EXAMPLE 2.12-1

Express the following L.P. problem in standard form :

$$\begin{aligned} \text{Maximize} \quad & Z = 7x_1 + 5x_2, \\ \text{subject to} \quad & 2x_1 + 3x_2 \le 20, \\ & 3x_1 + x_2 \ge 10, \\ & x_1, x_2 \ge 0. \end{aligned}$$

Solution. Introducing slack and surplus variables, the problem can be expressed in standard form as

$$\begin{aligned} \text{maximize} \quad & Z = 7x_1 + 5x_2, \\ \text{subject to} \quad & 2x_1 + 3x_2 + s_1 = 20, \\ & 3x_1 + x_2 - s_2 = 10, \\ & x_1, x_2, s_1, s_2 \ge 0. \end{aligned}$$

EXAMPLE 2.12-2

Express the following linear programming problem in the standard form :

$$\begin{aligned} \text{Maximize} \quad & Z = 3x_1 + 2x_2 + 5x_3, \\ \text{subject to} \quad & 2x_1 - 3x_2 \le 3, \\ & x_1 + 2x_2 + 3x_3 \ge 5, \\ & 3x_1 + 2x_3 \le 2, \\ & x_1 \ge 0, x_2 \ge 0. \end{aligned}$$

Solution. Here x_1 and x_2 are restricted to be non-negative, while x_3 is unrestricted.

Let us express x_3 as $x_3' - x_3''$, where $x_3' \ge 0$ and $x_3'' \ge 0$. Thus the above problem can be written as

$$\begin{aligned} \text{maximize} \quad & Z = 3x_1 + 2x_2 + 5x_3' - 5x_3'', \\ \text{subject to} \quad & 2x_1 - 3x_2 \le 3, \\ & x_1 + 2x_2 + 3x_3' - 3x_3'' \ge 5, \\ & 3x_1 + 2x_3' - 2x_3'' \le 2, \\ \text{where} \quad & x_1 \ge 0, x_2 \ge 0, x_3' \ge 0, x_3'' \ge 0. \end{aligned}$$

Introducing slack variables, the standard form is

$$\begin{aligned} \text{maximize} \quad & Z = 3x_1 + 2x_2 + 5x_3' - 5x_3'', \\ \text{subject to} \quad & 2x_1 - 3x_2 + s_1 = 3, \\ & x_1 + 2x_2 + 3x_3' - 3x_3'' - s_2 = 5, \\ & 3x_1 + 2x_3' - 2x_3'' + s_3 = 2, \end{aligned}$$

where $x_1 \ge 0$, $x_2 \ge 0$, $x_3' \ge 0$, $x_3'' \ge 0$, $s_1 \ge 0$, $s_2 \ge 0$ and $s_3 \ge 0$.

EXAMPLE 2.12-3

Express the following linear programming problem in the standard form :

Determine x_1, x_2, x_3 so as to

$$\begin{aligned} \text{maximize} \quad & Z = 3x_1 + 2x_2 + 5x_3, \\ \text{subject to} \quad & 2x_1 + 3x_2 - 2x_3 \le 40, \\ & 4x_1 - 2x_2 + x_3 \le 24, \\ & x_1 - 5x_2 - 6x_3 \ge 2, \\ & x_1 \ge 0. \end{aligned}$$

[P.T.U. B. Tech. (C.Sc.) 2010]

Solution. Here only x_1 is restricted to be non-negative, while x_2 and x_3 are unrestricted. Let us express

$$\begin{aligned} & x_1 \text{ as } = y_1, \text{ where } y_1 \ge 0, \\ & x_2 = y_2 - y_3, \text{ where } y_2, y_3 \ge 0, \\ \text{and} \quad & x_3 = y_4 - y_5, \text{where } y_4, y_5 \ge 0. \end{aligned}$$

Thus the given problem can be written as

$$\begin{aligned} \text{maximize} \quad & Z = 3y_1 + 2y_2 - 2y_3 + 5y_4 - 5y_5, \\ \text{subject to} \quad & 2y_1 + 3y_2 - 3y_3 - 2y_4 + 2y_5 \le 40, \\ & 4y_1 - 2y_2 + 2y_3 + y_4 - y_5 \le 24, \\ & y_1 - 5y_2 + 5y_3 - 6y_4 + 6y_5 \ge 2, \\ \text{where} \quad & y_1, y_2, y_3, y_4, y_5, \text{ all } \ge 0. \end{aligned}$$

Introducing the slack variables, the standard form is

$$\begin{aligned} \text{maximize} \quad & Z = 3y_1 + 2y_2 - 2y_3 + 5y_4 - 5y_5, \\ \text{subject to} \quad & 2y_1 + 3y_2 - 3y_3 - 2y_4 + 2y_5 + s_1 = 40, \\ & 4y_1 - 2y_2 + 2y_3 + y_4 - y_5 + s_2 = 24, \\ & y_1 - 5y_2 + 5y_3 - 6y_4 + 6y_5 - s_3 = 2, \\ \text{where} \quad & y_1, y_2, y_3, y_4, y_5, s_1, s_2, s_3, \text{ all } \ge 0. \end{aligned}$$

EXAMPLE 2.12-4

Reformulate the problem into standard form :

$$\begin{aligned} \textit{Minimize} \quad & Z = 2x_1 + 3x_2, \\ \textit{subject to} \quad & 2x_1 - 3x_2 - x_3 = -4, \\ & 3x_1 + 4x_2 - x_4 = -6, \\ & 2x_1 + 5x_2 + x_5 = 10, \\ & 4x_1 - 3x_2 + x_6 = 18, \\ \textit{where} \quad & x_3, x_4, x_5, x_6 \textit{ all} \ge 0. \end{aligned}$$

Solution. Here x_3, x_4, x_5, x_6 (which are all non-negative) are the slack variables. The decision variables x_1, x_2 are unrestricted in sign.

Putting $x_1 = y_1 - y_2$, $x_2 = y_3 - y_4$, $x_3 = y_5$, $x_4 = y_6$, $x_5 = y_7$ and $x_6 = y_8$, the problem in standard form is

$$\begin{aligned} \text{minimize} \quad & Z = 2y_1 - 2y_2 + 3y_3 - 3y_4, \\ \text{subject to} \quad & -2y_1 + 2y_2 + 3y_3 - 3y_4 + y_5 = 4, \\ & -3y_1 + 3y_2 - 4y_3 + 4y_4 + y_6 = 6, \\ & 2y_1 - 2y_2 + 5y_3 - 5y_4 + y_7 = 10, \\ & 4y_1 - 4y_2 - 3y_3 + 3y_4 + y_8 = 18, \\ \text{where} \quad & y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, \text{ all } \ge 0. \end{aligned}$$

Remark. When $x_1 = y_1 - y_2$, it can be seen that for any value of x_1, there will be an infinite number of combinations of (y_1, y_2) which satisfy this equation. However, if values of y_1 and y_2 are given, there will be only one value of x_1. Therefore, if an optimal solution to the new problem is obtained which contains specific values of $y_1, y_2, \ldots$ the corresponding unique values of $x_1, x_2, \ldots$ will also give an optimal solution for the given problem. Thus an optimal solution to the new problem is also an optimal solution to the original problem.

EXAMPLE 2.12-5

Express the following L.P. problem in standard form :

$$\begin{aligned} &\text{Maximize} && Z = 3x_1 + 5x_2 - 2x_3, \\ &\text{subject to} && x_1 + 2x_2 - x_3 \geq -4, \\ &&& -5x_1 + 6x_2 + 7x_3 \geq 5, \\ &&& 2x_1 + x_2 + 3x_3 = 10, \\ &&& x_1, x_2 \geq 0, \; x_3 \text{ unrestricted in sign.} \end{aligned}$$

Solution. Here only x_3 is unrestricted variable. Let us express $x_3 = x_4 - x_5$, where $x_4 \geq 0$, $x_5 \geq 0$. Thus the problem can be expressed as

$$\begin{aligned} &\text{maximize} && Z = 3x_1 + 5x_2 + 2x_4 - 2x_5, \\ &\text{subject to} && x_1 + 2x_2 - x_4 + x_5 \geq -4, \\ &&& -5x_1 + 6x_2 + 7x_4 - 7x_5 \geq 5, \\ &&& 2x_1 + x_2 + 3x_4 - 3x_5 = 10, \\ &&& x_1, x_2, x_4, x_5 \geq 0. \end{aligned}$$

Multiplying both sides of the first constraint by – 1, it takes the form

$$-x_1 - 2x_2 + x_4 - x_5 \leq 4.$$

Adding slack variable s_1 to this constraint and subtracting slack variable s_2 from the second constraint, the problem can be expressed in standard form as

$$\begin{aligned} &\text{maximize} && Z = 3x_1 + 5x_2 + 2x_4 - 2x_5, \\ &\text{subject to} && -x_1 - 2x_2 + x_4 - x_5 + s_1 = 4, \\ &&& -5x_1 + 6x_2 + 7x_4 - 7x_5 - s_2 = 5, \\ &&& 2x_1 + x_2 + 3x_4 - 3x_5 = 10, \\ &&& x_1, x_2, x_4, x_5, s_1, s_2 \geq 0. \end{aligned}$$

EXAMPLE 2.12-6

Express the following L.P. problem in the standard matrix form :

$$\begin{aligned} &\text{Maximize} && Z = 4x_1 + 2x_2 + 6x_3, \\ &\text{subject to} && 2x_1 + 3x_2 + 2x_3 \geq 6, \\ &&& 3x_1 + 4x_2 = 8, \\ &&& 6x_1 - 4x_2 + x_3 \leq 10, \\ &&& x_1, x_2, x_3 \geq 0. \end{aligned}$$

Solution. The problem can be expressed in the standard form and then represented in the matrix form as follows :

$$\begin{aligned} &\text{maximize} && Z = 4x_1 + 2x_2 + 6x_3 + 0x_4 + 0x_5, \\ &&& \quad = (4, 2, 6, 0, 0)\,[x_1, x_2, x_3, x_4, x_5], \\ &\text{subject to} && 2x_1 + 3x_2 + 2x_3 - x_4 = 6, \\ &&& 3x_1 + 4x_2 = 8, \\ &&& 6x_1 - 4x_2 + x_3 + x_5 = 10, \\ &&& x_1, x_2, \ldots, x_5 \geq 0. \end{aligned}$$

These constraints can be expressed as

$$\begin{pmatrix} 2 & 3 & 2 & -1 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 6 & -4 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ 10 \end{pmatrix}.$$

2.14 SOME IMPORTANT DEFINITIONS

Consider the general linear programming problem involving n variables and m constraints $(m \leq n)$:

Determine the values of variables $x_1, x_2, ..., x_n$ which

$$\begin{aligned} \text{maximize} \quad & Z = c_1x_1 + c_2x_2 + ... + c_n x_n, \\ \text{subject to} \quad & a_{11} x_1 + a_{12} x_2 + ... + a_{1n} x_n \leq b_1, \\ & a_{21} x_1 + a_{22} x_2 + ... + a_{2n} x_n \leq b_2, \\ & \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn} x_n \leq b_m, \\ \text{where} \quad & x_1, x_2, ..., x_n \geq 0. \end{aligned}$$

Introducing slack variables $x_{n+1}, x_{n+2}, ..., x_{n+m}$ in the constraints, it can be put in the following standard form :

$$\text{maximize} \quad Z = c_1x_1 + c_2x_2 + ... + c_n x_n, \qquad ... (2.13)$$

$$\text{subject to} \quad \left.\begin{aligned} & a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + x_{n+1} = b_1, \\ & a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n + x_{n+2} = b_2, \\ & \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n + x_{n+m} = b_m, \end{aligned}\right\} \qquad ... (2.14)$$

$$\text{where} \quad x_1, x_2, ..., x_n, x_{n+1}, x_{n+2}, ..., x_{n+m} \geq 0. \qquad ... (2.15)$$

1. *Solution*: A set of variables $[x_1, x_2, ..., x_{n+m}]$ is called a solution to L.P. problem if it satisfies the constraints (2.14).

2. *Feasible solution*: A set of variables $[x_1, x_2, ..., x_{n+m}]$ is called a feasible solution to L.P. problem if it satisfies the constraints (2.14) as well as non-negativity restrictions (2.15).

3. *Basic solution*: A solution obtained by setting any n variables (among $m + n$ variables) equal to zero and solving for remaining m variables (provided the determinant of the coefficients of these m variables is non-zero) is called a *basic solution*. These m variables (some of them may be zero) are called *basic variables* and the remaining n variables that have been put equal to zero each are called *non-basic variables*.

The number of basic solutions thus obtained will at most be ${}^{m+n}c_m = \dfrac{(m+n)!}{m!\,n!}$, which is the number of combinations of $m + n$ things taken m at a time.

4. *Basic feasible solution*: It is a basic solution that also satisfies the non-negativity restrictions (2.15). All variables in a basic feasible solution are ≥ 0. Every basic feasible solution of a problem is an extreme point of the convex set of feasible solutions and every extreme point is a basic feasible solution of the set of constraints.

5. *Non-degenerate basic feasible solution*: It is a basic feasible solution in which all the m basic variables are positive (> 0) and the remaining n variables are zero each.

6. *Degenerate basic feasible solution*: It is a basic feasible solution in which one or more of the m basic variables are equal to zero.

7. *Optimal basic feasible solution*: It is the basic feasible solution that also optimizes the objective function (2.13).

8. *Unbounded solution*: If the value of the objective function can be increased or decreased indefinitely, the solution is called unbounded solution.

Unless otherwise stated, solution means a feasible solution.

EXAMPLE 2.15-1

Solve example 2.9-1 by the trial and error method.

Solution

Linear programming model for the problem is

maximize $Z = 3x_1 + 4x_2$,

subject to $x_1 + x_2 \leq 450$,

$2x_1 + x_2 \leq 600$,

where $x_1, x_2 \geq 0$.

The problem is first expressed in standard form by the introduction of slack variables s_1 and s_2 as shown below :

maximize $Z = 3x_1 + 4x_2 + 0s_1 + 0s_2$,

subject to $x_1 + x_2 + s_1 = 450$,

$2x_1 + x_2 + s_2 = 600$,

$x_1, x_2, s_1, s_2 \geq 0$.

The slack variables represent dummy (fictitious or imaginary) products with zero profit per unit. Here total number of variables, $m + n = 4$ and number of constraints, $m = 2$. A basic solution can be obtained by setting any of the n (2) variables equal to zero and then solving the constraint equations. The total number of basic solutions will be

$$^4C_2 = \frac{4!}{2!(4-2)!} = \frac{4 \times 3 \times 2 \times 1}{(2 \times 1) \times (2 \times 1)} = 6.$$

Table 2.17 gives a summary of the characteristics of the various basic solutions.

