluto orbit with a speed of 591 m/s**.
      - The circular velocity is converted to km/s and used to compute radius using (3.4). Then, the period can be computed from radius.
      Answer: 26,510.0 s or 7 hr 21’ 50.0’’
8. A satellite is injected into orbit at an altitude of 500 km, with a total vehicle speed of 7.911 km/s. If the altitude is increasing at a rate of 664 m/s, determine
   a) **The specific mechanical energy of the orbit;**
      - Several strategies work. One is as follows: Altitude is converted to radius at the point of injection. The mechanical energy can be computed directly from the energy equation (3.17), knowing radius and velocity magnitude.
      Answer: -26.66 km²/s²
   b) **The eccentricity of the orbit;**
      - The radial velocity and velocity magnitude are used to compute perpendicular velocity. The angular momentum is computed from radius and perpendicular velocity. Semimajor axis is computed from the energy using (3.17). Eccentricity can then be computed from (3.11).
      Answer: 0.1157
   c) **The period of the orbit;**
      - The period is computed from the semimajor axis using (3.15).
      Answer: 6,432.5 s or 1 hr 47’ 12.5’’

9. For a certain Earth satellite the observed velocity at a true anomaly of $\theta = \frac{2\pi}{3}$ radians is 4.488 km/s. The eccentricity, $e$, of the elliptical orbit for this satellite is known to be 0.2. **Find the specific angular momentum, $h$, semimajor axis, $a$, and current position radius, $r$, for this orbit.**
   - The velocity component equations (2.23) and (2.24) can be applied together to get an expression for the velocity magnitude in terms of angular momentum, $h$, eccentricity, $e$, and true anomaly, $\theta$. The only unknown is specific angular momentum, which can then be computed. Once angular momentum is known, the perpendicular velocity component can be computed directly. Then, radial distance can be found from the definition of angular momentum. The semimajor axis can be computed using the energy equation (3.17) since distance and velocity magnitude are known.
   Answer: $h = 81,399.9$ km²/s $a = 17,315.7$ km $r = 18,470.1$ km

10. The period of revolution for a lunar satellite is 400 minutes. **Find the apoapsis altitude and speed at apoapsis** if the periapsis altitude is 135 km.
   - The period is converted to seconds, and then the semimajor axis can be computed from the period formula (3.15). The periapsis altitude is converted to radius by adding the lunar radius, 1737.4 km. The eccentricity is calculated from the periapsis radius and the semimajor axis using (3.9). The apoapsis radius is computed using (3.10), and converted to an altitude by subtracting the lunar radius. The angular momentum is computed from (3.11). The velocity at apoapsis is then computed from the angular momentum, dividing by radius at apoapsis:
   Answer: $z_a = 4,692.6$ km $v_a = 0.586$ km/s
11. A spacecraft establishing an orbit around Jupiter has a periapsis altitude of 34,000 km and a period of 266 days. **Determine the velocity difference at periapse compared to the local escape speed.**
   - The periapsis altitude is converted to radius by adding Jupiter’s equatorial radius, 71,492 km. The semimajor axis is computed from the period formula. The eccentricity is computed from the periapsis radius and semimajor axis using (3.9). The angular momentum is computed from (3.11). The velocity at periapsis is computed from angular momentum and the periapsis radius. The local escape velocity is computed from (3.19) and compared.
   Answers: $v_p = 48.9$ km/s  $\Delta V = 0.1085$ km/s

12. A spacecraft approaches Uranus with an aiming radius of 100,000 km and a relative speed, $v_\infty$, of 6.514 km/s. **Determine**
   a) **The altitude of closest approach**
      - The constant $a$ can be computed from the excess velocity using (3.28). The eccentricity can then be computed from the aiming radius using (3.27). The periapsis radius can then be computed using (3.9) and converted to altitude by subtracting the radius of Uranus, 25,559 km.
      Answer: 7,140.3 km
   b) **The eccentricity of the hyperbolic orbit**
      - Computed in the process for part (a). Answer: 1.239
   c) **The turning angle of the orbit**
      - The turning angle can be computed from (3.25) using eccentricity.
      Answer: 1.877 rad or 107.6°
   d) **The difference in velocity at periapsis compared to escape speed**
      - The speed at periapsis can be computed using (3.30), and compared to the escape speed.
      Answer: $v_p = 19.9$ km/s  $\Delta V = 1.095$ km/s

13. A spacecraft is delivered to a 200x25,000 km lunar orbit. At apolune, a 0.1 km/s $\Delta V$ is applied along a flight path angle of 10° relative to the local horizontal. **Determine the new period and eccentricity of the orbit after the burn.**
   - The periapsis and apoapsis altitudes given are converted to radii. The radii are used to compute the orbital parameters, including angular momentum. The velocity at apolune is computed, which lies in the horizontal direction. Since the flight path angle of the impulse added is nonzero, both the tangential and radial velocities are recomputed using the added $\Delta V$. From here, there are several methods to proceed. One easy way is to compute the new mechanical energy since the velocity magnitude has changed, and use the energy equation to compute the new semimajor axis. The period is then found using the period formula. The radial distance (the old apolune) and the tangential velocity can be used to compute angular momentum. From angular momentum and semimajor axis, the eccentricity is obtained.
   Answer: $T = 186,567.1$ s or 2 days 3 hr 49’ 27.1’’,  $e = 0.6434$. 