SOLID GEOMETRY

(Analytical Geometry of Three Dimensions)

[For B. A.; B. Sc. & B. Tech. Students of All Indian Universities]

(WITH OBJECTIVE TYPE QUESTIONS)

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PREFACE TO THE TWENTIETH EDITION

It gives me great pleasure in bringing out the Twentieth Edition of this book in such a short time.

The book has been thoroughly revised and number of new examples and articles selected from recent examination papers, have been added.

Besides giving due credit to the printers and publishers, I express my thanks to the professors and students for the appreciation and patronage of the book.

Suggestions for further improvement of the book will be highly appreciated.

—Author

PREFACE TO THE FIRST EDITION

The present book comprising the subject "Solid Geometry" is meant for the students appearing in the B. A.; B. Sc. & B. E. Examinations of All Indian Universities. Efforts have been made to make the treatment logical and simple.

I gratefully acknowledge my indebtedness to various authors and publishers whose books have been freely consulted during the preparation of this book.

I shall be grateful to the readers for pointing out errors and omissions that, inspite of all care, might have crept in.

I look forward to the suggestions from the readers for the improvement of the book.

—Author

Contents

Note– Please note that important questions are marked as* and very important questions as** in this book.

CHAPTER I

Systems of Co-ordinates

§ 1.01. Introduction. In plane analytical geometry, we know that the points are coplanar *i.e.* they lie in a single plane called the co-ordinate plane. The position of a point in this plane is determined with reference to two intersecting straight lines called the **co-ordinate axes** and their point of intersection is called the **origin** of co-ordinates. If these two axes of reference (generally we call them x and y axes) cut each other at right angle, they are called **rectangular axes** otherwise they are called **oblique axes.** These axes divide the co-ordinate plane in four quadrants.

But it is not possible for us to determine the position of all the points we can imagine (in space) with reference to these co-ordinates. If a point P lies on the co-ordinate plane, its position can be difinitely fixed with reference to the co-ordinanate axes (which have been chosen on this co-ordinate plane). But if P is any point in space, then it must not necessary lie on this co-ordinate plane and its position may be determined by its perpendicular distance, z say from this co-ordinate plane. Thus we find that to locate a point in space we require a third dimension z in addition to the two dimensions x and y. Thus we conclude that the geometry of all the bodies which occupy space *i.e.* are of three dimensions is called the geometry of three dimensions or solid geometry. Thus in solid geometry we discuss about the position of points in space.

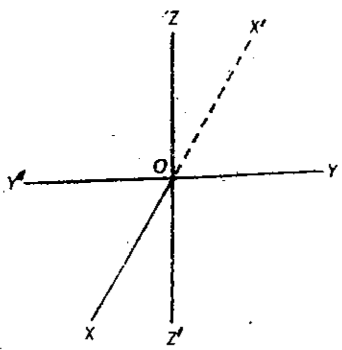
Here students should note that if the third dimension z is given the value zero, then the three dimensional system reduces to two dimensional system.

§ 1.02. Rectangular Cartesian Co-ordinates.

This system of co-ordinates is most commonly used in three dimensional analytical geometry.

Draw two mutually perpendicular lines $Y'OY$ and ZOZ' in the plane of the paper intersecting each other at O.

Through O imagine a third line $X'OX$ perpendicular to both the above lines, as shown in the adjoining figure.



(Fig. 1)

O is called, the origin and the set of these three mutually perpendicular lines $X'OX$, $Y'OY$ and ZOZ' are called the co-ordinate axes (rectangular). The co-ordinate axes XOX', YOY' and ZOZ' are called x-axis, y-axis and z-axis repectively.

Here OX is the positive direction of x-axis whereas OX' is its negative direction. Similarly OY and OZ are the positive directions and OY' and OZ' are negative directions of y and z-axes respectively.

These three axes taken in pairs give us three planes YOZ, ZOX and XOY and they are called yz, zx and xy-planes respectively. These planes are called **co-ordinate planes.**

§ 1.03. Co-ordinates of a point in space.

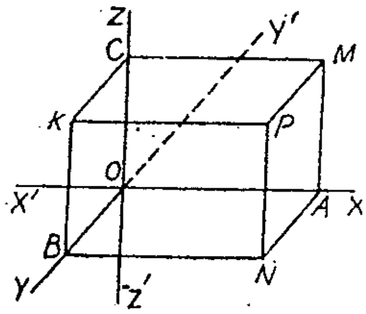
Let P be any point in space. Through P pass planes parallel to the three co-ordinate planes and cutting x, y and z axes in A, B and C respectively as shown in Fig. 2.

These planes, together with the co-ordinate planes form a rectangular parallelopiped.

The position of P relative to the co-ordinate system is given by its perpendicular distances from the co-ordinate planes and these distances are given by lengths OA, OB and OC.

Let $OA = a$, $OB = b$ and $OC = c$.

Then a, b, c are called x-co-ordinate, y-co-ordinate, and z-co-ordinate respectively of the point P. The point P is referred as (a, b, c).



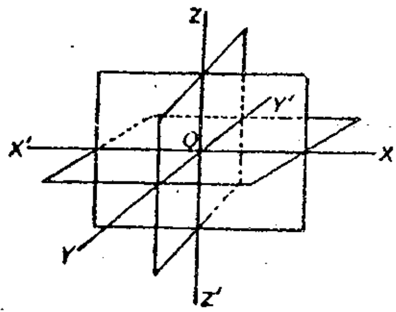
(Fig. 2)

Each of the above co-ordinates is measured from the origin O along the corresponding co-ordinate axis and is positive or negative according as its direction is the same or opposite to that of the positive direction of the axis as defined in § 1.02 above.

Here it should be noted that a point P in space has one and only one set of co-ordinates (a, b, c) say referred to one set of rectangular co-ordinate axes.

The co-ordinates of the origin O are $(0, 0, 0)$ and those of A, B, C, N, K and M in fig. 2 are $(a, 0, 0)$; $(0, b, 0)$; $(0, 0, c)$; $(a, b, 0)$; $(0, b, c)$ and $(a, 0, c)$ respectively.

The three co-ordinate planes divide space into eight parts (see fig. 3) and these parts are called **octants.** The sign of co-ordinates of a point determine the octant in which it lies.



(Fig. 3)

The signs for the eight octants are given in the tabular form below :

Octant	$OXYZ$	$OX'YZ$	$OX'Y'Z$	$OXY'Z$	$OXY'Z'$	$OX'YZ'$	$OX'Y'Z'$	$OXY'Z'$
x	+	−	−	+	+	−	−	+
y	+	+	−	−	−	+	−	−
z	+	+	+	+	−	−	−	−

The students should note in the above table that if negative side of an axis appears in octant then the corresponding co-ordinate is negative. For example in the octant $OX'Z'$ (which has X' and Z') the x and z-co-ordinaates are negative.

Solved Examples on § 1.02 – § 1.03.

Ex. 1. What is the locus of the point (i) whose x-co-ordinate is 5 and (ii) whose x-co-ordinate is 3 and y-co-ordinaate is 4 ?

Sol. (i) The required locus is $x = 5$ which represents a plane parallel to YOZ plane (or yz-plane) at a distance 5 from it.

(ii) The required locus is $x = 3$ and $y = 4$ which represents line parallel to z-axis.

Ex. 2. State the common property of the co-ordinates of points lying on (i) z-axis and (ii) xz-plane.

Sol. (i) The x and y-co-ordinates of all points on the z-axis are zero

i.e $\qquad x = 0 = y.$

(ii) The y-co-ordinates of all points on the xz-plane is zero *i.e.* $y = 0$. Hence the required common property is $y = 0$. **Ans.**

Ex. 3. In fig 2 Page 2 if OA = 2, OB = 3 and OC = 4.

(i) What are the equations to the planes PNAM, PMCK and PNBK ?

(ii) What equations are satisfied by the co-ordinates of any point on the line PN ?

Sol. (i) The y and z-co-ordinates of points on the plane $PNAM$ differ but all have the same x-co-ordinate *i.e.* the distance of all points on the plane $PNAM$ from the yz-plane remains constant and equal to OA *i.e.* 2. Hence the equation of the plane $PNAM$ is $x = 2$.

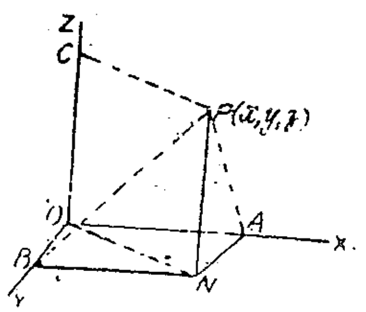
Similarly the equations of the planes $PMCK$ and $PNBK$ are $z = 4$ and $y = 3$ respectively.

(ii) All points on the line PN are equidistant from yz-plane and each of them is at a distance $BN = OA = 2$ from yz-plane *i.e.* for each point on the line PN we have $x = 2$. Similarly each point on the line PN is equidistant from the xz-plane and as such we have $y = OB = 3$. But the z-co-ordinaate of all points on the line PN are different. Hence all the points on the line PN satisfy the equations $x = 2$, $y = 3$. **Ans.**

Ex. 4. What are the perpendicular distance of the point (x, y, z) from the co-ordinaate axes ?

Sol. Let P be the point (x, y, z).

From P draw PN perpendicular to xy-plane (*i.e.* XOY plane).



(Fig. 4)

From N draw NA and NB perpendiculars to OX and OY respectively.
Join PA, PB and ON. Also from P draw PC perpendicular to z-axis.
Then $PN = z$, $NA = y$.
Then $OA = x = NB$.
Now the length of perpendicular from P to $OX = PA$ **(Note)**

$$= \sqrt{(PN^2 + NA^2)} = \sqrt{(z^2 + y^2)};$$

The length of perpendicular from P to $OY = PB$ **(Note)**

$$= \sqrt{(PN^2 + NB^2)} = \sqrt{(z^2 + x^2)}$$

The length of perpendicular from P to $OZ = PC = ON$ **(Note)**

$$= \sqrt{(OA^2 + NA^2)} = \sqrt{(x^2 + y^2)}$$

Exercises on § 1.02 – § 1.03

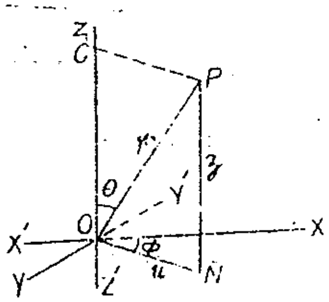
Ex. 1. State the common property of the co-ordinates of all points lying on (i) the xy-plane, (ii) the yz-plane, (iii) the xz-plane, (iv) the x-axis, (v) the y-axis and (vi) the z-axis.

Ans. (i) $z = 0$, (ii) $x = 0$, (iii) $y = 0$, (iv) $y = 0 = z$, (v) $x = 0 = z$, (vi) $x = 0 = y$.

Ex. 2. Show that the distance of the point (1, 2, 3) from the co-ordinate axes are $\sqrt{(13)}$, $\sqrt{(10)}$, $\sqrt{(5)}$.

****§ 1.04. Other methods of defining the position of any point P in space.**

(a) Cylindrical Co-ordinates. If XOX', YOY', ZOZ' are the rectangular axes and PN the perp. from P to the XOY plane the position of the point P can be determined if ON, the angle XON and NP are known. The measures of the quantities u, ϕ and z are the cylindrical co-ordinates of the point P.



(Fig. 5)

If the rectangular cartesian co-ordinates of P are (x, y, z) then those of N are $(x, y, 0)$ and we can easily have the following relations $\mathbf{x = u \cos \phi}$, $\mathbf{y = u \sin \phi}$ and $\mathbf{z = z}$ hence $u^2 = x^2 + y^2$ and $\phi = \tan^{-1}(y/x)$.

(b) Spherical Polar Co-ordinates.

The position of the point P can also be determined when OP, angle ZOP and angle XON are known.

The measures of these quantities r, θ, ϕ are known as spherical or three dimensional polar co-ordinates of the points P.

As before if the rectangular cartesian co-ordinates of P are (x, y, z), then we can have the following relations :

$$z = PN = OC = r \cos\theta \; ; \; u = ON = PC = r \sin\theta .$$

Also we have proved above $x = u \cos\phi$, $y = u \sin\phi$,

$\therefore \quad x = u \cos\phi = r \sin\theta \cos\phi$, $y = u \sin\phi = r \sin\theta \sin\phi$.

Thus we have $\mathbf{x = r \sin\theta \cos\phi}$, $\mathbf{y = r \sin\theta \sin\phi}$ **and** $\mathbf{z = r \cos\theta}$.

Also $r^2 = OP^2 = PN^2 + ON^2 = z^2 + u^2 = z^2 + (x^2 + y^2) = x^2 + y^2 + z^2$

and $$\tan\theta = \frac{u}{z} = \frac{\sqrt{(x^2+y^2)}}{z} \; ; \; \tan\phi = \frac{y}{x}$$

Solved Examples on § 1.04.

Ex. 1 (a). Transform the cartesian coordinates (1, 2, 3) of a point into spherical polar coordinates.

Sol. Let (r, θ, ϕ) be the polar co-ordinates of the point whose cartesian co-ordinates are given by $x = 1, y = 2, z = 3$.

Then from § 1.04 (b) above, we have

$$r^2 = x^2 + y^2 + z^2 = 1^2 + 2^2 + 3^2 = 14$$

or $$r = \sqrt{(14)}$$

Also $$\tan\theta = \frac{\sqrt{(x^2+y^2)}}{z} = \frac{\sqrt{(1^2+2^2)}}{3} = \frac{\sqrt{5}}{3}$$

and $$\tan\phi = y/x = 2/1 = 2.$$

$\therefore$ The required polar co-ordinates are

$$[\sqrt{(14)}, \tan^{-1}(\tfrac{1}{2}\sqrt{5}), \tan^{-1} 2].$$ **Ans.**

Ex. 1 (b). Find the cylindrical co-ordinates of the points whose rectangular co-ordinates are (2, 3, 5), (1, 2, 3) and (– 2, 5, 2).

(Bundelkhand 90)

Sol. Let (u, ϕ, z) be the cylindrical coordinates of the point (2, 3, 5) *i.e.* the point whose cartesian co-ordinates are $x = 2, y = 3, z = 5$.

Then from § 1.04 (a) Page 4, we have

$$u^2 = x^2 + y^2 = 2^2 + 3^2 = 13 \text{ or } u = \sqrt{(13)}$$

And $$\tan\phi = \frac{y}{x} = \frac{3}{5} \text{ or } \phi = \tan^{-1}\left(\frac{3}{5}\right)$$

$\therefore$ The required cylindrical co-ordinates are

$$[\sqrt{(13)}, \tan^{-1}(3/5), 5]$$ **Ans.**

Similarly find cylindrical co-ordinates for other points.

Ex. 2. Find (i) the cartesian, (ii) the cylindrical and (iii) the polar equations of the sphere whose centre is the origin and radius a.

Sol. (i) Let P be any point (x, y, z) on the sphere.

Then $a = OP = \sqrt{(PN^2 + ON^2)}$, see figure 5 Page 4 Ch. I.

$$= \sqrt{\{PN^2 + (OA^2 + NA^2)\}} = \sqrt{\{z^2 + (x^2 + y^2)\}}$$

or $$a^2 = x^2 + y^2 + z^2 \text{ i.e. } x^2 + y^2 + z^2 = a^2$$

is the required cartesian equation.

(ii) Let P be any point on the sphere.

Then $a = OP = \sqrt{(PN^2 + ON^2)}$, see figure 5 Page 4 Ch. I.

or $\quad a^2 = (z^2 + u^2)$ *i.e.* $u^2 + z^2 = a^2$,

is the required cylindrical equation.

(iii) Let $P(r, \theta, \phi)$ be any point on the sphere.

Then $a = OP = r$ *i.e.* $r = a$ is the required polar equation.

Ex. 3. Find (i) the cartesian, (ii) the cylindrical and (iii) the polar equation of the right circular cylinder whose axis is OZ and radius *a*.

Sol. Let $P(x, y, z)$ be any point on the cylinder.

Then the radius $a = PC = ON$ (see figure 5 Page 4 Ch. I)

$$= \sqrt{(OA^2 + NA^2)} = \sqrt{(x^2 + y^2)}$$

or $\quad x^2 + y^2 = a^2$ is the required cartesian equation of the cylinder.

(ii) Let $P(u, \phi, z)$ be any point on the cylinder.

Then radius $a = PC = ON$ (see fig. 5 Page 4 Ch. I)

or $\quad a = u$ *i.e.* $u = a$ is the required cylindrical equation.

(iii) Let $P(r, \theta, \phi)$ be any point of the cylinder.

Then radius $a = PC = OP \sin \theta$; see fig. 5 Page 4 Ch. I

or $\quad a = r \sin \theta$ *i.e.* $r \sin \theta = a$ is the required polar equation.

***Ex. 4. Find (i) the cartesian, (ii) cylindrical and (iii) the polar equations of the right circular cone whose vertex is O, axis OZ, and semi-vertical angle α.**

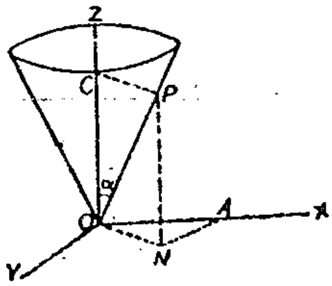
Sol. (i) Let $P(x, y, z)$ be any point on the surface of the cone. From P draw PN and PC perpendicular to xy-plane and OZ respectively. From N draw NA perpendicular to OX.

$\angle POC = \alpha$ (given)

Then $z = PN = OC$, and

$PC = ON = \sqrt{(OA^2 + NA^2)} = \sqrt{(x^2 + y^2)}$.

Now in $\Delta\, OPC$, $PC = OC \tan \alpha$



(Fig. 6)

or $\quad \sqrt{(x^2 + y^2)} = z \tan \alpha \quad$ or $\quad x^2 + y^2 = z^2 \tan^2 \alpha$

is the required cartesian equation of the cone.

(ii) Let $P(u, \phi, z)$ be any point on the cone.

Then as in case (i) above, we have $PC = OC \tan \alpha$

or $\quad ON = OC \tan \alpha, \because ON = PC = u$

or $\quad u = z \tan \alpha$ is the required cylindrical equation of the cone.

(iii) Let $P(r, \theta, \phi)$ be any point on the cone.

Then from figure 6 above it is evident that $\theta = \angle POC = \alpha$.

$\therefore \quad \theta = \alpha$ is required polar equation of the cone.

Ex. 5. Find (i) the cartesian, (ii) the cylindrical and (iii) the polar equation to the plane through OZ and making an angle α with the plane ZOX.

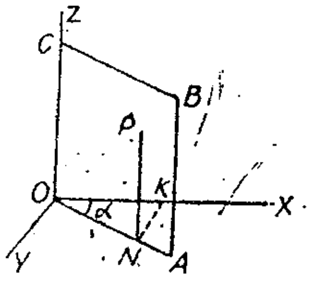
Sol. (i) Let $P(x, y, z)$ be any point on the plane $OABC$ passing through OZ and making an angle α with the zx-plane (*i.e.* ZOX plane).

From P draw PN perpendicular to xy-plane, meeting it in N.

From N draw NK perpendicular to OX.

Then $OK = x$, $NK = y$.

From the figure it is evident that



(Fig. 7)

$$\tan\alpha = \tan\angle NOK = \frac{NK}{OK} = \frac{y}{x}$$

or $\quad y = x\tan\alpha$ is the required cartesian equation of the plane $OABC$.

(ii) Let $P(u, \phi, z)$ be any point on the plane $OABC$.

Then ϕ = angle $AOX = \alpha$ (given).

$\therefore\ \phi = \alpha$ is the required cylindrical equation of the plane $OABC$.

(iii) Let $P(r, \theta, \phi)$ be any point on the plane $OABC$.

Proceed further as in case (ii) above.

Exercises on § 1.04

Ex. 1. Find the polar co-ordinates of the point (3, 4, 5) so that r may be positive.

(Hint. See Ex. 1 (a). Page 5) **Ans.** $[5\sqrt{2}, \pi/4, \tan^{-1}(4/3)]$

Ex. 2. What is the polar equation of the plane which is parallel to the xy-plane and is at a distance c from it ? What is the cylindrical equation of the plane parallel to the zx-plane at a distance b from it ?

Ans. $r\cos\theta = c$; $u\sin\phi = b$.

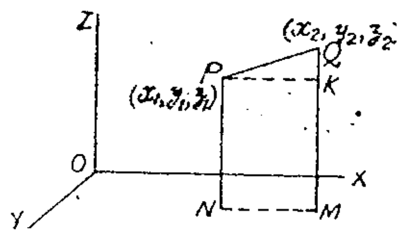
(*Kumaun 94*)

§ 1.05. Distance between two points.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the two points.

From P and Q draw PN and QM perpendiculars to xy-plane. Then the co-ordinates of N and M are $(x_1, y_1, 0)$ and $(x_2, y_2, 0)$ respectively.

The two dimensional co-ordinates of N and M referred to OX and OY are (x_1, y_1) and (x_2, y_2) respectively.



(Fig. 8)

$\therefore \quad NM^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. ...(i)

(See Author's Co-ordinate Geometry)

Now from P draw PK perpendicular to QM.

Then PK is parallel and equal to NM and $PN = KM$.

Now in $\Delta\, PKQ$, we have

$$PQ^2 = PK^2 + QK^2 = NM^2 + (QM - KM)^2, \quad \because PK = NM$$
$$= NM^2 + (QM - PN)^2, \qquad \because PN = KM$$
$$= \{(x_2 - x_1)^2 + (y_2 - y_1)^2\} + (z_2 - z_1)^2,$$
$$\because QM = z_2,\ PN = z_1 \text{ and from (i)}$$

$$\therefore \quad \mathbf{PQ = \sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}}}$$

i.e. **Distance between tow points.**

= √ [(difference of their x-co-ordinates)^2 + (difference of their y-co-ordinates)^2 + (difference of their z-co-ordinates)^2] **(Remember)**

Cor. The distance of the point $P(x_1, y_1, z_1)$ from the origin $O(0, 0, 0) = \sqrt{(x_1^2 + y_1^2 + z_1^2)}$.

Note : Working rule for proving a four sided figure to be :—

(i) a square :—Show that (a) four sides are equal and (b) the diagonals are equal.

(ii) a rhombus :—Show that (a) four sides are equal and (b) the diagonals are **not** equal.

(iii) a rectangle :—Show that (a) opposite sides are equal and (b) the diagonals are equal.

(iv) a parallelogram :—Show that (a) opposite sides are equal and (b) the diagonals are **not** equal.

Solved Examples on § 1.04 and § 1.05.

Ex. 1. (a) If A and B are the points (3, 4, 5) and (– 1, 3, – 7) then find the locus of P which moves so that PA^2 – PB^2 = 3.

Sol. Let P be (x, y, z). Then

$$PA^2 = (x-3)^2 + (y-4)^2 + (z-5)^2$$

and
$$PB^2 = (x+1)^2 + (y-3)^2 + (z+7)^2.$$

Given
$$PA^2 - PB^2 = 3.$$

or $\quad [(x-3)^2 + (y-4)^2 + (z-5)^2] - [(x+1)^2 + (y-3)^2 + (z+7)^2] = 3$

or $\quad 4x + y + 12z + 6 = 0$ is the required locus. **Ans.**

Ex. 1. (b) Find the locus of a point P which is at a distance r from the point (a, b, c).

Sol. Let P be (x, y, z). Then distance of $P(x, y, z)$ from the point $(a, b, c) = r$ (given)

i.e.
$$\sqrt{[(x-a)^2 + (y-b)^2 + (z-c)^2]} = r$$

or $(x-a)^2+(y-b)^2+(z-c)^2=r^2$ is the required locus. **Ans.**

****Ex. 2. (a) Find the locus of a point which moves so that the sum of its distances from the points (a, 0, 0) and (– a, 0, 0) is constant.**

Sol. Let $P(x, y, z)$ be the point whose locus is to be obtained. Let A and B be the points $(a, 0, 0)$ and $(-a, 0, 0)$ respectively.

Then according to the problem, we have

$$PA+PB=\text{constant}=2k \text{ (say)}. \qquad \textbf{(Note)}$$

or $\sqrt{[(x-a)^2+(y-0)^2+(z-0)^2]}+\sqrt{[(x+a)^2+(y-0)^2+(z-0)^2]}=2k$

or $\sqrt{[(x-a)^2+y^2+z^2]}=2k-\sqrt{[(x+a)^2+y^2+z^2]}$.

Squaring both sides, we have

$$(x-a)^2+y^2+z^2=4k^2+[(x+a)^2+y^2+z^2]-4k\sqrt{[(x+a)^2+y^2+z^2]}$$

or $$4k\sqrt{[(x+a)^2+y^2+z^2]}=4k^2+(x+a)^2-(x-a)^2$$

$$=4k^2+4ax, \text{ on simplyfying}$$

or $$\sqrt{[(x+a)^2+y^2+z^2]}=k+(ax/k)$$

Again squaring both sides we have

$$(x+a)^2+y^2+z^2=k^2+(a^2x^2/k^2)+2ax$$

or $$x^2+y^2+z^2+a^2=k^2+(a^2x^2/k^2)$$

or $x^2[1-(a^2/k^2)]+y^2+z^2=k^2-a^2$, which is the required locus. **Ans.**

Ex. 2. (b) Find the locus of the point the difference of whose distances from (2, 0, 0) and (– 2, 0, 0) is 1.

Sol. Do as Ex. 2 (a) above. **Ans.** $60x^2-4(y^2+z^2)=15$.

Ex. 3. (a) Show that (0, 7, 10), (– 1, 6, 6) and (– 4, 9, 6) form an isosceles right angled triangle.

Sol. Let A, B and C be the points $(0, 7, 10)$, $(-1, 6, 6)$ and $(-4, 9, 6)$ respectively.

Then $AB=\sqrt{[(0+1)^2+(7-6)^2+(10-6)^2]}=\sqrt{(1+1+16)}=3\sqrt{2}$

$BC=\sqrt{[(-1+4)^2+(6-9)^2+(6-6)^2]}=\sqrt{(9+9+0)}=3\sqrt{2}$

$CA=\sqrt{[(-4-0)^2+(9-7)^2+(6-10)^2]}=\sqrt{(16+4+16)}=6$

$\because$ $AB=BC$, so ΔABC is isosceles and as

$$AB^2+BC^2=(3\sqrt{2})^2+(3\sqrt{2})^2=18+18=36=CA^2,$$

so ΔABC is right angled triangle, $\angle B$ being a right angle.

Ex. 3. (b) Are the points (3, 6, 9), (10, 20, 30), (25, – 41, 5) the vertices of a right angled triangle ?

Sol. Do as Ex. 3 (a) above. **Ans.** No.

****Ex. 4. Find the co-ordinates of the point which is equidistant from the four points O, A, B and C, where O is the origin and A, B, C are the points on the axes of x, y, z respectively at distances a, b, c from the origin.**

Sol. According to the problem, the co-ordinates of the points O, A, B and C are respectively $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$.

Let $P(x, y, z)$ be the centre of the sphere. Then as O, A, B and C lie on the sphere, so P is equi-distant from O, A, B and C, *i.e.* $PO = PA = PB = PC$ *i.e.* $PO^2 = PA^2 = PB^2 = PC^2$.

From $PO^2 = PA^2$, we get $x^2 + y^2 + z^2 = (x - a)^2 + (y - 0)^2 + (z - 0)^2$

or $$2ax = a^2 \quad \text{or} \quad x = \frac{1}{2}a.$$

Similarly from $PO^2 = PB^2$ we get $y = \frac{1}{2}b$

and from $PO^2 = PC^2$, we get $z = \frac{1}{2}c$.

$\therefore$ The required centre of the sphere is $(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c)$. **Ans.**

Ex. 5. Prove that the four points whose co-ordinates are (5, – 1, 1), (7, – 4, 7), (1, – 6, 10), (– 1, – 3, 4) are the vertices of a rhombus.

Sol. Let the given points be $A(5, -1, 1)$, $B(7, -4, 7)$, $C(1, -6, 10)$ and $D(-1, -3, 4)$.

Then we have

$AB = \sqrt{[(5-7)^2 + (-1+4)^2 + (1-7)^2]} = \sqrt{(4+9+36)} = 7$;

$BC = \sqrt{[(7-1)^2 + (-4+6)^2 + (7-10)^2]} = \sqrt{(36+4+9)} = 7$;

$CD = \sqrt{[(1+1)^2 + (-6+3)^2 + (10-4)^2]} = \sqrt{(4+9+36)} = 7$

and $DA = \sqrt{[(-1-5)^2 + (-3+1)^2 + (4-1)^2]} = \sqrt{(36+4+9)} = 7$

Also length of diagonal AC

$= \sqrt{[(5-1)^2 + (-1+6)^2 + (1-10)^2]} = \sqrt{(16+25+81)} = \sqrt{(122)}$

and length of diagonal BD

$= \sqrt{[(7+1)^2 + (-4+3)^2 + (7-4)^2]} = \sqrt{(64+1+9)} = \sqrt{(74)}$

Thus we prove that $AB = BC = CD = DA$ *i.e.* four sides of the figure $ABCD$ are equal and the diagonals AC and BD are not equal.

$\therefore$ The four given points are the vertices of a rhombus.

[See Note (ii) at the end of § 1.05 Page 8].

Ex. 6. Find the distance of the point whose spherical polar co-ordinates are $(2\sqrt{2}, \frac{1}{4}\pi, \pi/6)$ from the point whose cartesian co-ordinates are $(2\sqrt{3}, -1, -4)$.

Sol. In § 1.04 Pages 4-5 we have read if (r, θ, ϕ) be the spherical polar co-ordinates of a point (x, y, z), then

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r\cos\theta$$

$\therefore$ If (x_1, y_1, z_1) be the cartesian co-ordinates of the point whose spherical polar co-ordinates are $(2\sqrt{2}, \frac{1}{4}\pi, \frac{1}{6}\pi)$, then we have

$x_1 = 2\sqrt{2} \sin\frac{1}{4}\pi \cos\frac{1}{6}\pi = 2\sqrt{2} \,.\, (1/\sqrt{2}) \,.\, \frac{1}{2}\sqrt{3} = \sqrt{3}$;

$y_1 = 2\sqrt{2} \sin\frac{1}{4}\pi \sin\pi/6 = 2\sqrt{2} \,.\, (1/\sqrt{2}) \frac{1}{2} = 1$;

$z_1 = 2\sqrt{2} \cos \frac{1}{4}\pi = 2\sqrt{2}\,(1/\sqrt{2}) = 2.$

$\therefore$ This point in cartesian co-ordinates is $(\sqrt{3}, 1, 2)$.

The other point is $(2\sqrt{3}, -1, -4)$. So the required distance

$= \sqrt{[(2\sqrt{3} - \sqrt{3})^2 + (-1-1)^2 + (-4-2)^2]}$

$= \sqrt{(3 + 4 + 36)} = \sqrt{(43)}$. **Ans.**

Ex. 7. Show that the points A (1, 2, 3), B (4, 0, 4) and C (– 2, 4, 2) are collinear. *(Rohilkhand 97)*

Sol. $AB = \sqrt{[(4-1)^2 + (0-2)^2 + (4-3)^2]} = \sqrt{(9+4+1)} = \sqrt{(14)}$

$BC = \sqrt{[(-2-4)^2 + (4-0)^2 + (2-4)^2]} = \sqrt{(36+16+4)}$

$= \sqrt{(56)} = 2\sqrt{(14)}$

and $AC = \sqrt{[(-2-1)^2 + (4-2)^2 + (2-3)^2]} = \sqrt{(9+4+1)} = \sqrt{(14)}$

$\because AB + BC = \sqrt{(14)} + \sqrt{(14)} = 2\sqrt{(14)} = AC$

$\therefore$ The points A, B, C are collinear. $\boxed{AB+BC=AC}$ Hence proved.

Exercises on § 1.04 and § 1.05

Ex. 1. Find the locus of the point $P(x, y, z)$, the difference of whose distances from $(0, 0, -4)$ and $(0, 0, 4)$ is 4. **Ans.** $x^2 + y^2 - 3z^2 + 12 = 0$

Ex. 2. Find the locus of the point the sum of whose distances from $(4, 0, 0)$ and $(-4, 0, 0)$ is equal to 10. **Ans.** $9x^2 + 25(y^2 + z^2) = 225$.

Ex. 3. Find the locus of the point $P(x, y, z)$ if $PA^2 + PB^2 = 2k^2$, where A and B are the points $(3, 4, 5)$ and $(-1, 3, -7)$.

Ans. $x^2 + y^2 + z^2 - 2x - 7y + 2z + 5 = k^2$

Ex. 4. A is the point $(-2, 2, 3)$ and B is the point $(13, -3, 13)$. A point $P(x, y, z)$ moves so that $3\,PA = 2\,PB$. Find the locus of P.

Ans. $x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0$

Ex. 5. The axes are rectangular and A, B are the points $(3, 4, 5)$, $(-1, 3, -7)$. A variable point P has co-ordinates (x, y, z). Find the locus of P if

$PA^2 - PB^2 = 2k^2$. **Ans.** $8x + 2y + 24z + 9 + 2k^2 = 0$

Ex. 6. Find the equation of the sphere whose centre is $(0, 1, -1)$ and radius 2. **Ans.** $x^2 + y^2 + z^2 - 2y + 2z = 2$

***Ex. 7.** Prove that the four points A, B, C, D whose coordinates are $(1, 1, 1)$, $(-2, 4, 1)$, $(-1, 5, 5)$ and $(2, 2, 5)$ are the vertices of a square.

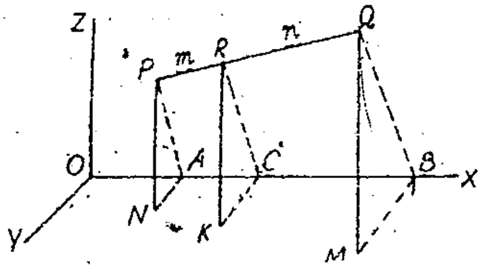
[Hint : Show that $AB = BC = CD = DA$ and $AC = BD$].

Ex. 8. Show that the point $D\,(-\frac{1}{2}, 2, 0)$ is the circumcentre of the triangle formed by the points $A(1, 1, 0)$, $B(1, 2, 1)$ and $C(-2, 2, -1)$.

[Hint : Show that $DA = DB = DC$].

Ex. 9. Show that the points $(1, 2, 3)$, $(-1, -2, -1)$, $(2, 3, 2)$ and $(4, 7, 6)$ are the vertices of a parallelogram.

§ 1.06. Section of the join of two points. *To find the coordinates of the point which divides the line joining two given points* $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$ *in a given ratio* $m:n$.



(Fig. 9)

Let $R(x, y, z)$ be the point which divides the line joining P and Q in the ratio $m:n$.

Let planes PNA, QMB and RKC through P, Q and R parallel to the plane YOZ meet the axis of x at A, B and C respectively.