S.No. of the basic solution	*Basic variables*	*Non-basic variables*	*Values of the basic variables given by the constraint equations*	*Value of the objective function*	*Is the solution feasible? (are all $x_j \geq 0$?)*	*Is the solution non-degenerate (are all basic variables > 0?)*	*Is the solution feasible and optimal?*
1	x_1, x_2	s_1, s_2	$x_1 + x_2 = 450$ $2x_1 + x_2 = 600$ $\therefore x_1 = 150, x_2 = 300$	1,650	Yes	Yes	No
2	x_1, s_1	x_2, s_2	$x_1 + s_1 = 450$ $2x_1 = 600$ $\therefore x_1 = 300, s_1 = 150$	900	Yes	Yes	No
3	x_1, s_2	x_2, s_1	$x_1 = 450$ $2x_1 + s_2 = 600$ $\therefore x_1 = 450, s_2 = -300$	—	No	No	No
4	x_2, s_1	x_1, s_2	$x_2 + s_1 = 450$ $x_2 = 600$ $\therefore x_1 = -150$	—	No	No	No
5	x_2, s_2	x_1, s_1	$x_2 = 450$ $x_2 + s_2 = 600$ $\therefore s_2 = 150$	1,800	Yes	Yes	Yes
6	s_1, s_2	x_1, x_2	$s_1 = 450, s_2 = 600$	0	Yes	Yes	No

Out of these solutions, solutions in which all basic variables x_j are ≥ 0 will be basic feasible; solutions in which all basic variables are > 0 will be non-degenerate basic feasible and the basic feasible solution that optimizes the objective function will be the optimal basic feasible solution. It may be seen that solutions 1, 2, 5 and 6 are basic feasible; they are also non-degenerate. Out of these, solution 5 gives the maximum value of the objective function Z and is, therefore, the optimal solution. Solutions 3 and 4 are infeasible and are to be discarded from consideration. Thus the optimal solution to the problem is

$x_1 = 0,\ x_2 = 450$; Z_{max} = ₹ 1,800.

EXAMPLE 2.15-2

Find all the basic solutions to the following problem :

$$\text{Maximize} \quad Z = x_1 + 3x_2 + 3x_3,$$
$$\text{subject to} \quad x_1 + 2x_2 + 3x_3 = 4,$$
$$2x_1 + 3x_2 + 5x_3 = 7.$$

Also find which of the basic solutions are

(i) basic feasible,

(ii) non-degenerate basic feasible, and

(iii) optimal basic feasible.

Solution. Since $m + n = 3$ and $m = 2$ in this problem, a basic solution can be obtained by setting any of the n variables equal to zero and then solving the constraint equations. The total number of basic solutions is

$$^3C_2 = \frac{3!}{2!1!} = 3.$$

Out of these solutions, solutions in which all basic variables (x_j) are ≥ 0 will be basic feasible; solutions in which all basic variables are > 0 will be non-degenerate basic feasible and the basic feasible solution that optimizes the objective function will be the optimal basic feasible solution. Table 2.18 gives a summary of the characteristics of the various basic solutions.

S.No. of the basic solution	*Basic variables*	*Non-basic variables*	*Values of the basic variables given by the constraint equations*	*Value of the objective function*	*Is the solution feasible? (are all x_j ≥ 0?)*	*Is the solution non-degenerate (are all basic variables > 0?)*	*Is the solution feasible and optimal?*
1	x_1, x_2	x_3	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$ $\therefore\ x_1 = 2, x_2 = 1$	5	Yes	Yes	Yes
2	x_1, x_3	x_2	$x_1 + 3x_3 = 4$ $2x_1 + 5x_3 = 7$ $\therefore\ x_1 = 1, x_3 = 1$	4	Yes	Yes	No
3	x_2, x_3	x_1	$2x_2 + 3x_3 = 4$ $3x_2 + 5x_3 = 7$ $\therefore\ x_2 = -1, x_3 = 2$	—	No	No	No

It may be seen that the first two solutions are basic feasible; they are also non-degenerate basic feasible solutions. The first solution, of course, is the optimal one.

Thus the optimal solution is $x_1 = 2,\ x_2 = 1,\ x_3 = 0$ with $Z_{max} = 5$.

EXAMPLE 2.15-3

A firm manufactures four different machine parts M_1, M_2, M_3 and M_4 made of copper and zinc. The amounts of copper and zinc required for each machine part, their exact availability and the profit earned from one unit of each machine part are as follows :

	M_1 (kg)	M_2 (kg)	M_3 (kg)	M_4 (kg)	Exact availability (kg)
Copper	5	4	2	1	100
Zinc	2	3	8	1	75
Profit (₹)	12	8	14	10	

How many of each part be manufactured to maximize profit ? For this problem find

(i) basic solutions,

(ii) basic feasible solutions,

(iii) non-degenerate basic feasible solutions, and

(iv) optimal basic feasible solution. [P.U.B.E. (Elect.) 1996]

Solution. Let x_1, x_2, x_3 and x_4 represent the quantities to be manufactured of machine parts M_1, M_2, M_3 and M_4 respectively. Then the linear programming problem is

maximize $Z = 12x_1 + 8x_2 + 14x_3 + 10x_4$,

subject to $5x_1 + 4x_2 + 2x_3 + x_4 = 100$,

$2x_1 + 3x_2 + 8x_3 + x_4 = 75$,

where x_1, x_2, x_3, x_4, all ≥ 0.

Here $m + n = 4$ and $m = 2$. A basic solution can be obtained by setting any of the ($n = 2$) non-basic variables equal to zero and then solving the constraint (containing the basic variables) equations. The total number of basic solutions is

$$^4C_2 = \frac{4!}{2!\,2!} = 6\,.$$

S.No. of the basic solution	Basic variables	Non-basic variables	Values of the basic variables given by the constraint equations	Value of the objective function	Is the solution feasible? (are all $x_j \geq 0$?)	Is the solution non-degenerate? (are all basic variables > 0?)	Is the solution feasible and optimal?
1	x_1, x_2	x_3, x_4	$5x_1 + 4x_2 = 100$ $2x_1 + 3x_2 = 75$ $\therefore\ x_1 = 0,\ x_2 = 25$	200	Yes	No	No
2	x_1, x_3	x_2, x_4	$5x_1 + 2x_3 = 100$ $2x_1 + 8x_3 = 75$ $\therefore\ x_1 = 325/18$ $x_3 = 175/36$	5,125/18	Yes	Yes	No
3	x_1, x_4	x_2, x_3	$5x_1 + x_4 = 100$ $2x_1 + x_4 = 75$ $\therefore\ x_1 = 25/3,$ $x_4 = 175/3$	2,050/3	Yes	Yes	Yes
			$4x_2 + 2x_3 = 100$				

4	x_2, x_3	x_1, x_4	$3x_2 + 8x_3 = 75$ $\therefore$ $x_2 = 25, x_3 = 0$ $4x_2 + x_4 = 100$	200	Yes	No	No
5	x_2, x_4	x_1, x_3	$3x_2 + x_4 = 75$ $\therefore$ $x_2 = 25, x_4 = 0$ $2x_3 + x_4 = 100$	200	Yes	No	No
6	x_3, x_4	x_1, x_2	$8x_3 + x_4 = 75$ $\therefore$ $x_3 = -25/6,$ $x_4 = 325/3$	—	No	No	No

From the table the following inferences can be drawn:

1. Basic solutions are no. 1, 2, 3, 4, 5 and 6.
2. Basic feasible solutions are no. 1, 2, 3, 4 and 5.
3. Non-degenerate basic feasible solutions are no. 2 and 3.
4. Optimal basic feasible solution is no. 3, which gives $x_1 = 25/3$, $x_2 = 0$, $x_3 = 0$, $x_4 = 175/3$ and $Z_{max} = 2{,}050/3$.

This trial and error method, however, suffers from the following inefficiencies :

1. In linear programming problems where m and n are large, solving numerous sets of simultaneous equations would be extremely cumbersome and time-consuming.

2. Scanning the profit column (value of objective function) in table 2.19, we find that its value changes from 200 to 5,125/18 to 2050/3 to 200 *i.e.*, there are ups and downs. There is need for a method that would ensure that successive solutions yield higher profit, culminating into the optimal one.

3. Some of the sets yield infeasible solutions. There should be means to detect such sets and not to solve them at all.

The simplex method, to be discussed soon, overcomes all these drawbacks.

EXAMPLE 2.15-4

Show, using matrix-vector notation, that the following system of linear equations has degenerate solutions :

$$2x_1 + x_2 - x_3 = 2,$$
$$3x_1 + 2x_2 + x_3 = 3.$$

[*Madurai B.Sc. (Math.) 1984*]

Solution. The given system of equations can be written as

$$\boldsymbol{Ax} = \boldsymbol{b},$$

where $$\boldsymbol{A} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Since $\boldsymbol{A}$ is of the order 2×3, there can be $^3C_2 = \frac{3!}{2!1!} = 3$ submatrices of the order 2×2.

They are

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}.$$

Any one of them can be taken as our basis matrix **B**. The variables not associated with the columns of these submatrices are respectively x_3, x_1 and x_2.

Considering $$\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix},$$

a basic solution to the given system is obtained by setting $x_3 = 0$ and solving the system

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

or $$2x_1 + x_2 = 2,$$
$$3x_1 + 2x_2 = 3,$$

which gives the basic solution to the problem :

$x_1 = 1, x_2 = 0$ (basic); $x_3 = 0$ (non-basic).

Similarly, considering the other two submatrices, we get solutions :

$x_2 = 5/3, x_3 = -1/3$ (basic); $x_1 = 0$ (non-basic) and

$x_1 = 1, x_3 = 0$ (basic); $x_2 = 0$ (non-basic).

Since in two of these basic feasible solutions, one basic variable is zero, they are degenerate solutions. The second solution [0, 5/3, – 1/3] is not feasible. Thus the remaining two degenerate basic feasible solutions are

(1, 0, 0) and (1, 0, 0).

EXAMPLE 2.15-5

Find all the basic feasible solutions of the equations

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3,$$
$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2.$$

[*Madurai B.Sc. (Appl. Math.) 1984; Delhi B.Sc. (Math.) 1983*]

Solution. The given system of equations can be presented in the following matrix form :

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b},$$

where $$\mathbf{A} = \begin{pmatrix} 2 & 6 & 2 & 1 \\ 6 & 4 & 4 & 6 \end{pmatrix}, \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Since **A** is of order 2×4, we can take any of the following $^4C_2 = \frac{4!}{2!2!} = 6$, 2×2

submatrices as our basis matrix **B**:

$$\begin{pmatrix} 2 & 6 \\ 6 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 6 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 6 & 6 \end{pmatrix}, \begin{pmatrix} 6 & 2 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 6 & 1 \\ 4 & 6 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 1 \\ 4 & 6 \end{pmatrix}.$$

The variables not associated with the first submatrix are x_3 and x_4. The basic solution to the given system is obtained by setting $x_3 = 0$, $x_4 = 0$ and solving the system.

$$\begin{pmatrix} 2 & 6 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Solving this system of equations, a basic solution to the given system of equations is

$x_1 = 0$, $x_2 = 1/2$ (basic); $x_3 = x_4 = 0$ (non-basic).

Similarly, the other five solutions are

$x_1 = -2$, $x_3 = 7/2$ (basic); $x_2 = x_4 = 0$ (non-basic),

$x_1 = 8/3$, $x_4 = -7/3$ (basic); $x_2 = x_3 = 0$ (non-basic),

$x_2 = 1/2$, $x_3 = 0$ (basic); $x_1 = x_4 = 0$ (non-basic),

$x_2 = 1/2$, $x_4 = 0$ (basic); $x_1 = x_3 = 0$ (non-basic),

$x_3 = 2$, $x_4 = -1$ (basic); $x_1 = x_2 = 0$ (non-basic).

Out of the these six solutions, solutions $(-2, 0, 7/2, 0)$, $(8/3, 0, 0, -7/3)$ and $(0, 0, 2, -1)$ are not feasible. Also in each of the remaining three basic feasible solutions, at least one of the basic variables is zero. Hence the degenerate basic feasible solutions of the given system are

$(0, 1/2, 0, 0)$, $(0, 1/2, 0, 0)$ and $(0, 1/2, 0, 0)$.

EXAMPLE 2.15-6

Is the following solution

$$x_1 = 1,\ x_2 = \frac{1}{2},\ x_3 = x_4 = x_5 = 0$$

a basic solution of the equations

$$x_1 + 2x_2 + x_3 + x_4 = 2,$$

$$x_1 + 2x_2 + \frac{1}{2}x_3 + x_5 = 2\ ?$$ [*Madurai B.Sc. (Appl. Math.) 1983*]

Solution

The given system of equations can be presented in the following matrix form :

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}, \text{ where } \quad \mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 1 & 0 \\ 1 & 2 & \frac{1}{2} & 0 & 1 \end{pmatrix},\ \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

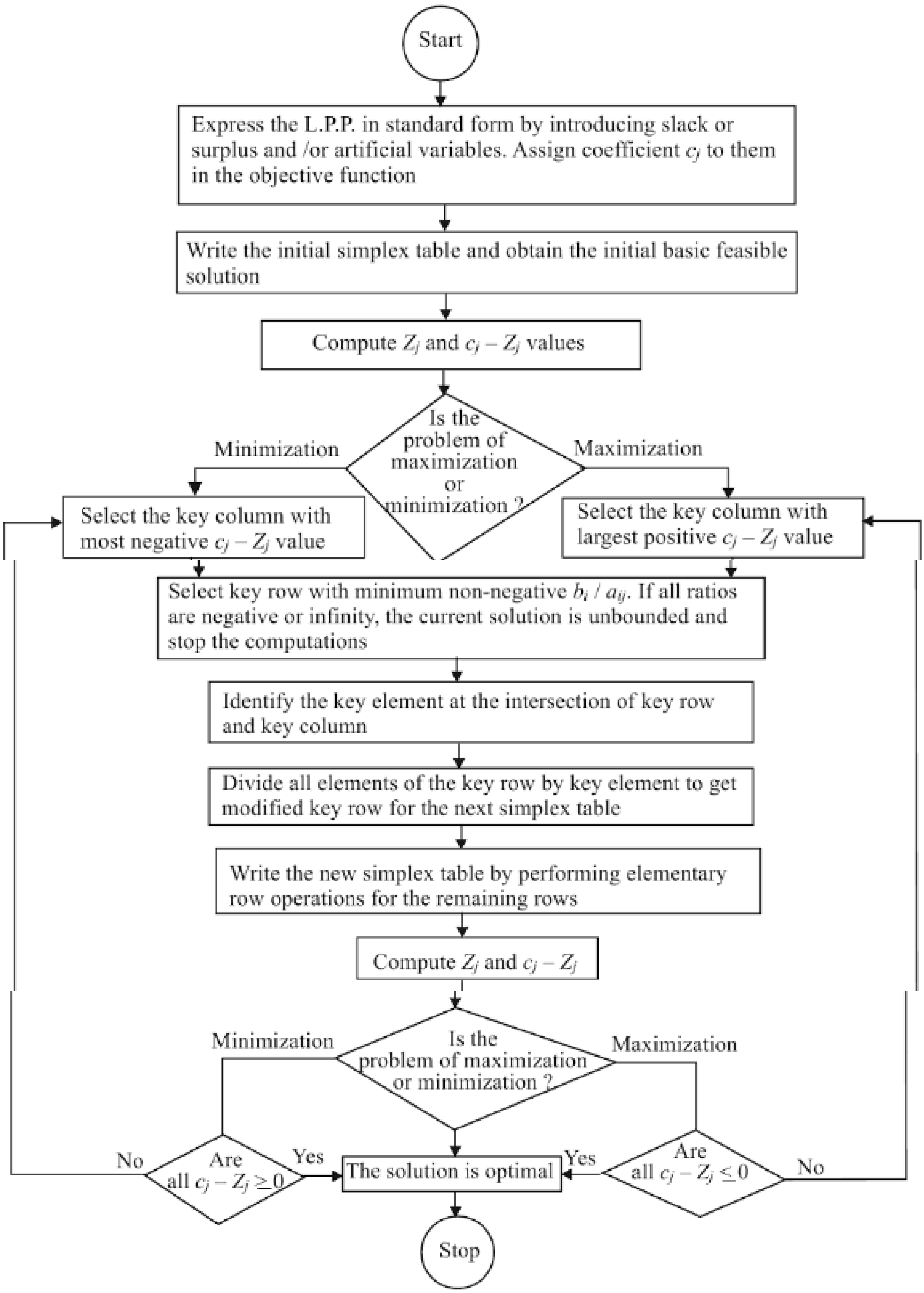
Since **A** is of order 2 × 5, we can take any of the following ${}^5C_2 = \frac{5!}{3!\,2!} = 10, 2 \times 2$ submatrices as our basis matrix **B** :

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Out of these ten submatrices, the first submatrix may be dropped from consideration. Considering the remaining submatrices, the solutions are

$x_1 = 2$, $x_3 = 0$ (basic); $x_2 = x_4 = x_5 = 0$ (non-basic),
$x_1 = 2$, $x_4 = 0$ (basic); $x_2 = x_3 = x_5 = 0$ (non-basic),
$x_1 = 2$, $x_5 = 0$ (basic); $x_2 = x_3 = x_4 = 0$ (non-basic),
$x_2 = 1$, $x_3 = 0$ (basic); $x_1 = x_4 = x_5 = 0$ (non-basic),
$x_2 = 1$, $x_4 = 0$ (basic); $x_1 = x_3 = x_5 = 0$ (non-basic),
$x_2 = 1$, $x_5 = 0$ (basic); $x_1 = x_3 = x_4 = 0$ (non-basic),
$x_3 = 4$, $x_4 = -2$ (basic); $x_1 = x_2 = x_5 = 0$ (non-basic),
$x_3 = 2$, $x_5 = 1$ (basic); $x_1 = x_2 = x_4 = 0$ (non-basic),
$x_4 = 2$, $x_5 = 2$ (basic); $x_1 = x_2 = x_3 = 0$ (non-basic).