Then as these planes divide any two straight lines proportionately, therefore, we have

$$\frac{AC}{CB} = \frac{PR}{RQ} = \frac{m}{n} \quad \text{or} \quad \frac{(OC - OA)}{(OB - OC)} = \frac{m}{n} \quad \text{or} \quad \frac{x - x_1}{x_2 - x} = \frac{m}{n}$$

or $\quad nx - nx_1 = mx_2 - mx \quad$ or $\quad x(n + m) = mx_2 + nx_1$

or
$$\mathbf{x} = \frac{\mathbf{nx_1 + mx_2}}{\mathbf{n + m}}$$

Similarly $\mathbf{y} = \dfrac{\mathbf{ny_1 + my_2}}{\mathbf{m + n}}$ **and** $\mathbf{z} = \dfrac{\mathbf{nz_1 + mz_2}}{\mathbf{n + m}}$

Note. If (m/n) is negative then R divides PQ externally and if it is positive then internally.

Cor. 1. The mid-point of the line PQ is

$$[\tfrac{1}{2}(x_1 + x_2), \tfrac{1}{2}(y_1 + y_2), \tfrac{1}{2}(z_1 + z_2)].$$

Cor. 2. General co-ordinates of a point on the line PQ.

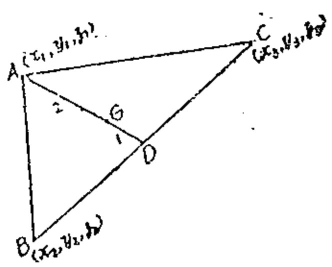
Let $m : n = \lambda : 1$, then the co-ordinates of the point R are

$$\left[\frac{x_1 + \lambda x_2}{1 + \lambda}, \frac{y_1 + \lambda y_2}{1 + \lambda}, \frac{z_1 + \lambda z_2}{1 + \lambda}\right].$$

Corresponding to every value of λ we shall get a point on the line PQ.

***§ 1.07. Centroid of a triangle.**

To find the co-ordinates of the centroid of a triangle whose vertices are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, *and* $C(x_3, y_3, z_3)$.



(Fig. 10)

The coordinates of D, the mid-point of BC are

$\{\frac{1}{2}(x_2+x_3), \frac{1}{2}(y_2+y_3), \frac{1}{2}(z_2+z_3)\}$

$\because$ The centroid G divides the median AD in the ratio 2 : 1.

$\therefore$ The co-ordinates of G are

$$\left[\frac{1 \cdot x_1 + 2 \cdot \frac{1}{2}(x_2+x_3)}{1+2}, \frac{1 \cdot y_1 + 2 \cdot \frac{1}{2}(y_2+y_3)}{1+2}, \frac{1 \cdot z_1 + 2 \cdot \frac{1}{2}(z_2+z_3)}{1+2}\right]$$

i.e. $$\left[\frac{1}{3}(x_1+x_2+x_3), \frac{1}{3}(y_1+y_2+y_3), \frac{1}{3}(z_1+z_2+z_3)\right]$$

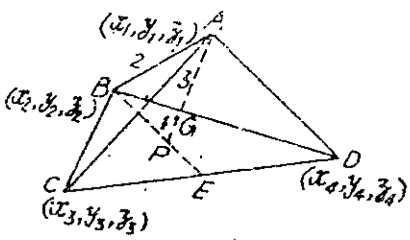
i.e. $$\left[\frac{\text{sum of } x\text{–co–ordinates}}{3}, \frac{\text{sum of } y\text{–co–ordinates}}{3}, \frac{\text{sum of } z\text{–co–ordinates}}{3}\right]$$ **(Remember)**

§ **1.08. Centroid of a tetrahedron.** *To find the co-ordinates of the centroid of the tetrahedron whose vertices are* $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ *and* $D(x_4, y_4, z_4)$.

We know from Statics that the centroid of the tetrahedron divides the line joining any vertex and the C.G. of the opposite face in the ratio 3 : 1.

...**(See Author's Statics)**

Let P be the C.G. of the triangle BCD. Join PA and divide it in the ratio 3 : 1. Let G be the point on AP such that $AG : GP = 3 : 1$, then G is the required centroid of the given tetrahedron.



(Fig. 11)

Now the co-ordinates of P, the centroid of $\Delta\, BCD$, are given by

$$\left[\frac{1}{3}(x_2+x_3+x_4), \frac{1}{3}(y_2+y_3+y_4), \frac{1}{3}(z_2+z_3+z_4)\right]$$

...See § 1.07 on Page 12.

$\therefore$ If $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of G, which divides AP in the ratio 3 : 1,

then $$\bar{x} = \frac{1 \cdot x_1 + 3 \cdot \frac{1}{3}(x_2+x_3+x_4)}{1+3}$$ ...See § 1.06 Page 12

$$= \frac{1}{4}(x_1+x_2+x_3+x_4)$$

Similarly $\bar{y} = \frac{1}{4}(y_1+y_2+y_3+y_4)$ and $\bar{z} = \frac{1}{4}(z_1+z_2+z_3+z_4)$.

Hence the co-ordinates of the centroid G of the tetrahedron are

$$\left[\frac{1}{4}(x_1+x_2+x_3+x_4), \frac{1}{4}(y_1+y_2+y_3+y_4), \frac{1}{4}(z_1+z_2+z_3+z_4)\right]$$

i.e. $\left[\dfrac{\text{sum of } x\text{–co–ordinates}}{4}, \dfrac{\text{sum of } y\text{–co–ordinates}}{4}, \dfrac{\text{sum of } z\text{–co–ordinates}}{4}\right]$ **(Remember)**

Solved Examples on § 1.06 – § 1.08.

Ex. 1. Find the co-ordinates of the point which divides the line joining (1, – 1, 2) and (2, 3, 7) in the ratio 2 : 3.

Sol. Let (x, y, z) be the required point, then

$$x = \frac{2\,(2) + (3)\,(1)}{2+3} = \frac{7}{5};\ y = \frac{2\,(3) + 3\,(-1)}{2+3} = \frac{3}{5}$$

and $$z = \frac{2\,(7) + 3\,(2)}{2+3} = \frac{20}{5} = 4.$$

$\therefore$ The required point is (7/5, 3/5, 4). **Ans.**

Ex. 2. The point P lies on the line whose end points are A (7, 2, 1) and B (10, 5, 7). If the y-co-ordinate of P is 4; find its other co-ordinates.

Sol. Let $P\,(x, y, z)$ divide the join of $A\,(7, 2, 1)$ and $B\,(10, 5, 7)$ in the ratio $m : n$.

Then $$y = \frac{m\,(5) + n\,(2)}{m+n}$$ ...See § 1.06 Page 12

But it is given that the y-co-ordinate of P is 4

$$\therefore\quad 4 = \frac{5m + 2n}{m+n} \text{ or } 4m + 4n = 5m + 2n \text{ or } m = 2n \text{ or } \frac{m}{n} = \frac{2}{1}$$

$$\therefore\quad x = \frac{m\,(10) + n\,(7)}{m+n} = \frac{2\,(10) + 1\,(7)}{2+1} = 9;$$

$$z = \frac{m\,(7) + n\,(1)}{m+n} = \frac{2\,(7) + 1\,(1)}{2+1} = 5$$ **Ans.**

***Ex. 3 (a). Given three collinear points A (3, 2, – 4), B (5, 4, – 6), C (9, 8, – 10), find the ratio in which B divides AC.** *(Rohilkhand 93, 90)*

Sol. Let B divide AC in the ratio $m : n$.

Then x-co-ordinate of $B = \dfrac{m\,.\,9 + n\,.\,3}{m+n}$ or $5 = \dfrac{9m + 3n}{m+n}$

or $5m + 5n = 9m + 3n$ or $4m = 2n$ or $m : n = 1 : 2$. **Ans.**

***Ex. 3 (b). Show that the three points A, B, C whose coordinates are respectively (– 2, 3, 5), (1, 2, 3) and (7, 0, – 1) are collinear. Also find the ratio in which point B divides the line AC.** *(Kanpur 94)*

Sol. Here $AB = \sqrt{[(1+2)^2 + (2-3)^2 + (3-5)^2]} = \sqrt{(9+1+4)} = \sqrt{(14)}$;

$$BC = \sqrt{[(7-1)^2 + (0-2)^2 + (-1-3)^2]} = \sqrt{(36+4+16)}$$
$$= \sqrt{(56)} = 2\sqrt{(14)}$$

and $$AC = \sqrt{[(7+2)^2 + (0-3)^2 + (-1-5)^2]} = \sqrt{(81+9+36)}$$

$= \sqrt{(126)} = 3\sqrt{(14)}$

$\because AB + BC = \sqrt{(14)} + 2\sqrt{(14)} = 3\sqrt{(14)} = AC$

$\therefore$ Points A, B, C are collinear.

Again let B divide AC in the ratio $m : n$.

Then x-coordinate of $B = \dfrac{m \cdot 7 + n \cdot (-2)}{m+n}$ or $1 = \dfrac{7m - 2n}{m+n}$

or $m + n = 7m - 2n$ or $6m = 3n$ or $m : n = 1 : 2$ **Ans.**

***Ex. 4. Find the ratio in which the co-ordinate planes divide the line joining the points (– 2, 4, 7), (3, – 5, 8).**

Sol. Let xy-plane divide the line joining the given points in the ratio $m : n$.

Then z-co-ordinate of the point where this line meets the xy-plane is zero, as the z-co-ordinate of any point on the xy-plane is zero

$\therefore \quad 0 = \dfrac{m(8) + n(7)}{m+n}$ or $8m + 7n = 0$ or $m : n = -7 : 8$. **Ans.**

Similarly we can find that yz-plane (for every point on which $x = 0$) and xz-plane (for every point on this plane $y = 0$) divides the line joining the given points in the ratio $2 : 3$ and $4 : 5$. **Ans.**

Ex. 5. Find the ratio in which the yz-plane divides the line joining the points (3, 5, – 7) and (– 2, 1, 8). Find also the points of division.

Sol. Let yz-plane divide the line joining the given points in the ratio $m : n$.

Then x-co-ordinate of the point where this line meets the yz-plane is zero, as the x-co-ordinate of any point on the yz-plane is zero.

$\therefore \quad 0 = \dfrac{m(3) + n(-2)}{m+n}$ or $3m - 2n = 0$ or $m : n = 2 : 3$. **Ans.**

Also if $(0, y_1, z_1)$ be this point of division, then

$$y_1 = \frac{2(5) + 3(1)}{2+3} = \frac{10+3}{5} = \frac{13}{5}$$

and

$$z_1 = \frac{2(-7) + 3(8)}{2+3} = \frac{-14+24}{5} = \frac{10}{5} = 2$$

$\therefore$ The required point is $(0, 13/5, 2)$. **Ans.**

***Ex. 6. From any point (1, – 2, 3) lines are drawn to meet the shpere $x^2 + y^2 + z^2 = 4$ and they are divided in the ratio 2 : 3. Prove that the points of section lie on a sphere.**

Sol. Let any line through $(1, -2, 3)$ meet the given sphere in (x_1, y_1, z_1).

Then
$$x_1^2 + y_1^2 + z_1^2 = 4. \qquad \text{...(i)}$$

Also let (x_2, y_2, z_2) be the point which divides the join of $(1, -2, 3)$ and (x_1, y_1, z_1) in the ration $2 : 3$.

Then $x_2 = \dfrac{2 \cdot x_1 + 3 \cdot 1}{2+3}$ or $x_1 = \dfrac{5x_2 - 3}{2}$;

$y_2 = \dfrac{2y_1 + 3(-2)}{2+3}$ or $y_1 = \dfrac{5y_2 + 6}{2}$;

$z_2 = \dfrac{2 \cdot z_1 + 3(3)}{2+3}$ or $z_1 = \dfrac{5z_2 - 9}{2}$

Substituting these values of x_1, y_1 and z_1 in (i) we get

$$\frac{1}{4}(5x_2-3)^2 + \frac{1}{4}(5y_2+6)^2 + \frac{1}{4}(5z_2-9)^2 = 4$$

or $$25x_2^2 + 25y_2^2 + 25z_2^2 - 30x_2 + 60y_2 - 90z_2 + 110 = 0$$

or $$5x_2^2 + 5y_2^2 + 5z_2^2 - 6x_2 + 12y_2 - 18z_2 + 22 = 0.$$

$\therefore$ The locus of the point of section (x_2, y_2, z_2) is

$5(x^2+y^2+z^2) - 6x + 12y - 18z + 22 = 0$, which represents a sphere.

Ex. 7. Find the distance of the point (1, 2, 0) from the point where the line joining (2, – 3, 1) and (3, – 4, – 5) cuts the plane 2x + y + z = 7.

Sol. Let the line joining the point (2, – 3, 1) and (3, – 4, – 5) meet the given plane in (x_1, y_1, z_1). Then as the point (x_1, y_1, z_1) lies on the plane

$2x + y + z = 7$, so we have $2x_1 + y_1 + z_1 = 7$. ...(i)

Also let the point (x_1, y_1, z_1) divide the line joining (2, – 3, 1) and (3, – 4, – 5) in the ratio $m : n$.

Then $x_1 = \dfrac{3m+2n}{m+n}$, $y_1 = \dfrac{-4m-3n}{m+n}$, $z_1 = \dfrac{-5m+n}{m+n}$ ...(ii)

Substituting these values in (i) we get

$2(3m+2n) + (-4m-3n) + (-5m+n) = 7(m+n)$

or $10m - 5n = 0$ or $n = 2m$

Substituting this value in (ii) we get

$$x_1 = \frac{3m+4m}{m+2m} = \frac{7}{3};\ y_1 = \frac{-4m-6m}{m+2m} = \frac{-10}{3};\ z_1 = \frac{-5m+2m}{m+2m} = -1$$

$\therefore$ The point where the line joining (2, – 3, 1) and (3, – 4, – 5) meets the plane $2x + y + z = 7$ is (7/3, – 10/3, – 1).

$\therefore$ the required distance

$=$ distance between (1, 2, 0) and (7/3, – 10/3, – 1)

$= \sqrt{[\{(7/3-1\}^2 + \{(-10/3)-2\}^2 + (-1-0)^2]}$

$= \sqrt{[(16/9) + (256/9) + 1]} = \sqrt{(281/9)} = \frac{1}{3}\sqrt{(281)}$. **Ans.**

****Ex. 8. Show that the plane ax + by + cz + d = 0 divides the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio**

$$-(ax_1 + by_1 + cz_1 + d)/(ax_2 + by_2 + cz_2 + d).$$

Sol. Let the given plane meet the line joining the given points in (x_3, y_3, z_3). Then $\quad ax_3 + by_3 + cz_3 + d = 0.$ ...(i)

Also let the point (x_3, y_3, z_3) divide the line joining the given points in the ratio $m : n$.

Then $x_3 = \dfrac{mx_1 + nx_2}{m+n}$; $y_3 = \dfrac{my_1 + ny_2}{m+n}$; $z_3 = \dfrac{mz_1 + nz_2}{m+n}$

Substituting these values in (i) we get

$$a\left(\frac{mx_1 + nx_2}{m+n}\right) + b\left(\frac{my_1 + ny_2}{m+n}\right) + c\left(\frac{mz_1 + nz_2}{m+n}\right) + d = 0$$

or $\quad a(mx_1 + nx_2) + b(my_1 + ny_2) + c(mz_1 + nz_2) + d(m+n) = 0$

or $\quad m(ax_1 + by_1 + cz_1 + d) + n(ax_2 + by_2 + cz_2 + d) = 0$

or $\quad \dfrac{n}{m} = -\left(\dfrac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}\right).$ Hence proved.

***Ex. 9. Find the ratios in which the sphere $x^2 + y^2 + z^2 = 504$ divides the line joining the points (12, – 4, 8) and (27, – 9, 18).**

Sol. Let the sphere meet the line through given points in (x_1, y_1, z_1).

Then $\quad x_1^2 + y_1^2 + z_1^2 = 504.$ ...(i)

Also let the point (x_1, y_1, z_1) divide the line joining the given points in the ratio $m : n$.

Then $x_1 = \dfrac{m(27) + n(12)}{m+n}$, $y_1 = \dfrac{m(-9) + n(-4)}{m+n}$, $z_1 = \dfrac{m(18) + n(8)}{m+n}$

Substituting these values of x_1, y_1, z_1 in (i) we get

$$\frac{(27m + 12n)^2}{(m+n)^2} + \frac{(-9m - 4n)^2}{(m+n)^2} + \frac{(18m + 8n)^2}{(m+n)^2} = 504$$

or $\quad 9(9m + 4n)^2 + (9m + 4n)^2 + 4(9m + 4n)^2 = 504(m+n)^2$

or $\quad 14(9m + 4n)^2 = 504(m+n)^2$, or $(9m + 4n)^2 = 36(m+n)^2$

or $\quad 9m + 4n = \pm 6(m+n)$, taking square root of both sides.

Taking + sign we get $3m = 2n$ or $m : n = 2 : 3$. **Ans.**

Taking – sign we get $15 = -10n$ or $m : n = -2 : 3$. **Ans.**

Exercises on § 1.06 – § 1.08

Ex. 1. Find the co-ordinates of the point which divides the join of (1, 2, 3) and (3, – 5, 6) in the ratio 3 : (– 5). **Ans.** (– 2, 25/2, – 3/2)

Ex. 2. Find the ratio in which the join of (2, 1, 5) and (3, 4, 3) is divided by the plane $x + y - z = \frac{1}{2}$. **Ans.** 5 : 7

Ex. 3. Find the ratio in which the xz-plane divides the join of (– 3, 4, – 8) and (5, – 6, 4). **Ans.** 2 : 1

Ex. 4. Given three collinear points A (3, 2, – 4), B (5, 4, – 6), C (9, 8, – 10), show that B divides AC in the ratio 1 : 2. *(Rohilkhand 90)*

EXERCISES ON CHAPTER I

Ex. 1. The point A (1, 2, 2) is one vertex of the rectangular parallelopiped formed by the co-ordinate planes, and the planes passing through A parallel to the co-ordinate planes. Find the co-ordinates of other seven vertices.

Ans. (0, 0, 0), (1, 0, 0), (1, 2, 0), (0, 0, 2), (0, 2, 0), (0, 2, 2) (2, 0, 2).

Ex. 2. The line joining the points (1, 8, – 1) and (4, – 4, 2) meets the zx and xy planes at P and Q respectively. Find the co-ordinates of P and Q.

Ans. P (3, 0, 1), Q (2, 4, 0)

Ex. 3. The sphere $x^2 + y^2 + z^2 - 2x + 6y + 14z + 3 = 0$ meets the line joining A (2, – 1, 4), B (5, 5, 5) in the points C and D. Prove that

$$AC : CB = - AD : DB = 1 : 2.$$

(Hint : See Ex. 9 Page 17)

Ex. 4. Find the ratio in which the join of the points (x_r, y_r, z_r), $r = 1, 2$ is divided by the plane $ax + by + cz + d = 0$.

(Hint : See Ex. 8 Page 17).

Ex. 5. Find the ratio in which yz-plane divides the line joining the points (– 2, 4, 7) and (3, – 5, 8). **Ans.** 2 : 3

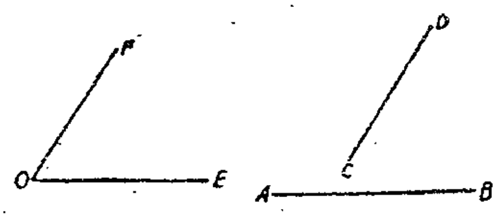
Ex. 6. Find the ratio in which the line joining the points (2, 4, 5) and (3, 5, – 4) is divided by the yz-plane. Find also the co-ordinates of the point at which the line meets the xy-plane.

CHAPTER II

Direction Cosines and Projection

§ 2.01. To find the angles between two non-coplanar lines.

Let AB and CD be two non-coplanar and non-intersecting lines. Take a point O and through O draw two lines OE and OF parallel to AB and CD respectively. Then the angle between AB and CD is equal to the angle EOF.

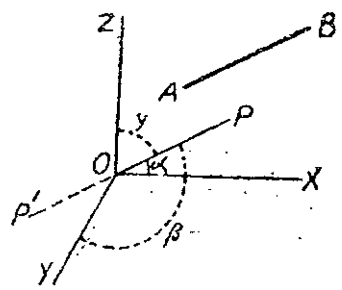


(Fig. 12)

§ 2.02. Direction cosines of a line. (*Kumaun 95; Purvanchal 97*)

If α, β, γ are the angles that a given line AB makes with the positive directions of x, y and z axes then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines (or d.c.'s) of the line AB. Generally the direction cosines are represented by l, m, n.

Here we should remember that *the angles which a given line AB makes with the coordinate axes are the same as made by the line OP drawn through the origin O parallel to the given line AB.* (See Fig 13 and § 2.01 above).



(Fig. 13)

These angles α, β, γ are called the **direction angles** of the line AB.

The direction cosines of the line BA are $\cos(\pi-\alpha), \cos(\pi-\beta)$ and $\cos(\pi-\gamma)$ *i.e.* $-\cos\alpha, -\cos\beta, -\cos\gamma$ since the direction angles of the line OP' through O parallel to BA are $\pi-\alpha, \pi-\beta$ and $\pi-\gamma$. **(Note)**

Here students should remember that angles α, β and γ are not coplanar.

Cor. d. c.'s of the co-ordinate axes : The x-axis makes angles $0, \pi/2$ and $\pi/2$ with x, y and z-axis respectively. So the direction cosines of the x-axis are $\cos 0, \cos\pi/2, \cos\pi/2$, *i.e.* $1, 0, 0$.

Similarly the d.c.'s of y and z-axes are $0, 1, 0$ and $0, 0, 1$ respectively.

**§ 2.03. *To prove that $l^2+m^2+n^2=1$, where l, m, n are the direction-cosines of a line AB.* (*Kumaun 96*)

Or

To show that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$, where $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of a line AB. (*Kanpur 93; Kumaun 92*)

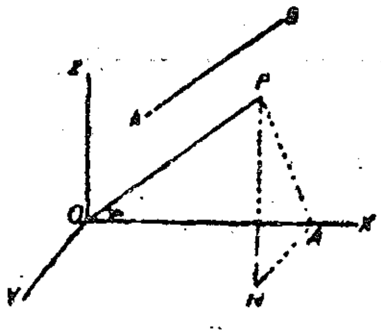
Through the origin O draw a line OP parallel to AB and of length r (say) so that direction cosines of the line OP are l, m, n or $\cos\alpha, \cos\beta, \cos\gamma$ (say).

Let the co-ordinates of P be (x, y, z). Draw PN perpendicular to the xy-plane and NA perpendicular to x-axis.

Then $OA = x$, $AN = y$ and $NP = z$.

$\therefore$ In the right angled triangle OAP, $\cos\alpha = \dfrac{OA}{OP}$ or $l = \dfrac{x}{r}$, $\quad \because \cos\alpha = l$

or $\quad x = lr.$



(Fig. 14)

Similarly $y = mr$, $z = nr$.

Also $OA^2 = x^2 + y^2 + z^2$

$\therefore \quad r^2 = (lr)^2 + (mr)^2 + (nr)^2$ or $\mathbf{l^2 + m^2 + n^2 = 1}$ ...(i)

or $$\mathbf{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.} \quad ...(ii)$$

i.e. **the sum of the squares of the direction cosines of a line is equal to unity.**

(Remember)

Cor. 1. From above we find that if l, m, n be the d.c.'s, of a line OP and $OP = r$, then the co-ordinates of the point P are **(lr, mr, nr).**

Cor. 2. Also if $OP = r$ and the co-ordinates of the point P (x, y, z), then the d.c.'s of the line OP are $x/r, y/r, z/r$.

***§ 2.04. Direction ratios.** *(Rohilkhand 94)*

Definition. If a, b, c be three numbers proportional to the actual direction cosines l, m, n of a given line then they are defined as direction ratios or direction numbers

Now from the above definition, we have

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm\frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{(a^2 + b^2 + c^2)}} = \pm\frac{1}{\sqrt{(a^2 + b^2 + c^2)}}, \quad \because l^2 + m^2 + n^2 = 1$$

or $$l = \pm\frac{a}{\sqrt{(a^2 + b^2 + c^2)}}, \quad m = \pm\frac{b}{\sqrt{(a^2 + b^2 + c^2)}}, \quad n = \pm\frac{c}{\sqrt{(a^2 + b^2 + c^2)}}$$

(Rohilkhand 94)

If AB be the line whose direction ratios are a, b, c then the direction cosines of AB are given by the + sign and those of the line BA by − sign. **(Note)**

Note 1. From above we conclude that if direction ratios a, b and c of a line are given we should divide each of them by $\sqrt{(a^2 + b^2 + c^2)}$ to get the corresponding direction cosines.

Note 2. The sum of the squares of the direction ratios of a line is **not equal** to unity.

Solved Examples on § 2.01 – § 2.04.

Ex. 1. (a) Find the direction cosines of a line that makes equal angles with the axes. *(Kanpur 93)*

Sol. Let l, m, n be the required direction cosines. Thus as the line makes equal angles with the axes, so we have $l = m = n$
(*i.e.* $\cos\alpha = \cos\beta = \cos\gamma$ or $\alpha = \beta = \gamma$).

Also we know $l^2 + m^2 + n^2 = 1$.

So here we have $l^2 + l^2 + l^2 = 1$ or $3l^2 = 1$ or $l = \pm 1/\sqrt{3}$.

The required direction cosines are $\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3}$. **Ans.**

[**Note** : Here **four** such lines are possible, and their d.c.'s are $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$; $-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$; $1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}$ and $1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3}$; as l, m, n and $-l, -m, -n$ are the d.c.'s of the same line].

Ex. 1 (b). Find the direction cosines of a line which is equally inclined to the positive directions of the axes. *(Kanpur 96)*

Sol. Do as Ex. 1 (a). above. Here l, m, n are all positive.

Ans. $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$.

***Ex. 2. If α, β, γ be the angles which a given line makes with the positive directions of the axes then prove that**

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$ *(Meerut 96P; Rohilkhand 96)*

Sol. As in § 2.03 Page 19 we can prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

or $$(1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$$

or $$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 3 - 1 = 2.$$

***Ex. 3. Find the d.c.'s l, m, n of two lines which are connected by the relation $l - 5m + 3n = 0$ and $7l^2 + 5m^2 - 3n^2 = 0$.**

(Garhwal 95; Kanpur 95; Kumaun 95)

Sol. Given $l - 5m + 3n = 0$ ...(i) $7l^2 + 5m^2 - 3n^2 = 0$. ...(ii)

Substituting the value of l from (i) in (ii) we get

$$7(5m - 3n)^2 + 5m^2 - 3n^2 = 0 \quad \text{or} \quad 180m^2 - 210mn + 60n^2 = 0$$

or $$6m^2 - 7mn + 2n^2 = 0 \quad \text{or} \quad (3m - 2n)(2m - n) = 0$$

or $$\frac{m}{n} = \frac{2}{3}, \frac{1}{2}$$

If $m/n = 2/3$, we have $\dfrac{m}{2} = \dfrac{n}{3} = \dfrac{5m - 3n}{5.2 - 3.3} = \dfrac{l}{1}$, from (i)

or $$\frac{l}{1} = \frac{m}{2} = \frac{n}{3} = \frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{(1^2 + 2^2 + 3^2)}} = \frac{1}{\sqrt{(14)}}$$

$\therefore$ The direction cosines of one line are

$$1/\sqrt{14},\ 2/\sqrt{14},\ 3/\sqrt{14}. \qquad \textbf{Ans.}$$

If $m/n = 1/2$, we have $\dfrac{m}{1} = \dfrac{n}{2} = \dfrac{5m-3n}{5.1-3.2} = \dfrac{l}{-1}$, from (i)

or $$\frac{l}{-1} = \frac{m}{1} = \frac{n}{2} = \frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{[(-1)^2+(1)^2+(2)^2]}} = \frac{1}{\sqrt{6}}$$

$\therefore$ The direction cosines of the other line are $-1/\sqrt{6},\ 1/\sqrt{6},\ 2/\sqrt{6}$. **Ans.**

Exercise on § 2.01 – § 2.04.

Ex. The direction ratios of a line are 2, 3, 4. What are its direction cosines ? **Ans.** $2/\sqrt{(29)},\ 3/\sqrt{(29)},\ 4/\sqrt{(29)}$

§ 2.05. Projection of a point on a line.

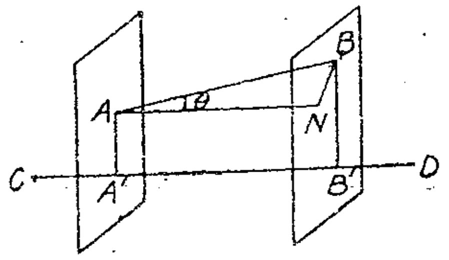
The projectin of a point P on a line AB is the foot N of the perpendicular PN from P on the line AB.

(Fig. 15)

N is also the same point where the line AB meets the plane through P and perpendicular to AB.

§ 2.06. Projection of a segment of a line on another line.

The projection of the segment AB of a given line on another line CD is the segment $A'B'$ of CD, where A' and B' are the projections of the points A and B on the line CD. Here we may also say that A' and B' are the points in which planes through A and B perpendicular to CD meet the line CD.



(Fig. 16)

To find the length of the projection $A'B'$.

Through A draw the line AN parallel to $A'B'$ meeting the plane through B perpendicular to CD in the point N.

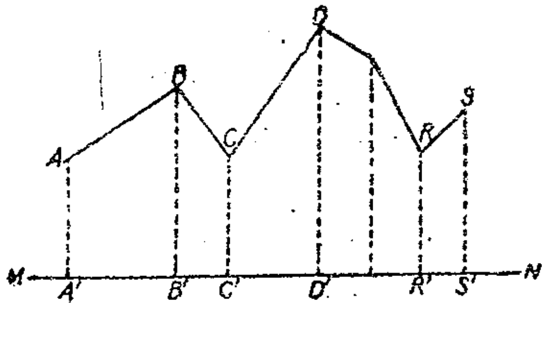
Then $$AN = A'B' \qquad \text{...(i)} \quad \textbf{(Note)}$$

If θ be the angle between the lines AB and CD, then AN being parallel to CD we have $\angle BAN = \theta$.

Also BN is a line through N and is lying in the plane which is perpendicular to the line CD and hence perpendicular to AN. $\therefore \angle ANB = 90°$.

$\therefore$ From (i) we get $A'B' = AN = AB \cos\theta$

§ 2.07 Projection of a broken line on a given line



(Fig. 17)

Let $A, B, C, D, \ldots, S$ be any number of points in space and let $A', B', C', \ldots S'$ be their projections on any line MN. Then as $A', B', C', D, \ldots, S'$ lie on the same straight line MN, so we have

$A'B' + B'C' + C'D' + \ldots + R'S' = A'S'$. ... (i)

But $A'B'$ is the projection of AB on the line MN.

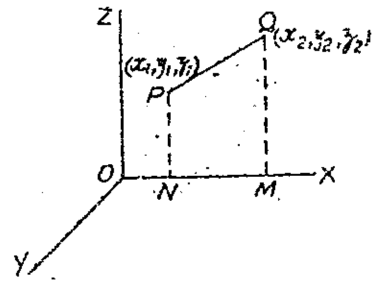
Hence from (i) we conclude that the sum of projections of $AB, BC, CD, \ldots\ldots, RS$ on the line MN = projection $A'S'$ of AS on the line MN.

§ 2.08. Direction cosines of the line joining the points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2). *(Bundelkhand 91)*

Let N and M be the projections of the points P and Q on x-axis.

Then $ON = x_1$ and $OM = x_2$

$\therefore \quad NM = OM - ON = x_2 - x_1$



(Fig. 18)

Also if OP makes angles α, β and γ with the axes of x, y and z respectively, then from § 2.06 Page 22 we know that $NM = PQ \cos \alpha$ or $(x_2 - x_1) = PQ \cos \alpha$

or $\quad \dfrac{x_2 - x_1}{\cos \alpha} = PQ$

Similarly projecting PQ on the y and z-axes we can find that

$$\frac{y_2 - y_1}{\cos \beta} = PQ = \frac{z_2 - z_1}{\cos \gamma}$$

So we have $\dfrac{x_2 - x_1}{\cos \alpha} = \dfrac{y_2 - y_1}{\cos \beta} = \dfrac{z_2 - z_1}{\cos \gamma}$

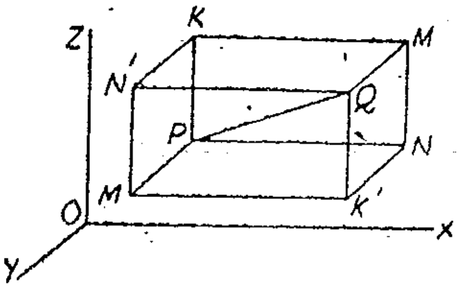
Hence the direction cosines of the line joining two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are proportional to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$ *i.e.* direction ratios of PQ are $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$ *i.e.* direction ratios of the line joining two points are difference of their x-co-ordinates, difference of their y-co-ordinates, difference of their z-co-ordinates. **(Remember)**

And actual direction cosines are $\dfrac{x_2 - x_1}{PQ}, \dfrac{y_2 - y_1}{PQ}, \dfrac{z_2 - z_1}{PQ}$,

where $PQ = \sqrt{\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\}}$

***§ 2.09. Projection of a line joining the points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) on another line whose direction cosines are l, m and n.**

Through P and Q draw planes parallel to the co-ordinate planes and thus form a rectangular parallelpiped as shown in the figure.



(Fig. 19)

Then $PN = x_2 - x_1$,

$NK' = y_2 - y_1$

and $K'Q = z_2 - z_1$

If the line whose direction cosine are given as l, m, n, makes angles α, β and γ with the co-ordinate axes, then $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

The projections of PN (which is parallel to x-axis) on the given line

$= PN \cdot \cos\alpha = (x_2 - x_1)\cos\alpha$. ... See § 2.06 Page 22

Similarly the projections of NK' (which is parallel to y-axis) and $K'Q$ (which is parallel to z-axis) on the given line are $NK' \cos\beta$ and $K'Q\cos\gamma$ respectively *i.e.* $(y_2 - y_1)\cos\beta$ and $(z_2 - z_1)\cos\gamma$ respectively.

Also from § 2.07 Page 23 we know that the sum of projections of PN, NK' and $K'Q$ on a line equal to the projections of PQ on that line and therefore the projections of PQ on the given line.

$= $ sum of projections of PN, NK' and $K'Q$ on that line.

$= (x_2 - x_1)\cos\alpha + (y_2 - y_1)\cos\beta + (z_2 - z_1)\cos\gamma$

$= (\mathbf{x_2} - \mathbf{x_1})\, l + (\mathbf{y_2} - \mathbf{y_1})\, m + (\mathbf{z_2} - \mathbf{z_1})\,\mathbf{n}$, from (i).

Cor. If P is a point (x_1, y_1, z_1) then the projection of OP on a line whose direction cosines are l_1, m_1, n_1 is $l_1 x_1 + m_1 y_1 + n_1 z_1$, where O is the origin.

Refer Fig 4 Page 4.

The projection of OP on the given line

$=$ projection of $OA +$ projection of $AN +$ projection of NP on that line

$= x_1 l_1 + y_1 m_1 + z_1 n_1$.

Solved Examples on § 2.05 to § 2.09.

Ex. 1. The co-ordinates of a point A are (2, 3, 6). Find the direction cosines of OA, where O is origin.

Sol. The direction cosines of OA (See § 2.08 Page 23) are

$$\frac{2-0}{\sqrt{(2^2+3^2+6^2)}}, \frac{3-0}{\sqrt{(2^2+3^2+6^2)}}, \frac{6-0}{\sqrt{(2^2+3^2+6^2)}}, \quad \because O \text{ is } (0, 0, 0)$$

or $2/7,\ 3/7,\ 6/7$.

****Ex. 2. The projections of a line on the co-ordinatte axes are 2, 3, 6. Find the length and the direction cosines of the line.** (*Rohilkhand* 97)

Sol. Let PQ be the line and its direction cosines be $\cos\alpha, \cos\beta$ and $\cos\gamma$. Then the projection of the line PQ on the co-ordinate axes are $PQ\cos\alpha$, $PQ\cos\beta$ and $PQ\cos\gamma$.

Then according to the problem, we have

$$PQ\cos\alpha = 2,\ PQ\cos\beta = 3 \text{ and } PQ\cos\gamma = 6. \quad \text{...(i)}$$

Squaring and adding these, we get

$$(PQ)^2[\cos^2\alpha+\cos^2\beta+\cos^2\gamma] = 2^2+3^2+6^2 = 4+9+36$$

or $$(PQ)^2[1] = 49, \quad \because \cos^2\alpha+\cos^2\beta+\cos^2\gamma = 1$$

or $$PQ = \sqrt{(49)} = 7.$$ **Ans.**

$\therefore$ From (i), we have $7\cos\alpha = 2$, $7\cos\beta = 3$, $7\cos\gamma = 6$

or $$\cos\alpha = 2/7,\ \cos\beta = 3/7 \text{ and } \cos\gamma = 6/7.$$ **Ans.**

Ex. 3. A and B are (2, 3, – 6), (3, – 4, 5). Find the direction cosines of OA, OB, BO, and AB where O is (0, 0, 0).

Sol. The direction ratios of OA are $(2-0), (3-0), (-6-0)$

i.e. $$2, 3, -6.$$

$\therefore$ The d.c.'s of OA are

$$\frac{2}{\sqrt{[2^2+3^2+(-6)^2]}}, \frac{3}{\sqrt{[2^2+3^2+(-6)^2]}}, \frac{-6}{\sqrt{[2^2+3^2+(-6)^2]}}$$

or $$2/7,\ 3/7,\ -6/7.$$ **Ans.**

Similarly the d.c.'s of OB are

$$\frac{3-0}{\sqrt{[(3-0)^2+(-4-0)^2+(5-0)^2]}}, \frac{-4-0}{\sqrt{[(3-0)^2+(-4-0)^2+(5-0)^2]}},$$

$$\frac{5-0}{\sqrt{[(3-0)^2+(-4-0)^2+(5-0)^2]}},$$

or $$\frac{3}{\sqrt{(50)}}, \frac{-4}{\sqrt{(50)}}, \frac{5}{\sqrt{(50)}}, \quad \text{or} \quad \frac{3}{5\sqrt{2}}, \frac{-4}{5\sqrt{2}}, \frac{1}{5\sqrt{2}}$$ **Ans.**

And the d.c.'s of BO are

$$\frac{3-0}{\sqrt{[(0-3)^2+(0+4)^2+(0-5)^2]}}, \frac{0-(-4)}{\sqrt{[(0-3)^2+(0+4)^2+(0-5)^2]}},$$

$$\frac{0-5}{\sqrt{[(0-3)^2+(0+4)^2+(0-5)^2]}}$$

or $$\frac{-3}{\sqrt{(50)}}, \frac{4}{\sqrt{(50)}}, \frac{-5}{\sqrt{(50)}}, \quad \text{or} \quad \frac{-3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{-1}{5\sqrt{2}}$$

Also the d.c.'s of AB are

$$\frac{3-2}{\sqrt{[(3-2)^2+(-4-3)^2+(5+6)^2]}}, \frac{-4-3}{\sqrt{[(3-2)^2+(-4-3)^2+(5+6)^2]}},$$

$$\frac{5-(-6)}{\sqrt{[(3-2)^2+(-4-3)^2+(5+6)^2]}},$$

or $$\frac{1}{\sqrt{(1+49+121)}}, \frac{-7}{\sqrt{(1+49+121)}}, \frac{11}{\sqrt{(1+49+121)}}$$

or $$1/\sqrt{(171)},\ -7/\sqrt{(171)},\ 11/\sqrt{(171)}.$$ **Ans.**

Ex. 4. Coordinattes of four points A, B, C, D are (1, 2, 3) (3, 5, 7), (2, 3, – 1) and (3, 4, – 3). Find the projection of AB on CD. (*Bundelkhand* 93)

Sol. Let l, m, n be the direction cosines of CD.

Then $$l = \frac{(3-2)}{\sqrt{[(3-2)^2+(4-3)^2+(-3+1)^2]}} = \frac{1}{\sqrt{6}},$$

$$m = \frac{(4-3)}{\sqrt{[(3-2)^2+(4-3)^2+(-3+1)^2]}} = \frac{1}{\sqrt{6}}$$

and $$n = \frac{(-3+1)}{\sqrt{[(3-2)^2+(4-3)^2+(-3+1)^2]}} = \frac{-2}{\sqrt{6}}$$

$\therefore$ The required projection of AB on Cd.

$= \text{"}(x_2-x_1)\, l + (y_2-y_1)\, m + (z_2-z_1)\, n\text{"}$ ...See § 2.09 Page 23

$= (3-1)(1/\sqrt{6}) + (5-2)(1/\sqrt{6}) + (7-3)(-2/\sqrt{6})$

$= (2/\sqrt{6}) + (3/\sqrt{6}) - (8/\sqrt{6}) = -3/\sqrt{6} = -\sqrt{(3/2)}$ **Ans.**

***Ex. 5. If A, B, C, D are the points (3, 4, 5), (4, 6, 3), (– 1, 2, 4) and (1, 0, 5), find the projection of CD on AB.**

Sol. Let l, m, n be the direction cosines of AB.

Then $$l = \frac{4-3}{\sqrt{[(4-3)^2+(6-4)^2+(3-5)^2]}} = \frac{1}{\sqrt{[(1^2+2^2+2^2)]}} = \frac{1}{3}.$$

Similarly $$m = \frac{6-4}{\sqrt{[(4-3)^2+(6-4)^2+(3-5)^2]}} = \frac{2}{3}$$

and $$n = \frac{3-5}{\sqrt{[(4-3)^2+(6-4)^2+(3-5)^2]}} = \frac{-2}{3}$$

$\therefore$ The projection of CD on AB

$= [1-(-1)]\, l + (0-2)\, m + (5-4)\, n,$ ...See § 2.09 Page 23

$= 2\left(\frac{1}{3}\right) - 2\left(+\frac{2}{3}\right) + 1\left(-\frac{1}{3}\right) = -\frac{4}{3} = \frac{4}{3}$, taking numerical value. **Ans.**

Exercises on § 2.05 — § 2.09

Ex. 1. If A, B are (2, 3, 5) (– 1, 3, 2), find the direction cosines of AB.

Ans. $(-1/\sqrt{2}, 0, -1/\sqrt{2})$

Ex. 2. Find the direction cosines of the line joining the origin to the point (3, 12, 4). (*Kumaun* 92)

Ans. (3/13, 12/13, 4/13).

***Ex. 3.** The projections of a line on the axes are 12, 4, 3. Find the length of the line and its direction cosines. **Ans.** 13 and 12/13, 4/13, 3/13.

Ex. 4. A (3, 1, 2), B (5, 1, 2), C (0, 1, 5) and D (0, – 1, – 5) are four points. Show that AB is perpendicular to CD.

[**Hint :** Prove that the projection of AB and CD is zero].

Ex. 5. Choose the correct answer :

The direction cosines of the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

(i) $\dfrac{x_1+x_2}{PQ}, \dfrac{y_1+y_2}{PQ}, \dfrac{z_1+z_2}{PQ}$; (ii) $\dfrac{x_2-x_1}{PQ}, \dfrac{y_2-y_1}{PQ}, \dfrac{z_2-z_1}{PQ}$;

(iii) $x_2 - x_1, y_2 - y_1, z_2 - z_1$; (iv) none of these.

[Hint : See § 2.08 Page 23]. **Ans.** (ii)

****§ 2.10. The angle between two lines.** *If* (l_1, m_1, n_1) *and* (l_2, m_2, n_2) *be the direction cosines of any two lines and* θ *be the angle between them, then*

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2.$$

(*Kanpur* 97; *Kumaum* 95, 93; *Meerut* 90*S*; *Rohilkhand* 92)

Through the origin O, draw two lines OA and OB parallel respectively to the given lines with d.c's (l_1, m_1, n_1) and (l_2, m_2, n_2).

Let $OB = r$ and the co-ordinates of B be (x_1, y_1, z_1).

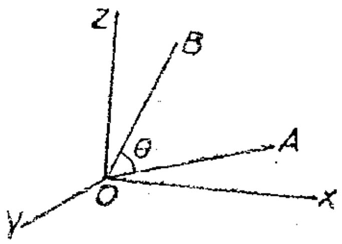
Then $x_1 = rl_2,\ y_1 = rm_2$ and $z_1 = rn_2$

And from § 2.09 cor. Page 24 we know that the projection of OB on OA

$$= l_1x_1 + m_1y_1 + n_1z_1$$

$$= l_1 \,.\, rl_2 + m_1 \,.\, rm_2 + n_1 \,.\, rn_2,$$ from (i)

$$= r(l_1l_2 + m_1m_2 + n_1n_2) \quad \text{...(ii)}$$



(Fig. 20)

Also from the figure 20 above it is evident that the projection of OB on OA

$$= OB\cos\theta = r\cos\theta, \quad \because \quad OB = r. \quad \text{...(iii)}$$

$\therefore$ From (ii) and (iii), we have $r\cos\theta = r(l_1l_2 + m_1m_2 + n_1n_2)$

or $$\cos\theta = \mathbf{l_1l_2 + m_1m_2 + n_1n_2} \quad \text{... (A)}$$

(*Rohilkhand* 92)

If instead of (l_1, m_1, n_1) and (l_2, m_2, n_2), the d.c.'s of the given lines are expressed as $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$ respectively, then the relation (A) takes the form

$$\mathbf{\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'} \quad \text{...(B)}$$

Cor. 1. If (a_1, b_1, c_1) and (a_2, b_2, c_2) be the direction ratios of the given lines, then their actual direction cosines are given by

$$\frac{a_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}, \frac{b_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}, \frac{c_1}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}}$$

and $$\frac{a_2}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}, \frac{b_2}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}, \frac{c_2}{\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

$\therefore$ The angle θ between the lines is given by

$$\cos\theta = \frac{\mathbf{a_1a_2 + b_1b_2 + c_1c_2}}{\sqrt{(\mathbf{a_1^2 + b_1^2 + c_1^2})} \,.\, \sqrt{(\mathbf{a_2^2 + b_2^2 + c_2^2})}} \quad \text{...(C)}$$

Cor. 2. We know the Lagrange's Identity

$$(\mathbf{l_1^2 + m_1^2 + n_1^2})(\mathbf{l_2^2 + m_2^2 + n_2^2}) - (\mathbf{l_1l_2 + m_1m_2 + n_1n_2})^2$$

$$\equiv (\mathbf{m_1n_2 - m_2n_1})^2 + (\mathbf{n_1l_2 - n_2l_1})^2 + (\mathbf{l_1m_2 - l_2m_1})^2$$

[Note : Students should commit this identity to memory].

Now $\sin^2\theta = 1 - \cos\theta$

$= 1 - (l_1l_2 + m_1m_2 + n_1n_2)^2$, from § 2.10 above

$= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2$,

since $l_1^2 + m_1^2 + n_1^2 = 1 = l_2^2 + m_2^2 + n_2^2$

$= (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$, by Lagrange's identity

$$\boxed{\therefore\ \mathbf{\sin\theta = \pm\sqrt{[(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2]}}} \quad \text{...(D)}$$

$$= \pm\sqrt{\left[\begin{vmatrix} m_1 & m_2 \\ n_1 & n_2 \end{vmatrix}^2 + \begin{vmatrix} n_1 & n_2 \\ l_1 & l_2 \end{vmatrix}^2 + \begin{vmatrix} l_1 & l_2 \\ m_1 & m_2 \end{vmatrix}^2\right]}$$

and $$\mathbf{\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\pm\sqrt{[\Sigma(m_1n_2 - m_2n_1)^2]}}{l_1l_2 + m_1m_2 + n_1n_2}}$$

substituting values of $\sin\theta$ and $\cos\theta$.

Cor. 3. If instead of direction cosines we have the direction ratios of the lines as in cor. 1 above, then we have

$$\sin\theta = \frac{\pm\sqrt{[(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2]}}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

and $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \pm\frac{\sqrt{[\Sigma(b_1c_2 - b_2c_1)^2]}}{a_1a_2 + b_1b_2 + c_1c_2}$$

***Cor. 4. Condition for perpendicularity of two lines.** (*Bundelkhand* 91)

If the two given lines whose d.c.'s are (l_1, m_1, n_1) and (l_2, m_2, n_2) are perpendicular to each other, then $\theta = 90°$ *i.e.* $\cos\theta = 0$ and from result (A) of § 2.10 Page 27 we have the required condition as

$$\mathbf{l_1l_2 + m_1m_2 + n_1n_2 = 0} \quad \text{...(E)}$$

If instead of d.c.'s we have the direction ratios of the lines as a_1, b_1, c_1 and a_2, b_2, c_2 then from result (C) above, we have

$$\mathbf{a_1a_2 + b_1b_2 + c_1c_2 = 0} \quad \text{...(F)}$$

***Cor. 5. Condition for parallelism of two lines.** (*Bundelkhand* 91)

If two given lines are parallel, then $\theta = 0$ *i.e.* $\sin\theta = 0$ and from result (D) above we have the required condition as

$$(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 = 0.$$

But this being the sum of squares of three quantities can be zero only if each of them is separately zero. **(Note)**

i.e. if $m_1n_2 - m_2n_1 - 0$, $n_1l_2 - n_2l_1 = 0$ and $l_1m_2 - l_2m_1 = 0$

i.e. if $\dfrac{m_1}{m_2} = \dfrac{n_1}{n_2} = \dfrac{l_1}{l_2} = \dfrac{\sqrt{(l_1^2 + m_1^2 + n_1^2)}}{\sqrt{(l_2^2 + m_2^2 + n_2^2)}} = 1$ **(Note)**

i.e. if $l_1 = l_2$, $m_1 = m_2$ and $n_1 = n_2$

i.e. **the d.c.'s of parallel lines are the same** **(Remember)**

If instead of d.c.'s we have the direction ratios of the given lines as a_1, b_1, c_1 and a_2, b_2, c_2 then proceeding as above with the help of cor. 3 above we get the required condition as

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

(Student's should remember that $a_1^2+b_1^2+c_1^2 \neq 1$)

i.e. **the direction ratios of parallel lines are proportional.**

Solved Examples on § 2.10

Ex. 1 (a) If points P, Q are (2, 3, – 6) and (3, – 4, 5), find the angle that OP makes with OQ.

Sol. The direction ratios of OP are $2-0, 3-0, -6-0$, *i.e.* 2, 3, – 6 and those of OQ are $3-0, -4-0, 5-0$ *i.e.* 3, – 4, 5

∴ If θ be the required anlge, then

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{(a_1^2+b_1^2+c_1^2)}\,.\,\sqrt{(a_2^2+b_2^2+c_2^2)}} \qquad \text{... § 2.10 (C) Page 28}$$

$$=\frac{2.3+3(-4)+(-6)5}{\sqrt{[2^2+3^2(-6)^2]}\,.\,\sqrt{[3^2+(-4)^2+5^2]}}=\frac{6-12-30}{\sqrt{(49)}\sqrt{(50)}}=\frac{-36}{7\times5\sqrt{2}}$$

or $\theta=\cos^{-1}[-18\sqrt{2}/35]$ **Ans.**

****Ex. 1 (b). Find the angle between the two lines whose direction cosines are $\cos\alpha, \cos\beta, \cos\gamma$ and $\cos\alpha', \cos\beta', \cos\gamma'$. Deduce the condition of perpendicularity and parallelism of the above lines.**

Sol. If θ be the required angle, then as in § 2.10 Pages 27-29 we can prove that $\cos\theta=\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma'$ ...(1)

If these two lines are perpendicular, then $\theta=90°$ and from (1) we get the **condition of perpendicularity** as

$$\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma'=0.$$

Again if two lines are parallel, then $\theta=0$ and from (1) we get the **condition of parallelism** as

$$\cos\alpha\cos\alpha'+\cos\beta\cos\beta'+\cos\gamma\cos\gamma'=1. \qquad \textbf{Ans.}$$

Otherwise if given lines are parallel, then their d.cosines are equal and so we have

$$\cos\alpha=\cos\alpha',\ \cos\beta=\cos\beta',\ \cos\gamma=\cos\gamma'$$

or $\alpha=\alpha', \beta=\beta', \gamma=\gamma'$

Ex. 2 (a). If A, B, C are (– 1, 3, 2), (2, 3, 5) and (3, 5, – 2); find the angles of the triangle ABC. (*Bundelkhand* 94; *Meerut* 93)

Sol. The direction ratios of the line AB are $\{2-(-1)\}, (3-3), (5-2)$ *i.e.* 3, 0, 3.

∴ The d.c.'s of the line AB are

$$\frac{3}{\sqrt{(3^2+0^2+3^2)}},\ \frac{0}{\sqrt{(3^2+0^2+3^2)}},\ \frac{3}{\sqrt{(3^2+0^2+3^2)}}$$

i.e. $\frac{3}{3\sqrt{2}}, \frac{0}{3\sqrt{2}}, \frac{3}{3\sqrt{2}}$, *i.e.* $\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$.

Similarly d.c.'s of the lines BC and AC are

$\frac{1}{3\sqrt{6}}, \frac{2}{3\sqrt{6}}, \frac{-7}{3\sqrt{6}},$ and $\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$ respectively.

$\therefore \cos A$ = cosine of the angle between AB and AC

$$= \text{“}l_1l_2 + m_1m_2 + n_1n_2\text{”}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{2}{3} + 0 \cdot \frac{1}{3} - \frac{1}{\sqrt{2}} \cdot \left(-\frac{2}{3}\right) = 0$$

Therefore $\angle A = 90^\circ$ **Ans.**

$\cos B$ = cosine of the angle between BC and BA

$$= \frac{1}{3\sqrt{6}}\left(-\frac{1}{\sqrt{2}}\right) + \frac{2}{3\sqrt{6}}(0) - \frac{7}{3\sqrt{6}}\left(-\frac{1}{\sqrt{2}}\right),$$ **Note** the d.c.'s of BA

and $\cos B = \frac{6}{3\sqrt{6} \cdot \sqrt{2}} = \frac{1}{\sqrt{3}}$ or $\angle B = \cos^{-1}(1/\sqrt{3})$.

Similarly we can find that $\angle C = \cos^{-1}(1/3)$ **Ans.**

Ex. 2 (b). Find the angles of triangle ABC whose vertices A, B and C are the points (1, 3, 5), (– 1, 2, 3) and (3, 4, – 2). (*Bundelkhand* 90)

Sol. Do as Ex. 2 (a) above. **Ans.** $\cos^{-1}[3/2\sqrt{(10)}]$, 90°, $\cos^{-1}[3/2\sqrt{2}]$

Ex. 3 (a). Find the angle between the lines whose direction ratios are (2, 3, 4) and (1, – 2, 1).

Sol. If θ be the angle between the given lines, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)} \cdot \sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$ ... See Cor. 1. Page 27

$$= \frac{2.1 + 3.(-2) + 4.1}{\sqrt{(2^2 + 3^2 + 4^2)} \cdot \sqrt{[1^2 + (-2)^2 + 1^2]}} = \frac{0}{\sqrt{(20)}\sqrt{(6)}} = 0$$

or $\theta = 90^\circ$ *i.e.* lines are at right angles.

Ex. 3 (b). Find the acute angle between the lines whose direction cosines are proportional to (2, 3, – 6), and (3, – 4, 5).

Sol. If θ b the angle between the given lines, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)} \cdot \sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$ ... See Cor. 1. Page 27

$$= \frac{2.3 + 3.(-4) + (-6).5}{\sqrt{(2^2 + 3^2 + (-6)^2)} \cdot \sqrt{[3^2 + (-4)^2 + 5^2]}} = \frac{-36}{35\sqrt{(2)}}$$

= negative. Hence θ is obtuse angle.

$\therefore$ Required acute angle $= \pi - \theta$, where $\cos\theta = (-36/35)\sqrt{(1/2)}$ **Ans.**

Ex. 4. If A, B, C, D are the points (3, 4, 5), (4, 6, 3), (– 1, 2, 4) and (1, 0, 5), find the angle between CD and AB.

Sol. The direction ratios of AB are $4 - 3, 6 - 4, 3 - 5$ or $1, 2, -2$.

And the direction ratios of CD are $1 + 1, 0 - 2, 5 - 4$ or $2, -2, 1$

$\therefore$ If θ be the required angle between AB and CD, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

$$= \frac{1.2 + 2.(-2) + (-2).1}{\sqrt{[1^2 + 2^2 + (-2)^2]}\sqrt{[2^2 + (-2)^2 + 1^2]}} = \frac{-4}{9}$$

= negative. Hence $\theta = \cos^{-1}(-4/9)$ and is an obtuse angle.

****Ex. 5. Show that the straight line whose direction cosines are given by the equations : $ul + vm + wn = 0$ $al^2 + bm^2 + cn^2 = 0$ are (α) perpendicular if $u^2(b+c) + v^2(c+a) + w^2(a+b) = 0$ and (β) parallel, if $(u^2/a) + (v^2/b) + (w^2/c) = 0$.**

(*Bundelkhand* 96; *Garhwal* 96, 94, 92, 91; *Gorakhpur* 91; *Kanpur* 93; *Kumaum* 94, 92; *Rohilkhand* 90)

Sol. The d.c.'s of the lines are given by

$$ul + vm + wn = 0 \quad \text{and} \quad al^2 + bm^2 + cn^2 = 0$$

Eliminating n between these, we get

$$al^2 + bm^2 + c\,[-(ul+vm)/w]^2 = 0$$

or $$(aw^2 + cu^2)\,l^2 + (bw^2 + cv^2)\,m^2 + 2cuvlm = 0$$

or $$(aw^2 + cu^2)\,(l/m)^2 + 2\,cuv\,(l/m) + (bw^2 + cv^2) = 0, \quad \text{...(i)}$$

dividing each term by m^2.

(α) Its two roots are l_1/m_1 and l_2/m_2, if the d.c.'s of the two lines be taken as (l_1, m_1, n_1) and (l_2, m_2, n_2).

$\therefore$ From (i), we have

$$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \text{product of the roots} = \frac{bw^2 + cv^2}{cu^2 + aw^2}$$

or $$\frac{l_1 l_2}{bw^2 + cv^2} = \frac{m_1 m_2}{cu^2 + aw^2} = \frac{n_1 n_2}{av^2 + bu^2}, \text{ by symmetry.}$$

$\therefore$ If the two lines are perpendicular, then we have

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \quad i.e. \quad (bw^2 + cv^2) + (cu^2 + aw^2) + (av^2 + bu^2) = 0.$$

or $$u^2(b+c) + v^2(c+a) + w^2(a+b) = 0 \qquad \text{Hence proved.}$$

(β) If the two lines are parallel, then their d.c.'s are equal are consequently the roots of (i) are equal, the condition for the same being

$$\text{"}b^2 = 4ac\text{"} \quad i.e. \quad (2cuw)^2 = 4\,(aw^2 + cu^2)\,(bw^2 + cv^2)$$

or $$c^2u^2v^2 = abw^4 + acw^2v^2 + bcu^2w^2 + c^2u^2v^2$$

or $$abw^4 + acw^2v^2 + bcu^2w^2 = 0 \text{ or } abw^2 + acv^2 + bcu^2 = 0$$

or $$\frac{w^2}{c} + \frac{v^2}{b} + \frac{u^2}{a} = 0, \text{ dividing each term by } abc.$$

Ex. 6 (a) Find the angle between the lines whose d.c.'s (l, m, n) satisfy the equations $l + m + n = 0$ and $2lm + 2nl - mn = 0$.

(*Kanpur* 94; *Purvanchal* 95, 93; *Rohilkhand* 93)

Sol. Eliminating n between the given relations, we get

$$2lm + 2(-l-m)\,l - m(-l-m) = 0$$

or $$2l^2 - lm - m^2 = 0 \quad \text{or} \quad (2l+m)(l-m) = 0 \quad \text{... (i)}$$

If $2l + m = 0$, then from $l + m + n = 0$, we get $n = l$

$\therefore$ we have $2l = -m = 2n$ or $\frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$.

$\therefore$ The direction cosines of one line are

$$\frac{1}{\sqrt{[1^2+(-2)^2+1^2]}}, \frac{-2}{\sqrt{[1^2+(-2)^2+1^2]}}, \frac{1}{\sqrt{[1^2+(-2)^2+1^2]}}$$

or $$1/\sqrt{6}, \ -2/\sqrt{6}, \ 1/\sqrt{6}. \quad \text{...(ii)}$$

If $l - m = 0$, then from $l + m + n = 0$, we get $n = -2m$.

$\therefore$ we have $l = m = -\frac{n}{2}$ or $\frac{l}{1} = \frac{m}{1} = \frac{n}{-2}$

$\therefore$ The direction cosines of the second line are

$$\frac{1}{\sqrt{[1^2+1^2+(-2)^2]}}, \frac{-2}{\sqrt{[1^2+1^2+(-2)^2]}}, \frac{1}{\sqrt{[1^2+1^2+(-2)^2]}}$$

or $$1/\sqrt{6}, \ 1/\sqrt{6}, \ -2/\sqrt{6}.$$

$\therefore$ If θ be the required angle, then

$\cos\theta =$ "$l_1l_2 + m_1m_2 + n_1n_2$"

$$= \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}} + \left(-\frac{2}{\sqrt{6}}\right) \cdot \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} \cdot \left(\frac{-2}{\sqrt{6}}\right) = \frac{-3}{6}$$

or $\cos\theta = -1/2$ or $\theta = 120^\circ$. **Ans.**

****Ex. 6 (b). Prove that the acute angle between the lines whose direction cosines are given by the relations $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$ is $\pi/3$.** *(Meerut 95)*

Sol. Eliminating n between the given relations, we get

$$l^2 + m^2 - (-l - m)^2 = 0 \quad \text{or} \quad lm = 0$$

$\therefore$ either $l = 0$ or $m = 0$

If $l = 0$, then from $l + m + n = 0$ we get $m = -n$

$\therefore$ we have $l = 0$, $m = -n$ or $\frac{l}{0} = \frac{m}{-1} = \frac{n}{1}$

$\therefore$ Direction cosines of one line are

$$\frac{0}{\sqrt{[0^2+(-1)^2+1^2]}} = -\frac{-1}{\sqrt{[0^2+(-1)^2+1^2]}} = \frac{1}{\sqrt{[0^2+(-1)^2+1^2]}}$$

or $$0, \ -1/\sqrt{2}, \ 1/\sqrt{2}$$

If $m = 0$, then from $l + m + n = 0$ we get $l = -n$

$\therefore$ we have $\frac{l}{1} = \frac{m}{0} = \frac{n}{-1}$

$\therefore$ Direction cosines of the second line as above are $1/\sqrt{2}, 0, -1/\sqrt{2}$

$\therefore$ If θ be the angle between these two lines, then

$\cos\theta =$ "$l_1l_2 + m_1m_2 + n_1n_2$"

$$= 0 \cdot (1/\sqrt{2}) + (-1/\sqrt{2}) \cdot 0 + (1/\sqrt{2})(-1/\sqrt{2})$$

or $\cos\theta = -1/2$ or $\theta = 120^\circ$

$\therefore$ Required acute angle between the lines

$$= 180^\circ - 120^\circ = 60^\circ \ \ i.e.\ \pi/3. \qquad \text{Proved}$$

***Ex. 7 (a). Find the angle between the two lines whose d.c.'s (l, m, n) satisfy the equatons l + m + n = 0 and $l^2 + m^2 + n^2 = 0$.** *(Bundelkhand 92)*

Sol. Eliminating n between the given relations, we get

$$l^2 + m^2 + (-l - m)^2 = 0 \qquad \text{or} \qquad l^2 + m^2 + l\,m = 0$$

or $$(l/m)^2 + (l/m) + 1 = 0, \text{ dividing each term by } m^2$$

Let its roots be l_1/m_1 and l_2/m_2, then

$$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \text{product of the roots} = -\frac{1}{1}$$

or $$\frac{l_1 l_2}{1} = \frac{m_1 m_2}{-1} \qquad \text{...(i)}$$

Again eliminating m between the given relations we get as above

$$l^2 + l\,n + n^2 = 0$$

or $$(l/n)^2 + (l/n) + 1 = 0, \text{ dividing each term by } n^2$$

If its roots be l_1/n_1 and l_2/n_2, then

$$\frac{l_1}{n_1} \cdot \frac{l_2}{n_2} = \text{product of the roots} = -\frac{1}{1}$$

or $$\frac{l_1 l_2}{1} = \frac{n_1 n_2}{-1} \qquad \text{...(ii)}$$

$\therefore$ From (i) and (ii) we get $\dfrac{l_1 l_2}{1} = \dfrac{m_1 m_2}{-1} = \dfrac{n_1 n_2}{-1}$

$\therefore$ If θ be the required angle, then $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

or $$\cos\theta = 1 - 1 - 1 = -1 \qquad \text{or} \qquad \theta = \pi. \qquad \textbf{Ans.}$$

****Ex. 7 (b). Prove that the straight lines whose direction cosines are given by relations al + bm + cn = 0 and fmn + gnl + hlm = 0 are perpendicular**

if $\dfrac{f}{a} + \dfrac{g}{b} + \dfrac{h}{c} = 0$ and parallel if $\sqrt{(af)} \pm \sqrt{(bg)} \pm \sqrt{(ch)} = 0$

(Gorakhpur 90; Kanpur 96; Meerut 91 S; Purvanchal 92, 90; Rohilkhand 94)

Sol. Let the d.c.'s of the two lines be (l_1, m_1, n_1) and (l_2, m_2, n_2).

Eliminating n between the given relations, we get

$$fm\,[-(al + bm)/c] + gl\,[-(al + bm)/c] + hlm = 0$$

or $$-afm - bfm^2 - agl^2 - bglm + chlm = 0$$

or $$ag\,(l/m)^2 + (af + bg - ch)\,(l/m) + bf = 0,$$

dividing each term by m^2.

Its roots are l_1/m_1 and l_2/m_2

$\therefore \dfrac{l_1}{m_1} \cdot \dfrac{l_2}{m_2} =$ Product of the roots $= \dfrac{bf}{ag}$

or $\dfrac{l_1 l_2}{bf} = \dfrac{m_1 m_2}{ag}$ or $\dfrac{l_1 l_2}{(f/a)} = \dfrac{m_1 m_2}{(g/b)} = \dfrac{n_1 n_2}{(h/c)}$, by symmetry

If the lines are perpendicular, then $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

or $$(f/a) + (g/b) + (h/c) = 0.$$ Hence proved.

If the lines are parallel, then their d.c.'s must be the same *i.e.* the roots of (i) must be equal, the condition for the same is "$b^2 = 4ac$"

i.e. $$(af + bg - ch)^2 = 4ag \,.\, bf \quad \text{...(ii)}$$

or $$af + bg - ch = \pm 2\sqrt{(af)}\sqrt{(bg)}$$

or $$af + bg \pm 2\sqrt{(af)}\sqrt{(bg)} = ch$$

or $$[\sqrt{(af)} \pm \sqrt{(bg)}]^2 = ch = [\sqrt{(ch)}]^2$$

or $\sqrt{(af)} \pm \sqrt{(bg)} \pm \sqrt{(ch)} = 0$ is the required conditon.

Also from (ii), we get $a^2f^2 + b^2g^2 + c^2h^2 + 2abfg - 2acfh - 2bcgh = 4abfg$.

or $$a^2f^2 + b^2g^2 + c^2h^2 - 2bcgh - 2cafh - 2abfg = 0.$$

***Ex. 8. If $(l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3)$ are the d.c.'s of three mutually perpendicular lines then prove that the line whose direction cosines are proportional to $l_1 + l_2 + l_3$, $m_1 + m_2 + m_3$, $n_1 + n_2 + n_3$ makes equal angls with them.**

Sol. As $(l_1, m_1, n_1), (l_2, m_2, n_2)$, and (l_3, m_3, n_3) are d.c.'s of three mutually perpendicular lines, therefore we have

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ ... (i) $\quad l_2 l_3 + m_2 m_3 + n_2 n_3 = 0$ ...(ii)

and $$l_1 l_3 + m_1 m_3 + n_1 n_3 = 0 \quad \text{...(iii)}$$

Also l_1, m_1, n_1 being d.c.'s of a line we have $l_1^2 + m_1^2 + n_1^2 = 1$.

Similarly $l_2^2 + m_2^2 + n_2^2 = 1$ and $l_3^2 + m_3^2 + n_3^2 = 1$ ...(iv)

Again $\sqrt{[(l_1 + l_2 + l_3)^2 + (m_1 + m_2 + m_3)^2 + (n_1 + n_2 + n_3)^2]}$

$= \sqrt{[(l_1^2 + m_1^2 + n_1^2) + (l_2^2 + m_2^2 + n_2^2) + (l_3^2 + m_3^2 + n_3^2)}$

$+ 2(l_1 l_2 + m_1 m_2 + n_1 n_2) + 2(l_2 l_3 + m_2 m_3 + n_2 n_3) + 2(l_3 l_1 + m_3 m_1 + n_3 n_1)$

expanding and rearranging terms

$= \sqrt{[1 + 1 + 1]}$, from (i), (ii), (iii) and (iv)

$= \sqrt{3}$.