Clearly none of these corresponds to the given solution, *viz.*, $x_1 = 1$, $x_2 = 1/2$, $x_3 = x_4 = x_5 = 0$. Hence the given solution is not basic.


Start
Express the L.P.P. in standard form by introducing slack or surplus and /or artificial variables. Assign coefficient c_j to them in the objective function
Write the initial simplex table and obtain the initial basic feasible solution
Compute Z_j and $c_j - Z_j$ values
Is the problem of maximization or minimization ?
Minimization
Maximization
Select the key column with most negative $c_j - Z_j$ value
Select the key column with largest positive $c_j - Z_j$ value
Select key row with minimum non-negative b_i / a_{ij}. If all ratios are negative or infinity, the current solution is unbounded and stop the computations
Identify the key element at the intersection of key row and key column
Divide all elements of the key row by key element to get modified key row for the next simplex table
Write the new simplex table by performing elementary row operations for the remaining rows
Compute Z_j and $c_j - Z_j$
Is the problem of maximization or minimization ?
Minimization
Maximization
Are all $c_j - Z_j \geq 0$
Are all $c_j - Z_j \leq 0$
No
Yes
The solution is optimal
Yes
No
Stop

EXAMPLE 2.16-2

Use simplex method to solve the following problem :

$$\text{Maximize } Z = 2x_1 + 5x_2,$$
$$\text{subject to} \quad x_1 + 4x_2 \leq 24,$$
$$3x_1 + x_2 \leq 21,$$
$$x_1 + x_2 \leq 9,$$
$$x_1, x_2 \geq 0.$$

Solution. It consists of the following steps (for details refer to example 2.16-1) :

Step 1. Express the problem in standard form

Introducing slack variables s_1, s_2 and s_3 the problem can be expressed in the following standard form :

$$\text{Maximize } Z = 2x_1 + 5x_2 + 0s_1 + 0s_2 + 0s_3,$$
$$\text{subject to} \quad x_1 + 4x_2 + s_1 = 24,$$
$$3x_1 + x_2 + s_2 = 21,$$
$$x_1 + x_2 + s_3 = 9,$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0.$$

The slack variables can be treated as imaginary products, contributing zero profits. Accordingly, they are assigned zero coefficients in the objective function.

Step 2. Find initial basic feasible solution

We shall start with a basic solution which we shall get by assuming that profit earned is zero. This will be so when non-basic variables x_1 and x_2 are each equal to zero. Setting $x_1 = 0$, $x_2 = 0$, the constraints yield the following initial basic feasible solution (*i. b. f. s.*):

$$s_1 = 24, s_2 = 21, s_3 = 9 \text{ and } Z = 0.$$

The above information can be expressed in the form of a table, called simplex table (table 2.23). The non-basic variables x_1 and x_2 are each zero. If any of them is made positive, Z will increase. This can be achieved by changing the basis of table 2.23 *i.e.*, by including x_1, x_2 in place of some basic variables (s_1, s_2 or s_3), which form the present basis.

TABLE 2.23

		c_j	2	5	0	0	0			
F.R.	c_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ	
	0	s_1	1	(4)	1	0	0	24	6	←
1/4	0	s_2	3	1	0	1	0	21	21	
1/4	0	s_3	1	1	0	0	1	9	9	
		Z_j	0	0	0	0	0	0		
		c_j-Z_j	2	5	0	0	0			
				↑K			*Initial basic feasible solution*			

Step 3. Perform optimality test

By performing the optimality test we can find whether the current feasible solution can be improved or not. This is done by computing c_j-Z_j, where $Z_j = \sum c_B a_{ij}$. Here a_{ij} is the matrix element in the ith row and jth column. If c_j-Z_j is positive under any column, the current feasible solution is not optimal and at least one better solution is possible. This is shown in table 2.23. Since c_j-Z_j is positive under x_1 and x_2-columns, *i.b.f.s.* is not optimal and can be improved.

Step 4. Iterate towards an optimal solution

Mark the key column, key row and key element as shown in table 2.23. x_2-column is the key column, s_1-row is the key row and (4) is the key element. x_2 is the incoming variable which replaces the outgoing variable s_1 in the next table (table 2.24). The incoming basic variable should appear *only* in the first (key) row with unit coefficient. Therefore, key element (4) is made 1 in that table. For this elements of s_1-row of table 2.23 are divided by 4 and written as elements of x_2-row in table 2.24. The intersectional elements 1 and 1 of key column x_2 are now made zero each in table 2.24. For this, first the elements of key row in table 2.23 are multiplied by a *proper multiple* (also called *fixed ratio*) and then are *subtracted* from elements of s_2-row. Proper multiple or fixed ratio (F.R.) is always equal to intersectional element divided by key element. This is repeated for s_3-row as well. The fixed ratios $\frac{1}{4}$, $\frac{1}{4}$ are entered in the first column of table 2.23 against s_2-row and s_3-row. These row operations lead to the following elements of s_2-row and s_3-row of table 2.24.

$$s_2\text{-row}: 3-\frac{1}{4} = \frac{11}{4},\ 1-\frac{4}{4}=0,\ 0-\frac{1}{4}=-\frac{1}{4},\ 1-\frac{0}{4}=1,\ 0-\frac{0}{4}=0,\ 21-\frac{24}{4}=15;$$

$$s_3\text{-row}: 1-\frac{1}{4}=\frac{3}{4},\ 1-\frac{4}{4}=0,\ 0-\frac{1}{4}=-\frac{1}{4},\ 0-\frac{0}{4}=0,\ 1-\frac{0}{4}=1,\ 9-\frac{24}{4}=3\ .$$

		c_j	2	5	0	0	0		
F.R.	c_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
$1/3$	5	x_2	$1/4$	1	$1/4$	0	0	6	24
$11/3$	0	s_2	$11/4$	0	$-1/4$	1	0	15	$60/11$
	0	s_3	$(3/4)$	0	$-1/4$	0	1	3	4 ←
		Z_j	$5/4$	5	$5/4$	0	0	30	
		c_j-Z_j	$3/4$	0	$-5/4$	0	0	*Second feasible solution*	
			↑K						

Step 5. Check second feasible solution for optimality

Z_j values and c_j-Z_j values for various variable-columns are calculated in table 2.24. Since value under x_1-column is positive, the second feasible solution is not optimal.

Step 6. Iterate towards an optimal solution

x_1-column is marked as the key column. x_1 is the incoming variable. Replacement ratios are $\frac{6}{1/4} = 24$, $\frac{15}{11/4} = \frac{60}{11}$, $\frac{3}{3/4} = 4$. Since 4 is the minimum non-negative ratio, s_3-row is marked as the key row. s_3 is the outgoing variable. It is replaced by x_1 in table 2.25. Elements of s_3-row in table 2.24 are multiplied by 4/3 to make the key element 1 in table 2.25 and the values are entered as the elements of x_1-row in this table. Next, the intersectional elements $\frac{1}{4}, \frac{11}{4}$ of x_1-column in table 2.24 are to be made zeros in table 2.25. To make $\frac{1}{4}$ as 0, elements of s_3-row of table 2.24 are multiplied by the fixed ratio $\frac{1/4}{3/4} = \frac{1}{3}$ and the values are then subtracted from the elements of x_2-row to get the new elements of x_2-row in table 2.25. These elements are

$\frac{1}{4}-\frac{1}{3}\left(\frac{3}{4}\right) = 0, 1-\frac{1}{3}(0) = 1, \frac{1}{4}-\frac{1}{3}\left(-\frac{1}{4}\right) = \frac{1}{4}+\frac{1}{12} = \frac{1}{3}, 0-\frac{1}{3}(0) = 0, 0-\frac{1}{3}(1) = -\frac{1}{3}, 6-\frac{1}{3}(3) = 5.$

Similarly, elements of s_2-row in table 2.25 will be

$\frac{11}{4}-\frac{11}{3}\left(\frac{3}{4}\right) = 0, 0-\frac{11}{3}(0) = 0, \ -\frac{1}{4}-\frac{11}{3}\left(-\frac{1}{4}\right) = -\frac{1}{4}+\frac{11}{12} = \frac{8}{12} = \frac{2}{3}, \ 1-\frac{11}{3}(0) = 1,$

$0-\frac{11}{3}(1) = -\frac{11}{3}, \ 15-\frac{11}{3}(3) = 4$. Table 2.25 is now completed.

	c_j	2	5	0	0	0	
c_B	Basis	x_1	x_2	s_1	s_2	s_3	b
5	x_2	0	1	$1/3$	0	$-1/3$	5
0	s_2	0	0	$2/3$	1	$-11/3$	4
2	x_1	1	0	$-1/3$	0	$4/3$	4
	Z_j	2	5	1	0	1	33
c_j-Z_j	0	0	-1	0	-1	*Third feasible solution (optimal solution)*	

Step 7. Check third feasible solution for optimality

Z_j-row and c_j–Z_j-row values are calculated in table 2.25. Since all c_j-Z_j values are negative or zero, third feasible solution is optimal. The optimal solution is given by

$$\left.\begin{aligned} x_1 &= 4, \\ x_2 &= 5, \\ s_2 &= 4, \end{aligned}\right\} \text{(basic)} \qquad \left.\begin{aligned} s_1 &= 0, \\ s_3 &= 0, \end{aligned}\right\} \text{(non-basic)}$$

$$Z = 33.$$

EXAMPLE 2.16-5

Solve by simplex method the following L.P. problem :

$$\text{Minimize } Z = x_1 - 3x_2 + 3x_3,$$
$$\text{subject to} \quad 3x_1 - x_2 + 2x_3 \le 7,$$
$$2x_1 + 4x_2 \ge -12,$$
$$-4x_1 + 3x_2 + 8x_3 \le 10,$$
$$x_1, x_2, x_3 \ge 0.$$

[Kuru. U. B.E. (Mech.) June, 2012; J.N.T.U. Hyderabad B.Tech. (Mech.) May, 2012; P.U. B.Com. Sept., 2005]

Solution

Step 1. Set up the problem in standard form

As the right-hand side of the second constraint is negative , it is made positive by multiplying the constraint on both sides by – 1. This will reverse the inequality sign also. Thus this constraint takes the form

$$-2x_1 - 4x_2 \le 12.$$

Introducing slack variables s_1, s_2 and s_3, the problem can be expressed in the following standard form :

$$\text{Minimize } Z = x_1 - 3x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3,$$
$$\text{subject to} \quad 3x_1 - x_2 + 2x_3 + s_1 = 7,$$
$$-2x_1 - 4x_2 + s_2 = 12,$$
$$-4x_1 + 3x_2 + 8x_3 + s_3 = 10,$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0.$$

Step 2. Find initial basic feasible solution

Putting $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ in the constraints we get *i.b.f.s.* as

$$s_1 = 7,\ s_2 = 12,\ s_3 = 10\ ;\ Z = 0.$$

		c_j	1	– 3	3	0	0	0		
F.R.	c_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
1/3	0	s_1	3	– 1	2	1	0	0	7	– 7
4/3	0	s_2	– 2	– 4	0	0	1	0	12	– 3
	0	s_3	– 4	(3)	8	0	0	1	10	10/3 ←
		Z_j	0	0	0	0	0	0	0	
		c_j-Z_j	1	– 3	3	0	0	0		
				↑K						

Step 3. Perform optimality test

$Z_j = \sum c_B a_{ij}$ and c_j-Z_j rows are added to table 2.33.

Note that the objective is to *minimize* Z. Therefore, *if any c_j-Z_j coefficient is negative the solution is not optimal. The column having most negative c_j-Z_j value will be the key column.* Accordingly, x_2-column is marked the key column. Replacement ratios are – 7, – 3, and $\frac{10}{3}$. As $\frac{10}{3}$ is the only non-negative ratio, s_3-row is marked the key row and (3) is the key element.