$\therefore$ The d.c.'s of the given line are

$$\frac{l_1 + l_2 + l_3}{\sqrt{3}}, \frac{m_1 + m_2 + m_3}{\sqrt{3}}, \frac{n_1 + n_2 + n_3}{\sqrt{3}}$$

If θ be the angle between this line and the line with d.c.'s l_1, m_1, n_1, then

$$\cos\theta = l_1\left(\frac{l_1 + l_2 + l_3}{\sqrt{3}}\right) + m_1\left(\frac{m_1 + m_2 + m_3}{\sqrt{3}}\right) + n_1\left(\frac{n_1 + n_2 + n_3}{\sqrt{3}}\right)$$

... See § 2.10 result (E) Page 28.

$$= (1/\sqrt{3})\,[(l_1^2 + m_1^2 + n_1^2) + (l_1 l_2 + m_1 m_2 + n_1 n_2) + (l_1 l_3 + m_1 m_3 + n_1 n_3)]$$

or $\quad \cos\theta = (1/\sqrt{3})\,[1 + 0 + 0]$ ... from (i), (iii) and (iv)

or $\quad \theta = \cos^{-1}(1/\sqrt{3})$

In a similar way we can prove that the line with d.c.'s $(l_1 + l_2 + l_3)/\sqrt{3}, (m_1 + m_2 + m_3)/\sqrt{3}, (n_1 + n_2 + n_3)/\sqrt{3}$ is inclined to line with d.c.'s l_2, m_2, n_2 and l_3, m_3, n_3 also at angle $\cos^{-1}(1/\sqrt{3})$. Hence proved.

****Ex. 9. If l_1, m_1, n_1 and l_2, m_2, n_2 are the d.c.'s of two lines, then the direction ratio of another which is perpendicular to both the given lines are $(m_1n_2 - m_2n_1), (n_1l_2 - n_2l_1), (l_1m_2 - l_2m_1)$.** (*Garhwal* 96)

Prove further if the given lines are at right angles to each other then these direction ratios are the acutal direction cosines. (*Purvanchal* 97)

Sol. Let l, m, n be the required d.c.'s of the line.

Then as this line is perpendicular to both the given lines, so we have

$$l\,l_1 + m\,m_1 + n\,n_1 = 0 \quad \text{and} \quad l\,l_2 + m\,m_2 + n\,n_2 = 0$$

Solving these, we get

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1},$$ which give the required direction ratios.

$\therefore$ The d.c.'s of this line are

$$\frac{m_1n_2 - m_2n_1}{\sqrt{[\Sigma(m_1n_2 - m_2n_1)^2]}}, \frac{n_1l_2 - n_2l_1}{\sqrt{[\Sigma(m_1n_2 - m_2n_1)^2]}}, \frac{l_1m_2 - l_2m_1}{\sqrt{[\Sigma(m_1n_2 - m_2n_1)^2]}} \quad \text{... (i)}$$

If θ be the angle between the two given lines whose d.c.'s are (l_1, m_1, n_1) and (l_2, m_2, n_2) then we know

$$\sin\theta = \sqrt{[\Sigma(m_1n_2 - m_2n_1)^2]}$$ ... See § 2.10 (D) Page 28.

It these lines are at right angles, then $\theta = 90^\circ$ or $\sin\theta = 1$ and therefore

$$\sqrt{[\Sigma(m_1n_2 - m_2n_1)^2]} = 1.$$

$\therefore$ From (i) the actual d.c.'s are

$(m_1n_2 - m_2n_1), (n_1l_2 - n_2l_1)$ and $(l_1m_2 - l_2m_1)$. Hence proved.

Ex. 10 (a). Show that the lines whose directon cosines are given by $l + m + n = 0$, $2mn + 3ln - 5lm = 0$ are perpendicular to one another.

(*Meerut* 96 *P*, 94)

Sol. Given $\quad l + m + n = 0$ ... (i); $\quad 2mn + 3ln - 5lm = 0$ ... (ii)

Eliminating n between (i) and (ii), we have

$$2m(-l - m) + 3l(-l - m) - 5lm = 0$$

or $\quad 3l^2 + 10lm + 2m^2 = 0$ or $3(l/m)^2 + 10(l/m) + 2 = 0$

or $\quad \dfrac{l}{m} = \dfrac{-10 \pm \sqrt{(100 - 24)}}{6} = \dfrac{-5 \pm \sqrt{(19)}}{3}$ ...(iii)

Let l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c.'s of the two lines, then from (iii) we have $\dfrac{l_1}{m_1}=\dfrac{-5+\sqrt{(19)}}{3}$ and $\dfrac{l_2}{m_2}=\dfrac{-5-\sqrt{(19)}}{3}$

If. $\dfrac{l_1}{m_1}=\dfrac{\sqrt{(19)}-5}{3}$, then $\dfrac{l_1}{\sqrt{(19)}-5}=\dfrac{m_1}{3}=k_1$ (say)

$\therefore\quad l_1=k_1\,[\sqrt{(19)}-5]\,;\; m_1=3k_1$

Also from (i) $n_1=-l_1-m_1=k_1\,(5-\sqrt{19})-3k_1=k_1\,(2-\sqrt{19})$.

$$\therefore \frac{l_1}{\sqrt{(19)}-5}=\frac{m_1}{3}=\frac{n_1}{2-\sqrt{(19)}} \qquad \text{...(iv)}$$

Similarly taking $\dfrac{l_2}{m_2}=\dfrac{-5-\sqrt{(19)}}{3}$ we can find that

$$\frac{l_2}{-\sqrt{(19)}-5}=\frac{m_2}{3}=\frac{n_2}{2+\sqrt{(19)}} \qquad \text{...(v)}$$

From (iv) and (v) we have the direction ratios of two lines and if they are at right angles then we must have

"$a_1a_2+b_1b_2+c_1c_2=0$" ... See § 2.10 Cor. 4 Page 28

Here $a_1a_2+b_1b_2+c_1c_2=[\sqrt{(19)}-5]\,[-\sqrt{(19)}-5]+3\times 3$

$+[2-\sqrt{(19)}]\,[2+\sqrt{(19)}]=(-5)^2-19+9+(2)^2-19=38-38=0$

Hence the two lines are at right angles. Hence proved.

Ex. 10 (b). Show that the lines where direction cosines are given by the equations 2l + 2m − n and mn + nl + lm = 0 are at right angles.

(*Meerut* 98, 97)

Sol. Eliminating n between given equations we get

$m\,(2l+2m)+(2l+2m)\,l+lm=0$ or $2l^2+2m^2+5lm=0$

or $2\,(l/m)^2+5\,(l/m)+2=0$, dividing each term by m^2

or $\dfrac{l}{m}=\dfrac{-5\pm\sqrt{[(5)^2-4\,(2)\,(2)]}}{2\,(2)}=\dfrac{-5\pm 3}{4}=-\dfrac{1}{2},-2$

Let l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c.'s of the two lines, then we have

$$l_1/m_1=-1/2\,,\quad l_2/m_2=-2 \qquad \text{...(i)}$$

If $\dfrac{l_1}{m_1}=-\dfrac{1}{2}$, then $\dfrac{l_1}{-1}=\dfrac{m_1}{2}=k_1$ (say)

or $$l_1=-k_1,\; m_1=2k_1$$

Also from $n=2l+2m$ we have $n_1=2l_1+2m_1$

or $$n_1=2\,(-k_1)+2\,(2k_1)=2k_1$$

$\therefore$ we have $l_1=-k_1,\; m_1=2k_1,\; n_1=2k_1$ ...(iii)

Again from (i), $\dfrac{l_2}{m_2}=\dfrac{-2}{1}$ or $\dfrac{l_2}{-2}=\dfrac{m_2}{1}=k_2$ (say)

or $$l_2=-2k_2,\; m_2=k_2$$

And from $n=2l+2m$ we have $n_2=2l_2+2m_2$

or $$n_2 = 2(-2k_2) + 2k_2 = -2k_2$$

$\therefore$ we have $l_2 = -2k_2,\ m_2 = k_2,\ n_2 = -2k_2$

$\therefore$ If θ be the angle between the given lines, then

$$\cos\theta = \text{“}a_1a_2 + b_1b_2 + c_1c_2\text{”} \quad \text{... See § 2.10 Cor 4 Page 28}$$
$$= (-k_1)(-2k_2) + (2k_1)(k_2) + (2k_1)(-2k_2)$$
$$= 2k_1k_2 + 2k_1k_2 - 4k_1k_2 = 0$$

Hence the given lines are at right angles.

Ex. 11 (a). Lines OA, OB are drawn from O with d.c.'s proportional to (1, − 2, − 1), (3, − 2, 3). Find the d.c.'s of the normal to the plane AOB.

Sol. Let a, b, c be the direction ratios of the required normal to the plane AOB.

Then as OA lies in this plane so it is perpendicular to the normal to this plane and consequently we have $a(1) + b(-2) + c(-1) = 0$ ...(i)

Similarly OB is also perpendicular to this normal and so we have

$$a(3) + b(-2) + c(3) = 0 \quad \text{...(ii)}$$

Solving (i) and (ii) simultaneously, we have

$$\frac{a}{(-2)(3) - (-1)(-2)} = \frac{b}{(-1)(3) - (1)(3)} = \frac{c}{(1)(-2) - (-2)(3)}$$

or $\dfrac{a}{-8} = \dfrac{b}{-6} = \dfrac{c}{4}$ or $a : b : c = 4 : 3 : -2$

Also $4^2 + 3^2 + (-2)^2 = 29$.

$\therefore$ Required d.c.'s are $4/\sqrt{(29)},\ 3/\sqrt{(29)},\ -2/\sqrt{(29)}$. **Ans.**

****Ex. 11 (b). The direction ratios of two lines are 1, − 2, − 2 and 0, 2, 1. Find the direction cosines of the line perpendicular to the above lines.** (*Bundelkhand* 92)

Sol. Let a, b, c be the direction ratios of the line whose direction cosines are required. Then as this line is perpendicular to the given lines so we have
$$a(1) + b(-2) + c(-2) = 0$$
and
$$a(0) + b(2) + c(1) = 0$$

Solving these simultaneously, we get

$$\frac{a}{(-2)(1) - (-2)(2)} = \frac{b}{(-2)(0) - (1)(1)} = \frac{c}{(1)(2) - (0)(-2)}$$

or $$a/2 = b/-1 = c/2 \quad i.e.\quad a : b : c = 2 : -1 : 2$$

$\therefore$ The required direction cosines are

$$\frac{2}{\sqrt{(2^2+1^2+2^2)}},\ \frac{-1}{\sqrt{(2^2+1^2+2^2)}},\ \frac{2}{\sqrt{(2^2+1^2+2^2)}} \quad i.e.\quad \frac{2}{3}, \frac{-1}{3}, \frac{2}{3}$$

Ans.

Ex. 11 (c). Find the direction cosines of the line which is perpendicular to the lines whose direction cosines are proportional to 3, − 1, 1 and − 3, 2, 4 (*Rohilkhand* 95)

Sol. Do as Ex. 11 (b) above. **Ans.** $\frac{-2}{\sqrt{(30)}}, \frac{-5}{\sqrt{(30)}}, \frac{1}{\sqrt{(30)}}$

***Ex. 12. Prove that the three lines drawn from a point with direction cosines proportional to (1, – 1, 1), (2, – 3, 0) and (1, 0, 3) are coplanar.**

Sol. Let PA, PB and PC be three given lines with d.c.'s proportional to $(1, -1, 1)$, $(2, -3, 0)$ and $(1, 0, 3)$.

Let l, m, n be the direction ratios of the normal to the plane APB. Then as PA and PB are perpendicular to this normal so we have

$$l \cdot 1 + m(-1) + n \cdot 1 = 0 \qquad \text{...(i)}$$

and
$$l \cdot 2 + m(-3) + n \cdot 0 = 0 \qquad \text{...(ii)}$$

Solving (i) and (ii), we have

$$\frac{l}{(-1)\,0 - 1.(-3)} = \frac{m}{1.2 - 1.0} = \frac{n}{1.(-3) - (-1)2} \quad i.e. \quad \frac{l}{3} = \frac{m}{2} = \frac{n}{-1} \qquad \text{...(iii)}$$

If PC also lies in this plane PAB, then PC must be at right angles to this normal whose direction ratios are given by (iii), the condition fo the same as

"$a_1a_2 + b_1b_2 + c_1c_2 = 0$" ... See § 2.10 Cor. 4 Page 28

Here "$a_1a + b_1b_2 + c_1c_2 = l \cdot 1 + m \cdot 0 + n \cdot 3$

$$= 3.1 + 2.0 + (-1) \cdot 3, \text{ from (iii)}$$
$$= 0.$$

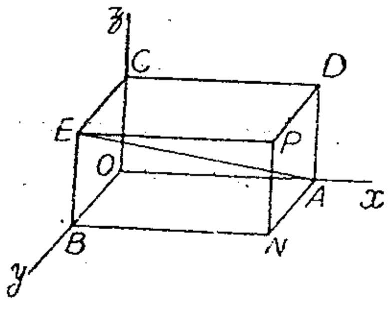
Hence proved.

****Ex. 13. If the edges of a rectangular parallepiped be a, b, c show that the angles between the four diagonals are given by**

$$\cos^{-1}\left[\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right]$$ (*Kanpur* 97; *Meerut* 96, 90)

Sol. Let one corner O of the rectangular parallelopiped be taken as origin and the three coterminus edges OA, OB and OC be taken as coordinate axes. Let $OA = a$, $OB = b$ and $OC = c$.

Then the co-ordinates of O, A, B, C, D, P, N and E are respectively $(0, 0, 0)$, $(a, 0, 0)$ $(0, b, 0)$, $(0, 0, c)$, $(a, 0, c)$, (a, b, c), $(a, b, 0)$ and $(0, b, c)$ as is evident from the adjoining figure.



(Fig. 21)

Here OP, CN, AE and BD are the diagonals.

The direction ratios of OP are $a - 0, b - 0, c - 0$ i.e. a, b, c.

Again direction ratios of CN are $(a - 0), (b - 0), (0 - c)$, i.e. $a, b, -c$.

Similarly we can find that the direction ratios of AE and BD are $-a, b, c$ and $a, -b, c$ respectively.

If θ be the angle between OP and CN, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\,.\,\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$ ... See § 2.10 (C) Page 28

or $$\cos\theta = \frac{a.a + b.b + c\,(-c)}{\sqrt{(a^2 + b^2 + c^2)}\,\sqrt{[a^2 + b^2 + (-c)^2]}} = \frac{a^2 + b^2 - c^2}{a^2 + b^2 + c^2}$$

Similarly we can find the angle between other pairs of diagonals and we have six such pairs out of these four diagonals and all these angles are given by

$$\cos^{-1}\left[\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right]$$ Hence proved.

****Ex. 14. A line makes angles $\alpha, \beta, \gamma\,\delta$ with the four diagonals of a cube; prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 4/3$.**

(*Avadh* 92; *Garhwal* 93; *Gorakhpur* 90; *Kanpur* 95, 93, 92; *Kumaun* 93; *Meerut* 92; *Purvanchal* 94; *Rohilkhand* 94, 92)

Sol. As in last example we can prove that the direction ratios of the diagonals OP, CN, AE and BD of the cube (whose edges are all equal *i.e.* $a = b = c$) are (a, a, a); $(a, a, -a)$; $(-a, a, a)$ and $(a, -a, a)$ respectively.

$\therefore$ d.c.'s of diagonals are

$$\frac{a}{\sqrt{(a^2 + a^2 + a^2)}}, \frac{a}{\sqrt{(a^2 + a^2 + a^2)}}, \frac{a}{\sqrt{(a^2 + a^2 + a^2)}} \text{ i.e. } \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$

In a similar way the d.c.'s of the diagonals CN, AE and BD are

$$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right); \left(\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ and } \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right).$$

Let l, m, n be d.c.'s of the line which makes angles $\alpha, \beta, \gamma, \delta$ with the diagonals of the cube.

Then $\cos\alpha = l\,(1/\sqrt{3}) + m\,(1/\sqrt{3}) + n\,(1/\sqrt{3}) = (l + m + n)/\sqrt{3}$

or $$\cos^2\alpha = \tfrac{1}{3}\,(l + m + n)^2.$$

Similarly $\cos\beta = l\,(1/\sqrt{3}) + m\,(1/\sqrt{3}) + n\,(-1/\sqrt{3}) = (l + m - n)/\sqrt{3}$

or $$\cos^2\beta = \tfrac{1}{3}\,(l + m - n)^2. \qquad ...(ii)$$

And in a similar mannerr we can prove that

$$\cos^2\gamma = \tfrac{1}{3}\,(-l + m + n)^2 \text{ and } \cos^2\delta = \tfrac{1}{3}\,(l - m + n)^2 \qquad ...(iii)$$

From (i), (ii) and (iii), we get $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$$= \frac{1}{3}\,[(l + m + n)^2 + (l + m - n)^2 + (-l + m + n)^2 + (l - m + n)^2]$$

$$= \frac{1}{3}\,[4\,(l^2 + m^2 + n^2)], \text{ on simplifying}$$

$$= 4/3, \text{ since } l^2 + m^2 + n^2 = 1.$$ Hence proved.

****Ex. 15. Find the angle between the diagonals of a cube.**

(*Bundelkhand* 94)

Sol. As in last example, we can prove that the d.c.'s of the diagonals OP and CN of the cube are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$ respectively.

Therefore if θ be the angle between OP and CN, then

$$\cos\theta = \text{``}l_1l_2 + m_1m_2 + n_1n_2\text{''} = \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right) = \frac{1}{3}$$

or $\qquad \theta = \cos^{-1}\left(\frac{1}{3}\right)$ **Ans.**

Ex. 16. The vertices of a triangle ABC are the points A (– 1, 2, – 3), B (5, 0, – 6) and C (0, 4, – 1) in order. Find the direction ratios of the bisectors of the angle BAC.

Sol. $AB = \sqrt{[(5+1)^2 + (0-2)^2 + (-6+3)^2]} = 7;$

$AC = \sqrt{[(0+1)^2 + (4-2)^2 + (-1+3)^2]} = 3.$

Let the internal bisector of $\angle BAC$ meet BC in D. Then

$$BD : DC = BA : CA = 7 : 3 \qquad \textbf{(Note)}$$

i.e. D divides BC internally in the ratio 7 : 3

$\therefore$ The co-ordinates of D are

$$\left[\frac{7.0+3.5}{7+3}, \frac{7.4+3.0}{7+3}, \frac{7(-1)+3(-6)}{7+3}\right] \quad i.e. \quad \left[\frac{3}{2}, \frac{14}{5}, -\frac{5}{2}\right]$$

$\therefore$ The d.c.'s of the line AD are proportional to $[(3/2)+1]$, $[(14/5)-2]$, $[(-5/2)+3]$ *i.e.* $5/2, 4/5, 1/2$ *i.e.* proportional to (25, 8, 5). **Ans.**

Again if the external bisector of the angle BAC meets BC in E, then E divides BC **externally in the ratio** 7 : – 3.

$\therefore$ The coordinates of E are

$$\left[\frac{7.0-3.5}{7-3}, \frac{7.4-3.0}{7-3}, \frac{7(-1)-3(-6)}{7-3}\right] \quad i.e. \quad \left[\frac{-15}{4}, 7, -\frac{11}{4}\right]$$

$\therefore$ The direction ratios of the line AE are

$$\left(\frac{-15}{4}+1, 7-2, -\frac{11}{4}+3\right) \quad i.e. \quad \left(\frac{-11}{4}, 5, \frac{23}{4}\right) \quad i.e. \quad (-11, 20, 23). \textbf{ Ans.}$$

***Ex. 17. Show that the equation to the right circular cone whose vertex is at the origin, whose axis has d.c.'s cos α, cos β, cos γ and whose semi-vertical angle is θ is**

$$(y\cos\gamma - z\cos\beta)^2 + (z\cos\alpha - x\cos\gamma)^2 + (x\cos\beta - y\cos\alpha)^2$$
$$= (x^2+y^2+z^2)\sin^2\theta.$$

Sol. Let $P(x, y, z)$ be any point on the cone (*i.e.* on the surface of the cone). The vertex of the cone is given as the origin $O(0, 0, 0)$.

Then the d.c.'s of the generator OP are

$$\frac{x-0}{\sqrt{[(x-0)^2+(y-0)^2+(z-0)^2]}}, \frac{y-0}{\sqrt{[(x-0)^2+(y-0)^2+(z-0)^2]}},$$

$$\frac{z-0}{\sqrt{[(x-0)^2+(y-0)^2+(z-0)^2]}}$$

i.e. $$\frac{x}{\sqrt{(x^2+y^2+z^2)}}, \frac{y}{\sqrt{(x^2+y^2+z^2)}}, \frac{z}{\sqrt{(x^2+y^2+z^2)}}.$$

Also the d.c.'s of the axis of the cone are given as $\cos\alpha$, $\cos\beta$, $\cos\gamma$ and the line OP is inclined at an angle θ to the axis.

$\therefore \sin^2\theta = \Sigma\,(m_1n_2 - m_2n_1)^2$ ... See § 2.10 (D) Page 28

$$= \Sigma\left[\frac{y\cos\gamma - z\cos\beta}{\sqrt{(x^2+y^2+z^2)}}\right]^2 = \frac{\Sigma\,(y\cos\gamma - z\cos\beta)^2}{(x^2+y^2+z^2)}$$

or $$(x^2+y^2+z^2)\sin^2\theta = (y\cos\gamma - z\cos\beta)^2 + (z\cos\alpha - x\cos\gamma)^2 + (x\cos\beta - y\cos\alpha)^2.$$

Hence proved.

***Ex. 18. If two pairs of opposite edges of a tetrahedron are perpendicular, show that the third pair is also perpendicular.** *(Kanpur 97)*

Sol. Let $OABC$ be the tetrahedron where, O, A, B and C are $(0, 0, 0)$, (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively.

Let two pairs of opposite edges viz. OA, BC and OC, AB be perpendicular.

The d.c.'s of OA, OC, BC and AB are proportional to (x_1, y_1, z_1), (x_3, y_3, z_3), $(x_3-x_2, y_3-y_2, z_3-z_2)$ and $(x_2-x_1, y_2-y_1, z_2-z_1)$ respectively.

As OA is perpendicular to BC, so

$$x_1(x_3-x_2) + y_1(y_3-y_2) + z_1(z_3-z_2) = 0 \qquad \text{... (i)}$$

As OC is perpendicular to AB, so

$$x_3(x_2-x_1) + y_3(y_2-y_1) + z_3(z_2-z_1) = 0 \qquad \text{... (ii)}$$

Adding (i) and (ii) we have $x_2(x_3-x_1) + y_2(y_3-y_1) + z_2(z_3-z_1) = 0$ which shows that the third pair of opposite edges viz. OB, AC are also perpendicular.

Hence proved.

Ex. 19. If in a tetrahedron OABC, $OA^2 + BC^2 = OB^2 + CA^2 = OC^2 + AB^2$, then its pairs of opposite edges are at right angles.

Sol. With the same coordinaes as in last example, we have

$$OA^2 + BC^2 = (x_1^2+y_1^2+z_1^2) + [(x_3-x_2)^2 + (y_3-y_2)^2 + (z_3-z_2)^2]$$
$$= (x_1^2+y_1^2+z_1^2) + (x_2^2+y_2^2+z_2^2) + (x_3^2+y_3^2+z_3^2) - 2(x_2x_3 + y_2y_3 + z_2z_3)$$

Similarly we can prove that

$$OB^2 + CA^2 = (x_1^2+y_1^2+z_1^2) + (x_2^2+y_2^2+z_2^2) + (x_3^2+y_3^2+z_3^2) - 2(x_1x_3 + y_1y_3 + z_1z_3)$$

$$OC^2 + AB^2 = (x_1^2+y_1^2+z_1^2) + (x_2^2+y_2^2+z_2^2) + (x_3^2+y_3^2+z_3^2) - 2(x_1x_2 + y_1y_2 + z_1z_2).$$

If $OA^2 + BC^2 = OB^2 + CA^2$, then we have

$$x_2x_3 + y_2y_3 + z_2z_3 = x_1x_3 + y_1y_3 + z_1z_3$$

or $x_3(x_2 - x_1) + y_3(y_2 - y_1) + z_3(z_2 - z_1) = 0$, which shows that OC is perpendicular to AB *i.e.* a pair of opposite edges of a tetrahedron $OABC$ are perpendicular. In a similar manner we can prove for other pairs.

Ex. 20. If a pair of opposite edges ofa tetrahedron be perpendicular, then prove that the distances between the mid-poins of the other two pairs of opposite edges are equal.

Sol. Let $OABC$ be the tetrahedron, then with the same notations as in Ex. 18 Page 41 we find that if OA is perpendicular to BC, then

$$x_1(x_3 - x_2) + y_1(y_3 - y_2) + z_1(z_3 - z_2) = 0 \quad \text{...(i)}$$

Also the mid-points of OB and CA are

$(\frac{1}{2}x_2, \frac{1}{2}y_2, \frac{1}{2}z_2)$ and $\left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}, \frac{z_1 + z_3}{2}\right)$ respectively.

$\therefore$ square of the distance between the mid-points of OB and CA

$$= \{\tfrac{1}{2}(x_1 + x_3) - \tfrac{1}{2}x_2\}^2 + \{\tfrac{1}{2}(y_1 + y_3) - \tfrac{1}{2}y_2\}^2 + \{\tfrac{1}{2}(z_1 + z_3) - \tfrac{1}{2}z_2\}^2$$

$$= \frac{1}{4}[(x_1 + x_3 - x_2)^2 + (y_1 + y_3 - y_2)^2 + (z_1 + z_3 - z_2)^2] \quad \text{...(ii)}$$

Similarly the square of the distance between the mid-points of OC

and $AB = \frac{1}{4}[(x_1 + x_2 - x_3)^2 + (y_1 + y_2 - y_3)^2 + (z_1 + z_2 - z_3)^2]$ ...(iii)

If these distances are equal, then from (ii) and (iii) we have

$$(x_1 + x_3 - x_2)^2 + (y_1 + y_3 - y_2)^2 + (z_1 + z_3 - z_2)^2$$
$$= (x_1 + x_2 - x_3)^2 + (y_1 + y_2 - y_3)^2 + (z_1 + z_2 - z_3)^2$$

or
$$\{(x_1 + x_3 - x_2)^2 - (x_1 + x_2 - x_3)^2\} + \{(y_1 + y_3 - y_2)^2 - (y_1 + y_2 - y_3)^2\} + \dots = 0$$

or
$$\{2x_1 . 2(x_3 - x_2)\} + \{2y_1 . 2(y_3 - y_2)\} + \{2z_1 . 2(z_3 - z_2)\} = 0$$

or $x_1(x_3 - x_2) + y_1(y_3 - y_2) + z_1(z_3 - z_2) = 0$, which is true by virtue of (i).

Hence proved.

Exercises on § 2.10

Ex. 1. If points A and B are $(2, 3, 4)$ and $(1, -2, 1)$ respectively, then prove that OA is perpendicular to OB, where O is $(0, 0, 0)$.

Ex. 2. Find the angle between the lines whose direction ratios are $1, 1, 2$ and $\sqrt{3} - 1, -\sqrt{3} - 1, 4$.

Ex. 3. Find the angle between the lines whose directon cosines are given by the relations $3l + m + 5n = 0$ and $6mn - 2nl + 5lm = 0$. **Ans.** $\cos^{-1}(1/6)$

Ex. 4. Find the angle between the lines whose direction cosines are given by the equations $3l + m + 6n = 0$, $6mn - 2ln + lm = 0$. *(Rohilkhand 95)*

Ex. 5. Show that the pair of lines whose direction cosines are given by the equations $l + 2m + 3n = 0$ and $mn - 4nl + 3lm = 0$ are at right angles.
(*Bundelkhand* 93)

Ex. 6. If θ is the angle between the lines whose direction cosines are proportional to 5, – 12, 13 and – 3, 4, 5; then find the value of tan θ.
Ans. $\sqrt{(4226)}/65$

Ex. 7. Obtain the direction cosines of the diagonal of a cube through one corner, taking the coordinatte axes along the edges of the cube through that corner. **Ans.** $(1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3})$.

Ex. 8. Prove that the equation to the right circular cone whose vertex is the point (2, – 3, 5) and whose axis is the line which makes equal angles with the co-ordinatte axes and semivertical angle 30° is

$$4\,[(y - z + 8)^2 + (z - x - 3)^2 + (z - y - 5)^2] = 3\,[(x - 2)^2 + (y + 3)^2 + (z - 5)^2]$$

Miscellaneous Solved Examples

****Ex. 1. Prove that three concurrent lines with direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) are coplanar, if**

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$$
(*Kumaun* 90)

Sol. Let l, m, n be the d.c's of the normal to the plane which contains the two concurrent lines with d.c.'s (l_1, m_1, n_1) and (l_2, m_2, n_2). Then we have

$$l\,l_1 + m\,m_1 + n\,n_1 = 0 \qquad ...(i)$$

and
$$l\,l_2 + m\,m_2 + n\,n_2 = 0 \qquad ...(ii)$$

If the third line with d.c.'s (l_3, m_3, n_3) also lies on this plane then the line with d.c.'s (l, m, n) is at right angles to this third line and so we have

$$l\,l_3 + m\,m_3 + n\,n_3 = 0. \qquad ...(iii)$$

Eliminating l, m, n from (i), (ii) and (iii) we have

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0,$$ as the required condition.

Ex. 2. Show that the points (0, 4, 1), (2, 3, – 1) (4, 5, 0) and (2, 6, 2) are the vertices of a square. (*Kumaun* 96)

Sol. Let A, B, C, D be respectively the points (0, 4, 1), (2, 3, – 1), (4, 5, 0) and (2, 6, 2).

Find the lengths of sides AB, BC, CD, DA and diagonals AC, BD of the quadrilateral $ABCD$.

Then prove that $AB = BC = CD = DA = 3$ and $AC = BD = 3\sqrt{2}$

$\because$ sides of quad. $ABCD$ are equal in length and its diagonals are also equal in length, so it represents a square.

Ex. 3 (a). Find the area of the triangle OAB where, O, A and B are (0, 0, 0), (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

Sol. The area of the triangle $OAB = \frac{1}{2} OA.OB \sin \angle OAB.$ ...(i)

Now $OA = \sqrt{(x_1^2 + y_1^2 + z_1^2)}$, $OB = \sqrt{(x_2^2 + y_2^2 + z_2^2)}$

and $\sin \angle AOB = \dfrac{\sqrt{[(y_1z_2 - y_2z_1)^2 + (z_1x_2 - z_2x_1)^2 + (x_1y_2 - x_2y_1)^2]}}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}\sqrt{(x_2^2 + y_2^2 + z_2^2)}}$

since the d.c.'s of the lines OA and OB are

$$\frac{x_1}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}}, \frac{y_1}{\sqrt{(\Sigma x_1^2)}}, \frac{z_1}{\sqrt{(\Sigma x_1^2)}} \text{ and } \frac{x_2}{\sqrt{(x_2^2 + y_2^2 + z_2^2)}}, \frac{y_2}{\sqrt{(\Sigma x_2^2)}}, \frac{z_2}{\sqrt{(\Sigma x_2^2)}}$$

$\therefore$ from (i), the required area of $\Delta\, OAB$

$$= \frac{1}{2}\sqrt{(x_1^2 + y_1^2 + z_1^2)} . \sqrt{(x_2^2 + y_2^2 + z_2^2)} \times \frac{\sqrt{[\Sigma (y_1z_2 - y_2z_1)^2]}}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}\sqrt{(x_2^2 + y_2^2 + z_2^2)}}$$

$$= \frac{1}{2}\sqrt{[(y_1z_2 - y_2z_1)^2 + (z_1x_2 - z_2x_1)^2 + (x_1y_2 - x_2y_1)^2]}.$$ **Ans.**

Ex. 3 (b). A plane makes intercepts OA, OB, OC whose measurres are a, b, c on the axes OX, OY, OZ. Find the area of the triangle ABC.

Sol. The co-ordinates of A, B, C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

Now the area of the triangle $ABC = \frac{1}{2} CA.CB \sin \angle ACB.$ ...(i)

Now $CA = \sqrt{[(0-a)^2 + (0-0)^2 + (c-0)^2]} = \sqrt{(a^2 + c^2)}$

$CB = \sqrt{[(0-0)^2 + (0-b)^2 + (c-0)^2]} = \sqrt{(b^2 + c^2)}$

Also direction cosines of the lines CA and CB are

$$\frac{0-a, 0-0, c-0}{\sqrt{\{(0-a)^2 + (0-0)^2 + (c-0)^2\}}} \text{ i.e. } \frac{-a, 0, c}{\sqrt{(a^2 + c^2)}}$$

and

$$\frac{0-0, 0-b, c-0}{\sqrt{\{(0-0)^2 + (0-b)^2 + (c-0)^2\}}} \text{ i.e. } \frac{0, -b, c}{\sqrt{(b^2 + c^2)}}$$

$\sin ACB =$ "$\sqrt{[\Sigma (m_1n_2 - m_2n_1)^2]}$"

$$= \frac{\sqrt{[(0.c + b.c)^2 + (c.0 + c.a)^2 + (a.b - 0.0)^2]}}{\sqrt{(a^2 + c^2)} . \sqrt{(b^2 + c^2)}}$$

$$= \frac{\sqrt{(b^2c^2 + c^2a^2 + c^2b^2)}}{\sqrt{(a^2 + c^2)} . \sqrt{(b^2 + c^2)}}$$

$\therefore$ From (i), the required area of $\Delta\, ABC = \frac{1}{2} CA.CB \sin \angle ACB$

$$= \frac{1}{2}\sqrt{(a^2 + c^2)} . \sqrt{(b^2 + c^2)} \frac{\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}}{\sqrt{(a^2 + c^2)} . \sqrt{(b^2 + c^2)}}$$

$$= \frac{1}{2}\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}.$$ **Ans.**

Ex. 3 (c). Find the area of the triangle included between the plane $x+y+z=5$ and the co-ordinate planes. (*Purvanchal* 93)

Sol. If the given plane $x+y+z=5$ or $\frac{x}{5}+\frac{y}{5}+\frac{z}{5}=1$ meets the coordinate area in A, B and C, then we have $A\ (5, 0, 0)$, $B\ (0, 5, 0)$ and $C\ (0, 0, 5)$

Now do as Ex. 3 (b) above. Here $a=b=c=5$ **Ans.** $(25\sqrt{3})/2$ sq. units

Ex. 3 (d) Show that the line joining the points (1, 2, 3) and (4, 5, 7) is parallel to the line joining the points (– 4, 3, – 6) and (2, 9, 2).

Sol. The direction ratios of the line joining (1, 2, 3) and (4, 5, 7) are

$$4-1, 5-2, 7-3 \quad \text{or} \quad 3, 3, 4 \qquad \text{... (i)}$$

The direction ratios of the line joining (– 4, 3, – 6) and (2, 9, 2) are

$$2-(-4), 9-3, 2-(-6) \quad \text{or} \quad 6, 6, 8 \quad \text{or} \quad 3, 3, 4. \qquad \text{... (ii)}$$

From (i) and (ii) we find that the direction ratios of the two lines are the same. Hence they are parallel.

Ex. 3 (e). Show that the line joining the points (1, 2, 3), (– 1, – 2, – 3) is parallel to the line joining the points (2, 3, 4), (5, 9, 13) and perpendicular to the line joining the points (– 2, 1, 5), (3, 3, 2). (*Rohilkhand* 96)

Sol. The direction ratios of the line joining (1, 2, 3) and (– 1, – 2, – 3) are $-1-1, -2-2, -3-3$ *i.e.* $-2, -4, -6$ *i.e.* $1, 2, 3$ ... (i)

The direction ratios of the line joining the points (2, 3, 4) and (5, 9, 13) are $5-2,\ 9-3,\ 13-4$ *i.e.* $3, 6, 9$ *i.e.* $1, 2, 3$ ... (ii)

From (i) and (ii) we conclude that these two lines are parallel.

Also the direction ratios of the line joining the points (– 2, 1, 5) and (3, 3, 2) are $3+2, 3-1, 2-5$ *i.e.* $5, 2, -3$ ... (iii)

Now as $(1)(5)+(2)(2)+(3)(-3)=5+4-9=0$

So the lines whose d.c.'s are given by (i) and (iii) are perpendicular.

Ex. 4. O, A, B, C are four points not necessarily lying in the same plane and such that OA ⊥ BC and OB ⊥ CA. Prove that OC ⊥ AB.

Sol. Let O, A, B, C be the points $(0, 0, 0)$, (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) respectively. Then direction ratios of the lines OA, OB and OC are x_1-0, y_1-0, z_1-0; x_2-0, y_2-0, z_2-0 and x_3-0, y_3-0, z_3-0 *i.e.* x_1, y_1, z_1; x_2, y_2, z_2 and x_3, y_3, z_3 respectively. ...(i)

Also the direction ratios of AB, BC and CA are respectively $x_2-x_1, y_2-y_1, z_2-z_1$; $x_3-x_2, y_3-y_2, z_3-z_2$ and $x_1-x_3, y_1-y_3, z_1-z_3$. ...(ii)

Now if $OA \perp BC$, then from (i) and (ii), we get

$$x_1(x_3-x_2)+y_1(y_3-y_2)+z_1(z_3-z_2)=0$$

or
$$x_1x_3-x_1x_2+y_1y_3-y_1y_2+z_1z_3-z_1z_2=0 \qquad \text{...(iii)}$$

And if $OB \perp CA$, then from (i) and (ii), we get

$$x_2(x_1-x_3)+y_2(y_1-y_3)+z_2(z_1-z_3)=0$$

or
$$x_2x_1-x_2x_3+y_2y_1-y_2y_3+z_2z_1-z_2z_3=0 \qquad \text{...(iv)}$$

Adding (iii) and (iv), we get

$$x_1x_3-x_2x_3+y_1y_3-y_2y_3+z_1z_3-z_2z_3=0$$

or $$x_3(x_2-x_1)+y_3(y_2-y_1)+z_3(z_2-z_1)=0$$

i.e. the lines whose direction ratios are x_3, y_3, z_3 and $x_2-x_1,\ y_2-y_1,\ z_2-z_1$ are perpendicular.

i.e. $OC \perp AB$, from (i) and (ii) Hence proved.

Ex. 5 (a). Find the direction cosines of the line perpendicular to a pair of mutually perpendicular lines with their direction cosines as (l_1, m_1, n_1) and (l_2, m_2, n_2) respectively.

Sol. Let l, m, n be the required direction cosines of the line which is perpendicular to given lines, then $l\,l_1+m\,m_1+n\,n_1=0$ ... (i)

and $$l\,l_2+m\,m_2+n\,n_2=0 \quad \text{... (ii)}$$

Also as the given lines are mutually perpendicular so we have

$$l_1\,l_2+m_1\,m_2+n_1\,n_2=0 \quad \text{... (iii)}$$

Solving (i) and (ii) simultaneously we get

$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}$$

$$=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{[(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2]}}$$

$$=\frac{1}{\sqrt{[(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2)-(l_1l_2+m_1m_2+n_1n_2)^2]}}$$

$\because l^2+m^2+m^2=1$ and by Lagrange's Identity (See Page 27)

$$=\frac{1}{\sqrt{[(1)(1)-(0)]}}, \text{ from (iii) and } \Sigma l_1^2=1=\Sigma l_2^2$$

$=1.$

$\therefore l=m_1n_2-m_2n_1,\ m=n_1l_2-n_2l_1,\ n=l_1m_2-l_2m_1.$ **Ans.**

Ex. 5 (b). (l_1, m_1, n_1) and (l_2, m_2, n_2) are the d.c.'s of two intersecting lines. Show that all lines through the intersection of these two whose direction cosines are proportional to $l_1+\lambda l_2,\ m_1+\lambda m_2,\ n_1+\lambda n_2$ are coplanar with them.

Sol. Let l, m, n be the d.c.'s of the normal to the plane containing the two given intersecting lines. Then we have $l\,l_1+m\,m_1+n\,n_1=0$ ... (i)

and $$l\,l_2+m\,m_2+n\,n_2=0 \quad \text{... (ii)}$$

Multiplying (ii) by λ and adding to (i), we get

$l(\lambda l_2+l_1)+m(\lambda m_2+m_1)+n(\lambda n_2+n_1)=0$, which shows that the lines with d.c.'s (l, m, n) and $(l_1+\lambda l_2, m_1+\lambda m_2,\ n_1+\lambda n_2)$ are perpendicular and as such all lines having d.c.'s $(l_1+\lambda l_2, m_1+\lambda m_2,\ n_1+\lambda n_2)$ are coplanar with the given two intersecting lines. Hence proved.

***Ex. 6. (l_1, m_1, n_1) and (l_2, m_2, n_2) are the d.c.'s of two concurrent lines, show that the d.c.'s of two lines bisecting the angles between them are proportional to $(l_1 \pm l_2,\ m_1 \pm m_2,\ n_1 \pm n_2)$**

Sol. Draw two lines OP and OQ through the origin O parallel to the given concurrent lines.

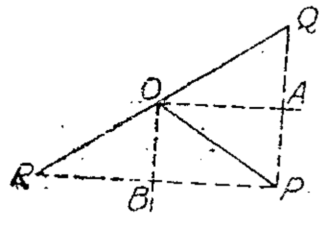
Mark off $OP=r$ and $OQ=r$. Then as (l_1, m_1, n_1) and (l_2, m_2, n_2) are the d.c.'s of the lines OP and OQ, therefore the co-ordinates of P and Q are (l_1r, m_1r, n_1r) and (l_2r, m_2r, n_2r) respectively.

Produce QO and mark $OR=OQ=r$, then the co-ordinates of R are

$(-l_2r, -m_2r, -n_2r)$.

(Note)

Let A and B be the mid-points of PQ and PR then OA and OB are the internal and external bisectors of the $\angle QOP$.



The co-ordinates of A and B are

$[\frac{1}{2}(l_1r + l_2r), \frac{1}{2}(m_1r + m_2r), \frac{1}{2}(n_1r + n_2r)]$

and $[\frac{1}{2}(l_1r - l_2r), \frac{1}{2}(m_1r - m_2r), \frac{1}{2}(n_1r - n_2r)]$

respectively.

(Fig. 22)

$\therefore$ The direction ratios of OA and OB, where O is the origin (0, 0, 0) are

$$[\frac{1}{2}(l_1 + l_2)\, r, \frac{1}{2}(m_1 + m_2)\, r, \frac{1}{2}(n_1 + n_2)\, r]$$

and
$$[\frac{1}{2}(l_1 - l_2)r, \frac{1}{2}(m_1 - m_2)r, \frac{1}{2}(n_1 - n_2)r].$$

i.e. d.c.'s of OA and OB are proportional to

$[(l_1 + l_2), (m_1 + m_2), (n_1 + n_2)]$ and $[(l_1 - l_2), (m_1 - m_2)\ (n_1 - n_2)]$ respectively.

Ex. 7. Find the direction ratios of the lines bisecting the angles between the lines whose direction ratios are l_1, m_1, n_1 and l_2, m_2, n_2 and the angle betwen these lines is θ.

Sol. In Ex. 6 above we haave proved that the direction ratios of the inernal bisector of the angle between the lines whose d.c.'s are (l_1, m_1, n_1) and (l_2, m_2, n_2) are given by $[l_1 + l_2,\ m_1 + m_2,\ n_1 + n_2]$

$\therefore$ The direction cosines of the internal bisector are

$$\frac{l_1 + l_2}{\sqrt{[(l_1 + l_2)^2 + (m_1 + m_2)^2 + (n_1 + n_2)^2]}},$$
$$\frac{m_1 + m_2}{\sqrt{[(l_1 + l_2)^2 + (m_1 + m_2)^2 + (n_1 + n_2)^2]}}, \frac{n_1 + n_2}{\sqrt{[\ldots]}}$$

Now $\sqrt{[(l_1 + l_2)^2 + (m_1 + m_2)^2 + (n_1 + n_2)^2]} = \sqrt{[\Sigma l_1^2 + \Sigma l_2^2 + 2\,\Sigma l_1 l_2]}$

$$= \sqrt{(1 + 1 + 2\cos\theta)},\ \because \Sigma l_1 l_2 = \cos\theta$$
$$= \sqrt{[2(1 + \cos\theta)]} = 2\cos\frac{1}{2}\theta$$

The d.c.'s of the internal bisector are

$$\frac{l_1 + l_2}{2\cos\frac{1}{2}\theta}, \frac{m_1 + m_2}{2\cos\frac{1}{2}\theta}, \frac{n_1 + n_2}{2\cos\frac{1}{2}\theta} \qquad \textbf{Ans.}$$

In a similar manner we can prove that the d.c.'s of the external bisector are

$$\frac{l_1 - l_2}{2\sin\frac{1}{2}\theta}, \frac{m_1 - m_2}{2\sin\frac{1}{2}\theta}, \frac{n_1 - n_2}{2\sin\frac{1}{2}\theta}, \qquad \textbf{Ans.}$$

where $\sqrt{[(l_1 - l_2)^2 + (m_1 - m_2)^2 + (n_1 - n_2)^2]} = \sqrt{[\Sigma l_1^2 + \Sigma l_2^2 - 2\,\Sigma l_1 l_2]}$

$$= \sqrt{[1 + 1 - 2\cos\theta]} = \sqrt{[2(1 - \cos\theta)]} = 2\sin\frac{1}{2}\theta.$$

****Ex. 8. The direction cosines of a variable line in two adjacent positions are l, m, n; $l+\delta l$, $m+\delta m$, $n+\delta n$, show that the small angle $\delta\theta$ between the two positions is given by $(\delta\theta)^2=(\delta l)^2+(\delta m)^2+(\delta n)^2$.**

(*Garhwal* 90; *Kumaun* 91; *Kanpur* 97; *Meerut* 92P)

Sol. As (l, m, n) and $(l+\delta l,\ m+\delta m,\ n+\delta n)$ are the d.c.'s of lines so we have

$$l^2+m^2+n^2=1 \qquad \text{...(i)}$$

and

$$(l+\delta l)^2+(m+\delta m)^2+(n\,\delta n)^2=1 \qquad \text{...(ii)}$$

Subtracting (i) from (ii) we get

$$2(l\,\delta l+m\,\delta m+n\,\delta n)+[(\delta l)^2+(\delta m)^2+(\delta n)^2]=0$$

or

$$2\Sigma(l\,\delta l)=-\Sigma(\delta l)^2 \qquad \text{...(iii)}$$

Also as $\delta\theta$ is the angle between these lines so we have

$$\cos\delta\theta=l(l+\delta l)+m(m+\delta m)+n(n+\delta n)$$

$$=(l^2+m^2+n^2)+(l\,\delta l+m\,\delta m+n\,\delta n)$$

$$=1-\tfrac{1}{2}[\Sigma(\delta l)^2],\ \text{from (i) and (iii)}$$

or $\frac{1}{2}[\Sigma(\delta l)^2]=1-\cos\delta\theta=2\sin^2(\frac{1}{2}\delta\theta)]=2(\frac{1}{2}\delta\theta)^2$, $\because \sin(\frac{1}{2}\delta\theta)=\frac{1}{2}\delta\theta$

or $(\delta l)+(\delta m)^2+(\delta n)^2=(\delta\theta)^2$ Hence proved.

Exercises on Chapter II

Ex. 1. Find the direction ratios of a line perpendicular to two lines whose direction ratios are 1, 2, 3 and -2, 1, 4. **Ans.** 1, -2, 1

Ex. 2. Find the area of the triangle whose vertices are (1, 2, 3), $(-2, 1, -4)$ and $(3, 4, -2)$. **Ans.** $\frac{1}{2}\sqrt{(1218)}$

***Ex. 3.** l_r, m_r, n_r $(r=1, 2, 3)$ are the direction cosines of three mutually perpendicular lines and also.

$$\frac{a}{l_1}+\frac{b}{m_1}+\frac{c}{n_1}=0,\ \frac{a}{l_2}+\frac{b}{m_2}+\frac{c}{n_2}=0,$$

then prove that $\frac{a}{l_3}+\frac{b}{m_3}+\frac{c}{n_3}=0$ (*Rohilkhand* 91)

Ex. 4. Which are the direction cosines of a line ?

(i) (2/3, 2/3, 2/3); (ii) (1/3, 1/3, 1/3);

(iii) $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$; (iv) $(1/\sqrt{3}, 1/\sqrt{3}, 2/\sqrt{3})$ **Ans.** (iii)

Ex. 5. The condition of orthogonality for the two lines whose d.c.'s are l_1, m_1, n_1 and l_2, m_2, n_2 is

(i) $l_1^2+m_1^2+n_1^2$ and $l_2^2+m_2^2+n_2^2=1$; (ii) $l_1l_2+m_1m_2+n_1n_2=0$;

(iii) $l_1/l_2=m_1/m_2=n_1/n_2$ (iv) $l_1l_2+m_1m_2+n_1n_2=1$.

Ans. (ii)

CHAPTER III

The Plane

§ 3.01 The Plane.

Definition. A plane is defined as the surface which is such that the line joining any two points on it lies wholly on it.

General Equation of the first degree. *To show that the general equation of the first degree in x, y, z rrepresents a plane.* (*Gorakhpur* 90; *Kanpur* 92)

The general equation of the first degree is

$$Ax + By + Cz + D = 0 \quad \text{...(i)}$$

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be the co-ordinates of any two points on the locus given by (i)

Then we have $$Ax_1 + By_1 + Cz_1 + D = 0 \quad \text{...(ii)}$$

and $$Ax_2 + By_2 + Cz_2 + D = 0 \quad \text{...(iii)}$$

Multiplying (iii) by k and adding to (ii) we have

$$A(x_1 + kx_2) + B(y_1 + ky_2) + C(z_1 + kz_2) + D(1 + k) = 0$$

or $$A\left[\frac{x_1 + kx_2}{1 + k}\right] + B\left[\frac{y_1 + ky_2}{1 + k}\right] + C\left[\frac{z_1 + kz_2}{1 + k}\right] + D = 0 \quad \text{...(iv)}$$

This relation shows that the point

$$\left[\frac{x_1 + kx_2}{1 + k}, \frac{y_1 + ky_2}{1 + k}, \frac{z_1 + kz_2}{1 + k}\right] \text{ lies on (i).}$$

But from § 1.06 Cor. 2 Chapter I we know that this point divides the join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $k : 1$

Since k can have any value, so each point on the line PQ lies on (i) *i.e.* the line wholly lies on (i).

Also a plane is defined as a surface such that the line joining any two points on it lies wholly on it. Hence by the definition of the plane as given above we conclude that (i) represents a plane.

Hence the general equation of first degree in x, y, z viz $Ax + By + Cz + D = 0$ represents a plane.

Cor. 1. The equation of the plane through the origin is given by $Ax + By + Cz = 0$ *i.e.* if $D = 0$ in (i) then the plane passes through the origin.

Cor 2. General equation of a plane through a given point (x_1, y_1, z_1).

(*One Point Form*)

Let $$Ax + By + Cz + D = 0 \quad \text{...(i)}$$

be the required plane.

If it passes through (x_1, y_1, z_1) then

$$Ax_1 + By_1 + Cz_1 + D = 0 \quad \text{...(ii)}$$

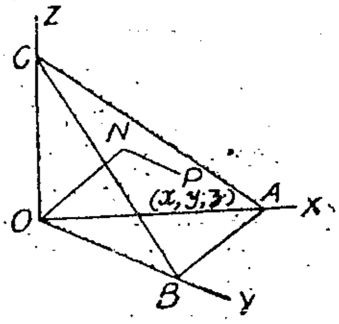
Subtracting (ii) from (i), we hvae

$$\boxed{A(x - x_1) + B(y - y_1) + C(z - z_1) = 0,}$$

which is the required equation.

§ 3.02 Normal form of the equation of a plane. (*Agra* 90; *Kumaun* 93)

Let ABC be the given plane and let ON be the normal from O to the plane ABC. Let the direction cosines of ON be $\cos\alpha, \cos\beta, \cos\gamma$ and let $ON = p$. Then the co-ordinates on N are $(p\cos\alpha, p\cos\beta, p\cos\gamma)$.



(Fig. 1)

Let $P(x, y, z)$ be any point on the plane ABC. Join PN, then NP is a line passing through N and lying on the plane ABC.

Therefore ON and NP are at right angles.

The direction cosines of NP are proportional to $(x - p\cos\alpha)$, $(y - p\cos\beta)$ and $(z - p\cos\gamma)$.

$\because$ ON and NP are at right angles, so we have

$$(x - p\cos\alpha) \,.\, \cos\alpha + (y - p\cos\beta) \,.\, \cos\beta + (z - p\cos\gamma)\cos\gamma = 0$$

or $\quad x\cos\alpha + y\cos\beta + z\cos\gamma = p(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = p\,(1)$

or $$\mathbf{x\cos\alpha + y\cos\beta + z\cos\gamma = p.} \qquad \text{...(i)}$$

This equation, satisfied by the co-ordinates of every point on the plane, represents the plane.

If l, m, n be the d.c's of the normal ON, then the equation of the plane ABC will be $\boxed{\mathbf{lx + my + nz = p.}}$

Here p is always taken as positive. **(Note)**

§ 3.03 Intercept form of the equation of the plane. (*Kumaun* 95)

Let the plane ABC cut the axes OX, OY and OZ at A, B and C respectively. Let $OA = a$, $OB = b$ and $OC = c$ Let ON be the normal from O to the plane ABC the direction cosines of ON be $\cos\alpha$, $\cos\beta$ and $\cos\gamma$.

From the figure 1 above it is evident that

$$ON = OA\cos\alpha \;\; i.e. \;\; p = a\cos\alpha \;\text{ or }\; \cos\alpha = p/a.$$

Similarly we can prove that $\cos\beta = p/b$ and $\cos\gamma = p/c$.

Substituting these values in the result (i) of § 3.02 above, we get

$$x(p/a) + y(p/b) + z(p/c) = p$$

or $$\boxed{\mathbf{(x/a) + (y/b) + (z/c) = 1.}}$$

which is the required intercept form of the equation of the plane.

Cor. The equation $Ax + By + Cz + D = 0$ can be written as

$$\frac{x}{-(D/A)} + \frac{y}{-(D/B)} + \frac{z}{-(D/C)} = 1.$$

which represents a plane making intercepts $-D/A, -D/B, -D/C$ on x, y and z-axes respectively.

§ 3.04 Reduction of the general equation to the normal form.

(*Rohilkhand 93*)

If $Ax+By+Cz+D=0$ and $x\cos\alpha+y\cos\beta+z\cos\gamma=p$ represent the same plane, then we have

$$\frac{\cos\alpha}{A}=\frac{\cos\beta}{B}=\frac{\cos\gamma}{C}=\frac{-p}{D}, \quad ...(iii)$$

which shows that the d.c.'s **of the normal to the plane** $\mathbf{Ax+By+Cz+D=0}$ **are proportional to A, B, C.** **(Remember)**

Also from (i) we have

$$\frac{\cos\alpha}{A}=\frac{\cos\beta}{B}=\frac{\cos\gamma}{C}=\frac{-p}{D}=\pm\frac{\sqrt{(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)}}{\sqrt{(A^2+B^2+C^2)}}$$

$$=\pm\frac{1}{\sqrt{(A^2+B^2+C^2)}}$$

Now if D is positive, p being a positive number we have

$$p=\frac{D}{\sqrt{(A^2+B^2+C^2)}};\ \cos\alpha=\frac{-A}{\sqrt{(A^2+B^2+C^2)}};$$

$$\cos\beta=\frac{-B}{\sqrt{(A^2+B^2+C^2)}} \quad \text{and} \quad \cos\gamma=\frac{-C}{\sqrt{(A^2+B^2+C^2)}}$$

If D is negative, we should change the sign of $\sqrt{(A^2+B^2+C^2)}$.

Solved Examples on § 3.01 to § 3.04

Ex. 1 (a). Find the intercepts of the plane $\mathbf{2x-3y+z=12}$ on the co-ordinate axes. (*Kumaun* 96)

Sol. The given equation can be written as

$$\frac{x}{6}+\frac{y}{-4}+\frac{z}{12}=1, \text{ dividing each term by } 12$$

$\therefore$ The required intercepts as 6, – 4, 12 (See § 3.03 P. 2).

Ex. 1 (b). Find the intercepts made on the co-ordinate axes by the plane $\mathbf{x+2y-2z=9}$. Find also the direction cosines of the normal to the plane.

Sol. The given equation can be written as

$$\frac{x}{9}+\frac{y}{(9/2)}+\frac{z}{(-9/2)}=1.$$

$\therefore$ The required intercepts are 9, 9/2 and $-9/2$. (See § 3.03)

Again the direction ratios of the normal to the given planes are the coefficientts of x, y, z in the equation of the plane.

i.e. $1, 2, -2.$ (See § 3.04 above)

$\therefore$ The d.c.'s of the normal are

$$\frac{1}{\sqrt{[1^2+2^2+(-2)^2]}},\ \frac{2}{\sqrt{[1^2+2^2+(-2)^2]}},\ \frac{-2}{\sqrt{[1^2+2^2+(-2)^2]}}$$

i.e. $1/3,\ 2/3,\ -2/3.$ **Ans.**

Ex. 1 (c) Establish the equation of a plane passing through the points (a, 0, 0), (0, b, 0), (0, 0, c).

Sol. From the coordinates of the given points, we find that the intercepts made by the required plane on the axes are a, b and c. **(Note)**

$\therefore$ Required equation is $(x/a)+(y/b)+(z/c)=1$. **Ans.**

Ex. 1 (d). Find the coordinates of the point of intersection of the plane (x/a) + (y/b) + (z/c) = 1 with the coordinate axes. (*Purvanchal* 96)

Sol. Given plane is $(x/a)+(y/b)+(z/c)=1$ ...(i)

Also on x-axis we have $y=0,\ z=0$

$\therefore$ Putting $y=0,\ z=0$ in (i) we find that $x/a=1$ or $x=a$

Hence (i) meets x-axis in $(a, 0, 0)$ **Ans.**

Similarly we can find that (i) meets y and z-axes at $(0, b, 0)$ and $(0, 0, c)$ respectively. **Ans.**

****Ex. 2. A variable plane moves so that the sum of reciprocals of its intercepts on the three co-ordinate axes is constant. Show that it passes through a fixed point.**

Sol. Let the equation of the variable plane be

$$(x/a)+(y/b)+(z/c)=1 \quad \text{...(i)}$$

The intercepts made by this plane on the co-ordinate axes are a, b and c.

$\therefore$ According to the problem we have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\text{constant}=\frac{1}{\lambda}\ \text{(say)}$$

or
$$(\lambda/a)+(\lambda/b)+(\lambda/c)=1 \quad \text{...(ii)}$$

From (i) and (ii) we observe that the point $(\lambda, \lambda, \lambda)$ satisfies the equation (i) of the plane and hence the plane (i) passess through the fixed point $(\lambda, \lambda, \lambda)$.

Hence proved.

****Ex. 3. A plane meets the co-ordinate axes in A, B, C such that the centroid of the triangle ABC is the point (p, q, r), show that equation of the plane is (x/p) + (y/q) + (z/r) = 3.** (*Avadh* 93; *Meerut* 94, 91)

Sol. Let the equation of the plane be $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$ ...(i)

This plane meets the co-ordinate axes in $A\,(a, 0, 0)$, $B\,(0, b, 0)$ and $C\,(0, 0, c)$ respectively.

$\therefore$ The co-ordinates of the centroid of the triangle ABC are

$$[\tfrac{1}{3}(a+0+0), \tfrac{1}{3}(0+b+0), \tfrac{1}{3}(0+0+c)] \quad \text{or} \quad (\tfrac{1}{3}a, \tfrac{1}{3}b, \tfrac{1}{3}c)$$

But it is given that the centroid is the point (p, q, r).

$\therefore$ We have $\frac{1}{3}a=p,\ \frac{1}{3}b=q$ and $\frac{1}{3}c=r$ or $a=3p,\ b=3q,\ c=3r$.

Substituting these values of a, b and c in (i) we have the required equation as

$$\frac{x}{3p}+\frac{y}{3q}+\frac{z}{3r}=1 \quad \text{or} \quad \frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3.$$

Ex. 4. A plane makes intercepts $-6, 3, 4$ upon the co-ordinate axes. What is the length of perpendicular from the origin on it ?

Sol. As the plane makes intercepts $-6, 3, 4$ upon the co-ordinate axes, so its equation is $(x/-6)+(y/3)+(z/4)=1$ or $-2x+4y+3z=12$ ...(i)

Comparing (i) with the normal form

$x \cos \alpha + y \cos \beta + z \cos \gamma = p$ we have $\dfrac{\cos \alpha}{-2}=\dfrac{\cos \beta}{4}=\dfrac{\cos \gamma}{3}=\dfrac{p}{12}$

which gives $\cos \alpha = -(1/6)\,p$, $\cos \beta = (1/3)\,p$, $\cos \gamma = (1/4)\,p$.

Also $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$, so we get

$$\left(-\frac{1}{6}p\right)^2+\left(\frac{1}{3}p\right)^2+\left(\frac{1}{4}p\right)^2=1 \quad \text{or} \quad p^2\left[\frac{1}{36}+\frac{1}{9}+\frac{1}{16}\right]=1$$

or $\quad p^2\left[\dfrac{4+16+9}{144}\right]=1 \quad$ or $\quad p^2=\dfrac{144}{29} \quad$ or $\quad p=\dfrac{12}{\sqrt{(29)}}$ **Ans.**

Ex. 5. If D be the point $(2, 3, -1)$ find the equation to the plane through D at right angles to the line OD, where O is the origin.

Sol. The direction ratios of OD are $(2-0, 3-0, -1-0)$

or $(2, 3, -1)$. ...(i)

Now the equation of any plane through $D\,(2, 3, -1)$ is

$$A\,(x-2)+B\,(y-3)+C\,(z+1)=0$$

...See § 3.01 Cor. 2 Page 1. Ch. III

If this plane is perpendicular to OD, then the normal to it must be parallel to OD *i.e.* the direction ratios of its normal must be proportional to those of OD.

i.e. $\quad \dfrac{A}{2}=\dfrac{B}{3}=\dfrac{C}{-1}=k$ (say) or $A=2k$, $B=3k$, $C=-k$.

$\therefore$ From (ii) the required equation is

$2\,(x-2)+3\,(y-3)-(z+1)=0$ or $2x+3y-z=14$. **Ans.**

Ex. 6. O is the origin and A is the point (a, b, c). Find th· direction cosines o. joi· OA and deduce the equation of the p h A at right angles to OA. (... npur 92)

Sol. The direction ratios of OA are $a-0, b-0, c-0$ ... a, b, c

$\therefore$ The d.c's of OA are

$$\frac{a}{\sqrt{(a^2+b^2+c^2)}}, \frac{b}{\sqrt{(a^2+b^2+c^2)}} \frac{c}{\sqrt{(a^2+b^2+c^2)}}.$$ **Ans.**

Now the equation of any plane through $A\ (a, b, c)$ is

$$A\ (x-a)+B\ (y-b)+C\ (z-c)=0 \qquad ...(iii)$$

The direction ratios of its normal are A, B and C. If the plane given by (i) is perpendicular to OA, then the direction ratios of its normal must be proportional to those of OA.

i.e. $$A/a=B/b=C/c$$

$\therefore$ From (i) the required equation is

$$a\ (x-a)+b\ (y-b)+c\ (z-c)=0 \qquad \textbf{Ans.}$$

Exercises on § 3.01 to § 3.04.

Ex. 1. A plane meets the co-ordinate axes in A, B, C which are such that the centroid of ΔABC is the point (2, 3, 4). Find the equation of the plane.

[Hint. See Ex. 3 Page 4 Ch. III] **Ans.** $6x+4y+3z=6$

Ex. 2. Find the intercepts the plane $3x-y-4z=0$ makes on the axes.

Ans. Each intercept is zero, as it passes through (0, 0, 0).

Ex. 3. Does the point $(4, -6, 0)$ lie on the plane which intersects the positive x, y and z- axes at distance 2, 3, 5 units respectively. **Ans.** No.

Ex. 4. Show that the general equation of the first degree in x, y, z represents a plane. Also write the equation of the plane in any other two forms. (*Kanpur* 92)

Ex. 5. Show that at least three conditions are required to determine the equation of the plane. Also write down the equation of the plane in three different forms. (*Kanpur* 93)

[**Hint.** There are three arbitary constants in the equation of a plane (any form), hence three conditions are required to obtain these constants. For three forms see § 3.01, § 3.02 and § 3.03 Pages 1-2 of this chapter.]

****§ 3.05 Angle between two planes.**

Let the two planes be $A_1x+B_1y+C_1z+D=0$

and $A_2x+B_2y+C_2z+D=0$.

The direction ratios of the normals to these planes are (A_1, B_1, C_1) and (A_2, B_2, C_2). Therefore the direction cosines of these normals are

$$\left[\frac{A_1}{\sqrt{(A_1^2+B_1^2+C_1^2)}}, \frac{B_1}{\sqrt{(\Sigma A_1^2)}}, \frac{C_1}{\sqrt{(\Sigma A_1^2)}}\right];$$

and $$\left[\frac{A_2}{\sqrt{(A_2^2+B_2^2+C_2^2)}}, \frac{B_2}{\sqrt{(\Sigma A_2^2)}}, \frac{C_2}{\sqrt{(\Sigma A_2^2)}}\right]$$

If θ be the angle between these planes then θ is the angle between their normals and consequently, we have

$$\cos\theta=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{(A_1^2+B_1^2+C_1^2)}\,.\,\sqrt{(A_2^2+B_2^2+C_2^2)}} \qquad ...(i)$$

Condition of perpendicularity of two planes.

If given planes are at right angles to each other than $\theta = 90^\circ$

or $\cos\theta = 0$ and so from (i) we have $\boxed{A_1 A_2 + B_1 B_2 + C_1 C_2 = 0}$ ...(ii)

Condition of paralleism of two planes.

If the given planes are parallel then their normals are also parallel *i.e.* the d.c.'s of the normals are proportional.

i.e. $$\boxed{\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}}.$$ ...(iii)

Solved Examples on § 3.05.

***Ex. 1. Find the angle between the planes $3x - 4y + 5z = 0$ and $2x - y - 2z = 5$.** (*Kanpur* 94)

Sol. The required angle θ between the given planes is the angle between their normals.

Now as the direction ratios of the normal to a plane are the coefficients of x, y and z in its equation, so the direction ratios of the normals to the given planes are $3, -4, 5$ and $2, -1, -2$ respectively.

$$\therefore \cos\theta = \frac{\text{“}A_1 A_2 + B_1 B_2 + C_1 C_2\text{”}}{\sqrt{(A_1^2 + B_1^2 + C_1^2)} \cdot \sqrt{(A_2^2 + B_2^2 + C_2^2)}}$$

$$= \frac{3.2 + (-4)(-1) + 5(-2)}{\sqrt{[3^2 + (-4)^2 + 5^2]} \cdot \sqrt{[2^2 + (-1)^2 + (-2)^2]}}$$

$$= 0$$

i.e. the given planes are at right angles. **Ans.**

***Ex. 2 (a). Prove that the equation to plane through (α, β, γ) parallel to the plane $ax + by + cz = 0$ is $ax + by + cz = a\alpha + b\beta + c\gamma$.** (*Agra* 91; *Kanpur* 93)

Sol. The equation to any plane parallel to $ax + by + cz = 0$ is

$$ax + by + cz = k \qquad \text{...(i)}$$

If this plane passes through (α, β, γ) then we have

$$a\alpha + b\beta + c\gamma = k. \qquad \text{.(ii)}$$

Substituting the value of k from (ii) in (i), we get the required equation as

$$ax + by + cz = a\alpha + b\beta + c\gamma.$$ Hence proved.

Ex. 2 (b). Find the equation of the plane through the point (1, 2, –1) and parallel to the plane $2x + 3y - 4z + 5 = 0$.

Sol. Equation of any plane parallel to the given plane is

$$2x + 3y - 4z + k = 0 \qquad \text{...(i)}$$

If it passes through $(1, 2, -1)$, then we have

$$2(1) + 3(2) - 4(-1) + k = 0 \quad \text{or} \quad k = -12$$

$\therefore$ From (i), the required plane is $2x + 3y + 4z = 12$. **Ans.**

Ex. 2 (c). Find the equation of the plane through (1, 2, 3) parallel to

$$3x + 4y - 5z = 0.$$

Sol. Do as Ex. 2 (b) above. **Ans.** $3x + 4y - 5z + 4 = 0$

Ex. 2 (d). Find the equation of the plane through the point $(1, 2, -3)$ and parallel to the plane $2x + 3y - 4z + 5 = 0$. (*Bundelkhand* 94)

Sol. Do as Ex. 2 (b) above. **Ans.** $2x + 3y - 4z = 20$

Ex. 2 (e). Find the equation of the plane through the point $(2, 3, 4)$ and parallel to the plane $5x - 6y + 7z = 3$. (*Kumaum* 94)

Sol. Do as Ex. 2 (b) above. **Ans.** $5x - 6y + 7z = 20$

Ex. 2 (f). Find the equation of the plane through $(0, 1, -2)$ parallel to the plane $2x - 3y + 4z = 0$. (*Rohilkhand* 97)

Sol. Do as Ex. 2 (b) above. **Ans.** $2x - 3y + 4z + 11 = 0$

****Ex. 3. Find the equation of the plane through the points $(2, 2, 1)$ and $(9, 3, 6)$ and perpendicular to the plane $2x + 6y + 6z = 9$.**

(*Agra* 92; *Bundelkhand* 95; *Kanpur* 97; *Purvanchal* 97, *Rohilkhand* 96)

Sol. Let the equation of the plane be $Ax + By + Cz + D = 0$. ...(i)

If it passes through (2, 2, 1), then $2A + 2B + C + D = 0$ ...(ii)

If it passes through (9, 3, 6) then $9A + 3B + 6C + D = 0$ ...(iii)

If it is perpendicular to the plane $2x + 6y + 6z = 9$, then

$$2A + 6B + 6C = 0. \quad \text{(See 3.05 Page 6 Ch. III)}$$

or
$$A + 3B + 3C = 0. \quad \text{...(iv)}$$

Subtracting (ii) from (iii), we get $7A + B + 5C = 0$. ...(v)

Solving (iv) and (v), we get $\dfrac{A}{3} = \dfrac{B}{4} = \dfrac{C}{-5} = k$ (say).

$$\therefore \quad A = 3k, \; B = 4k, \; C = -5k$$

Substituting these values in (ii), we get $D = -2k$.