Step 4. Iterate towards an optimal solution

Variable s_3 is replaced by x_2 by performing the row operations in the usual ways. Solution will become optimal when all c_j-Z_j coefficients become *zero or positive*. Iterations yield the following tables :

TABLE 2.34

F.R.	c_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b	θ	
		c_j	1	− 3	3	0	0	0			
	0	s_1	$(5/3)$	0	$14/3$	1	0	$1/3$	$31/3$	$31/3$	←
$22/5$	0	s_2	$-22/3$	0	$32/3$	0	1	$4/3$	$76/3$	$-38/11$	
$4/5$	− 3	x_2	$-4/3$	1	$8/3$	0	0	$1/3$	$10/3$	$-5/2$	
		Z_j	4	− 3	− 8	0	0	− 1	− 10		
		c_j-Z_j	− 3	0	11	0	0	1			
			↑K								

c_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b	
	c_j	1	− 3	3	0	0	0		
1	x_1	1	0	$14/5$	$3/5$	0	$1/5$	$31/5$	
0	s_2	0	0	$156/5$	$22/5$	1	$14/5$	$354/5$	
− 3	x_2	0	1	$32/5$	$4/5$	0	$3/5$	$58/5$	
	Z_j	1	− 3	$-82/5$	$-9/5$	0	$-8/5$	$-143/5$	
	c_j-Z_j	0	0	$97/5$	$9/5$	0	$8/5$		*Optimal solution*

∴ Optimal solution to the problem is

$$x_1 = \frac{31}{5},\ x_2 = \frac{58}{5},\ x_3 = 0\ ;\ Z_{min} = -\frac{143}{5}.$$

EXAMPLE 2.16-6

A food processing company produces three canned fruit products : mixed fruit, fruit cocktail and fruit delight. The main ingredients in each product are pears and peaches. Each product is produced in lots and must go through three processes, mixing, canning and packaging. The resource requirement for each product and each process are shown in the following L.P. formulation :

Maximize $Z = 10x_1 + 6x_2 + 8x_3,$ *(profit, ₹)*

subject to

$20x_1 + 10x_2 + 16x_3 \leq 320,$ *(pears, kg)*

$10x_1 + 20x_2 + 16x_3 \leq 400,$ *(peaches, kg)*

$x_1 + 2x_2 + 2x_3 \leq 43,$ *(mixing, hr.)*

$x_1 + x_2 + x_3 \leq 60,$ *(canning, hr.)*

$2x_1 + x_2 + x_3 \leq 40,$ *(packaging, hr.)*

$x_1, x_2, x_3 \geq 0.$

Final simplex table 2.35(*a*)

c_B	Basic Variables B	c_j / Solution values $b\ (=X_b)$	10 x_1	6 x_2	8 x_3	0 s_1	0 s_2	0 s_3	0 s_4	0 s_5
10	x_1	8	1	0	$8/15$	$1/15$	$1/30$	0	0	0
6	x_2	16	0	1	$8/15$	$-1/30$	$1/15$	0	0	0
0	s_3	3	0	0	$2/15$	0	$-1/10$	1	0	0
0	s_4	36	0	0	$-1/15$	$1/30$	$1/30$	0	1	0
0	s_5	8	0	0	$-8/15$	$1/10$	0	0		1
	Z_j		10	6	$128/15$	$7/15$	$1/15$	0	0	0
	$c_j - Z_j$		0	0	$-8/15$	$-7/15$	$-1/15$	0	0	0

On the basis of above information, answer the following questions :

(*i*) *Is the above solution feasible ?*

(*ii*) *Is the above solution optimal ? If yes, what is it ?*

(*iii*) *Is the above solution unbounded ?*

(*iv*) *Is the above solution degenerate ?*

(*v*) *Does't have multiple solutions ?*

(*vi*) *Determine the amount of used and unused resources.* [P.U. M.B.A. Feb., 2009]

Solution

(*i*) Yes, the above solution is feasible as values of all the variables are non-negative (≥ 0).

(*ii*) Yes, since $c_j - Z_j$ coefficients for the given maximization problem are either negative or zero, the solution is optimal.

The optimal solution is x_1 = 8 kg, x_2 = 16 kg, x_3 = 0, s_1 = 0, s_2 = 0; s_3 = 3 hrs., s_4 = 36 hrs., s_5 = 8 hrs. and Z_{max} = ₹ (10 × 8 + 6 × 16 + 8 × 0) = ₹ 176.

(*iii*) The above solution is not unbounded since values of all variables as well as Z_{max} are finite.

(*iv*) No, the above solution is non-degenerate since none of the basic variables has zero value.

(*v*) No, it does not have multiple solutions since $c_j - Z_j$ coefficients under none of the non-basic variables x_3, s_1 and s_2 have zero values.

(*vi*) Since s_1 = 0, all the 320 kg of pears are used in making the canned fruit products. Since s_2 = 0, all the 400 kg of peaches are also used. Since s_3 = 3 hrs., mixing process remains unused for 3 hours. Since s_4 = 36 hrs., time available for canning process remains unused for 36 hours. Since s_5 = 8 hrs., time available for packaging process remains unused for 8 hours.

EXAMPLE 2.17-1

Food X contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and costs 12 paise per gram. Food Y contains 8 units of vitamin A per gram and 12 units of vitamin B per gram and costs 20 paise per gram. The daily minimum requirement of vitamin A and vitamin B is 100 units and 120 units respectively. Find the minimum cost of product mix by the simplex method. [*P.U. B. Com. April, 2007*]

Solution. Let x_1 and x_2 be the grams of food X and Y to be purchased. Then the problem can be formulated as follows :

$$\begin{aligned} \text{Minimize } Z &= 12x_1 + 20x_2, \\ \text{subject to } \quad & 6x_1 + 8x_2 \geq 100, \\ & 7x_1 + 12x_2 \geq 120, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Step 1. Express the problem in standard form

Slack variables s_1 and s_2 are *subtracted* from the left-hand sides of the constraints to convert them to equations. These variables are also called *negative slack variables* or *surplus variables*. Variable s_1 represents units of vitamin A in product mix *in excess* of the minimum requirement of 100, s_2 represents units of vitamin B in product mix *in excess* of requirement of 120. Since they represent 'free' foods, the cost coefficients associated with them in the objective function are zeros. The problem, therefore, can be written as follows :

$$\begin{aligned} \text{Minimize } Z &= 12x_1 + 20x_2 + 0s_1 + 0s_2, \\ \text{subject to } \quad & 6x_1 + 8x_2 - s_1 = 100, \\ & 7x_1 + 12x_2 - s_2 = 120, \\ & x_1, x_2, s_1, s_2 \geq 0. \end{aligned}$$

Step 2. Find initial basic feasible solution

Putting $x_1 = x_2 = 0$, we get $s_1 = -100$, $s_2 = -120$ as the first basic solution but it is not feasible as s_1 and s_2 have negative values that do not satisfy the non-negativity restrictions. Therefore, we introduce artificial variables A_1 and A_2 in the constraints, which take the form

$$6x_1 + 8x_2 - s_1 + A_1 = 100,$$
$$7x_1 + 12x_2 - s_2 + A_2 = 120,$$
$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0.$$

Now artificial variables with values greater than zero violate the equality in constraints established in step 1. Therefore, A_1 and A_2 should not appear in the final solution. To achieve this, they are assigned a large unit penalty (a large positive value, + M) in the objective function, which can be written as

$$\text{minimize } Z = 12x_1 + 20x_2 + 0s_1 + 0s_2 + MA_1 + MA_2.$$

Problem, now, has six variables and two constraints. Four of the variables have to be zeroised to get initial basic feasible solution to the 'artificial system'. Putting $x_1 = x_2 = s_1 = s_2 = 0$, we get

$$A_1 = 100, A_2 = 120, Z = 220\ M.$$

Note that we are starting with a very heavy cost (compare it with zero profit in maximization problem) which we shall minimize during the solution procedure. Table 2.36 represents the problem and its solution.

Step 3. Perform optimality test

TABLE 2.36

		c_j	12	20	0	0	M	M		
F.R.	c_B	Basis	x_1	x_2	s_1	s_2	A_1	A_2	b	θ
2/3	M	A_1	6	8	– 1	0	1	0	100	25/2
	M	A_2	7	(12)	0	– 1	0	1	120	10 ←
		Z_j	13M	20M	– M	– M	M	M	220 M	
		c_j-Z_j	12-13M	20-20M	M	M	0	0		
				↑K					*Initial solution*	

Since c_j-Z_j is *negative* under x_1, x_2-columns, initial solution is not optimal and can be improved. c_j-Z_j is most negative under x_2-column. x_2-column is the key column, A_2-row is the key row and (12) is the key element. *Since A_2 is leaving variable, column A_2 is deleted from the next tables.*

Step 4. Iterate towards on optimal solution

Performing iterations results in the following tables:

TABLE 2.37

		c_j	12	20	0	0	M		
F.R.	c_B	Basis	x_1	x_2	s_1	s_2	A_1	b	θ
	M	A_1	$(4/3)$	0	-1	$2/3$	1	20	$15 \leftarrow$
$7/16$	20	x_2	$7/12$	1	0	$-1/12$	0	10	$120/7$
		Z_j	$\frac{35}{3}+\frac{4}{3}M$	20	$-M$	$-\frac{5}{3}+\frac{2}{3}M$	M	$200 + 20M$	
		c_j-Z_j	$\frac{1}{3}-\frac{4}{3}M$ ↑K	0	M	$\frac{5}{3}-\frac{2}{3}M$	0		

c_B	Basis	x_1	x_2	s_1	s_2	b
12	x_1	1	0	/	$1/2$	15
20	x_2	0	1	$7/16$	$-3/8$	$5/4$
	Z_j	12	20	$-1/4$	$-3/2$	205
	c_j-Z_j	0	0	$1/4$	$3/2$	

∴ Optimal solution is

$x_1 = 15$, $x_2 = 5/4$; $Z_{min} = 205$ Paise = ₹ 2.05.

Hence 15 grams of food X and $\frac{5}{4}$ grams of food Y should be the required product mix with minimum cost of ₹ 2.05.

EXAMPLE 2.17-3

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4,$

subject to $x_1 + 2x_2 + 3x_3 = 15,$
$2x_1 + x_2 + 5x_3 = 20,$
$x_1 + 2x_2 + x_3 + x_4 = 10,$
$x_1, x_2, x_3, x_4 \geq 0.$

[*R.T.M. Nagpur U.B.E. (Mech.) 2011; Dec., 2003; J.N.T.U. Hyderabad B.Tech. May, 2011; M.D.U.B.E. (Mech.) Dec., 2006; P.T.U.B.E. (Mech.) 2010; May, 2006, C.Sc., 2009; P.U.B.B.A., 2001*]

Solution

Step 1. Introducing artificial variables A_1, A_2, A_3, the given problem in standard form is

maximize $Z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2 - MA_3,$

subject to $x_1 + 2x_2 + 3x_3 + 0x_4 + A_1 + 0A_2 + 0A_3 = 15,$
$2x_1 + x_2 + 5x_3 + 0x_4 + 0A_1 + A_2 + 0A_3 = 20,$
$x_1 + 2x_2 + x_3 + x_4 + 0A_1 + 0A_2 + A_3 = 10,$
$x_1, x_2, x_3, x_4, A_1, A_2, A_3 \geq 0.$

Step 2. Initial basic (non-degenerate) solution to the artificial system is

$x_1 = x_2 = x_3 = x_4 = 0,$
$A_1 = 15,$
$A_2 = 20,$
$A_3 = 10,$
$Z = -45\,M.$

Table 2.41 represents this solution.

TABLE 2.41

	c_j	1	2	3	– 1	– M	– M	– M		
c_B	Basis	x_1	x_2	x_3	x_4	A_1	A_2	A_3	b	θ
– M	A_1	1	2	3	0	1	0	0	15	5
– M	A_2	2	1	(5)	0	0	1	0	20	4 ←
– M	A_3	1	2	1	1	0	0	1	10	10
	Z_j	– 4M	– 5M	– 9M	– M	– M	– M	– M	– 45M	
	$c_j–Z_j$	1 + 4M	2 + 5M	3 + 9M	– 1 + M	0	0	0		
				↑						*Initial solution*

Since $c_j–Z_j$ is positive under some variable columns, table 2.41 is not optimal.

Step 3. Performing iterations to get an optimal solution results in the following tables :

TABLE 2.42

	c_j	1	2	3	– 1	– M	– M		
c_B	Basis	x_1	x_2	x_3	x_4	A_1	A_3	b	θ
– M	A_1	$-1/5$	$(7/5)$	0	0	1	0	3	$15/7$ ←
3	x_3	$2/5$	$1/5$	1	0	0	0	4	20
– M	A_3	$3/5$	$9/5$	0	1	0	1	6	$10/3$
	Z_j	$\frac{6-2M}{5}$	$\frac{3-16M}{5}$	3	– M	– M	– M	12–9M	
	$c_j–Z_j$	$\frac{-1+2M}{5}$	$\frac{7+16M}{5}$	0	– 1 + M	0	0		
			↑						*Second solution*

TABLE 2.43

c_j	1	2	3	−1	−M			
c_B	Basis	x_1	x_2	x_3	x_4	A_3	b	θ
2	x_2	$-1/7$	1	0	0	0	$15/7$	∞
3	x_3	$3/7$	0	1	0	0	$25/7$	∞
−M	A_3	$6/7$	0	0	(1)	1	$15/7$	$15/7$ ←
	$Z_j = \Sigma c_B a_{ij}$	$\frac{7-6M}{7}$	2	3	−M	−M	$\frac{105-15M}{7}$	
	$c_j - Z_j$		0	0	−1+M	0		
					↑		*Third solution*	

TABLE 2.44

	c_j	1	2	3	−1		
c_B	Basis	x_1	x_2	x_3	x_4	b	θ
2	x_2	$-1/7$	1	0	0	$15/7$	−15
3	x_3	$3/7$	0	1	0	$25/7$	$25/3$
−1	x_4	$(6/7)$	0	0	1	$15/7$	$5/2$ ←
	$Z_j = \Sigma c_B a_{ij}$	$1/7$	2	3	−1	$90/7$	
	$c_j - Z_j$	$6/7$	0	0	0		
		↑				*4th basic feasible solution*	

TABLE 2.45

	c_j	1	2	3	−1	
c_B	Basis	x_1	x_2	x_3	x_4	b
2	x_2	0	1	0	$1/6$	$5/2$
3	x_3	0	0	1	$-1/2$	$5/2$
1	x_1	1	0	0	$7/6$	$5/2$
	$Z_j = \Sigma c_B a_{ij}$	1	2	3	0	15
	$c_j - Z_j$	0	0	0	−1	
					Optimal basic feasible solution	

∵ $c_j - Z_j$ is either zero or negative under all columns, the optimal basic feasible solution has been obtained. Optimal values are

$$x_1 = \frac{5}{2},\ x_2 = \frac{5}{2},\ x_3 = \frac{5}{2},\ x_4 = 0\ .$$

Also $A_1 = A_2 = A_3 = 0$ and $Z_{max} = 15$.

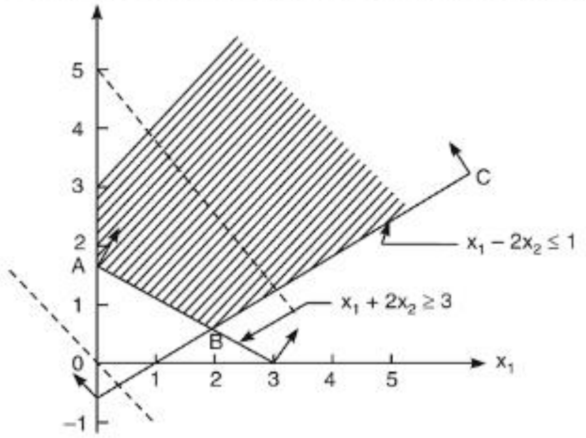
EXAMPLE 2.10-3

$$\begin{aligned} \text{Maximize } Z &= 5x_1 + 4x_2, \\ \text{subject to } \quad & x_1 - 2x_2 \le 1, \\ & x_1 + 2x_2 \ge 3, \\ & x_1, x_2 \ge 0. \end{aligned}$$

[P.T.U.B.E. 2001]

Solution

The solution space satisfying the constraints $x_1 - 2x_2 \le 1$, $x_1 + 2x_2 \ge 3$ and the non-negativity conditions $x_1 \ge 0$, $x_2 \ge 0$ is shown shaded in Fig. 2.16. This shaded convex region is unbounded.