Hence from (i) the required equation is

$$3kx + 4ky - 5kz - 9k = 0 \quad \text{or} \quad 3x + 4y - 5z = 9. \quad \textbf{Ans.}$$

Exercises on § 3.05

Ex. 1. Show that the angle between the planes $2x - y + z = 6$ and $x + y + 2z = 3$ is $\pi/3$.

Ex. 2. Find the equation of the plane through the points $(1, -2, 4)$, $(3, -4, 5)$ and perpendicular to the plane $x + y - 2z = 6$.

Ans. $3x + 5y + 4z - 9 = 0$

Ex. 3. Find the equation of the plane through the point $(2, -3, 4)$ and parallel to the plane $2x - 6y - 7z = 6$. **Ans.** $2x - 6y - 7z + 6 = 0$

Ex. 4. Find the equation of the plane through the points $(1, 1, 1)$ and $(2, 3, 4)$ and perpendicular to the plane $x + y + z = 1$. **Ans.** $x - 2y + z = 0$.

Ex. 5. Find the equation of the plane through $(-1, 1, 1)$ and $(1, -1, 1)$ and perpendicular to the plane $x + 2y + 2z = 5$. (*Bundelkhand* 92; *Kumaun* 92)

(Hint : See Ex. 3 above)

**§ 3.06 Plane through three given points.

To find the equation of the plane through (x_1, y_1, z_1), $(x_2\, y_2, z_2)$ *and* (x_3, y_3, z_3).

Let the equation of the plane be $Ax + By + Cz + D = 0$. ...(i)

If it passes through the given points, then we have

$$Ax_1 + By_1 + Cz_1 + D = 0 \quad \text{...(ii)}$$

$$Ax_2 + By_2 + Cz_2 + D = 0 \quad \text{...(iii)}$$

and $$Ax_3 + By_3 + Cz_3 + D = 0 \quad \text{...(iv)}$$

Eliminating A, B, C and D from (i), (ii), (iii) and (iv), we have the required equation as

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Solved Examples on § 3.06

***Ex. 1. Find the equation to the plane through the points $(1, 1, 0), (1, 2, 1)$ and $(-2, 2, -1)$.**

Sol. Let the required plane be $Ax + By + Cz + D = 0$ ...(i)

If (i) passes through $(1, 1, 0)$, then $A + B + D = 0$ ...(ii)

If (i) passes through $(1, 2, 1)$, then $A + 2B + C + D = 0$ ...(iii)

If (i) passes through $(-2, 2, -1)$, then $-2A + 2B - C + D = 0$ ...(iv)

From (ii) and (iii) we get $B + C = 0$ or $B = -C$ ...(v)

From (iii) and (iv) we get $3A + 2C = 0$ or $A = -(2/3)\,C$ ...(vi)

$\therefore$ From (ii), (v), (vi) we get

$$-(2/3)\,C - C + D = 0 \quad \text{or} \quad D = (5/3)\,C \quad \text{...(viii)}$$

Substituting values of A, B, D from (v), (vi), (vii) in (i) we get the required equation as

$$-(2/3)\,Cx - Cy + Cz + (5/3)\,C = 0$$

or $$-2x - 3y + 3z + 5 = 0 \quad \text{or} \quad 2x + 3y - 3z = 5$$ **Ans.**

Ex. 2. Show that the four points $(0, -1, 0)$, $(2, 1, -1)$, $(1, 1, 1)$ and $(3, 3, 0)$ are coplanar. *(Agra 92)*

Sol. As in Ex. 1 above we can can prove that the equation of the plane through $(0, -1, 0)$, $(2, 1, -1)$ and $(1, 1, 1)$ is

$$4x - 3y + 2z = 3. \quad \text{(To be proved in Exam.)}$$

It is evident that the point $(3, 3, 0)$ lies on this plane as the co-ordinates of this point satisfies the equation of the plane.

Hence the given four points are coplanar.

Ex. 3 (a). Find the equation of the plane which passes through three points $(4, 5, 1)$, $(3, 9, 4)$ and $(-4, 4, 4)$. *(Kanpur 94)*

Hence show that the four points $(0, -1, -1)$, $(4, 5, 1)$, $(3, 9, 4)$ and $(-4, 4, 4)$ lie on a plane.

Sol. Let the plane through $(4, 5, 1)$, $(3, 9, 4)$ and $(-4, 4, 4)$ be

$$Ax + By + Cz + D = 0 \quad \text{...(i)}$$

If (i) passes through $(4, 5, 1)$, then $4A + 5B + C + D = 0$ ...(ii)

If (i) passes through $(3, 9, 4)$, then $3A + 9B + 4C + D = 0$ ...(iii)

If (i) passes through $(-4, 4, 4)$, then $-4A+4B+4C+D=0$ ...(iv)

From (ii) and (iii) we get $A-4B-3C=0$ ...(v)

From (ii) and (iv) we get $7A+5B=0$. ...(vi)

From (vi) we get $A=-(5/7)B$ ...(vii)

$\therefore$ From (v) we get $-(5/7)B-4B-3C=0$

or $$(11/7)B+C=0 \quad \text{or} \quad C=-(11/7)B \qquad ...(viii)$$

$\therefore$ From (iv) we have $D=4A-4B-4C$

or $$D=[4(-5/7)-4-4(-11/7)]B$$
$$=[-20-28+44](B/7)=(-4/7)B \qquad ...(ix)$$

$\therefore$ From (i), (vii), (viii) and (ix) we get the equation of the required plane as

$$(-5/7)Bx+By+(-11/7)Bz-(4/7)B=0$$

or $$5x-7y+11z+4=0$$ **Ans.**

Also as $5(0)-7(-1)+11(-1)+4=7-11+4=0$,

so the fourth point $(0,-1,-1)$ lies on the above plane.

Ex. 3 (b). Find the direction cosines of any normal to the plane passing through the points $(0,-1,-1)$, $(4,5,1)$, $(3,9,4)$, $(-4,4,4)$.

Sol. Let the plane through $(0,-1,-1)$, $(4,5,1)$, and $(3,9,4)$ be

$$Ax+By+Cz+D=0 \qquad ...(i)$$

If (i) passes through $(0,-1,-1)$ then $-B-C+D=0$ ...(ii)

If (i) passes through $(4, 5, 1)$ then $4A+5B+C+D=0$ ....(iii)

If (i) passes through $(3, 9, 4)$ then $3A+9B+4C+D=0$ ...(iv)

From (ii) and (iii) we get $2A+3B+C=0$ ...(v)

From (ii) and (iv) we get $3A+10B+5C=0$ ...(vi)

From (v) and (vi) eliminating C we get

$$7A+5B=0 \quad \text{or} \quad A=-(5/7)B \qquad ...(vii)$$

$\therefore$ From (v) we get

$$2(-5/7)B+3B+C=0 \quad \text{or} \quad C=-(11/7)B \qquad ...(viii)$$

$\therefore$ From (ii) we get $D=B+C=B-(11/7)B=-(4/7)B$ ...(ix)

$\therefore$ From (i), (vii), (viii), (ix) we get the equation of the plane as

$$(-5/7)Bx+By+(-11/7)Bz+(-4/7)B=0$$

or $$5x-7y+11z+4=0$$

$\because$ The coordinates of the point $(-4, 4, 4)$ satisfy it, so this plane passes through the four given points.

$\therefore$ The direction ratios of any normal to this plane are the coefficients of x, y, z in its equation *i.e.* $5, -7, 11$

Also $\sqrt{[5^2+(-7)^2-(11)^2]}=\sqrt{(25+49+121)}=\sqrt{(195)}$.

$\therefore$ The required direction cosines are

$$5/\sqrt{(195)},\ -7/\sqrt{(195)},\ 11/\sqrt{(195)}$$ **Ans.**

Exercises on § 3.06

Ex. 1. Find the equation to the plane through the points $(0,-1,0)$; $(2,1,-1)$ and $(1,1,1)$. **Ans.** $4x-3y+2z=3$

Ex. 2. Find the equation of the plane through the points $(2, 5, -3)$; $(-2, -3, 5)$ and $(5, 3, -3)$. **Ans.** $2x + 3y + 4z - 7 = 0$

Ex. 3. Find the equation of the plane containing the points $A(1, 0, 1)$, $B(3, 1, 2)$ and $C(1, -1, 2)$. **Ans.** $x - y - z = 0$

Ex. 4. Show that the four points $(-1, 4, -3), (3, 2, -5), (-3, 8, -5)$ and $(-3, 2, 1)$ are coplanar. Find the equation of the plane containing them.

(*Meerut* 91)

***§ 3. 07. Equations of some particular planes.**

(a) Equations to the co-ordinate planes.

(i) yz-plane : Any point on yz-plane will have its x-co- ordinate as zero, hence, the equation of the yz-plane is $\mathbf{x = 0}$.

(ii) xz-plane : Its equation is $\mathbf{y = 0}$

and **(iii) xy-plane :** Its equation is $\mathbf{z = 0}$

(b) Equations to the plane parallel to co-ordinate planes.

Any point on a plane parallel to yz-plane at a distance a from it will have its x-co-ordinate as a. Therefore, the equation $\mathbf{x = a}$ represents a plane parallel to yz-plane and at a distance a from it.

Similarly the equation to the plane parallel to xz-plane at a distance b from it is $\mathbf{y = b}$ and that of the plane parallel to xy-plane at a distance c from it is $\mathbf{z = c}$.

(c) Equations to the plane perpendicular to the co-ordinate planes.

The equation of the yz-plane is $x = 0$

or $$1 . x + 0 . y + 0 . z = 0 \quad ...(i)$$

Let the equation to the plane perpendicular to yz-plane *i.e.* to the plane given by (i) be $$Ax + By + Cz + D = 0 \quad ...(ii)$$

Since (i) and (ii) are at right angles, so we have

$A . 1 + B . 0 + C . 0 = 0$ or $A = 0$ ...See § 3.05 (ii) P. 7 Ch. III.

$\therefore$ From (ii) the equation to the plane perpendicular to yz-plane is

$$By + Cz + D = 0.$$

Similarly the equations to the planes perpendicular to zx and xy-planes are $Ax + Cz + D = 0$ and $Ax + By + D = 0$ respectively. (Note)

(d) Equations of the planes perpendicular to co-ordinate axes.

Any plane perpendicular to x-axes is evidently parallel to yz-plane and its equation is $x = a$, as proved in part (b) above.

Similarly the equations to the planes perpendicular to y and z-axes are $y = b$ and $z = c$ respectively.

§ 3.08 Equation to the plane through the line of intersection of two given planes.

Let the two given planes be $$P_1 = A_1 x + B_1 y + C_1 z + D_1 = 0 \quad ...(i)$$

and $$P_2 = A_2 x + B_2 y + C_2 z + D_2 = 0 \quad ...(ii)$$

Then $P_1 + \lambda P_2 = 0$

or $$(A_1 x + B_1 y + C_1 z + D_1) + \lambda (A_2 x + B_2 y + C_2 z + D_2) = 0 \quad \text{...(iii)}$$
for all values of λ represents a plane through the line of intersection of the plane (i) and (ii) since if any point (x_1, y_1, z_1) satisfies both (i) and (ii) then for all values of λ it satisfies (iii) also.

In this way points which are common to (i) and (ii) *i.e.* all points which lie on the line of intersection of (i) and (ii) will lie on (iii). Hence the equation (iii) which is a first degree equation in x, y and z represents a plane through the line of intersection of (i) and (ii).

In case $P_1 = 0$ and $P_2 = 0$ are parallel, then $P_1 + \lambda P_2 = 0$ represents a plane parallel to them.

§ 3.09 Condition for a line to lie on or perpendicular to a given plane. (*Kanpur* 94)

Let the plane be $Ax + By + Cz + D = 0$ ...(i)

Let the direction ratios of the line be l, m, n.

If the line lies on the plane (i), then this line must be at right angles to the normal to the plane (i) whose direction ratios are A, B, C and the condition for the same is
$$\mathbf{Al + Bm + Cn = 0} \quad \text{...(ii)}$$

If the line is perpendicular to the plane (i), then it must be parallel to the normal to the plane and the condition for the same is
$$\frac{\mathbf{A}}{\mathbf{l}} = \frac{\mathbf{B}}{\mathbf{m}} = \frac{\mathbf{C}}{\mathbf{n}} \quad \text{...(iii)}$$

Solved Examples on § 3.07 to § 3.09

Ex. 1 (a). Equation of yz-plane is

(i) x = 0; (ii) x = 1; (iii) y = 0; (i\, ̷ = 0.

Hint : See § 3.07 Page 11 Ch. III **Ans.** (i)

Ex. 1 (b). Find the equation of the plar ·ng through the line of intersection of the plane 2x + 3y − 4z = 1, 3x ̓ = 0 and the point (0, 1, 1).

Sol. The equation of any plane through section of the given planes is
$$(2x + 3y - 4z - 1) + \lambda (3x - y + z + 2) \quad \text{...(i)}$$

If this plane passes through the point (0, 1, 1), t we have
$$(2.0 + 3.1 - 4.1 - 1) + \lambda (3.0 - 1 + 1 \; z) = 0$$
or $$-2 + \lambda (2) = 0 \text{ or } \lambda = 1.$$

Substituting this value of λ in (i), we have the required equation as
$$(2x + 3y - 4z - 1) + (3x - y + z$$
or $$5x + 2y - 3z + 1 = 0$$ s.

Ex. 1 (c). Find the equation of the pla ough the line of intersection of the planes 2x + 3y − 4z = 1, 3x − = 0 and passing through the origin.

Sol. Do as Ex. 1 (b) above. Ans. $7x + 5y - 7z = 0$

Ex. 1 (d). Find the equation of the plane through the intersection of the plane $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$ and the point (4, 4, 4).

Sol. Do as Ex. 1 (b) above. **Ans.** $29x + 23y + 17z = 276$

Ex. 1 (e). Find the equation of the plne passing through the intersection of the planes $2x + y + 2z = 9$, $4x - 6y - 4z = 1$ and the point (3, 2, 1). (*Rohilkhand* 97)

Sol. Do as Ex. 1 (b) above. **Ans.** $14x - y + 6z = 46$

***Ex. 2 (a). Find the equation of the plane which contains the line of intersectio· of ·· planes $?_{,} + 3z - 4 = 0$ and $2x + y - ? + 5$ ·· and is perpendicular to the plane $5x + 3y + 6z + 8 = 0$.**

Sol. The equation of any plane through the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$

is $$(x + 2y + 3z - 4) + \lambda (2x + y - z + 5) = 0$$

or $$(1 + 2\lambda) x + (2 + \lambda) y + (3 - \lambda) z + (5\lambda - 4) = 0 \qquad ...(ii)$$

If this plane is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ then we have "$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$"

i.e. $$5 (1 + 2\lambda) + 3 (2 + \lambda) + 6 (3 - \lambda) = 0 \text{ or } \lambda = -29/7.$$

Substituting this value of λ in (i) we have the required equation as

$$51x + 15y - 50z + 173 = 0. \qquad \textbf{Ans.}$$

Ex. 2 (b). The plane $x - 2y + 3z = 0$ is rotated through a right angle about its line of intersection with the plane $2x + 3y - 4z - 5 = 0$. Find the equation of the plane in its new position. (Civils 2008)

Sol. Here we are to find the equation of the plane through the line of intersection of the planes $x - 2y + 3z = 0$ and $2x + 3y - 4z - 5 = 0$ and at right angles to the plane $x - 2y + 3z = 0$. **(Note)**

Now the equation of the plane through the line intersection of the planes

$$x - 2y + 3z = 0 \text{ and } 2x + 3y - 4z - 5 = 0 \text{ is}$$

$$(x - 2y + 3z) + \lambda (2x + 3y - 4z - 5) = 0$$

or $$(1 + 2\lambda) x + (3\lambda - 2) y + (3 - 4\lambda) z = 5\lambda \qquad ...(i)$$

If this plane is perpendicular to the plane $x - 2y + 3z = 0$, then we have

'$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$'

i.e. $$1 (1 + 2\lambda) - 2 (3\lambda - 2) + 3 (3 - 4\lambda) = 0 \quad \text{or} \quad \lambda = 7/8$$

Substituting the value of λ in (i), the required equation is

$$\left(1 + \frac{7}{4}\right) x + \left(\frac{21}{8} - 2\right) y + \left(3 - \frac{7}{2}\right) z = \frac{35}{8}$$

or $$22x + 5y - 4z = 35 \qquad \textbf{Ans.}$$

***Ex. 3 (a). Find the equation of the plane through the line of intersection of the planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ and perpendicular to xy-plane (i.e. $z = 0$).** (*Purvanchal* 90)

Sol. The equation of any plane through the line of intersection of the given planes is

$$(ax+by+cz+d)+\lambda(a'x+b'y+c'z+d')=0$$

or $$(a+\lambda a')x+(b+\lambda b')y+(c+\lambda c')z+(d+\lambda d')=0 \quad ...(i)$$

Also the equation of xy-plane is $z=0$ *i.e.* $0.x+0.y+1.z=0$ ...(ii)

If the plane (i) and (ii) are at right angles, then we have

$$\text{“}a_1 a_2+b_1 b_2+c_1 c_2=0\text{”}$$

i.e. $(a+\lambda a').0+(b+\lambda b').0+(c+\lambda c').1=0$ or $\lambda=-c/c'$

Substituting this value of λ in (i) we have the required equation as

$$c'(ax+by+cz+d)-c(a'x+b'y+c'z+d')=0$$

or $$(ac'-a'c)x+(bc'-b'c)y+(dc'-d'c)=0 \quad \textbf{Ans.}$$

***Ex. 3 (b) Find the equation of the plane through the line of intersection of the planes $ax+by+cz+d=0$ and $a'x+b'y+c'z+d'=0$ and perpendicular to the plane $lx+my+nz=p$.**

Sol. The equation of any plane through the line of intersection of the given planes is $(a+\lambda a')x+(b+\lambda b')y+(c+\lambda c')z+(d+\lambda d')=0$ ...(i)

...See Ex. 3 (a) above.

If this plane is perpendicular to the plane $lx+my+nz=p$, then

$$\text{“}a_1a_2+b_1b_2+c_1c_2=0\text{”}$$

i.e. $$(a+\lambda a')l+(b+\lambda b')m+(c+\lambda c')n=0$$

or $$\lambda=-(al+bm+cn)/(a'l+b'm+c'n)$$

Substituting this value of λ in (i) we get the required equation as

$$(ax+by+cz+d)(a'l+b'm+c'n)$$
$$=(a'x+b'y+c'z+d')(al+bm+cn) \quad \textbf{Ans.}$$

***Ex. 3 (c). Find the equation of the plane passing through the lines of intersection of the planes $2x-y=0$ and $3x-y=0$ and perpendicular to the plane $4x+5y-3z=8$.** (*Gorakhpur* 91; *Kumaun* 90; *Rohilkhand* 96, 93)

Sol. The equation of the plane through the intersection of the given planes is $$(2x-y)+\lambda(3x-y)=0 \quad ...(i)$$

or $$2x-(\lambda+1)y+3\lambda z=0$$

If this plane is perpendicular to the plane $4x+5y-3z=8$, then we have

$4.2+5(-\lambda-1)-3.(3\lambda)=0,$ ...See § 3.05 (ii) P. 7 Ch. III

or $8-5\lambda-5-9\lambda=0$ or $14\lambda=3$ or $\lambda=3/(14)$

Substituting this value of λ in (i), the required equation is

$$(2x-y)+(3/14)=0 \quad \text{or} \quad 28x-17y+9z=0. \quad \textbf{Ans.}$$

Ex. 3 (d). Find the equation of the plane containing the line of intersection of the planes $x+2y+3z=4$ and $2x+y-z+5=0$ and is perpendicular to the plane $5x+3y+6z+8=0$ (*Avadh* 95, 91)

Sol. The equation of any plane through the line of intersection of the given planes is $(x+2y+3z-4)+\lambda(2x+y-z+5)=0$

or $$(1+2\lambda)x+(2+\lambda)y+(3-\lambda)z+(5\lambda-4)=0 \quad ...(i)$$

If this plane is perpendicular to the plane $5x+3y+6z+8=0$, then we have $5(1+2\lambda)+3(2+\lambda)+6(3-\lambda)=0$.. See § 3.05 (ii) P. 7 Ch. III

or $5+10\lambda+6-3\lambda+18-6\lambda=0$ or $7\lambda+29=0$

or $\lambda = -29/7$

Substituting this value of λ in (i), the required equation is

$$\left(1-\frac{58}{7}\right)x+\left(2-\frac{29}{7}\right)y+\left(3+\frac{29}{7}\right)z+\left(\frac{-145}{7}-4\right)=0$$

or $$51x+15y-50z+173=0$$ **Ans.**

***Ex. 4. Find the equation of the plane through the intersection of the planes $x+y+z=1$ and $2x+3y-z+4=0$ which is parallel to**

(a) the x-axis (*Avadh* 90; *Purvanchal* 95) ; **(b) the y-axis.**

Sol. The equation of any plane through the intersection of the given planes is $(x+y+z-1)+\lambda(2x+3y-z+4)=0$ ...(i)

or $(1+2\lambda)x+(1+3\lambda)y+(1-\lambda)z-(1-4\lambda)=0$

(a) If this plane is parallel to x-axis, then it is perpendicular to yz-plane *i.e.* $x=0$ or $1.x+0.y+0.z=0$ and the condition for the same is

$$1.(1+2\lambda)+0\,(1+3\lambda)+0.(1-\lambda)=0 \text{ or } \lambda=-\frac{1}{2}.$$

$\therefore$ From (i), the required equation is

$$(x+y+z-1)-\tfrac{1}{2}(2x+3y-z+4)=0 \quad\text{or}\quad y-3z+6=0.$$ **Ans.**

(b) If this plane is parallel to y-axis, then it is perpendicular to zx-plane *i.e.* $y=0$ or $0.x+1.y+0.z=0$ and the condition for the same is

$$0.(1+2\lambda)+1.(1+3\lambda)+0.(1-\lambda)=0 \text{ or } \lambda=-1/3$$

From (i), the required equation is

$$(x+y+z-1)-(1/3)(2x+3y-z+4)=0 \quad\text{or}\quad x+4z=7.$$ **Ans.**

Ex. 5. Find the equation to the plane through the line of intersection of the planes $ax+by+cz+d=0$, $a'x+b'y+c'z+d'=0$ and parallel to x-axis. (*Agra* 90)

Sol. The equation of any plane through the line of intersection of the given planes is $(ax+by+cz+d)+\lambda(a'x+b'y+c'z+d')=0$ ...(i)

or $(a+\lambda a')x+(b+\lambda b')y+(c+\lambda c')z+(d+\lambda d')=0.$

If this plane is parallel to x-axis, then it is perpendicular to yz-plane *i.e.* $x=0$ *i.e.* $1.x+0.y+0.z=0$ and the condition for the same is

$$1.(a+\lambda a')+0.\,(b+\lambda b')+0.(c+\lambda c')=0$$

or $$a+\lambda a'=0 \quad\text{or}\quad \lambda=-a/a'.$$

$\therefore$ From (i), the required equation is

$$(ax+by+cz+d)-(a/a')(a'x+b'y+c'z+d')=0$$

or $$a'(by+cz+d)=a(b'y+c'z+d').$$ **Ans.**

***Ex. 6 (a). Find the equation of the plane through the point $(2,-3,1)$ and normal to the line joining the points $(3,4,-1)$ and $(2,-1,5)$.**

(*Bundelkhand* 94; *Garhwal* 96)

Sol. The direction ratios of the line joining the points $(3,4,-1)$ and $(2,-1,5)$ are $3-2$, $4-(-1)$, $-1-5$ or $1, 5, -6$.

Also the equation of any plane perpendicular to this line (*i.e.* the normal to the plane is parallel to this line) is

$$1.x + 5.y - 6.z = k, \text{ where } k \text{ is constant.}$$

If this plane passes through $(2, -3, 1)$, then

$$1.2 + 5(-3) - 6.1 = k \quad \text{or} \quad k = -19$$

$\therefore$ The required plane is $x + 5y - 6z + 19 = 0$. **Ans.**

Ex. 6 (b). Find the equation to a plane through the point $(2, -3, 4)$ and normal the line joining the points $(3, 4, -1)$ and $(2, -1, 6)$. (*Kanpur* 90)

Sol. Do as Ex. 6 (a) above. **Ans.** $x + 5y - 7z + 41 = 0$

***Ex. 7 (a). Find the equation of plane through (α, β, γ) and perpendicular to the line joining this point to the origin.**

Sol. The direction ratios of the line joining (α, β, γ) and $(0, 0, 0)$ are α, β, γ.

Also the equation of any plane through (α, β, γ) is

$$A(x - \alpha) + B(y - \beta) + C(z - \gamma) = 0 \quad \text{...(i)}$$

If this plane is perpendicular to the line joining (α, β, γ) to $(0, 0, 0)$, then the d-ratios of the normal to this plane are α, β, γ

i.e. $\quad A/\alpha = B/\beta = C/\gamma = k$ (say) **(Note)**

or $\quad A = k\alpha,\ B = k\beta,\ C = k\gamma.$

Substituting these values in (i) we get the required equation as

$$\alpha(x - \alpha) + \beta(y - \beta) + \gamma(z - \gamma) = 0.$$

i.e. $\quad \alpha x + \beta y + \gamma z = \alpha^2 + \beta^2 + \gamma^2.$ **Ans.**

Ex. 7 (b). If the axes are rectangular and P is the point $(2, 3, -1)$, find the equation to the plane through P at right angles to OP. (*Meerut* 93)

Sol. The direction ratios of the line OP are

$$2 - 0, 3 - 0, -1 - 0 \ \ i.e.\ \ 2, 3, -1 \quad \text{...(i)}$$

Also the equation of any plane through $P(2, 3, -1)$ is

$$A(x - 2) + B(y - 3) + C(z + 1) = 0 \quad \text{...(ii)}$$

The d.ratios of the normal to this plane are A, B, C which must be the same as of the line OP, if this plane is perpendicular to OP.

$\therefore \quad A/2 = B/3 = C/(-1) = k$ (say)

Then $A = 2k,\ B = 3k,\ C = -k$.

$\therefore$ From (ii), the required equation of plane is

$$2k(x - 2) + 3k(y - 3) - k(z + 1) = 0 \quad \text{or} \quad 2x + 3y - z = 14. \quad \textbf{Ans.}$$

Ex. 8. Find the equation of the plane through $(4, -1, 2)$ and perpendicular to the line joining $(1, -5, 10)$ and $(2, 3, 4)$. Also find the angles which it makes with the co-ordinate planes.

Sol. The direction ratios of the line joining $(1, -5, 10)$ and $(2, 3, 4)$ are $1 - 2,\ -5 - 3,\ 10 - 4$ *i.e.* $-1, -8, 6$.

Also the equation of any plane through $(4, -1, 2)$ is

$$A(x - 4) + B(y + 1) + C(z - 2) = 0 \quad \text{...(i)}$$

The normal to this plane is parallel to the line joining $(1, -5, 10)$ and $(2, 3, 4)$ *i.e.* parallel to the line with direcion ratios $-1, -8, 6$.

$$\therefore \quad \frac{A}{-1} = \frac{B}{-8} = \frac{C}{6} = k \text{ (say)} \qquad \textbf{(Note)}$$

$$\therefore A = -k,\ B = -8k,\ C = 6k$$

Substituting in (i), the required equation is

$$-k(x-4) - 8k(y+1) + 6k(z-2) = 0 \quad \text{or} \quad x + 8y - 6z + 16 = 0. \quad \textbf{Ans.}$$

If α be the angle which this plane makes with *yz*-plane, then α is the angle between the normals to this plane and the *yz*-plane.

The direction ratios of the normal to this plane are $1, 8, -6$ and those of *yz*-plane are $1, 0, 0$

$$\therefore \quad \cos\alpha = \frac{1.1 + 8.0 + (-6).0}{\sqrt{(1^2 + 8^2 + 6^2)}\sqrt{(1^2 + 0^2 + 0^2)}} = \frac{1}{\sqrt{(101)}}$$

or $\qquad \alpha = \cos^{-1}[1/\sqrt{(101)}]$

Similarly we can find the angles which this plane makes with *zx* and *xy*-planes.

Ex. 9 (a). Find the equation of the plane through $(2, 3, -4)$ and $(1, -1, 3)$ and parallel to the x-axis. (*Avadh* 93; *Gorakhpur* 91)

Sol. The equation of any plane through $(2, 3, -4)$ is

$$A(x-2) + B(y-3) + C(z+4) = 0. \qquad \text{...(i) (Note)}$$

If it passes through $(1, -1, 3)$, then

$$A(1-2) + B(-1-3) + C(3+4) = 0 \quad \text{or} \quad A + 4B - 7C = 0 \qquad \text{...(ii)}$$

If the plane (i) is parallel to *x*-axis, then it is perpendicular to *yz*-plane *i.e.*

$x = 0 \quad$ *i.e.* $\quad 1.x + 0.y + 0.z = 0$ **(Note)**

$$\therefore \quad 1A + 0.B + 0.C = 0 \quad \text{or} \quad A = 0. \qquad \text{...(iii)}$$

$\therefore$ From (ii), $4B - 7C = 0$ or $B = (7/4)\,C$.

$\therefore$ From (i) the required plane is $(7/4)(y-3) + (z+4) = 0$.

or $\qquad 7y + 4z - 5 = 0 \qquad$ **Ans.**

Ex. 9 (b). Find the equation of a plane through $(2, 3, -4)$ and $(1, -1, 3)$ and parallel to z-axes. (*Avadh* 94)

Sol. Equation of any plane through $(2, 3, -4)$ is

$$A(x-2) + B(y-3) + C(z+4) = 0 \qquad \text{...(i)}$$

If it passes through $(1, -1, 3)$, then

$$A(1-2) + B(-1-3) + C(3+4) = 0 \quad \text{or} \quad A + 4B - 7C = 0 \qquad \text{...(ii)}$$

If the plane (i) is parallel to *z*-axis, then it is perpendicular to *xy*-plane

i.e. $\quad z = 0 \quad$ *i.e.* $\quad 0.x + 0.y + 1.z = 0$

$\therefore A.0 + B.0 \quad -7C(1) = 0 \quad$ or $\quad C = 0 \qquad$ ...(iii)

$\therefore$ From (ii) we get $A + 4B = 0$ or $A = -4B$

$\therefore$ From (i), the required plane is $-4B(x-2) + B(y-3) + 0(z+4) = 0$

or $\qquad -4x + 8 + y - 3 \quad$ or $\quad 4x - y = 5 \qquad$ Ans.

Ex. 10. Find the equation of the plane through $(2, 2, 1)$ and $(1, -2, 3)$ and parallel to x-axis.

Sol. Do as Ex. 9 above **Ans.** $y+2z-4=0$.

Ex. 11 (a). Find the equation of the plane passing through the points $(2,-3,1),(-1,1,-7)$ and perpendicular to the plane

$$\mathbf{x-2y+5z+1=0.}$$ (*Gorakhpur* 92)

Sol. Let the required plane be $ax+by+cz+d=0$ ...(i)

If it passes through the points $(2,-3,1)$ and $(-1,1,-7)$

then we have $$2a-3b+c+d=0 \qquad ...(ii)$$

and $$-a+b-7c+d=0 \qquad ...(iii)$$

Also if the plane (i) is perpendicular to given plane $x-2y+5z+1=0$

then $$1.a-2.b+5.c=0 \quad \text{or} \quad a-2b+5c+0.d=0 \qquad ...(iv)$$

Eliminating a, b, c, d from (i), (ii), (iii) and (iv), the required equation is

$$\begin{vmatrix} x & y & z & 1 \\ 2 & -3 & 1 & 1 \\ -1 & 1 & -7 & 1 \\ 1 & -2 & 5 & 0 \end{vmatrix}=0$$

or $4x+7y+2z+11=0$, on simplifying. **Ans.**

***Ex. 11 (b). Find the equation of the plane through the point (2, 2, 1) and (9, 3, 6) and perpendicular to the given plane $2x+6y+6z=9$.**

(*Agra* 92; *Avadh* 95; *Kumaun* 90; *Purvanchal* 94, 92; *Rohilkhand* 90)

Sol. Let the required plane be $ax+by+cz+d=0$

If it passes through (2, 2, 1) and (9, 3, 6), we have

$$2a+2b+c+d=0 \qquad ...(ii)$$

and $$9a+3b+6c+d=0 \qquad ...(iii)$$

Also if the plane (i) is perpendicular to the given plane, then

$$2a+6b+6c=0 \quad \text{or} \quad a+3b+3c+0d=0 \qquad ...(iv)$$

Eliminating a, b, c, d from (i), (ii), (iii) and (iv), the required equation is

$$\begin{vmatrix} x & y & z & 1 \\ 2 & 2 & 1 & 1 \\ 9 & 3 & 6 & 1 \\ 1 & 3 & 3 & 0 \end{vmatrix}=0$$

or $3x+4y-5z=9$, on simplifying. (See Ex. 3. P. 8 also) **Ans.**

Ex. 12. Find the equation of the plane perpendicular to the yz-plane and passing through the points $(1,-2,4)$ and $(3,-4,5)$.

Sol. Let the required plane be $ax+by+cz+d=0$ ...(i)

The equation of the yz-plane is $x=0$ *i.e.* $1.x+0.y+0.z=0$ ...(ii)

Since (i) and (ii) are at right angles, so we have

$$1.a+0.b+0.c=0 \quad \text{or} \quad a=0.$$

$\therefore$ From (i), the equation of the plane perpendicular to yz-plane is

$$by+cz+d=0 \qquad ...(iii)$$

As $(1,-2,4)$ and $(3,-4,5)$ lie on (iii), so we have

$$-2b+4c+d=0 \quad \text{and} \quad -4b+5c+d=0$$

Solving these simultaneously, we have

$$\frac{b}{4-5}=\frac{c}{-4+2}=\frac{d}{-10+16} \quad \text{or} \quad \frac{b}{-1}=\frac{c}{-2}=\frac{d}{6}$$

From (iii), we have the required equation as $y+2z-6=0$. **Ans.**

***Ex. 13. Obtain the equation of the plane that bisects the segment joining the points (1, 2, 3) and (3, 4, 5) at right angles.** (*Kumaun* 96)

Sol. The coordinates of the mid-point of the join of (1, 2, 3) and (3, 4, 5) is $\{\frac{1}{2}(1+3), \frac{1}{2}(2+4), \frac{1}{2}(3+5)\}$ *i.e.* (2, 3, 4).

Also the direction ratios of the line joining the given points are $3-1$, $4-2$, $5-3$ *i.e.* 2, 2, 2.

Now the equation of any plane through (2, 3, 4) is

$$A(x-2)+B(y-3)+C(z-4)=0 \quad \text{...(i)}$$

...See § 3.01 Cor. 2 Page 1 Ch. III

If it is perpendicular to the join of the given points, then the normal to this plane must be parallel to the line joining the given points and therefore, we must have $\frac{A}{2}=\frac{B}{2}=\frac{C}{2}$ *i.e.* $A=B=C$

∴ From (i), the required equation is

$$A(x-2)+A(y-3)+A(z-4)=0 \quad \text{or} \quad x+y+z=9$$ **Ans.**

****Ex. 14. Find the equation of the plane which passes through the point $(-1, 3, 2)$ and is perpendicular to each of the two planes $x+2y+2z=5$ and $3x+3y+2z=8$.**

(*Bundelkhand* 93; *Garhwal* 92; *Rohilkhand* 94)

Sol. The equation of any plane through $(-1, 3, 2)$ is

$$A(x+1)+B(y-3)+C(z-2)=0. \quad \text{...(i)}$$

If this plane (i) is perpendicular to the plane $x+2y+2z=5$, then we have $A.1+B.2+C.2=0$ or $A+2B+2C=0$. ...(ii)

If the plane (i) is perpendicular to the plane $3x+3y+2z=8$, then we have $A.3+B.3+C.2=0$ or $3A+3B+2C=0$. ...(iii)

∴ From (ii) and (iii), we get

$$\frac{A}{4-6}=\frac{B}{6-2}=\frac{C}{3-6} \quad \text{or} \quad \frac{A}{2}=\frac{B}{-4}=\frac{C}{3}$$

Substituting these proportionate values of A, B, C in (i), we get the required equation as $2(x+1)-4(y-3)+3(z-2)=0$

or $$2x-4y+3z+8=0$$ **Ans.**

Exercises on § 3.07 – § 3.08

Ex. 1. Find equation of the plane through the intersection of the planes $x+y+z=6$ and $2x+3y+4z+5=0$ and the point (1, 1, 1).

Ans. $20x+23y+26z=69$.

Ex. 2. Find the equation of a plane through the origin and through the intersection of the planes $x+2y+3z=4$ and $4x+3y+2z+1=0$.

Ans. $17x+14y+10z=0$.

Ex. 3. Find the equation of the plane passing through the line of intersection of the planes $x+2y+2z=4$, $2x+y-z+5=0$ and perpendicular to the plane $4x+5y-3z=8$. **Ans.** $3y+5z=13$.

Ex. 4. A plane passes through the point $(1,-2,5)$ and is perpendicular to the line joining the origin to the point $(3,1,-1)$. Find the equation of the plane. **Ans.** $3x+y-z=-4$

Ex. 5. Obtain the equation of the plane that passes through the mid-point of the join of the points $(-2,2,-3)$ and $(6,4,5)$ and is perpendicular to the join. **Ans.** $4x+y+4z=15$.

Ex. 6. Find the equation of the plane passing through the line of intersection of the planes $x+y+z=6$ and $2x+3y+4z+5=0$ and perpendicular to the ~~join~~ plane $4x+5y-3z=8$. **Ans.** $x+7y+13z+96=0$

Ex. 7. Find the equation of the plane through the line of intersection of the planes $x+2y+3z=4$ and $2x+y-z=-5$ and perpendicular to the plane $5x+3y+9z=-8$. **Ans.** $36x+15y-25z+103=0$

Ex. 8. Find the equation of the plane through the point $(-1,3,2)$ and perpendicular to the planes

$x+2y+3z=5$, $3x+3y+z=9$. **Ans.** $7x-8y+3z+25=0$

Ex. 9. Find the equation of the plane which passes through the point $(2,1,4)$ and is perpendicular to each of the planes $9x-7y+6z+48=0$ and $x+y-z=0$. **Ans.** $x+15y+16z=81$

Ex. 10 Find the equation of the plane which is parallel to Ox and passes through the points $(2,3,1)$ and $(4,5,3)$. **Ans.** $y-z=2$

***Ex. 11.** Find the equation of the plane bisecting the join of $(2,4,6)$ and $(4,6,8)$ at right angles. (**Hint** : See Ex. 13 Page 19) **Ans.** $x+y+z=15$

Ex. 12. Find the equations of the planes which pass through the points $(2,3,4)$ and $(4,-5,3)$ and are parallel to x, y and z axes respectively.

Ex. 13. Find the equation of the plane through the point $(2,5,-8)$ and perpendicular to each of the planes $2x-3y+4z+1=0$, $4x+y-2z+6=0$.
Ans. $x+10y+7z+4=0$ (*Avadh* 92)

§ 3.10 Position of a point with respect to a plane.

To find the condition for two given points to lie on the same or opposite sides of a given plane.

Let the equation of the plane be $Ax+By+Cz+D=0$ ...(i)

Let this plane divide the join of two given points $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ in the ratio $\lambda:1$ at the point R, then the co-ordinates of R (See § 1.06 Cor. 2 Chapter I) are, $\left[\dfrac{x_1+\lambda x_2}{1+\lambda},\dfrac{y_1+\lambda y_2}{1+\lambda},\dfrac{z_1+\lambda z_2}{1+\lambda}\right]$

and λ is positive or negative according as R divides PQ internally or externally *i.e.* according as P and Q are on the opposite or same side of the plane (i).

Since R lies on the plane (i), we have

$$A\left[\frac{x_1+\lambda x_2}{1+\lambda}\right]+B\left[\frac{y_1+\lambda y_2}{1+\lambda}\right]+C\left[\frac{z_1+\lambda z_2}{1+\lambda}\right]+D=0$$

or $$A(x_1+\lambda x_2)+B(y_1+\lambda y_2)+C(z_1+\lambda z_2)+D(1+\lambda)=0$$

or $$\lambda(Ax_2+By_2+Cz_2+D)+(Ax_1+By_1+Cz_1+D)=0$$

or $$\lambda=-\frac{Ax_1+By_1+Cz_1+D}{Ax_2+By_2+Cz_2+D} \quad \text{...(ii)}$$

If P and Q are on the same side of (i), then λ is negative and from (ii) it follows that the expressions $Ax_1+By_1+Cz_1+D$ and $Ax_2+By_2+Cz_2+D$ should have the same sign.

Again if P and Q are on the opposite sides on the plane (i), then λ is positive and from (ii) itt follows that the expressions $Ax_1+By_1+Cz_1+D$ and $Ax_2+By_2+Cz_2+D$ should have opposite signs.

Hence we conclude that the **points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ lie on the same or opposite sides of the plane $Ax+By+Cz+D=0$ according as the expressions $Ax_1+By_1+Cz_1+D$ and $Ax_2+By_2+Cz_2+D$ are of the same or opposite signs.**

Cor. The point (x_1, y_1, z_1) and the origin $(0, 0, 0)$ lie on the same or opposite sides of the plane $Ax+By+Cz+D=0$ according as $Ax_1+By_1+Cz_1+D$ is positive or negative provided D is positive.

****§ 3.11 Perpendicular distance of a point from a given plane.**

Case I Normal form : *To find the perpendicular distance of the point $P(x_1, y_1, z_1)$ from the plane $lx+my+nz=p$,* ...(i)
where l, m, n and p have their usual meanings.

The equation of any plane parallel to (i) is $lx+my+nz=p_1$ ...(ii)

As it passes through $P(x_1, y_1, z_1)$, so we have

$$lx_1+my_1+nz_1=p_1. \quad \text{...(iii)}$$

Let a perpendicular be drawn from the origin O to the planes given by (i) and (ii) meeting them in N and K; so that $ON=p$ and $OK=p_1$. Also P lies on (ii), and let PM be the required perpendicular from P on the plane (i), then

$$PM=KN=OK-ON=p_1-p=lx_1+my_1+nz_1-p, \text{ from (iii)}$$

Note 1. Working rule for finding out the length of the perpendicular from the point $P(x_1, y_1, z_1)$ to the plane $lx+my+nz-p=0$ is that we should *substitute the co-ordinates of P in the left hand side expression of the equation to the plane* (right hand side being zero.)

Note 2. If however the point $P(x_1, y_1, z_1)$ and the origin O are on the same side of the plane $lx+my+nz=p$, then perpendicular distance of P from this plane $=-(lx_1+my_1+nz_1-p)$ (Note)

Case II. General form : *To find perpendicular distance of the point* $P(x_1, y_1, z_1)$ *from the plane* $Ax + By + Cz + D = 0$.

Writing this equation in normal form as in § 3.02 P. 2 Ch. III we have the required length of the perpendicular from (x_1, y_1, z_1) to this plane as $\pm \dfrac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{(A^2 + B^2 + C^2)}}$, where the sign of the radical is to be taken as positive or negative according as D is positive or negative.

Note. Working Rule for finding the length of perpendicular from point P to a plane whose equation is given in the general form is that we should *substitute the co-ordinates of the point in the equation of the plane* (right hand side being zero) *and divide the result by*

$\sqrt{}$[*sum of the squares of coefficients of* x, y, z *in the equation of the plane*].

Cor. To find the **distance between two parallel planes**, we should either choose a point on one the planes and find the length of perpendicular from this point to the other plane, or find out 'p' the perpendicular distance of each plane from the origin and retaining their signs subtract them.

Solved Examples on § 3.10 and § 3.11

Ex. 1. Find the distance of the point P (x', y', z') from the plane $x \cos \alpha + y \cos \beta + z \cos \gamma = p$. *(Meerut 90 S)*

Sol. Do as § 3.11 Case I Page 21 Chapter III

Ans. $x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p$

Ex. 2 (a). Find the distance of the point (1, 2, 0) from the plane $4x + 3y + 12z + 16 = 0$.

Sol. The given plane is $4x + 3y + 12z + 16 = 0$

$\therefore$ The required distance $= \dfrac{4(1) + 3(2) + 12(0) + 16}{\sqrt{[4^2 + 3^2 + (12)^2]}} = \dfrac{26}{13} = 2.$ **Ans.**

Ex. 2 (b). Find the distance of the point (4, 3, 5) from xz-plane.

Sol. The given plane is xz-plane *i.e.* $y = 0$ *i.e.* $0.x + 1.y + 0.z = 0$

$\therefore$ The required distance $= \dfrac{0.4 + 1.3 + 0.5}{\sqrt{(0^2 + 1^2 + 0^2)}} = 3.$ **Ans.**

Ex. 3. Show that the points (1, 1, 1) and (– 3, 0, 1) are on opposite sides and equidistant from the plane $3x + 4y - 12z + 13 = 0$.

Sol. The given plane is $3x + 4y - 12z + 13 = 0$. ...(i)

Its distance from (1, 1, 1) is $\dfrac{3.1 + 4.1 - 12.1 + 13}{\sqrt{(3^2 + 4^2 + 12^2)}} = \dfrac{8}{\sqrt{(169)}} = \dfrac{8}{13}$

Its distance from (– 3, 0, 1) is $\dfrac{3.(-3) + 4 \cdot 0 - 12.1 + 13}{\sqrt{(3^2 + 4^2 + 12^2)}} = \dfrac{-8}{\sqrt{(169)}}$

$= 8/13$ (numerically)

Hence the given points are equidistant from the given plane. Moreover on substituting the co-ordinates of the given points in the L.H.S. of (i) we have results of opposite signs, hence the given points are on opposite sides of plane (i).

Ex. 4. Find the distance of the points (3, 4, 7) and (2, – 3, – 5) from the plane x + 2y – 2z = 9. Are the points on the same side of the plane?

Sol. The given plane is $x + 2y - 2z - 9 = 0.$ ...(i)

Its distance from (3, 4, 7) is $\dfrac{3 + 2.4 - 2.7 - 9}{\sqrt{(1^2 + 2^2 + 2^2)}} = \dfrac{-12}{\sqrt{9}} = 4.$

Its distance from (2, – 3, – 5) is $\dfrac{2 + 2(-3) - 2(-5) - 9}{\sqrt{(1^2 + 2^2 + 2^2)}} = \dfrac{-3}{\sqrt{9}} = 1$

(Students should note that the distance is always positive).

Also substituting the co-ordinates of the given points in the L.H.S. of (i), we have results of the same sign (negative), hence given points are on the same side of the given plane (i).

Ex. 5 (a). Find the distance between the planes

ax + by + cz + d = 0 and ax + by + cz + e = 0.

Sol. Let $(0, 0, z_1)$ be a point on the plane $ax + by + cz + d = 0$

Then $a(0) + b(0) + c(z_1) + d = 0$ or $cz_1 = -d$ ...(i)

Now the distance between the given planes (which are parallel, as their equations differ only in the constant terms).

= the length of perpendicular from $(0, 0, z_1)$ to the plane

$ax + by + cz + e = 0$

$= \dfrac{a(0) + b(0) + c(z_1) + e}{\sqrt{(a^2 + b^2 + c^2)}} = \dfrac{-d + e}{\sqrt{(a^2 + b^2 + c^2)}}$, from (i). **Ans.**

***Ex. 5 (b). Find the distance between the parallel planes.**

2x – 2y + z + 3 = 0 and 4x – 4y + 2z + 5 = 0. (*Bundelkhand* 93)

Sol. The given planes are $2x - 2y + z + 3 = 0$, $4x - 4y + 2z + 5 = 0$.

These can be written in the normal form as

$$\frac{2x - 2y + z + 3}{\sqrt{(2^2 + (-2)^2 + 1^2)}} = 0 \quad \text{and} \quad \frac{4x - 4y + 2z + 5}{\sqrt{\{4^2 + (-4)^2 + 2^2\}}} = 0$$

or $(2/3)x - (2/3)y + \frac{1}{3}z + 1 = 0$ and $(2/3)x - (2/3)y + \frac{1}{3}z + (5/6) = 0.$

∴ The lengths of perpendiculars from origin to these planes are 1 and 5/6 respectively, and these planes are on the same side of the origin.

∴ The required distance $= 1 - (5/6) = 1/6.$

Ex. 5 (c). Find the distance between the planes 2x – y + 2z + 6 = 0 and 4x – 2y + 4z + 9 = 0. (*Avadh* 95)

Sol. Do as Ex. 5 (b) above. **Ans.** 1/2.

Ex. 5 (d). Find the shortest distance between the planes

2x + y – 2z = 12 and 4x + 2y – 4z + 15 = 0. (*Bundelkhand* 90)

Sol. Let $(x_1, 0, 0)$ be a point on the plane $2x + y - 2z = 12$.

Then $2x_1 + 0 - 2(0) = 12$ or $x_1 = 6$ ...(i)

Now the distance between the given planes

$$2x + y - 2z - 12 = 0 \quad \text{and} \quad 2x + y - 2z + (15/2) = 0 \quad \text{(Note)}$$

(which are parallel, as their equations differ only in constant terms)

= Length of perpendiculars from the point $(x_1, 0, 0)$ *i.e.* $(6, 0, 0)$ to the plane $2x + y - 2z + (15/2) = 0$

$$= \frac{2(6) + 0 - 2(0) + (15/2)}{\sqrt{[2^2 + 1^2 + (-2)^2]}} = \frac{(29/2)}{3} = \frac{29}{6} \quad \text{Ans.}$$

Ex. 6. Find the locus of the point whose distance from (1, 0, 0) is twice its distance from the plane $3x + 4y - z + 2 = 0$.

Sol. Let (x_1, y_1, z_1) be the point whose locus is required.

The distance of (x_1, y_1, z_1) from the point $(1, 0, 0)$

$$= \sqrt{[(x_1 - 1)^2 + y_1^2 + z_1^2]}.$$

The distance of (x_1, y_1, z_1) from the given plane

$$= \frac{3x_1 + 4y_1 - z_1 + 2}{\sqrt{\{3^2 + 4^2 + (-1)^2\}}} = \frac{3x_1 + 4y_1 - z_1 + 2}{\sqrt{(26)}}$$

Then according to the problem, we have

$$\sqrt{[(x_1 - 1)^2 + y_1^2 + z_1^2]} = 2\,[(3x_1 + 4y_1 - z_1 + 2)/\sqrt{(26)}]$$

Squaring and simplifying we get

$$5x_1^2 + 19y_1^2 - 11z_1^2 + 48x_1y_1 - 12x_1z_1 - 16y_1z_1 - 50x_1 + 32y_1 - 8z_1 - 5 = 0.$$

$\therefore$ Required locus of (x_1, y_1, z_1) is

$$5x^2 + 19y^2 - 11z^2 + 48xy - 12xz - 16yz - 50x + 32y - 8z - 5 = 0.$$

Ex. 7. The sum of the distances of any numbr of fixed points from a plane is zero; show that the plane always passes through a fixed point.

Sol. Let the fixed points be $(a_1, b_1, c_1), (a_2, b_2, c_2) \ldots, (a_m, b_m, c_m)$ and the equation of the given plane be $lx + my + nz = p$. ...(i)

Now according to the problem, the sum of the distances of the above points from the plane (i) is zero, so we have

$$(la_1 + mb_1 + nc_1 - p) + (la_2 + mb_2 + nc_2 - p) + \ldots$$
$$\ldots + (la_m + mb_m + nc_m - p) = 0$$

...See § 3.11 Case I Page 21 Ch. III

or $$l(a_1 + a_2 + \ldots + a_m) + m(b_1 + b_2 + \ldots + b_m)$$
$$+ n(c_1 + c_2 + \ldots + c_m) - mp = 0 \quad \text{(Note)}$$

or $$l\left(\frac{\Sigma a_1}{m}\right) + m\left(\frac{\Sigma b_1}{m}\right) + n\left(\frac{\Sigma c_1}{m}\right) = p.$$

which shows that the plane (i) passes through the point $\left(\frac{\Sigma a_1}{m}, \frac{\Sigma b_1}{m}, \frac{\Sigma c_1}{m}\right)$, which is fixed as $a's$, $b's$, $c's$ and m are fixed. Hence proved.

Ex. 8. Find the locus of a point the sum of the squares of whose distances from the planes $x + y + z = 0$, $x - z = 0$, $x - 2y + z = 0$ is 9.

(*Rohilkhand* 94)

Sol. Let (x_1, y_1, z_1) be the point whose locus is required.

Its distance from $x+y+z=0$ is $\dfrac{x_1+y_1+z_1}{\sqrt{(1^2+1^2+1^2)}}$ i.e. $\dfrac{x_1+y_1+z_1}{\sqrt{3}}$

Its distance from $x-z=0$ is $\dfrac{x_1-z_1}{\sqrt{[1^2+(-1)^2]}}$ i.e. $\dfrac{x_1-z_1}{\sqrt{2}}$

Its distance from $x-2y+z=0$ is $\dfrac{x_1-2y_1+z_1}{\sqrt{(1^2+(-2)^2+1^2)}}$ i.e. $\dfrac{x_1-2y_1+z_1}{\sqrt{6}}$

$\therefore$ According to the given problem, we have

$$\left[\frac{x_1+y_1+z_1}{\sqrt{3}}\right]^2+\left[\frac{x_1-z_1}{\sqrt{2}}\right]^2+\left[\frac{x_1-2y_1+z_1}{\sqrt{6}}\right]^2=9$$

or $\quad 6x_1^2+6y_1^2+6z_1^2=54$ or $x_1^2+y_1^2+z_1^2=9$, on simplifying.

$\therefore$ The required locus of (x_1, y_1, z_1) is $x^2+y^2+z^2=9$. **Ans.**

Ex. 9. Find the locus of a point the sum of squares of whose distances from planes $x+y+z=0$, $x-2y+z=0$ is equal to the square of the distance from the plane $x=z$.

Sol. Let (x_1, y_1, z_1) be the point whose locus is required.

Its distance from the plane $x+y+z=0$ is given by

$$p_1=\frac{x_1+y_1+z_1}{\sqrt{[1^2+1^2+1^2]}}=\frac{x_1+y_1+z_1}{\sqrt{3}} \qquad ...(i)$$

Its distance from the plane $x-2y+z=0$ is given by

$$p_2=\frac{x_1-2y_1+z_1}{\sqrt{[1^2+(-2)^2+1^2]}}=\frac{x_1-2y_1+z_1}{\sqrt{6}} \qquad ...(ii)$$

And its distance from the plane $x-z=0$ is given by

$$p_3=\frac{x_1-z_1}{\sqrt{[1^2+0^2+(-1)^2]}}=\frac{x_1-z_1}{\sqrt{2}} \qquad ...(iii)$$

Given that $p_1^2+p_2^2=p_3^2$

i.e. $\quad \frac{1}{3}(x_1+y_1+z_1)^2+\frac{1}{6}(x_1-2y_1+z_1)^2=\frac{1}{2}(x_1-z_1)^2$

or $\quad 2(x_1+y_1+z_1)^2+(x_1-2y_1+z_1)^2=3(x_1-z_1)^2$

or $\quad x_1^2(2+1-3)+y_1^2(2+4)+z_1^2(2+1-3)+2x_1y_1(2-2)$
$\qquad\qquad +2y_1z_1(2-2)+2z_1x_1(2+1+3)=0$

or $\quad 6y_1^2+12z_1x_1=0 \quad$ or $\quad y_1^2+2x_1z_1=0$

$\therefore$ The required locus of (x_1, y_1, z_1) is $y^2+2xz=0$. **Ans.**

Exercises on § 3.10 – § 3.11

Ex. 1. Find length of the perpendicular drawn from the origin to the plane $6x-3y+2z=14$.

Ans. 2

Ex. 2. Find the distance of the point $P(\alpha, \beta, \gamma)$ from the plane $ax+by+cz+d=0$. **Ans.** $(a\alpha+b\beta+c\gamma+d)/\sqrt{(a^2+b^2+c^2)}$

Ex. 3. Find the locus of a point whose distance from the origin is 7 times its distance from the plane $2x+3y-6z=2$.

Ans. $3x^2+8y^2+35z^2+12xy-24xz-36yz-8x-12y+24z=-4$

§ 3.12 Equation of the planes bisecting the angles between the given planes. (*Garhwal* 93)

Let the equation of the given planes be

$a_1x+b_1y+c_1z+d_1=0$ ...(i) and $a_2x+b_2y+c_2z+d_2=0$ ...(ii)

Let (x, y, z) be any point on the plane bisceting the angle between the given planes, then this point (x, y, z) must be equidistant from the given planes (i) and (ii).

i.e.

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}}=\pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}} \quad ...(iii)$$

is the required equation of the planes bisecting hte angles between (i) and (ii). Taking d_1 and d_2 as positive, the positive sign of the radical give the plane bisecting that angle between the given planes which contains the origin, while the negative sign gives the planes bisecting the other angle.

Note 1. If we are to find which of the two planes given by (iii) represents the plane bisecting the acute or obtuse angle between the given planes (i) and (ii), we should find the values of $\cos\theta$, where θ is the angle between the bisecting plane and any of the given planes. Then find the value of $\tan\theta$ from that of $\cos\theta$. In case the value of $\tan\theta>1$ then $\theta>\frac{1}{4}\pi$ and hence this bisecting plane bisects the obtuse angle between the given planes. Similarly if $\tan\theta<1$ *i.e.* $\theta<\frac{1}{4}\pi$ then this bisecting plane bisects the acute angle between the given planes.

***Note 2.** If the origin lies in the acute angle between the given planes (i) and (ii) then the angle between the normals to the planes (i) and (ii) is obtuse and hence the value of the cosine of this angle is negative.

$\therefore$ $\boxed{a_1a_2+b_1b_2+c_1c_2=\text{negative}}$ **is the condition for the origin to lie in the acute angle between the given planes.**

Similarly if the origin lies in the obtuse angle between the given planes, then the angle between the normals to the planes is acute and hence the value of the cosine of this angle is positive.

i.e. $\boxed{a_1a_2+b_1b_2+c_1c_2=\text{positive,}}$ **is the condition for the origin to lie in the obtuse angle between the given planes.**

Solved Examples on § 3.12

Ex. 1. Show that the origin lies in the acute angle between the planes $x+2y+2z-9=0$ and $4x-3y+12z+13=0$. Find the planes bisecting the angles between them, and the one which bisects the acute angle. (*Kanpur* 95)

Sol. The given planes are

$-x-2y-2z+9=0$ ...(i) $\quad 4x-3y+12z+13=0$ ...(ii)

Hence "$a_1a_2+b_1b_2+c_1c_2$" $=(-1)\,4+(-2)\,(-3)+(-2)\,(12)$
$=-4-6-24=$ negative.

Hence the origin lies in the acute angle between the given planes (see § 3.12 Note 2 Page 26).

The equation of the planes bisecting the angles between the given planes is

$$\frac{-x-2y-2z+9}{\sqrt{\{(-1)^2+(-2)^2+(-2)^2\}}}=\pm\frac{4x-3y+12z+13}{\sqrt{\{4^2+(-3)^2+(12)^2\}}}$$

or $\quad 13\,(-x-2y-2z+9)=\pm 3\,(4x-3y+12z+13)$

or $\quad 25x+17y+62z-78=0\,;\ x+35y-10z-156=0.$ **Ans.**

Let θ be the angle between the planes (i) and $25x+17y+62z-78=0$.

Then $\cos\theta=\dfrac{(-1)\,(25)+(-2)\,(17)+(-2)\,(62)}{\sqrt{\{(-1)^2+(-2)^2+(-2)^2\}}\,\sqrt{\{(25)^2+(17)^2+(62)^2\}}}$

$=-61/\sqrt{(4758)}$, on simplifying.

$\therefore\quad \tan\theta=-\{\sqrt{(1037)}\}/61$ *i.e.* $\tan\theta<1$ *i.e.* $\theta<\frac{1}{4}\pi$.

$\therefore$ The plane $2x+17y+62z-78=0$ bisects the acute angle between the given planes. (See § 3.12 Note 1 Page 26 Chapter III)

***Ex. 2. Find the bisector of the acute angle between the planes**
$2x-y+2z+3=0$ and $3x-2y+6z+8=0$ (*Rohilkhand* 92)

Sol. The given planes are $2x-y+2z+3=0$ ...(i)

and $\quad 3x-2y+6z+8=0$ ...(ii)

Also "$a_1a_2+b_1b_2+c_1c_2$" $=2.3+(-1)\,(-2)+2.6=$ positive

$\therefore$ The origin lies in the obtuse angle between the planes

(See § 3.12 Note 2 Page 26 Chapter III)

The equations of the planes bisecting the angles between the given planes

are $$\frac{2x-y+2z+3}{\sqrt{(2^2+1^2+2^2)}}=\pm\frac{3x-2y+6z+8}{\sqrt{(3^2+2^2+6^2)}}$$

or $\quad 7\,(2x-y+2z+3)=\pm\,(3x-2y+6z+8)$ ...(iii)

Taking + ve sign we get the equation of the plane bisecting angle between the planes (i) and (ii) in which origin lies *i.e.* the obtuse angle between the given planes.

Hence taking the –ve sign in (iii) we get the equation of the plane bisecting that angle between the planes (i) and (ii) in which origin does not lie *i.e.* the acute angle between the given planes.

$\therefore$ From (iii) we have the required equation as

$7\,(2x-y+2z+3)=-3\,(3x-2y+6z+8)$ or $23x-13y+32z+45=0$

Ex. 3. Show that the plane $14x - 8y + 13 = 0$ bisects the obtuse angle between angles $3x + 4y - 5z + 1 = 0$ and $5x + 12y - 13z = 0$.

Sol. The given planes are $3x + 4y - 5z + 1 = 0$ ...(i)

and $5x + 12y - 13z = 0$ ...(ii)

Also "$a_1a_2 + b_1b_2 + c_1c_2$" $= 3.5 + 4.12 + (-5)(-13) =$ positive

The origin lies in the obtuse angle between the planes.

(See § 3.12 Note 2 Page 26 Chapter III)

The equations of the planes bisecting the angles between the given planes (i) and (ii) are

$$\frac{3x + 4y - 5z + 1}{\sqrt{[3^2 + 4^2 + (-5)^2]}} = \pm \frac{5x + 12y - 13z}{\sqrt{[5^2 + 12^2 (-13)^2]}}$$

or $$13(3x + 4y - 5z + 1) = \pm 5(5x + 12y - 13z) \quad \text{...(iii)}$$

Taking + ve sign we get the equation of the plane bisecting the angle between the planes (i) and (ii) in which origin lies *i.e.* obtuse angle between tthe given planes.

$\therefore$ From (iii) we get the required equation as

$$13(3x + 4y - 5z + 1) = +5(5x + 12y - 13z)$$

or $$39x + 52y - 65z + 13 = 25x + 60y - 65z$$

or $$14x - 8y + 13 = 0.$$

Hence proved.

Exercises on § 3.12

Ex. 1. Find the equation of the plane which bisects the acute angle between the planes $3x - 4y + 12z = 26$ and $x + 2y - 2z = 9$

Ans. $22x + 14y + 10z = 195$

Ex. 2. Find the equation of the plane that bisects the angle between the planes $3x - 6y + 2z + 5 = 0$ and $4x - 12y + 3z = 3$ which contains the origin.

Ans. $67x + 162y + 47z + 44 = 0$

Ex. 3. Prove that the origin lies in acute angle between the planes $x + 2y + 2z = 9$, $4x - 3y + 12z + 8 = 0$. Find the bisector plane of the obtuse angle.

[Hint : See Exs. 1 and 3 Page 26-28 of Ch. III].

Some Important Solved Examples.

Ex. 1. Find the equation of the planes through the intersection of the planes $x + 3y + 6 = 0$; $3x - y - 4z = 0$ and whose perpendicular distance from the origin in unity.

Sol. The equation of any plane through the intersection of the given planes is

$$(x + 3y + 6) + \lambda(3x - y - 4z) = 0$$

or $$(1 + 3\lambda)x + (3 - \lambda)y - 4\lambda z + 6 = 0 \quad \text{...(i)}$$

The length of perpendicular from the origin *i.e.* (0, 0, 0) to (i) is given as unity, so we have

$$1 = \frac{6}{\sqrt{\{(1 - 3\lambda)^2 (3 - \lambda)^2 + (-4\lambda)^2\}}}$$

or $(1 + 3\lambda)^2 + (3 - \lambda)^2 + 16\lambda^2 = 36$ or $26\lambda^2 = 26$ or $\lambda = \pm 1$

Substituting these values of λ in (i) we have the required equations as

$$2x + y - 2z + 3 = 0 \quad \text{and} \quad x - 2y - 2z - 3 = 0 \qquad \text{Ans.}$$

****Ex. 2. The plane lx + my = 0 is rotated through an angle α about its line of intersection with the plane z = 0. Prove that equation to the plane in its new position is $lx + my \pm z\sqrt{(l^2 + m^2)} \tan \alpha = 0$.** (*Kanpur* 93; *Meerut* 98)

Sol. The equation of any plane through the line of intersection of the plane $lx + my = 0$ and $z = 0$ is $\quad lx + my + \lambda z = 0$. ...(i)

It is given that the angle between the plane $lx + my = 0$ and (i) is α, so the angle between their normal is $(\pi - \alpha)$. **(Note)**

Also the d.c's of their normals are $l, m, 0$ and l, m, λ respectively.

$$\therefore \tan(\pi - \alpha) = \pm \frac{\sqrt{[\Sigma (m_1 n_2 - m_2 n_1)]}}{\Sigma l_1 l_2} \qquad \text{... See § 2.10 cor. 2 Ch. II}$$

$$= \pm \frac{\sqrt{[(m\lambda - 0)^2 + (0 - l\lambda)^2 + (lm - ml)^2]}}{l^2 + m^2 + 0}$$

$$= \pm \frac{\sqrt{[\lambda^2 (l^2 + m^2)]}}{(l^2 + m^2)} = \pm \frac{\lambda}{\sqrt{(l^2 + m^2)}}$$

or $$-\tan \alpha = \pm \frac{\lambda}{\sqrt{(l^2 + m^2)}} \quad \text{or} \quad \lambda = \pm \sqrt{(l^2 + m^2)} \tan \alpha$$

$\therefore$ From (i) the required equation is $lx + my \pm z\sqrt{(l^2 + m^2)} \tan \alpha = 0$.

Ex. 3. The plane x − 2y + 3z = 0 is rotated through a right angle about its line of intersection with the plane 2x + 3y − 4z − 5 = 0, find the equation of the plane in its new position. (IAS 2008) (*Gorakhpur* 92)

Sol. The equation of any plane through the line of intersection of the plane $x - 2y + 3z = 0$ and $2x + 3y - 4z - 5 = 0$ is

$$(x - 2y + 3z) + \lambda (2x + 3y - 4z - 5) = 0$$

or $$(1 + 2\lambda) x + (3\lambda - 2) y + (3 - 4\lambda) z - 5\lambda = 0 \qquad \text{...(i)}$$

It is given that the angle between the plane $x - 2y + 3z = 0$ and (i) is a right angle, so the angle between their normals is a right angle. Also the d.r.'s of their normals are $1, -2, 3$ and $1 + 2\lambda, 3\lambda - 2, 3 - 4\lambda$

$\therefore 1 . (1 + 2\lambda) - 2(3\lambda - 2) + 3(3 - 4\lambda) = 0$

or $1 + 2\lambda - 6\lambda + 4 + 9 - 12\lambda = 0$ or $16\lambda = 14$ or $\lambda = 7/8$

$\therefore$ From (i) the required equation is

$[1 + 2(7/8)] x + [3(7/8) - 2] y + [3 - 4(7/8)] z - 5(7/8) = 0$

or $$22x + 5y - 4z - 35 = 0 \qquad \text{Ans.}$$

§ 3.13. The equation of pair of planes.

Let $l_1 x + m_1 y + n_1 z + p_1 = 0$ and $l_2 x + m_2 y + n_2 z + p_2 = 0$ be the equations of the two planes.

Then the equation $(l_1 x + m_1 y + n_1 z + p_1)(l_2 x + m_2 y + n_2 z + p_2) = 0$ represents a pair of planes as it is satisfied by all points which lie on either

$$l_1x + m_1y + n_1z + p_1 = 0 \quad \text{or} \quad l_2x + m_2y + n_2z + p_2 = 0.$$

**§ 3.14. *To find the condition that the general homogeneous equation of second degree in x, y and z viz.*

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad \text{...(i)}$$

represents a pair of planes. Also find the angle between them.

(*Agra* 91; *Avadh* 92; *Garhwal* 90; *Kanpur* 96; *Purvanchal* 96)

Let the equations of the planes represented by (i) be

$$l_1x + m_1y + n_1z = 0 \quad \text{and} \quad l_2x + m_2y + n_2z = 0$$

These equations do not contain the constant terms as otherwise their product will not be homogeneous.

So we have $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$

$$\equiv (l_1x + m_1y + n_1z)(l_2x + m_2y + n_2z) = 0$$

Comparing the coefficients of x^2, y^2, z^2, yz, zx and xy we get

$$l_1l_2 = a;\ m_1m_2 = b;\ n_1n_2 = c;\ m_1n_2 + m_2n_1 = 2f;$$

$$n_1l_2 + n_2l_1 = 2g \quad \text{and} \quad l_1m_2 + l_2m_1 = 2h. \qquad \text{...(ii)}$$

The required condition is obtained by eliminating l_1, m_1, n_1 and l_2, m_2, n_2 from the relations (ii).