The objective function, when Z = 0, gives the equation $5x_1 + 4x_2 = 0$, or $\frac{x_1}{x_2} = -\frac{4}{5}$. The corresponding point (–4, 5) is plotted, which when joined with origin, gives the plot of the dotted line $5x_1 + 4x_2 = 0$. As Z is increased from zero, this dotted line moves to the right, parallel to itself. Since we are interested in maximizing Z, we increase the value of Z till the dotted line passes through the farthest corner of the shaded region from the origin. As it is not possible to get the farthest corner for the shaded convex region, the maximum value of Z cannot be found as it occurs at infinity only. The problem, therefore, has an unbounded solution.

EXAMPLE 2.10-4

$$\begin{aligned} \text{Maximize } Z &= -4x_1 + 3x_2, \\ \text{subject to } \quad & x_1 - x_2 \le 0, \\ & x_1 \le 4, \\ & x_1, x_2 \ge 0. \end{aligned}$$

[P.U. B.Com. Jan., 2005; Sept., 2005]

Solution

The solution space satisfying the constraints $x_1 - x_2 \le 0$, $x_1 \le 4$ and meeting the non-negativity restrictions $x_1 \ge 0$, $x_2 \ge 0$ is shown shaded in Fig. 2.17. *The line $x_1 - x_2 = 0$ is drawn by joining points (0, 0) and say (1, 1). This line has a slope of 45° and since $x_1 - x_2 \le 0$ i.e., $x_1 \le x_2$, the arrowhead associated with the line is in the upward direction.*

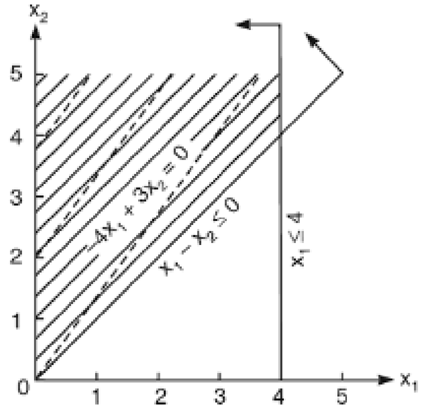


Fig. 2.17

To plot the line $Z = -4x_1 + 3x_2$, we assume Z = 0, giving $-4x_1 + 3x_2 = 0$ or $\frac{x_1}{x_2} = \frac{3}{4}$. The corresponding point (3, 4) is obtained, which when joined with origin, represents the dotted line $-4x_1 + 3x_2 = 0$. Lines are then drawn parallel to this line for increasing value of Z. Clearly, Z can be made large arbitrarily and the problem has no finite maximum value of Z.

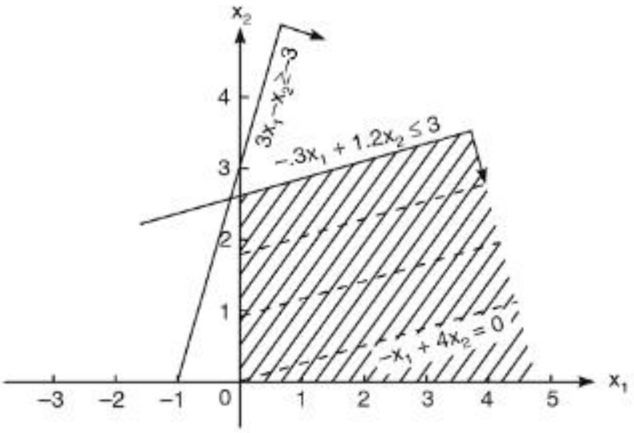
The problem, therefore, has an unbounded solution. *Value of variable x_1 is limited to 4, while value of variable x_2 can be increased indefinitely.*

EXAMPLE 2.10-5

$$\begin{aligned} \text{Maximize } Z &= -x_1 + 4x_2, \\ \text{subject to } \quad 3x_1 - x_2 &\ge -3, \\ -0.3x_1 + 1.2x_2 &\le 3, \\ x_1, x_2 &\ge 0. \end{aligned}$$

Solution

The solution space satisfying the constraints $3x_1 - x_2 \ge -3$, $-0.3x_1 + 1.2x_2 \le 3$ and meeting the non-negativity restrictions $x_1 \ge 0$, $x_2 \ge 0$ is shown shaded in Fig. 2.18.



Note that since the first constraint is $3x_1 - x_2 \geq -3$ (R.H.S. is negative), the direction of the arrowhead associated with this line is towards the origin.

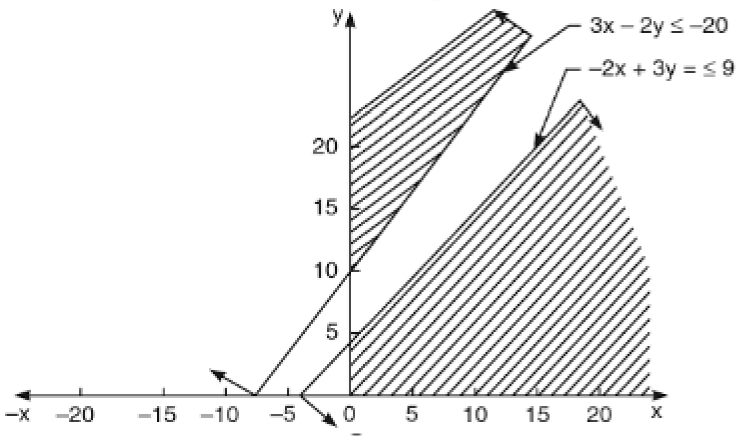
For Z = 0, the objective function becomes $-x_1 + 4x_2 = 0$, which yields $x_1/x_2 = 4/1$. Thus the dotted line passing through origin O(0, 0) and the point (4, 1) represents $-x_1 + 4x_2 = 0$. The value of Z can be increased by drawing lines parallel to this line and the maximum value is limited by the upper edge of the shaded figure. *Thus the optimum value of Z is 10. However, values of variables x_1, x_2 can be made arbitrarily large.* Further, any point (x_1, x_2) lying on the upper edge of the region of feasible solutions, which extends to infinity, yields the same optimal value of Z = 10 for the objective function. *Note that this problem has unbounded feasible region. But since value of Z is finite, it does not have an unbounded solution.*

EXAMPLE 2.10-6

$$\begin{aligned} \textit{Maximize } Z &= 3x + 2y, \\ \textit{subject to} \quad -2x + 3y &\leq 9, \\ 3x - 2y &\leq -20, \\ x, y &\geq 0. \end{aligned}$$

Solution

Fig. 2.19 indicates two shaded regions, one satisfying the constraint $-2x + 3y \leq 9$ and the other satisfying the constraint $3x - 2y \leq -20$. These two shaded regions in the first quadrant do not overlap with the result that there is no point (x, y) common to both the shaded regions. The problem cannot be solved graphically (or by any other method of solving L.P. problems) *i.e.*, the feasible solution to the problem does not exist or the problem has infeasible solution.



EXAMPLE 2.10-7

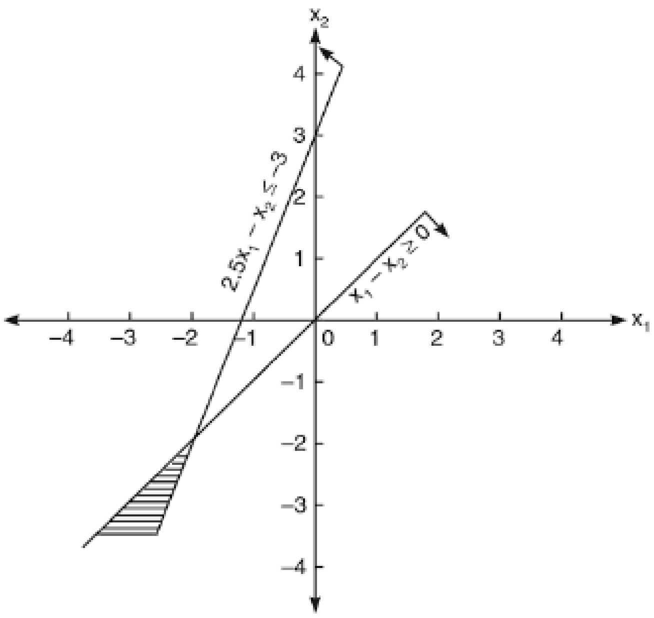
$$\begin{aligned} \textit{Maximize } Z &= 3x_1 + 4x_2, \\ \textit{subject to} \quad x_1 - x_2 &\geq 0, \\ 2.5x_1 - x_2 &\leq -3, \\ x_1, x_2 &\geq 0. \end{aligned}$$

[*P.U. B.E. (Mech.) Dec., 1982*]

Solution

The solution space satisfying the constraints $x_1 - x_2 \geq 0$, $2.5x_1 - x_2 \leq -3$ is shown shaded in Fig. 2.20.

Any point within this region satisfies the constraints but not the non-negativity restrictions. Thus although the constraints are consistent, the problem does not possess a feasible solution.



EXAMPLE 2.10-8

Maximize $Z = 5x_1 + 8x_2$,

subject to $3x_1 + 5x_2 = 18$,

$5x_1 + 3x_2 = 14$,

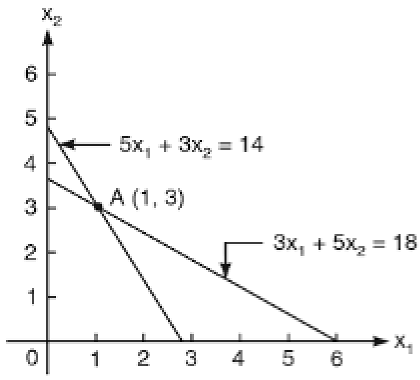
$x_1, x_2 \geq 0$.

Solution

Fig. 2.21 shows the graphical solution. The feasible region reduces to the point A (1, 3). Thus *the problem has just a single solution*

$$x_1 = 1, x_2 = 3, Z = 5 + 24 = 29.$$

As there is nothing to be maximized, such a problem is not of much intrest from point of view of operations research.



Evidently, there were two variables x_1 and x_2 in the above examples and the problems were, therefore, two-dimensional and were simple to be represented (by the two axes lying in a plane) and solved graphically. Now, as the number of variables increases to 3, 4, ... we come across 3-dimensional, 4-dimensional, ... problems which become quite laborious to be solved by graphical methods. In such cases *simplex technique* helps us in

(*i*) starting with a feasible solution,

(*ii*) searching optimal solution in a systematic way.

The Hungarian Method: The following algorithm applies the above theorem to a given $n \times n$ cost matrix to find an optimal assignment.

Step 1. Subtract the smallest entry in each row from all the entries of its row.

Step 2. Subtract the smallest entry in each column from all the entries of its column.

Step 3. Draw lines through appropriate rows and columns so that all the zero entries of the cost matrix are covered and the *minimum* number of such lines is used.

Step 4. *Test for Optimality:* (i) If the minimum number of covering lines is n, an optimal assignment of zeros is possible and we are finished. (ii) If the minimum number of covering lines is less than n, an optimal assignment of zeros is not yet possible. In that case, proceed to Step 5.

Step 5. Determine the smallest entry not covered by any line. Subtract this entry from each uncovered row, and then add it to each covered column. Return to Step 3.

Example 2: A construction company has four large bulldozers located at four different garages. The bulldozers are to be moved to four different construction sites. The distances in miles between the bulldozers and the construction sites are given below.

Bulldozer \ Site	A	B	C	D
1	90	75	75	80
2	35	85	55	65
3	125	95	90	105
4	45	110	95	115

How should the bulldozers be moved to the construction sites in order to minimize the total distance traveled?

Step 1. Subtract 75 from Row 1, 35 from Row 2, 90 from Row 3, and 45 from Row 4.

$$\begin{bmatrix} 90 & 75 & 75 & 80 \\ 35 & 85 & 55 & 65 \\ 125 & 95 & 90 & 105 \\ 45 & 110 & 95 & 115 \end{bmatrix} \sim \begin{bmatrix} 15 & 0 & 0 & 5 \\ 0 & 50 & 20 & 30 \\ 35 & 5 & 0 & 15 \\ 0 & 65 & 50 & 70 \end{bmatrix}$$

Step 2. Subtract 0 from Column 1, 0 from Colum 2, 0 from Column 3, and 5 from Column 4.

$$\begin{bmatrix} 15 & 0 & 0 & 5 \\ 0 & 50 & 20 & 30 \\ 35 & 5 & 0 & 15 \\ 0 & 65 & 50 & 70 \end{bmatrix} \sim \begin{bmatrix} 15 & 0 & 0 & 0 \\ 0 & 50 & 20 & 25 \\ 35 & 5 & 0 & 10 \\ 0 & 65 & 50 & 65 \end{bmatrix}$$

Step 3. Cover all the zeros of the matrix with the minimum number of horizontal or vertical lines.

$$\begin{bmatrix} 15 & 0 & 0 & 0 \\ 0 & 50 & 20 & 25 \\ 35 & 5 & 0 & 10 \\ 0 & 65 & 50 & 65 \end{bmatrix}$$

Step 4. Since the minimal number of lines is less than 4, we have to proceed to Step 5.

Step 5. Note that 5 is the smallest entry not covered by any line. Subtract 5 from each uncovered row.

$$\begin{bmatrix} 15 & 0 & 0 & 0 \\ 0 & 50 & 20 & 25 \\ 35 & 5 & 0 & 10 \\ 0 & 65 & 50 & 65 \end{bmatrix} \sim \begin{bmatrix} 15 & 0 & 0 & 0 \\ -5 & 45 & 15 & 20 \\ 30 & 0 & -5 & 5 \\ -5 & 60 & 45 & 60 \end{bmatrix}$$

Now add 5 to each covered column.

$$\begin{bmatrix} 15 & 0 & 0 & 0 \\ -5 & 45 & 15 & 20 \\ 30 & 0 & -5 & 5 \\ -5 & 60 & 45 & 60 \end{bmatrix} \sim \begin{bmatrix} 20 & 0 & 5 & 0 \\ 0 & 45 & 20 & 20 \\ 35 & 0 & 0 & 5 \\ 0 & 60 & 50 & 60 \end{bmatrix}$$

Now return to Step 3.

Step 3. Cover all the zeros of the matrix with the minimum number of horizontal or vertical lines.

$$\begin{bmatrix} 20 & 0 & 5 & 0 \\ 0 & 45 & 20 & 20 \\ 35 & 0 & 0 & 5 \\ 0 & 60 & 50 & 60 \end{bmatrix}$$

Step 4. Since the minimal number of lines is less than 4, we have to return to Step 5.

Step 5. Note that 20 is the smallest entry not covered by a line. Subtract 20 from each uncovered row.

$$\begin{bmatrix} 20 & 0 & 5 & 0 \\ 0 & 45 & 20 & 20 \\ 35 & 0 & 0 & 5 \\ 0 & 60 & 50 & 60 \end{bmatrix} \sim \begin{bmatrix} 20 & 0 & 5 & 0 \\ -20 & 25 & 0 & 0 \\ 35 & 0 & 0 & 5 \\ -20 & 40 & 30 & 40 \end{bmatrix}$$

Then add 20 to each covered column.

$$\begin{bmatrix} 20 & 0 & 5 & 0 \\ -20 & 25 & 0 & 0 \\ 35 & 0 & 0 & 5 \\ -20 & 40 & 30 & 40 \end{bmatrix} \sim \begin{bmatrix} 40 & 0 & 5 & 0 \\ 0 & 25 & 0 & 0 \\ 55 & 0 & 0 & 5 \\ 0 & 40 & 30 & 40 \end{bmatrix}$$

Now return to Step 3.

Step 3. Cover all the zeros of the matrix with the minimum number of horizontal or vertical lines.

$$\begin{bmatrix} 40 & 0 & 5 & 0 \\ 0 & 25 & 0 & 0 \\ 55 & 0 & 0 & 5 \\ 0 & 40 & 30 & 40 \end{bmatrix}$$

Step 4. Since the minimal number of lines is 4, an optimal assignment of zeros is possible and we are finished.

$$\begin{bmatrix} 40 & 0 & 5 & \boxed{0} \\ 0 & 25 & \boxed{0} & 0 \\ 55 & \boxed{0} & 0 & 5 \\ \boxed{0} & 40 & 30 & 40 \end{bmatrix}$$

Since the total cost for this assignment is 0, it must be an optimal assignment.

Here is the same assignment made to the original cost matrix.

$$\begin{bmatrix} 90 & 75 & 75 & \boxed{80} \\ 35 & 85 & \boxed{55} & 65 \\ 125 & \boxed{95} & 90 & 105 \\ \boxed{45} & 110 & 95 & 115 \end{bmatrix}$$

So we should send Bulldozer 1 to Site D, Bulldozer 2 to Site C, Bulldozer 3 to Site B, and Bulldozer 4 to Site A.

Theorem

The feasible set of an LP problem is convex.

Proof

Write the problem as minimize $c^T x$ s.t. $Ax = b$, $x \geq 0$, and let X_b be the feasible set. Suppose $x, y \in X_b$, so $x, y \geq 0$ and $Ax = Ay = b$. Consider $z = \lambda x + (1 - \lambda)y$ for $0 \leq \lambda \leq 1$. Then $z_i = \lambda x_i + (1 - \lambda)y_i \geq 0$ for each i. So $z \geq 0$. Secondly,

$$Az = A(\lambda x + (1 - \lambda)y) = \lambda Ax + (1 - \lambda)Ay = \lambda b + (1 - \lambda)b = b.$$

So $z \in X_b$ and hence X_b is convex.

EXAMPLE 3.6-5 (Profit Maximization Problem)

A company manufacturing air coolers has two plants located at Mumbai and Kolkata with a capacity of 200 units and 100 units per week respectively. The company supplies the air coolers to its four show rooms situated at Ranchi, Delhi, Lucknow and Kanpur which have a maximum demand of 75, 100, 100 and 30 units respectively. Due to the differences in raw material cost and transportation cost, the profit per unit in rupees differs which is shown in the table below.

TABLE 3.149

	Ranchi	Delhi	Lucknow	Kanpur
Mumbai	90	90	100	110
Kolkata	50	70	130	85

Plan the production programme so as to maximize the profit. The company may have its production capacity at both plants partly or wholly unused. *[P.U.B. Com. April, 2007]*

Solution. It consists of the following steps:

Step I: Make the Transportation Matrix

For the given data, the transportation matrix is as shown below:

TABLE 3.150

	Ranchi	Delhi	Lucknow	Kanpur	Supply
Mumbai	90	90	100	110	200 Total supply = 300 units
Kolkata	50	70	130	85	100 Total demand = 305 units
Demand	75	100	100	30	

Thus, supply and demand are not balanced. As the demand is more than supply, a dummy source is introduced to meet the extra demand and zero profit coefficients are introduced since nothing is produced at the dummy source and, therefore, nothing can be sold. The resulting matrix is shown in table 3.151.

TABLE 3.151

	Ranchi	Delhi	Lucknow	Kanpur	Supply
Mumbai	90	90	100	110	200
Kolkata	50	70	130	85	100
Dummy source	0	0	0	0	5 *Balanced maximization problem*
Demand	75	100	100	30	

Step II: Find Initial Basic Feasible Solution

We shall use Vogel's approximation method to find the initial feasible solution. This method consists of substeps 1, 2, 3 and 4 already explained in example 3.5-1.

Note that we are dealing with maximization problem. Hence we shall enter the difference between the *highest* and the *second highest* elements in each row to the right of the row and the difference between the highest and the second highest elements in each column below the corresponding column. Each of these differences represents the *unit profit lost* for not allocating to the highest profit cell. Thus, while making allocations, at first we select cell (2, 3) with highest entry in row 2 which corresponds to the highest difference of [45]. Same is true for other allocations in table 3.152.

TABLE 3.152

	Ranchi	Delhi	Lucknow	Kanpur	Supply
Mumbai	90 (70)	90 (100)	100	110 (30)	200/170/70/0 [10] [20] [0]
Kolkata	50	70	130 (100)	85	100/0 [45] ←
Dummy source	0 (5)	0	0	0	5/0 [0] [0] [0]
Demand	75/5/0	100/0	100/0	30/0	
	[40]	[20]	[30]	[25]	
	[90]	[90]		[110]	
	[90]	[90]		↑	
	↑	↑			

Step III: Perform Optimality Test

Required number of allocations = $m + n - 1 = 3 + 4 - 1 = 6$.

Actual number of allocations = 5.

Therefore we allocate very small positive number ε to cell (1, 3) [cell having *maximum profit* out of vacant cells] so that the number of allocations becomes 6. This is shown in table 3.153. These 6 allocations are in independent positions. Therefore optimality test can be performed. This consists of substeps 1, 2, 3, 4 and 5, details of which are given in example 3.5-1.

TABLE 3.153

	Ranchi	Delhi	Lucknow	Kanpur	Supply
Mumbai	90 (70)	90 (100)	100 (ε)	110 (30)	200
Kolkata	50	70	130 (100)	85	100
Dummy source	0 (5)	0	0	0	5
Demand	75	100	100	30	

Initial basic feasible solution

TABLE 3.154

u_i \ v_j	0	0	10	20
90	90	90	100	110
120			130	
0	0			

Matrix of (u_i, v_j) for allocated cells

→

TABLE 3.155

u_i \ v_j	0	0	10	20
90	·	·	·	·
120	120	120	·	140
0	·	0	10	20

Matrix of $(u_i + v_j)$ for vacant cells

TABLE 3.156

·	·	·	·
–70	–50	·	–55
·	0	–10	–20

Cell evaluation matrix

Since all cell values are *either negative or zero* (maximization problem), the initial basic feasible solution of table 3.153 is optimal. The demand at Ranchi is left unsatisfied by 5 units. The profit corresponding to the above scheme is

$$Z_{max} = ₹\,[90 \times 70 + 90 \times 100 + 110 \times 30 + 130 \times 100 + 0 \times 5] = ₹\,31{,}600.$$

EXAMPLE 3.6-6

A company has factories at four different places, which supply warehouses A, B, C, D and E. Monthly factory capacities are 200, 175, 150 and 325 units respectively. Monthly warehouse requirements are 110, 90, 120, 230 and 160 units respectively. Unit shipping costs are given in table 3.157. The costs are in rupees.

TABLE 3.157

To / From	A	B	C	D	E
1	13	--	31	8	20
2	14	9	17	6	10
3	25	11	12	17	15
4	10	21	13	--	17

Shipment from 1 to B and from 4 to D is not possible. Determine the optimum distribution to minimize shipping costs.

[H.P.U.B. Tech. (Mech.) June, 2010; P.U.B.E.(Mech.) 2002; B.E.(Prod.) 2001; B.E.(Elect.) 2003; Karn.U. B.E.(Mech.) 1999]

Solution

Step I: Set up the Transportation Table

Here, total capacity = 200 + 175 + 150 + 325 = 850 units,

total demand = 110 + 90 + 120 + 230 + 160 = 710 units.

Since demand is less than the capacity, a dummy warehouse *d* is created to absorb this additional capacity of 140 units. The associated cost elements will be all zero. Further to avoid allocation in cells (1, B) and (4, D), a very heavy penalty, + M is allocated to these cells. The modified transportation matrix is shown in table 3.158.

TABLE 3.158

To / From	A	B	C	D	E	d	*Capacity*
1	13	M	31	8 (200)	20	0	200/0(8) (5)
2	14	9	17	6 (30)	10 (145)	0	175/145/0 (6) (3) (3) (1)
3	25	11 (10)	12	17	15	0 (140)	150/10/0 (11)← (1) (1) (1) (1) (1)
4	10 (110)	21 (80)	13 (120)	M	17 (15)	0	325/215/200/80/0 (10) (3) (3) (3) (3) (4)
Demand	110/0	90 /80/0	120/0	230 /30/0	160 /15/0	140/0 (0)	
	(3)	(2)	(1)	(2)	(5)		
	(3)	(2)	(1)	(2)	(5)		
	(4)	(2)	(1)	(11)	(5)		
	(4)	(2)	(1)	↑	(5)		
	(15)	(10)	(1)		↑		
	↑	(10)	(1)		(2)		
		↑			(2)		

Initial basic feasible solution

Step II: Find the Initial Basic Feasible Solution

Using Vogel's approximation method, the initial basic feasible solution is obtained. This is shown in table 3.158.

MODI method is now used to find optimal solution. This solution obtained after a few iterations is shown in table 3.159.

TABLE 3.159

From \ To	A	B	C	D	E	d	Capacity
1	13	M	31	8 (200)	20	0	200
2	14	9	17	6 (30)	10 (145)	0	175
3	25	11 (90)	12 (45)	17	15 (15)	0	150
4	10 (110)	21	13 (75)	M	17	0 (140)	325
Demand	110	90	120	230	160	140	

Example 11.8 A company has three factories *A*, *B* and *C* which supply units to warehouses *X*, *Y* and *Z* every month. The capacities of the factories are 60, 70 and 80 units at *A*, *B* and *C* respectively. The requirements of *X*, *Y* and *Z* per month are 50, 80 and 80 units respectively. Transportation cost per unit in rupees are given in the following table. Find out the minimum cost of transportation

	X	*Y*	*Z*
A	8	7	5
B	6	8	9
C	9	6	5

Solution The transportation table for the given data is

	X	*Y*	*Z*	Availability
A	8	7	5	60
B	6	8	9	70
C	9	6	5	80
Requirement	50	80	80	210 balanced

We find an initial basic feasible solution by least cost rule. We get the initial solution as follows:

The least cost is 5 in (1, 3) cell and (3, 3) cell. Select (1, 3) cell and allot 60 unit. We have,

	X	*Y*	*Z*	
A	– / 8	– / 7	60 / 5	~~60~~
B	6	8	9	70
C	9	6	5	80
	50	80	~~80~~ 20	

The least cost in the reduced table is 5 in (3, 3) cell. Allot 20 units there. We get

	X	*Y*	*Z*	
A	– / 8	– / 7	60 / 5	~~60~~
B	6	8	– / 9	70
C	9	6	20 / 5	~~80~~ 60
	50	80	~~80~~ ~~20~~	

(2, 1) cell and (3, 2) cell both have the minimum cost 6. Select (2, 1) cell for an allotment of 50 units. The table becomes

	X	*Y*	*Z*	
A	– / 8	– / 7	60 / 5	~~60~~
B	50 / 6	8	– / 9	~~70~~ 20
C	– / 9	6	20 / 5	~~80~~ 60
	~~50~~	80	~~80~~ ~~20~~	

Now we can allot 20 units to (2, 2) cell and 60 units to (3, 2) cell. The initial solution is

	X	*Y*	*Z*	
A	– / 8	– / 7	60 / 5	$u_1 = -2$
B	50 / 6	20 / 8	– / 9	$u_2 = 0$
C	– / 9	60 / 6	20 / 5	$u_3 = -2$
	$v_1 = 6$	$v_2 = 8$	$v_3 = 7$	

$m + n - 1 = 5$ and we have 5 basic cells. Defining simplex multipliers u_i and v_j for the rows and columns we get the equations for the basic cells,

$$u_1 + v_3 = 5 \qquad u_3 + v_2 = 6$$
$$u_2 + v_1 = 6 \qquad u_3 + v_3 = 5$$
$$u_2 + v_2 = 8$$

Assign zero value to u_2 we get the solutions

$u_1 = -2 \qquad v_1 = 6$

$u_2 = 0 \qquad v_2 = 8$

$u_3 = -2 \qquad v_3 = 7$

The relative cost factors for the non-basic cells are

$\bar{c}_{11} = c_{11} - u_1 - v_1 = 4 \qquad \bar{c}_{12} = c_{12} - u_1 - v_2 = 1$

$\bar{c}_{23} = c_{23} - u_2 - v_3 = 2 \qquad \bar{c}_{31} = c_{31} - u_3 - v_1 = 5$

All $\bar{c}_{ij} \geq 0$. Hence this solution is itself optimal.

The optimal solution is

		(Rs)
$A \to Z$:	60 units Cost	300
$B \to X$:	50 units Cost	300
$B \to Y$:	20 units Cost	160
$C \to Y$:	60 units Cost	360
$C \to Z$:	20 units Cost	100

Total cost = Rs 1220

Illustration 13: (Trans-shipment problem) A travelling salesman has to visit five cities. He wishes to start from a particular city, visit each city, one by one and then return to his starting point. The travelling cost, in thousands of rupees, to each city from a particular city is given below.

Which sequence of visits of the salesman minimises his total cost?.

To City

From City	A	B	C	D	E
A	∞	2	5	7	1
B	6	∞	3	8	2
C	8	7	∞	4	7
D	12	4	6	∞	5
E	1	3	2	8	∞

Solution: In order to find an optimum assignment/solution to a given travelling salesmen problem, first we follow the same procedure as followed in the routine assignment problem. If optimum solution also satisfies the additional constraints, then the given travelling salesmen problem obtain optimality.

M	2	5	7	1
6	∞	3	8	2
8	7	∞	4	7
12	4	6	∞	5
1	3	2	8	∞

Step 1: Assign an infinitely large element (∞ or M) in each of the squares, along the diagonal line in the cost matrix.

M	1	4	6	0
4	∞	1	6	0
4	3	∞	0	0
8	0	2	∞	1
0	2	1	7	∞

Step 2: Row-wise transformation Identify the least element in each row and subtract that element from all the elements of that row.

M	1	3	6	0
4	∞	0	6	0
4	3	∞	0	3
8	0	1	∞	1
0	2	0	7	∞

	A	B	C	D	E
A	∞	1	3	6	[0]
B	4	∞	[0]	6	0
C	4	3	∞	[0]	3
D	8	[0]	1	∞	1
E	[0]	2	0	7	∞

Step 3: Column-wise transformation From the transformed row matrix, identify the least element in each column and deduct that element from all the elements of that particular column.

Optimum assignment table	
A → E	1
B → C	3
C → D	4
D → B	4
E → A	1
	13

Step 4 : Assignment In the matrix derived from Step 2, start making allocations. Consider the matrix row by row. Identify any single zero in any row and encircle it.

This optimum assignment table shows the solution to the assignment problem but not to the travelling salesman problem. The following sequence can be identified from the above optimum solution table.

Sequence of the salesman

$$A \rightarrow E \rightarrow A, B \rightarrow C \rightarrow D \rightarrow B$$

This represents that the salesman should start his journey from city A to E and then come back to city without visiting cities B, C, D. Here this violates the additional restriction that the salesman can visit each city only once. Hence, the problem can be improved.

Optimum Sequence

0	[1]	3	6	0
4	∞	[0]	6	0
4	3	∞	[0]	3
8	0	1	∞	[1]
[0]	2	0	7	∞

Case 1: Now we examine the optimum assignment table to find the next best solution, that satisfies the additional restriction.

First, identify the next minimum element in the assignment table and try to find the effect of placing such an element in the solution. Here, the next minimum element is 1. So we make the unit assignment in the cell (A, B) instead of zero assignment in the cell (A, E) and delete the row A and column B to avoid any further assignment. Henceforth we proceed in the unit assignment as a routine assignment problem.

Sequence of the salesman is $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow A$

∞	[1]	3	6	0
4	∞	0	6	[0]
4	3	∞	[0]	3
8	0	[1]	∞	1
[0]	2	0	7	∞

Case 2: Instead of an assignment in the cell (D, E), we make a unit assignment in the cell (D, C) and proceed in the usual manner. The resulting solution table in given alongside:

Here the following sequence can be identified from the above assignment table:

$$A \rightarrow B \rightarrow D, C \rightarrow D \rightarrow C, E \rightarrow A$$

Since it violates the additional restriction and produces no feasible solution, which gives cost less than 15000, the best solution is:

$$A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow A.$$

Optimum Sequence

0	[1]	3	6	0
4	∞	[0]	6	0
4	3	∞	[0]	3
8	0	1	∞	[1]
[0]	2	0	7	∞

Sequence (in '1000)

$A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow A = 2 + 3 + 4 + 5 + 1 = 15$

EXAMPLE 25.12

Express the following LPP in standard form and determine the vertices algebraically.
Maximize

$$u = 4x + 3y$$

subject to constraints

$$x + y \leq 4$$
$$-x + y \leq 2$$
$$\text{and} \quad x, y \geq 0.$$

Solution. Introducing the slack variables r and t, the standard form (slack form) of the given LPP is
Maximize

$$u = 4x + 3y$$

subject to constraints

$$x + y + r = 4$$
$$-x + y + t = 2$$
$$\text{and } x, y, r, t \geq 0.$$

We have four variables and two constraint equations. Thus for a basic solution, we put 4--2 = 2 variables equal to zero and solve algebraically the constraint equations for the remaining two variables. There are $4_{C_2} = 6$ various possibilities of putting two variables equal to zero. Thus, we have the following table:

Non-basic variables	Basic variables	Basic solution	Feasibility
$x = y = 0$	r, t	$r = 4$, $t = 2$	Feasible

$x = r = 0$	y, t	$y = 4$, $t = -2$	Non-feasible
$x = t = 0$	y, r	$y = 2$, $r = 2$	Feasible
$y = r = 0$	x, t	$x = 4$, $t = 6$	Feasible
$y = t = 0$	x, r	$x = -2$, $r = 6$	Non-feasible
$r = t = 0$	x, y	$x = 1$, $y = 3$	Feasible

The feasible solution yields the following four vertices:

$$(0,0),\ (0,2),\ (4,0),\ (1,3).$$

Example 11.9 Solve the following transportation problem:

	X	Y	Z	Availability
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80
	50	80	80	210 balanced

Solution Applying VAM method we obtain an initial basic feasible solution, given below

	X	Y	Z
A	– 8	– 7	60 3
B	50 3	8	20 9
C	– 11	80 3	– 5

We have 4 basic cells but we should have (3 + 3 – 1 = 5) 5 basic cells. Select any non-basic cell from the set {(1, 2), (2, 2), (3, 1) and (3, 3)} which does not form a loop with the existing basic cells. Let us choose (2, 2) cell and make it a basic cell by allotting an infinitesimal quantity $\in$ to it. Now we get an initial basic feasible solution having 5 basic cells.

Introduce simplex multipliers u_1, u_2, u_3 for the rows and v_1, v_2, v_3 for the columns. We get 5 equations $u_i + v_j = c_{ij}$ corresponding to the basic cells.

$u_1 + v_3 = 3$ $\qquad$ $u_2 + v_3 = 9$
$u_2 + v_1 = 3$ $\qquad$ $u_3 + v_2 = 3$
$u_2 + v_2 = 8$

	X	Y	Z	
A	8	7	60 / 3	$u_1 = -6$
B	50 / 3	∈ / 8	20 / 9	$u_2 = 0$
C	– / 11	80 / 3	5	$u_3 = -5$
	$v_1 = 3$	$v_2 = 8$	$v_3 = 9$	

Set $u_2 = 0$. We get the solution

$u_1 = -6$ $\qquad$ $v_1 = 3$
$u_2 = 0$ $\qquad$ $v_2 = 8$
$u_3 = -5$ $\qquad$ $v_3 = 9$

The relative cost factors for the non-basic cells are $C_{11} = 8 + 6 - 3 = 11$

$\bar{c}_{12} = 7 + 6 - 8 = 5$ $\qquad$ $\bar{c}_{31} = 11 + 5 - 3 = 13$

$\bar{c}_{33} = 5 + 5 - 9 = 1$

$\bar{c}_{ij}$ are all non negative. Hence this solution is optimal. Take $\in$ to be zero. The optimal solution is

(Rs)
$A \to Z$: 60 units Cost 180
$B \to X$: 50 units Cost 150

$B \to Z$: 20 units Cost 180
$C \to Y$: 80 units Cost 240
Total Cost = 750

Example 7.10 A firm plans to purchase at least 200 quintals of scrap containing high quality metal X and low quality metal Y. It decides that the scrap to be purchased must contain at least 100 quintals of metal X and not more than 35 quintals of metal Y. The firm can purchase the scrap from two suppliers (A and B) in unlimited quantities. The percentage of X and Y metals in terms of weight in the scrap supplied by A and B is given below.

Metals	*Supplier A*	*Supplier B*
X	25%	75%
Y	10%	20%

The price of A's scrap is Rs 200 per quintal and that of B is Rs 400 per quintal. The firm wants to determine the quantities that it should buy from the two suppliers so that the total cost is minimized.

[*Delhi Univ., MBA, 1998, 2001*]

Solution Let us consider the following decision variables:

x_1 and x_2 = quantity (in quintals) of scrap purchased from suppliers A and B, respectively.

Then LP model of the given problem can be expressed as:

Minimize $Z = 200x_1 + 400x_2$

subject to the constraints

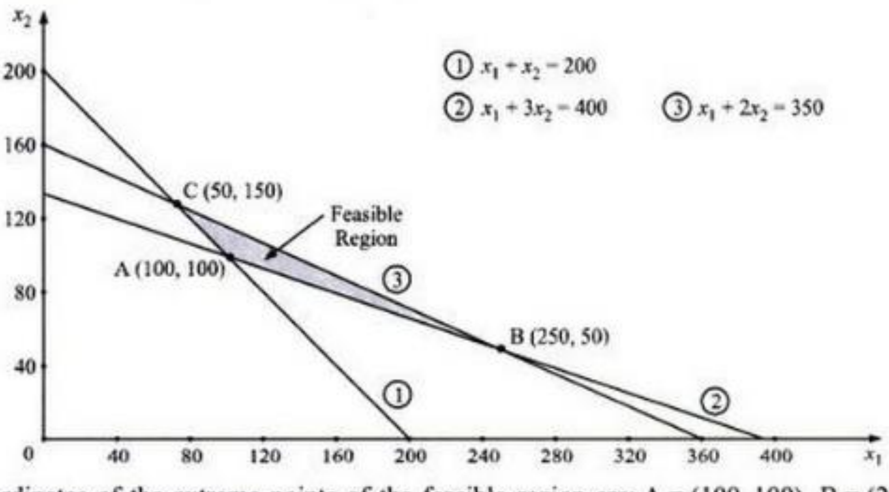
(i) $x_1 + x_2 \geq 200$ (Maximum purchase)

(ii) $\frac{x_1}{4} + \frac{3x_2}{4} \geq 100$ or $x_1 + 3x_2 \geq 400$

$\frac{x_1}{10} + \frac{x_2}{5} \geq 35$ or $x_1 + 2x_2 \leq 350$ } (Scrap containing X and Y metals)

and $x_1, x_2 \geq 0$.

The constraints are plotted on a graph as shown in Fig. 7.10. The feasible region is shown by the shaded area and is bounded by the corners, A, B and C.



The coordinates of the extreme points of the feasible region are: A = (100, 100), B = (250, 50), and C = (50, 150). The value of objective function at each of these extreme points is given below:

Extreme Point	*Coordinates* (x_1, x_2)	*Objective Function Value* $Z = 200x_1 + 400x_2$
A	(100, 100)	200(100) + 400(100) = 60,000
B	(250, 50)	200(250) + 400(50) = 70,000
C	(50, 150)	200(50) + 400(150) = 70,000

Since Z has the minimum value at the extreme point A (100, 100), the solution to the given problem is: $x_1 = 100$, $x_2 = 100$ and Min Z = Rs 60,000. That is, the firm should buy 100 quintals of scrap each from suppliers A and B in order to minimize the total cost of purchase.

Ex. 37. *A soft drink plant has two bottling machines A and B. It produces and sells 8 ounce and 16 ounce bottles. The following data is available :*

Machine	*8 Ounce*	*16 Ounce*
A	100*/minute*	40*/minute*
B	60*/minute*	75*/minute*

The machines can be run 8 *hrs. per day,* 5 *days per week. Weekly production of the drinks cannot exceed* 3,00,000 *ounces and the market can absorb* 25,000 *eight ounce bottles and* 7,000 *sixteen ounce bottles per week. Profit on these bottles is* 15 *paise and* 25 *paise per bottle respectively. The planner wishes to maximize his profit subject to all the production and marketing restrictions. Formulate it as a linear programming problem and solve graphically.*

Sol. Formulation of the problem as L.P.P.

Let the planner produce x_1 and x_2 numbers of bottles of 8 and 16 ounces respectively per week.

Then his profit in Rs. is $Z = 0{\cdot}15x_1 + 0{\cdot}25x_2$.

Total production of soft drink per week to fill up these bottles $= 8x_1 + 16x_2$.

But production of soft drink cannot exceed 3,00,000 ounces

$\therefore \quad 8x_1 + 16x_2 \le 3{,}00{,}000.$

Total time taken to produce these bottles on machine A

$$= \frac{x_1}{100 \times 60} + \frac{x_2}{40 \times 60} \text{ hours.}$$

on machine B

$$= \frac{x_1}{60 \times 60} + \frac{x_2}{75 \times 60} \text{ hours.}$$

But the machines can run for $8 \times 5 = 40$ hours per week

$$\therefore \quad \frac{x_1}{100 \times 60} + \frac{x_2}{40 \times 60} \le 40 \quad \text{and} \quad \frac{x_1}{60 \times 60} + \frac{x_2}{75 \times 60} \le 40$$

or $\quad 2x_1 + 5x_2 \le 4{,}80{,}000 \quad$ and $\quad 5x_1 + 4x_2 \le 7{,}20{,}000.$

Since market can absorb 25,000 eight ounce bottles and 7,000 sixteen ounce bottles,

$\therefore \quad x_1 \le 25{,}000, \; x_2 \le 7{,}000.$

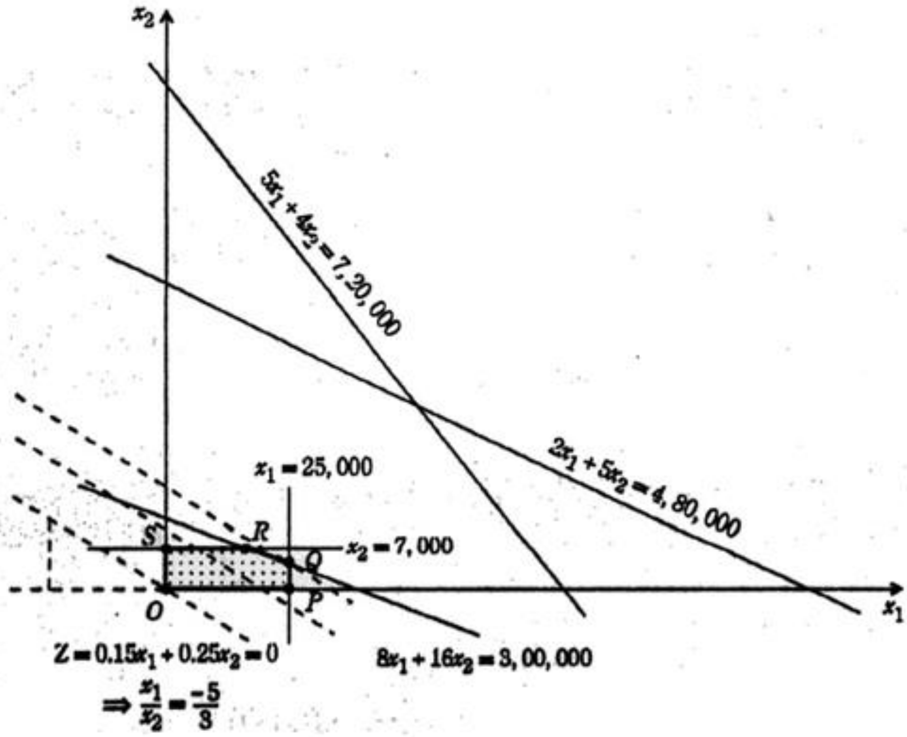
Since number of bottles cannot be negative

$\therefore \quad x_1 \ge 0, \; x_2 \ge 0.$

Hence the L.P.P. formulated for the given problem is as follows.

Max. $Z = 0{\cdot}15x_1 + 0{\cdot}25x_2$

s.t.
$$8x_1 + 16x_2 \le 3{,}00{,}000$$
$$2x_1 + 5x_2 \le 4{,}80{,}000$$
$$5x_1 + 4x_2 \le 7{,}20{,}000$$
$$x_1 \le 25{,}000$$
$$x_2 \le 7{,}000 \qquad \text{and} \quad x_1, x_2 \ge 0.$$



Solution of the problem. By Iso-profit method. Proceeding stepwise, the permissible region (the set of all points satisfying all constraints and the non-negative restrictions) consists of the shaded region *OPQRSO*.

Now we draw dotted line through the origin corresponding to $Z=0$. Drawing lines parallel to this line away from the origin we see that the farthest line (for Max. Z) in the permissible region passes through the vertex *Q* (25,000, 6250) of the convex polygon, which is the point of intersection of the lines $8x_1+16x_2=3{,}00{,}000$ and $x_1=25{,}000$.

Hence the optimal solution is

$$x_1=25000,\ x_2=6250$$

and Max. $Z=0{\cdot}15\times 25000+0{\cdot}25\times 6250=\text{Rs. } 5312{\cdot}50$.

By corner point method (Verification). Solving simultaneously the equations of the corresponding intersecting lines, the coordinates of the vertices of the convex polygon are

$$O\,(0,0),\ P\,(25{,}000,0),\ Q\,(25000,\ 6250),\ R\,(23500,\ 7000),\ S\,(0,\ 7000).$$

The values of objective function Z at these corners are as follows :

Point	$Z = 0{\cdot}15x_1 + 0{\cdot}25x_2$ in Rs.
O (0, 0)	$Z = 0{\cdot}15 \times 0 + 0{\cdot}25 \times 0 = 0$
P (25000, 0)	$Z = 0{\cdot}15 \times 25000 + 0 = 3750$
Q (25000, 6250)	$Z = 0{\cdot}15 \times 25000 + 0{\cdot}25 \times 6250 = 5312{\cdot}50$ (Max.)
R (23500, 7000)	$Z = 0{\cdot}15 \times 23500 + 0{\cdot}25 \times 7000 = 5275$
S (0, 7000)	$Z = 0 + 0{\cdot}25 \times 7000 = 1750$

Hence $Z =$ Rs. 5312·50 is maximum when $x_1 = 25000$ and $x_2 = 6250$.

Hence the planner should produce 25000 and 6250 bottles of 8 and 16 ounces respectively and then his profit will be maximum of Rs. 5312·50.

Example 1

A company has three plants at locations A, B, C which supply to warehouses located at D, E, F, G, and H. Monthly plant capacities are 800, 500 and 900 units, respectively. Monthly warehouse requirements are 400, 400, 500, 400 and 800 units, respectively. Unit transportation costs (in Rs) are given below. Determine an optimum distribution for the company in order to minimise the total transportation cost.

table 4.60

From \ To	D	E	F	G	H
A	5	8	6	6	3
B	4	7	7	6	5
C	8	4	6	6	4

Solution: In this problem, $\Sigma a_i = 2200$ units, whereas $\Sigma b_j = 2500$. So, the problem is an unbalanced one.

table 4.61

	D	E	F	G	H	Plant capacity
A	5	8	6	6	3	800
B	4	7	7	6	5	500
C	8	4	6	6	4	900
P	0	0	0	0	0	300
Requirements	400	400	500	400	800	

So, introduce a dummy plant P having all transportation costs equal to zero and having the plant availability equal to (2500 – 2200) = 300 units. The modified table is shown in Table 4.62.

Using VAM the following IBFS is obtained.

table 4.62

	D	E	F	G	H	
A	5	8	[500] 6	6	[300] 3	800
B	[400] 4	7	7	[100] 6	[ε] 5	$500 + \varepsilon = 500$
C	8	[400] 4	6	6	[500] 4	900
P	0	0	0	[300] 0	0	300
	400	400	500	400	$800 + \varepsilon = 800$	

Since the number of occupied cells = 7 which is less than $(m + n - 1) = 8$, the solution is degenerate.

To make the number of allocations equal to 8 introduce a infinitesimal quantity ε in the independent cell (2, 5). Now, test the current solution for optimality, by MODI's method.

Starting iteration:

table 4.63

(3) 5	(5) 8	[500] $-\theta$ 6	(2) 6	[300] $+\theta$ 3	$u_1 = -2$
[400] 4	(2) 7	(–1) 7	[100] $+\theta$ 6	[ε] $-\theta$ 5	$u_2 = 0$
(5) 8	[400] 4	(–1) 6	(1) 6	[500] 4	$u_3 = -1$
(2) 0	(1) 0	(–2) $+\theta$ 0	[300] $-\theta$ 0	(1) 0	$u_4 = -6$
$v_1 = 4$	$v_2 = 5$	$v_3 = 8$	$v_4 = 6$	$v_5 = 5$	

Here, $\theta = \min(500, \varepsilon, 300) = \varepsilon$. So, enter the non-basic cell (4, c) and leave the basic cell (2, 5).

First iteration:

table 4.64

(1) 5	(5) 8	500–ε, −θ, 6	(0) 6	300+ε, +θ, 3	$u_1 = 0$
400, 4	(4) 7	(1) 7	100+ε, 6	(2) 5	$u_2 = 0$
(3) 8	400, 4	(–1) 6	(–1) +θ 6	500, −θ, 4	$u_3 = 1$
(2) 0	(3) 0	ε, +θ, 0	300–ε, −θ, 0	(3) 0	$u_4 = -6$
$v_1 = 4$	$v_2 = 3$	$v_3 = 6$	$v_4 = 6$	$v_5 = 3$	

$\theta = \min(500 - \varepsilon, 500 - \varepsilon, 300 - \varepsilon) = 300 - \varepsilon.$

Second iteration:

table 4.65

(2) 5	(5) 8	200, −θ, 6	(1) 6	600+2ε, +θ, 3	$u_1 = -1$
400, 4	(3) 7	(0) 7	100+ε, 6	(1) 5	$u_2 = 0$
(4) 8	400, 4	(–1) +θ 6	300–ε, 6	200–ε, −θ, 4	$u_3 = 0$
(3) 0	(3) 0	300, 0	(1) 0	(3) 0	$u_4 = -7$
$v_1 = 4$	$v_2 = 4$	$v_3 = 7$	$v_4 = 6$	$v_5 = 4$	

Here, $\theta = \min(200, 200 - \varepsilon) = 200 - \varepsilon$

The third iteration is:

table 4.66

(1) 5	(4) 8	ε 6	(0) 6	$800+\varepsilon$ 3	$u_1 = 0$
400 4	(3) 7	(1) 7	$100+\varepsilon$ 6	(2) 5	$u_2 = 0$
(4) 8	400 4	$200-\varepsilon$ 6	$300-\varepsilon$ 6	(1) 4	$u_3 = 0$
(2) 0	(2) 0	300 0	(0) 0	(3) 0	$u_4 = -6$
$v_1 = 4$	$v_2 = 4$	$v_3 = 6$	$v_4 = 6$	$v_5 = 3$	

Since all the net evaluations are non-negative, the optimum solution is
$x_{13} = 0$, $x_{15} = 800$, $x_{21} = 400$, $x_{24} = 100$, $x_{32} = 400$, $x_{33} = 200$, $x_{34} = 300$, $x_{43} = 300$.

The optimum transportation cost is given by

$$z = \text{Rs } 0 \times 6 + 800 \times 3 + 400 \times 4 + 100 \times 6 + 400 \times 4 + 200 \times 6 + 300 \times 6 + 300 \times 0 = \text{Rs } 9200.$$

Example 4

A product is produced by four factories F_1, F_2, F_3 and F_4. Their unit production cost are Rs 2, 3, 1 and 5, respectively. Production capacity of the factories are 50, 70, 30 and 50 units, respectively. The product is supplied to four stores S_1, S_2, S_3 and S_4 the requirements of which are 25, 35, 105 and 20, respectively. Unit cost of transportation are given in Table 4.102. Find the transportation plan such that the total production and transportation cost is minimum.

table 4.77

	S_1	S_2	S_3	S_4
F_1	2	4	6	11
F_2	10	8	7	5
F_3	13	3	9	12
F_4	4	6	8	3

table 4.78

	S_1	S_2	S_3	S_4	Capacity
F_1	4	6	8	13	50
F_2	13	11	10	8	70
F_3	14	4	10	13	30
F_4	9	11	13	8	50
Demand	25	35	105	20	

Solution: Form the transportation table which consists of both production and transportation costs (see Table 4.78). The total capacity = 200 and the total demand = 185. The problem is an unbalanced one which is converted into a balanced one by adding a dummy store S_5 with cost 0 and supply in excess is given to this store.

table 4.79

(25) 4	(5) 6	(20) 8	13	0	50
13	11	(50) 10	(20) 8	0	70
14	(30) 4	10	13	0	30
9	11	(35) 13	8	(15) 0	50
25	35	105	20	15	

The initial basic feasible solution is obtained by least cost method. The solution is non-degenerate. The total transportation cost is

$$z = \text{Rs } [4 \times 25 + 6 \times 5 + 8 \times 20 + 10 \times 50 + 8 \times 20 + 4 \times 30 + 13 \times 55 + 0 \times 15] = \text{Rs } 1525.$$

Optimum solution by MODI's method is obtained. The final iteration now becomes:

table 4.80

25 4	5 6	20 8	(10) 13	(5) 0	$u_1 = 0$
(7) 13	(3) 11	70 10	5 8	(3) 0	$u_2 = 2$
(12) 14	30 4	(4) 10	(12) 13	(7) 0	$u_3 = -2$
(0) 9	(1) 11	15 13	(0) 8	15 0	$u_4 = 5$
$v_1 = 4$	$v_2 = 6$	$v_3 = 8$	$v_3 = 3$	$v_5 = -5$	

Since all $d_{ij} \geq 0$, the solution is optimum but an alternative solution exists with transportation cost as Rs 1465.

Ex. 7. *A manufacturer produces three models (I, II and III) of a certain product. He uses two types of raw materials (A and B) of which 4000 and 6000 units respectively are available. The raw material requirements per unit of the three models are given below :*

Raw Material	*Requirement per unit of given model*		
	I	*II*	*III*
A	2	3	5
B	4	2	7

The labour time for each unit of model I is twice that of model II and three times that of model III. The entire labour force of the factory can produce the equivalent of 2500 units of model I. A market survey indicates that the minimum demand of the three models are 500, 500 and 375 units respectively. However, the ratio of the number of units produced must be equal to 3 : 2 : 5. Assume that the profits per unit of models I, II and III are rupees 60, 40 and 100 respectively. Formulate the problem as a L.P.P. in order to determine the number of units of each product which will maximize profit.

Sol. Let the manufacturer produce x_1, x_2, x_3 units of models I, II and III respectively. Since the profit per unit on model I, II and III is Rs. 60, Rs. 40 and Rs. 100 respectively, the objective function is to maximize the profit :

$$Z = 60x_1 + 40x_2 + 100x_3. \qquad \ldots(1)$$

The raw materials used and available give rise to the following constraints respectively :

$$2x_1 + 3x_2 + 5x_3 \le 4000 \qquad \ldots(2)$$

$$4x_1 + 2x_2 + 7x_3 \le 6000 \qquad \ldots(3)$$

Now if t be the labour time required for one unit of model I, then the time required for one unit of model II will be $\frac{1}{2}t$ and that for the model III will be $\frac{1}{3}t$. As the factory can produce 2500 units of model I, so the restriction on the production time will be

$$t x_1 + \frac{1}{2} t x_2 + \frac{1}{3} t x_3 \le 2500\, t \quad \text{i.e.,} \quad 6x_1 + 3x_2 + 2x_3 \le 15{,}000. \qquad \ldots(4)$$

Now according to the market demand, we must have

$$x_1 \ge 500,\ x_2 \ge 500,\ x_3 \ge 375. \qquad \ldots(5)$$

Further the ratio of the number of units of different types of models is 3 : 2 : 5

i.e., $\quad x_1 = 3k,\ x_2 = 2k,\ x_3 = 5k.$

These give rise to the constraints

$$\frac{1}{3}x_1 = \frac{1}{2}x_2 \quad \text{and} \quad \frac{1}{2}x_2 = \frac{1}{5}x_3 \qquad \ldots(6)$$

Thus, the linear programming problem is as follows :

Maximize $Z = 60x_1 + 40x_2 + 100x_3$

subject to the conditions : $2x_1 + 3x_2 + 5x_3 \leq 4000$

$$4x_1 + 2x_2 + 7x_3 \leq 6000$$

$$6x_1 + 3x_2 + 2x_3 \leq 15{,}000$$

$$2x_1 = 3x_2,\ 5x_2 = 2x_3 \quad \text{and} \quad x_1 \geq 500,\ x_2 \geq 500,\ x_3 \geq 375.$$

Note. x_1, x_2, x_3 will automatically be non-negative due to (5). So we need not to mention that $x_1, x_2, x_3 \geq 0$. Even if we write then these will be redundant constraints.

Example 5.11 A manufacturer of jeans is interested in developing an advertising campaign that will reach four different age groups. Advertising campaigns can be conducted through TV, radio and magazines. The following table gives the estimated cost in paise per exposure for each age group according to the medium employed. In addition, maximum exposure levels possible in each of the media, namely, TV, radio and Maganizes are 40, 30 and 20 millions, respectively. Also, the minimum desired exposures within each age group, namely 13–18,19–25, 36 and older, are 30, 25, 15, and 10 millions. The objective is to minimise the cost of obtaining the minimum exposure level in each age group.

Media	*Age Groups*			
	13-18	*19-25*	*26-35*	*36 and older*
TV	12	7	10	10
Radio	10	9	12	10
Magazines	14	12	9	12

(i) Formulate the above as a transportation problem, and find the optimal solution.
(ii) Solve this problem if the policy is to provide at least 4 million exposures through TV in the 13–18 age group and at least 8 million exposures through TV in the age group 19–25. *(CA, May, 1991)*

(i) The given problem is formulated as a transportation problem and shown in Table 5.42. With the aggregate of 'maximum exposures available' equal to 90 million and 'minimum exposures required' totalling as 80 million, the problem is unbalanced. Hence, a dummy category to age-groups is added to make the problem balanced.

Table 5.42 *Advertising Campaign Schedule*

Media	*Age Groups*					*Max. Exposures available (in millions)*
	13–18	*19–25*	*26–35*	*36 & older*	*Dummy*	
TV	12	7	10	10	0	40
Radio	10	9	12	10	0	30
Magazines	14	12	9	12	0	20
Min. Exposures (in millions)	30	25	15	10	10	90

The initial solution using VAM is presented in Table 5.43. The solution is degenerate because there are 6 occupied cells against the required number of $7(= 5 + 3 - 1)$. Degeneracy is resolved by placing an ε in the cell represented by last column of the second row.

The solution is tested for optimality and is found to be non-optimal. Thus, a closed path is drawn as shown.

Table 5.43 *Initial Solution: Non-optimal*

Media	Age-groups 13–18	19–25	26–35	36 & over	Dummy	Max. Exp.	u_i
TV	12 (−1)	7 [25]	10 [5]	10 [10]	+0 (1)	40	0
Radio	10 [30]	9 (−3)	12 (−3)	10 (−1)	0 [ε]	30	−1
Magazines	14 (−4)	12 (−6)	+ 9 [10]	12 (−3)	− 0 [10]	20	−1
Min. Exp	30	25	15	10	10	90	
v_j	11	7	10	10	1		

The improved solution is given in Table 5.44. It is tested and found to be optimal. It may be noted, however, it is not unique since the cell (2, 4) $\Delta_{ij} = 0$.

The solution is:

Through TV : 25 m people to reach in the age group 19–25, 10 m people to reach in the age group 36 and over

Through Radio : 30 m people to reach in the age group 13–18

Through Magazines : 15 m people to reach in the age group 26–35

Total minimum cost = Rs 71 lakh

(ii) Since the optimal solution requires reaching 25 m people in the age group 19–25 through TV, it entirely meets the requirement of at least 8 m such exposures. Now, to ensure 4 m exposures through TV in the age group 13–18, we place a 4 in the cell (1, 1) and draw a closed path and make adjustments in the cells which lie on it. The cells include (1, 5), (2, 5) and (2, 1). The resulting solution is provided in Table 5.45.

Table 5.44 *Improved Solution: Optimal*

Media	Age-groups 13–18	19–25	26–35	36 & over	Dummy	Max. Exp.
TV	12	7 [25]	10	10 [10]	0 [5]	40
Radio	10 [30]	9	12	10	0 [ε]	30
Magazines	14	12	9 [15]	12	0 [5]	20
Min. Exp	30	25	10	10	10	90

Table 5.45 *Revised Solution*

Media	*Age-groups*					*Max. Exp.*
	13–18	*19–25*	*26–35*	*36 & over*	*Dummy*	
TV	4 12	25 7	10	10 10	1 0	40
Radio	26 10	9	12	10	4 0	30
Magazines	14	12	15 9	12	5 0	20
Min. Exp	30	25	15	10	10	90

The revised plan is:

Through TV : 4 m people to reach in the age group 13–18, 25 m people to reach in the age group 19–25, 10 m people to reach in the age group 36 and above

Through Radio : 26 m people to reach in the age group 13–18

Through Magazines : 15 m people to reach in the age group 26–35

This would involve a total cost of Rs 71.8 lakh.