Now consider the product of two zero determinants

$$\begin{vmatrix} l_1 & l_2 & 0 \\ m_1 & m_2 & 0 \\ n_1 & n_2 & 0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} l_2 & l_1 & 0 \\ m_2 & m_1 & 0 \\ n_2 & n_1 & 0 \end{vmatrix} \qquad \textbf{(Note)}$$

Multiplying these two determinants we get

$$\begin{vmatrix} 2l_1l_2 & l_1m_2 + l_2m_1 & l_1n_2 + l_2n_1 \\ l_1m_2 + l_2m_1 & 2m_1m_2 & m_1n_2 + m_2n_1 \\ l_1n_2 + l_2n_1 & m_1n_2 + m_2n_1 & 2n_1n_2 \end{vmatrix} = 0$$

ans substituting the value of l_1l_2, $l_1m_2 + l_2m_1$ etc. from (ii) we have

$$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0 \quad \text{or} \quad \boxed{\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0}$$

which on expanding reduces to

$\mathbf{abc + 2fgh - af^2 - bg^2 - ch^2 = 0}$, which is the required condition.

****Angles between the planes.**

If θ be the angle between the planes represented by (i) *i.e.* between the planes $l_1x + m_1y + n_1z = 0$ and $l_2x + m_2y + n_2z = 0$,

then
$$\tan\theta = \frac{\sqrt{[\Sigma(m_1n_2 - m_2n_1)^2]}}{\Sigma\, l_1l_2} \qquad \text{...(iii)}$$

Also $\Sigma\,(m_1n_2 - m_2n_1)^2 = \Sigma\,\{(m_1n_2 + m_2n_1)^2 - 4\,m_1m_2n_1n_2\}$

$$= \Sigma\,\{(2f)^2 - 4bc\}, \text{ from relations (ii)}$$

or $\quad \Sigma (m_1 n_2 - m_2 n_1)^2 = \Sigma [4(f^2 - bc)] = 4 [(f^2 - bc) + (g^2 - ca) + (h^2 - ab)]$

$\therefore$ From (ii) and (iii), $\tan \theta = \dfrac{\sqrt{[4 (f^2 + g^2 + h^2 - bc - ca - ab)]}}{a+b+c}$

or $$\theta = \tan^{-1} \left[\frac{2 \sqrt{(f^2 + g^2 + h^2 - bc - ca - ab)}}{a+b+c} \right] \quad \text{...(iv)}$$

Condition of perpendicularity of the planes given by (i).

If the planes given by (i) are perpendicular then

$$\theta = \frac{1}{2}\pi \quad \text{or} \quad \tan \theta = \tan \frac{1}{2} \pi = \infty$$

$\therefore$ From (iv) the condition of perpendicularly is $\mathbf{a + b + c = 0}$

i.e. **sum of coefficient of x^2, y^2, z^2 is zero.**

Solved Examples on § 3.14.

****Ex. 1 (a). Prove that the equation**

$$2x^2 - 6y^2 - 12z^2 + 18yz + 2xz + xy = 0$$

represents a pair of planes and find the angle between them.

(*Avadh* 90; *Bundelkhand* 91; *Garhwal* 95; *Meerut* 96*P*, 93)

Sol. Here $a = 2,\ b = -6,\ c = -12;\ 2f = 18,\ 2g = 2$ and $2h = 1$.

$\therefore abc + 2fgh - af^2 - bg^2 - ch^2$

$= 2(-6)(-12) + 2(9)(1)(\frac{1}{2}) - 2(9)^2 - (-6)(1)^2 - (-12)(\frac{1}{2})^2$

$= 144 + 9 - 162 + 6 + 3 = 0$

Hence the given equation represents a pair of planes.

Angle between the planes : If θ be the required angle, then

$\tan \theta = \dfrac{2 \sqrt{(f^2 + g^2 + h^2 - bc - ca - ab)}}{a+b+c}$ ... See § 3.14 (iv) above

$$= \frac{2 \sqrt{(9^2 + 1^2 + \frac{1}{4} - 72 + 24 + 12)}}{2 + (-6) + (-12)} = \frac{2\sqrt{(185/4)}}{16} = \frac{\sqrt{(185)}}{16}$$

$\therefore \cos \theta = \sqrt{\{1/(1 + \tan^2 \theta)\}}$

$$= \sqrt{\left\{\frac{1}{1 + (185/256)}\right\}} = \frac{16}{21} \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{16}{21}\right)$$

Ex. 1 (b). Show that the equation $6x^2 + 4y^2 - 10z^2 + 3yz + 4zx - 11xy = 0$ represents a pair of planes. Find the angle between them.

Sol. Do as Ex. 1 (a) above. **Ans.** $90°$

****Ex. 2. Show that $\dfrac{a}{y-z} + \dfrac{b}{z-x} + \dfrac{c}{x-y} = 0$ represents a pair of planes. Find angle between them also.** (*Agra* 92)

Sol. The given equation can be written as

$$a(z-x)(x-y) + b(y-z)(x-y) + c(y-z)(z-x) = 0$$

or $\quad ax^2 + by^2 + cz^2 - (b + c - a)\, yz - (c + a - b)\, zx - (a + b - c)\, xy = 0 \qquad ...(i)$

If it represents a pair of planes we should have

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \quad i.e. \quad \begin{vmatrix} a & -\frac{1}{2}(a+b-c) & -\frac{1}{2}(c+a-b) \\ -\frac{1}{2}(a+b-c) & b & -\frac{1}{2}(b+c-a) \\ -\frac{1}{2}(c+a-b) & -\frac{1}{2}(b+c-a) & c \end{vmatrix} = 0$$

or
$$\begin{vmatrix} -2a & a+b-c & c+a-b \\ a+b-c & -2b & b+c-a \\ c+a-b & b+c-a & -2c \end{vmatrix} = 0$$

Adding all the columns to first we find the det. on the left vanishes and hence the given equation represents a pair of planes.

Also comparing (i) with the equation

$$ax^2 + by^2 + cz^2 + 2\,fyz + 2\,gzx + 2\,hxy = 0$$

we find 'a' $= a$, 'b' $= b$, 'c' $= c$.

'f' $= \frac{1}{2}(a - b - c)$, 'g' $= \frac{1}{2}(b - c - a)$, 'h' $= \frac{1}{2}(c - a - b)$

$\therefore$ Required angle $= \tan^{-1}\left[\dfrac{2\sqrt{(f^2 + g^2 + h^2 - bc - ca - ab)}}{(a + b + c)}\right] \qquad ...(ii)$

Now $f^2 + g^2 + h^2 - bc - ca - ab$

$$= \frac{1}{4}(a - b - c)^2 + \frac{1}{4}(b - c - a)^2 + \frac{1}{4}(a - b - c)^2 - bc - ca - ab$$

$$= \frac{1}{4}[(a - b - c)^2 + (b - c - a)^2 + (c - a - b)^2 - 4bc - 4ca - 4ab]$$

$$= \frac{1}{4}[3\,(a^2 + b^2 + c^2) - 6\,(ab + bc + ca)]$$

$$= (3/4)\,[a^2 + b^2 + c^2 - 2ab - 2bc - 2ca]$$

Substituting this value in (ii), we have the required angle

$$= \tan^{-1}\left[\frac{\sqrt{\{3\,(a^2 + b^2 + c^2 - 2ab - 2bc - 2ca)\}}}{a + b + c}\right] \qquad \textbf{Ans.}$$

Ex. 3. Prove that $\dfrac{3}{y - z} + \dfrac{4}{z - x} + \dfrac{5}{x - y} = 0$ represents a pair of planes.

Sol. The given equations can be rewritten as

$$3\,(z - x)\,(x - y) + 4\,(y - z)\,(x - y) + 5\,(y - z)\,(z - x) = 0$$

or $\quad 3\,(xz - zy - x^2 + xy) + 4\,(yx - y^2 - zx + zy) + 5\,(yz - xy - z^2 + zx) = 0$

or $\quad 3x^2 + 4y^2 + 5z^2 - 6yz - 4zx - 2xy = 0$

If it represents a pair of planes, then we should have

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0 \quad i.e. \quad \begin{vmatrix} 3 & -1 & -2 \\ -1 & 4 & -3 \\ -2 & -3 & 5 \end{vmatrix} = 0 \qquad ...(i)$$

Now $\begin{vmatrix} 3 & -1 & -2 \\ -1 & 4 & -3 \\ -2 & -3 & 5 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ 11 & 4 & -11 \\ -11 & -3 & 11 \end{vmatrix}$, replacing C_1, C_3 by C_1+3C_2 and C_3-2C_2 respectively.

$$= \begin{vmatrix} 11 & -11 \\ -11 & 11 \end{vmatrix}, \text{ expanding w. r. to } R_1$$

$$= (11)(11) - (-11)(-11) = 0.$$

Hence (i) is satisfied and as such the given equation represents a pair of planes.

Exercises on § 3.14

Ex. 1. Prove that the equaton $4x^2+8y^2+z^2-6yz+5zx-12xy=0$ represents two planes and find the angle between them. **Ans.** $\tan^{-1}\{\sqrt{(29)}/13\}$

Ex. 2. Prove that the equation $2x^2+6y^2-12z^2+6yz+2zx+7xy=0$ represents a pair of planes and find the angle between them. **Ans.** $\cos^{-1}(4/27)$

(*Bundelkhand* 92)

§ 3.15. Projections on a plane.

Let the projection of the area S enclosed by the curve ABC... on a plane be S' enclosed by the curve $A'B'C'$...., when A', B', C'... are the feet of the perpendiculars from A, B, C,... on the plane of projection.

Then from Pure Solid Geometry, we have $\mathbf{S'=S\cos\theta}$, where θ is the angle between the plane of area S and the plane of projection. **(Remember)**

Theorem I. *If the projections of an area on the co-ordinate planes be A_x, A_y and A_z; respectively, then* $\boxed{A^2=A_x^2+A_y^2+A_z^2.}$

Let l, m, n be the direction cosines of the normal to the plane of area A. Also x-axis is the normal to the yz-plane.

$\therefore$ $l=$ cosine of the angle between the normals of the plane of area A and yz-plane

$\therefore$ $A_x = A \cdot l$ .... See § 3.15 above

Similarly $A_y = A.m$ and $A_z = A.n$

$$\therefore A_x^2+A_y^2+A_z^2 = A^2l^2+A^2.n^2+A^2n^2 = A^2(l^2+m^2+n^2) = A^2,$$

as $l^2+m^2+n^2=1.$ Hence proved.

Theorem II. *The projection of any plane area A on the given plane P is equal to the sum of the projections A_x, A_y and A_z on the plane P, where A_x, A_y and A_z are the projections of the area A on the yz, zx and xy planes respectively.*

Let l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c.'s of the normal to the plane of area A and the given plane P respectively. Then

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2, \quad \text{...(i)}$$

where θ is the angle between the plane P and that of area A.

Also by definition $A_x = A.\, l_1, A_y = A.\, m_1$ and $A_z = A.\, n_1$ ...(ii)

(See Theorem I above).

Let A_1 be the projection of area A on the plane P, then

$$A_1 = A\cos\theta = A\,(l_1 l_2 + m_1 m_2 + n_1 n_2), \text{ from (i)}$$
$$= (Al_1)\, l_2 + (Am_1)\, m_2 + (An_1)\, n_2$$

or $$A_1 = A_x l_2 + A_y m_2 + A_z n_2, \text{ from (ii).} \quad \text{...(iii)}$$

Also the projection of area A_x, which lies on yz-plane, on the plane $P = A_x l_2$, since d.c.'s of the normal to the plane P are l_2, m_2, n_2.

Similarly the projections of the areas A_y and A_z (which lie on zx and xy planes respectively) on the given plane P are $A_y m_2$ and $A_z n_2$ respectively.

Hence from (iii) we conclude that A_1, the projection of area A on the plane P, is equal to the sum of the projections of areas A_x, A_y and A_z on the given plane P.

****§ 3.16. The area of a triangle, the co-ordinattes of whose vertces are given.**

Let $A\,(x_1, y_1, z_1)$, $B\,(x_2, y_2, z_2)$ and $C\,(x_2, y_3, z_3)$ be the vertices of a triangle ABC. Let Δ be the area of the triangle ABC and l, m, n be the direction-cosines of the normal to the plane of this triangle.

Let A_x, A_y, A_z be the projections of the point A on yz, zx and xy-planes. Then the coordinates of A_x, A_y and A_z are $(0, y_1, z_1)$, $(x_1, 0, z_1)$ and $(x_1, y_1, 0)$ respectively.

Similarly if B_x, B_y, B_z and C_x, C_y, C_z are the projections of the points B and C on the yz, zx and xy-planes respectively then we have $B_x\,(0, y_2, z_2)$; $B_y\,(x_2, 0, z_2)$; $B_z\,(x_2, y_2, 0)$; $C_x\,(0, y_3, z_3)$; $C_y\,(x_3, 0, z_3)$ and $C_z\,(x_3, y_3, 0)$ respectively.

Let Δ_x, Δ_y, Δ_z be the projections of the area Δ of the triangle ABC on the coordinate planes. Then we have

$$\Delta_x = \Delta\,.\,l\,;\ \Delta_y = \Delta\,.\,m \text{ and } \Delta_z = \Delta\,.\,n$$

Also from § 3.15 Theorem I Page 33 Ch. III we have

$$\Delta_x^2 + \Delta_y^2 + \Delta_z^2 = \Delta^2 l^2 + \Delta^2 m^2 + \Delta^2 n^2 = \Delta^2 (l^2 + m^2 + n^2)$$
$$= \Delta^2, \quad \because \quad l^2 + m^2 + n^2 = 1$$

or $$\Delta^2 = \Delta_x^2 + \Delta_y^2 + \Delta_z^2. \quad \text{...(i)}$$

where $\Delta_x =$ The projection of the area Δ on the yz-plane

$=$ area of the triangle $A_x B_x C_x$; where A_x, B_x and C_x are the points $(0, y_1, z_1)$, $(0, y_2, z_2)$ and $(0, y_3, z_3)$ respectively.

or $\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix}$ **...(See Author's Coordinate Geometery.)**

Similarly $\Delta_y =$ area of the triangle $A_y B_y C_y$, and $\Delta_z =$ area of triangle $A_z B_z C_z$ and so we have

$$\Delta_y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} \quad \text{and} \quad \Delta_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Substituting these values of Δ_x, Δ_y and Δ_z in (i) we can evaluate Δ.

Solved Examples on § 3.16

***Ex. 1. Find the area of the triangle whose vertices area A (1, 2, 3), B (2, – 1, 1) and C (1, 2, – 4).** *(Meerut 96)*

Sol. The coordinates of the projections of A, B, C on the yz-plane are (0, 2, 3) (0, – 1, 1) and (0, 2, – 4) respectively

$\therefore$ $\Delta_x =$ area of projection of $\Delta\, ABC$ on yz-plane

$$= \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & 1 & 1 \\ 2 & -4 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 0 & 5 & 3 \\ -1 & 1 & 1 \\ 0 & -2 & 3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 5 & 3 \\ -2 & 3 \end{vmatrix} = 21/2$$

Similarly the projections of A, B and C on zx and xy-planes are (1, 0, 3), (2, 0, 1), (1, 0, – 4) and (1, 2, 0) (2, – 1, 0) (1, 2, 0) respectively. Also let Δ_y and Δ_z be the areas of the projection of the triangle ABC on zx and xy planes respectively. Then

$$\Delta_y = \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 1 \\ 1 & -4 & 1 \end{vmatrix} = 7/2 \text{ (numerically)}; \ \Delta_z = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$\therefore$ The required area $= \sqrt{[\Delta_x^2 + \Delta_y^2 + \Delta_z^2]}$ ...See § 3.16 Page 34 Ch III

$= \sqrt{[(21/2)^2 + (7/2)^2 + (0)^2]} = \sqrt{(490/4)} = (7/2)\sqrt{(10)}$ **Ans.**

Ex. 2. A plane makes intercepts OA = a, OB = b and OC = c respectively on the co-ordinate axes. Find the area of the triangle ABC. *(Kanpur 94)*

Sol. Since A, B and C lie on x, y and z axes respectively so the coordinates of A, B and C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

$\therefore$ The projections of the triangle ABC on yz, zx and xy-planes are triangles BOC, COA and AOB respectively. Let Δ, Δ_x, Δ_y and Δ_z be the areas of the triangles ABC, BOC, COA and AOB respectively.

Then $\Delta_x =$ area of $\Delta\, BOC$, where B and C lie on y and z-axes respectively $= \frac{1}{2} OB.OC = \frac{1}{2} bc$.

Similarly $\Delta_y =$ area of $\Delta\, COA = \frac{1}{2}\, ca$ and $\Delta_z =$ area of $\Delta\, AOB = \frac{1}{2}\, ab$

$$\therefore \qquad \Delta^2 = \Delta_x^2 + \Delta_y^2 + \Delta_z^2 = (\tfrac{1}{2}\, bc)^2 + (\tfrac{1}{2}\, ca)^2 + (\tfrac{1}{2}\, ab)^2.$$

$\therefore$ Required area of $\Delta\, ABC = \Delta = \frac{1}{2} \sqrt{(b^2 c^2 + c^2 a^2 + a^2 b^2)}$. **Ans.**

Ex. 3. Prove that the area of the triangle whose vertices are the points (0, 0, 0), (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{([\Sigma\, y_1 z_2 - y_2 z_1)^2]}$

Sol. Let $A\,(0, 0, 0)$, $B\,(x_1, y_1, z_1)$ and $C\,(x_2, y_2, z_2)$ be the vertices of the given triangle. The coordinates of the projections of A, B, C on the yz-plane are $(0, 0, 0)$, $(0, y_1, z_1)$ and $(0, y_2, z_2)$ respectively.

$\therefore$ $\Delta_x =$ area of projection of $\Delta\, ABC$ on yz-plane

$$= \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} = \frac{1}{2} (y_1 z_2 - y_2 z_1)$$

Similarly the projections of A, B and C on zx and xy-planes are $(0, 0, 0)$, $(x_1, 0, z_1)$, $(x_2, 0, z_2)$ and $(0, 0, 0)$, $(x_1, y_1, 0)$, $(x_2, y_2, 0)$ respectively. Also let Δ_y and Δ_z be the areas of projections of the triangle ABC on zx and xy-planes respectively.

Then $$\Delta_y = \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \end{vmatrix} = \frac{1}{2} (x_1 z_2 - x_2 z_1)$$

and $$\Delta_z = \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = \frac{1}{2} (x_1 y_2 - x_2 y_1)$$

$\therefore$ The required area $= \sqrt{(\Delta_x^2 + \Delta_y^2 + \Delta_z^2)} = \frac{1}{2} \sqrt{[\Sigma\, (y_1 z_2 - y_2 z_1)^2]}$

Hence proved.

Ex. 4. Find the area of the triangle included between the plane $2x - 3y + 4z = 12$ and the co-ordinate planes. (*Agra* 91; *Kanpur* 91)

Sol. The equation of the given plane can be written as

$$\frac{x}{6} + \frac{y}{(-4)} + \frac{z}{3} = 1, \text{ dividing each term by 12.}$$

$\therefore$ If the given plane meets the co-ordinate axes viz. x, y and z axes in A, B and C respectively, then the coordinates of A, B and C are given by $(6, 0, 0)$, $(0, -4, 0)$ and $(0, 0, 3)$ respectively.

Let $\Delta, \Delta_x, \Delta_y$ and Δ_x be the areas of the triangle ABC, BOC, COA and AOB respectively.

Then as in last example, we can prove that

$$\Delta_x = \text{ area of } \Delta\, BOC = \frac{1}{2} OB \,.\, OC = \frac{1}{2}(4)(3) = 6;$$

$$\Delta_y = \text{ area of } \Delta\, COA = \frac{1}{2} OC \,.\, OA = \frac{1}{2}(3)(6) = 9 \text{ and}$$

$$\Delta_z = \text{ area of } \Delta\, AOB = \frac{1}{2} OA \,.\, OB = \frac{1}{2}(6)(4) = 12$$

$$\therefore \text{ The required area} = \sqrt{[(\Delta_x^2 + \Delta_y^2 + \Delta_z^2)]} = \sqrt{[(6)^2 + (9)^2 + (12)^2]}$$
$$= \sqrt{[36 + 81 + 144]} = \sqrt{(261)} = 3\sqrt{(29)}.$$

****Ex. 5. Through a point P (α, β, γ) a plane is drawn at right angles to OP to meet the axes in A, B, C. Prove that the area of the triangle ABC is $p^5/(2\alpha\beta\gamma)$, where OP = p.** *(Kanpur* 97; *Meerut* 98, 91, 90)

Sol. The equation of the plane through $P(\alpha, \beta, \gamma)$ at right angles to the line OP is $\alpha(x-\alpha) + \beta(y-\beta) + \gamma(z-\gamma) = 0$

or $\quad x\alpha + y\beta + z\gamma = \alpha^2 + \beta^2 + \gamma^2 = p^2. \quad \because \quad OP^2 = p^2 = \alpha^2 + \beta^2 + \gamma^2$

or $$\frac{x}{(p^2/\alpha)} + \frac{y}{(p^2/\beta)} + \frac{z}{(p^2/\gamma)} = 1.$$

If this plane meets the co-ordinates axes in A, B and C, then the co-ordinates of A, B and C are

$$\left(\frac{p^2}{\alpha}, 0, 0\right), \left(0, \frac{p^2}{\beta}, 0\right) \text{ and } \left(0, 0, \frac{p^2}{\gamma}\right)$$

Let $\Delta_x, \Delta_y, \Delta_z$ be the projections of $\Delta\, ABC$ on the yz, zx and xy-planes, then as in last example we have

$$\Delta_x = \text{ area of } \Delta\, BCO = \frac{1}{2} \,.\, OB \,.\, OC = \frac{1}{2}\frac{p^2}{\beta}\frac{p^2}{\gamma};$$

$$\Delta_y = \text{ area of } \Delta\, COA = \frac{1}{2} \,.\, OC \,.\, OA = \frac{1}{2} \,.\, \frac{p^2}{\gamma}\frac{p^2}{\alpha};$$

and $$\Delta_z = \text{ area of } \Delta\, AOB = \frac{1}{2} \,.\, OA \,.\, OB = \frac{1}{2} \,.\, \frac{p^2}{\alpha}\frac{p^2}{\beta}.$$

$\therefore$ The required area of $\Delta\, AOB$

$$= \sqrt{[\Delta_x^2 + \Delta_y^2 + \Delta_z^2]} = \sqrt{\left[\left(\frac{p^4}{2\beta\gamma}\right)^2 + \left(\frac{p^4}{2\gamma\alpha}\right)^2 \left(\frac{p^4}{2\alpha\beta}\right)^2\right]}$$

$$= \frac{p^4}{2}\sqrt{\left[\frac{1}{(\beta\gamma)^2} + \frac{1}{(\gamma\alpha)^2} + \frac{1}{(\alpha\beta)^2}\right]} = \frac{p^4}{2\alpha\beta\gamma}\sqrt{\left[\alpha^2 + \beta^2 + \gamma^2\right]}$$

$$=\frac{p^4}{2\alpha\beta\gamma}\cdot\sqrt{(p^2)}, \quad \because p^2=\alpha^2+\beta^2+\gamma^2$$

$$=p^5/(2\alpha\beta\gamma).$$

Hence proved.

Exercises on § 3.06

Ex. 1. Find the area of the triangle whose vertices are $A(4, 3, -2)$; $B(3, 0, 1)$ and $C(2, -1, 3)$.

Ans. $\frac{1}{2}\sqrt{(14)}$

Ex. 2. Find the area of the triangle whose vertices are $A(1, 2, 3)$, $B(-2, 1, 4)$ and $C(3, 4, -2)$.

Ans. $\frac{1}{2}\sqrt{(194)}$

MISCELLANEOUS SOLVED EXAMPLES

Ex. 1. Through a point P (1, 2, 2) a plane is drawn at right angles to OP to meet the axes in A, B, C; find the area of the triangle ABC.

Sol. The direction ratios of the line OP are $1-0, 2-0, 2-0$ *i.e.* $1, 2, 2$.

Also the equation of any plane through $P(1, 2, 2)$ is

$$A(x-1)+B(y-2)+C(z-2)=0. \quad \text{...(i)}$$

If it is perpendicular to OP, the normal to this plane (i) is parallel to OP and so $A/1=B/2=C/2$ **(Note)**

$\therefore$ From (i) the equation of the plane through $P(1, 2, 2)$ and perpendicular to OP is given by

$$1(x-1)+2(y-2)+2(z-2)=0 \quad \text{or} \quad x+2y+2z=9$$

or

$$(x/9)+[y/(9/2)]+[z/(9/2)]=1.$$

$\therefore$ The intercepts made by this plane on coordinate axes are $9, 9/2, 9/2$ and hence the coordinates of A, B and C are $(9, 0, 0)$, $(0, 9/2, 0)$ and $(0, 0, 9/2)$ respectively.

The coordinates of the projection of A, B, C on the yz- plane are $(0, 0, 0)$, $(0, 9/2, 0)$ and $(0, 0, 9/2)$ respectively. **(Note)**

$\therefore \Delta_x=$ area of projection of ΔABC on yz-plane.

$$=\frac{1}{2}\begin{vmatrix}0 & 0 & 1\\ \frac{9}{2} & 0 & 1\\ 0 & \frac{9}{2} & 1\end{vmatrix}=\frac{1}{2}\times\frac{9}{2}\times\frac{9}{2}=81/8$$

Similarly the projections of A, B and C on zx and xy-planes are $(9, 0, 0)$, $(0, 0, 0)$, $(0, 0, \frac{9}{2})$ and $(9, 0, 0)$, $(0, \frac{9}{2}, 0)$, $(0, 0, 0)$ respectively. Also let Δ_y and Δ_z be the area of projections of ΔABC on zx and xy-planes respectively. Then

$$\Delta_y=\frac{1}{2}\begin{vmatrix}9 & 0 & 1\\ 0 & 0 & 1\\ 0 & \frac{9}{2} & 1\end{vmatrix}=81/4 \text{ (numerically)}; \quad \Delta_z=\frac{1}{2}\begin{vmatrix}9 & 0 & 1\\ 0 & \frac{9}{2} & 1\\ 0 & 0 & 1\end{vmatrix}=81/4$$

$\therefore$ The required area $= \sqrt{[\Delta_x^2 + \Delta_y^2 + \Delta_z^2]}$

$$= \sqrt{\left[\left(\frac{81}{2}\right)^2 + \left(\frac{81}{4}\right)^2 + \left(\frac{81}{4}\right)^2\right]} = \left(\frac{81}{4}\right)\sqrt{\left[\frac{1}{4} + 1 + 1\right]}$$

$$= \frac{81}{4} \times \frac{3}{2} = \frac{243}{8} = 30\frac{3}{8} \text{ sq. units}$$ **Ans.**

Ex. 2. Find the equation of the plane through the point (1, 2, 3) perpendicular to plane $x + 2y + 3z = 1$ and parallel to z-axis. (*Meerut* 97)

Sol. Let the required equation of the plane be

$$ax + by + cz + d = 0 \quad \text{...(i)}$$

If it passes through (1, 2, 3), then $a + 2b + 3c + d = 0$ ...(ii)

If the plane (i) is perpendicular to the given plane, then

$$a.1 + b.2 + c.3 = 0 \quad \text{or} \quad a + 2b + 3c = 0 \quad \text{...(iii)}$$

If the plane (i) is parallel to z-axis, then the normal is perpendicular to z-axis whose d.c.'s are 0, 0, 1.

Then $\quad a.0 + b.0 + c.1 = 0$ ...(iv)

Eliminating a, b, c, d from (i) to (iv), we get

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = 0 \quad \text{or} \quad 2x - y = 0.$$ **Ans.**

Ex. 3. Prove that the locus of a point, which moves so that the sum of its distances from any number of fixed planes is constant is a plane.

Sol. Let $P(\alpha, \beta\, \gamma)$ be the moving point and the equations of the fixed planes be

$$l_1 x + m_1 y + n_1 z = p_1$$
$$l_2 x + m_2 y + n_2 z = p_2$$
$$\dots\dots\dots\dots\dots$$
$$l_r x + m_r y + n_r z = p_r.$$

Given that the sum of the distances of the point $P(\alpha, \beta, \gamma)$ from these planes is constant $= k$, say

Then $(l_1 \alpha + m_1 \beta + n_1 \gamma - p_1) + (l_2 \alpha + m_2 \beta + n_2 \gamma - p_2)$

$$+ \dots + (l_r \alpha + m_r \beta + n_r \gamma - p_r) = k.$$

or $\quad \alpha(l_1 + l_2 + \dots + l_r) + \beta(m_1 + m_2 + \dots + m_r)$

$$+ \gamma(n_1 + n_2 + \dots + n_r) - (p_1 + p_2 + \dots + p_r) = k$$

or $\quad \alpha(\Sigma l_1) + \beta(\Sigma m_1) + \gamma(\Sigma n_1) = (\Sigma p_1) + k = \lambda$ (say)

$\therefore$ The locus of $P(\alpha, \beta, \gamma)$ is $x(\Sigma l_1) + y(\Sigma m_1) + z(\Sigma n_1) = \lambda$, which represents a plane as it is a first degree equation in x, y, z. Hence proved.

***Ex. 4. A variable plane passes through a fixed point (a, b, c) and meets the axes of reference in A, B and C.**

Show that he locus of the point of intersection of the planes through A, B and C parallel to the co-ordinate planes is

$(a/x)+(b/y)+(c/z)=1$. (*Kanpur* 92; *Kumaun* 95; *Purvanchal* 97)

Sol. Let the variable plane be $(x/\alpha)+(y/\beta)+(z/\gamma)=1$. ...(i)

If (i) passes through the point (a, b, c) then

$$(a/\alpha)+(b/\beta)+(c/\gamma)=1. \qquad ...(ii)$$

The plane (i) meets the co-ordinates axes in $A(\alpha, 0, 0)$, $B(0, \beta, 0)$ and $C(0, 0, \gamma)$.

The planes through A, B and C parallel to the co-ordinate planes are

$$x=\alpha,\ y=\beta \quad \text{and} \quad z=\gamma \qquad ...(iii)$$

The required locus is obtained by eliminating α, β and γ between (ii) and (iii) and is given by $(a/x)+(b/y)+(c/z)=1$ Hence proved.

***Ex. 4 (a). A variable plane, which remains at a constant distance 3p from the origin, cuts the co-ordinate axes at A, B, and C. Find the locus of the centroid of the triangle ABC.** (*Meerut* 92 *P*, 90 *S* ; *Rohilkhand* 95)

Sol. Let the equation of the variable plane be $(x/a)+(y/b)+(z/c)=1$ ...(i)

It is given that this plane is at a distance $3p$ from $(0, 0, 0)$.

$$\therefore 3p=\frac{1}{\sqrt{[(1/a)^2+(1/b)^2+(1/c)^2]}} \quad \text{or} \quad \frac{1}{9p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \qquad ...(ii)$$

Also the plane (i) meets the axes in A, B and C. So the co-ordinates of A, B and C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

Let (x, y, z) be the centroid of ΔABC, then

$x=\frac{1}{3}(a+0+0)=\frac{1}{3}a$. Similarly $y=\frac{1}{3}b$ and $z=\frac{1}{3}c$

or $$a=3x,\ b=3y,\ c=3z.$$

Substituting these values of a, b and c in (ii), we get the required locus of the centroid as $\frac{1}{9p^2}=\frac{1}{9x^2}+\frac{1}{9y^2}+\frac{1}{9z^2}$ or $x^{-2}+y^{-2}+z^{-2}=p^{-2}$ **Ans.**

***Ex. 4 (b). A variable plane, which remains at a constant distance p from the origin, cuts the cordinate axes at A, B and C. Show that the locus of the centroid of Δ ABC is $x^{-2}+y^{-2}+z^{-2}=9p^{-2}$.** (*Avadh* 91; *Meerut* 95)

Sol. Do as Ex. 4 (a) above.

****Ex. 4 (c) A variable plane is at a constant distance p from the origin and meets the axes in A, B and C. Show that the locus of the centroid of the tetrahedron OABC is $x^{-2}+y^{-2}+z^{-2}=16p^{-2}$.**

(*Agra* 90; *Buldelkhand* 95; *Kanpur* 97, 91; *Kumaum* 90; *Meerut* 92; *Purvanchal* 96)

Sol. Let the equation of the variable plane be

$$(x/a)+(y/b)+(z/c)=1 \qquad ...(i)$$

It is given that this plane is at a distance p from (0, 0, 0).

$$\therefore p=\frac{1}{\sqrt{\{(1/a)^2+(1/b)^2+(1/c)^2\}}} \quad \text{or} \quad \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

Also the plane (i) meets the axes in A, B and C. So the co-ordinates of O, A, B and C are (0, 0, 0), $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

Let (x, y, z) be the centroid of the tetrahedron $OABC$, then

$$x=\frac{1}{4}(0+a+0+0) \qquad \text{... See § 1.08 Ch. I}$$

or $$x=\frac{1}{4}a, \quad \text{Similarly} \quad y=\frac{1}{4}b \text{ and } z=\frac{1}{4}c$$

or $$a=4x,\ b=4y,\ c=4z.$$

Substituting these values of a, b and c in (ii), we have the required locus as

$$\frac{1}{p^2}=\frac{1}{16x^2}+\frac{1}{16y^2}+\frac{1}{16z^2} \text{ or } x^{-2}+y^{-2}+z^{-2}=16p^{-2}.$$

****Ex. 4 (d). A variable plane at a constant distance p from the origin meets the axes is A, B and C. Through A, B, C planes are drawn parallel to the co-ordinate planes. Show that locus of their points of intersection is**

$$\mathbf{x^{-2}+y^{-2}+z^{-2}=p^{-2}} \qquad (Kumaun\ 91)$$

Sol. Let the equation of the variable plane be

$$(x/a)+(y/b)+(z/c)=1. \qquad \text{...(i)}$$

The plane (i) meets the axes in A, B and C whose co-ordinates are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

Also the distance of the plane (i) from (0, 0, 0) is given as p, so we have,

$$p=\frac{1}{\sqrt{\{(1/a)^2+(1/b^2)+(1/c^2)\}}} \quad \text{or} \quad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2} \qquad \text{...(iii)}$$

The planes through A, B anc C parallel to the co- ordinates planes are given by $x=a$, $y=b$ and $z=c$ respectively. ...(iii)

The required locus is obtained by eliminating a, b, c from the equations of these planes and the relation (ii).

Substituting the values of a, b, c from (iii) in (ii), we have the required locus as $x^{-2}+y^{-2}+z^{-2}=p^{-2}$. Hence proved.

****Ex. 5. Two systems of rectangular axes have the same origin. If a plane cuts them at distances a, b, c and a′, b′, c′ from the origin show that**

$$\mathbf{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}} \qquad (Kanpur\ 97)$$

Sol. Let the equation of the plane referred to the first system of axes be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, the axes being Ox, Oy, Oz.

Again the equation of same plane referred to the second system of axes be $\frac{X}{a'}+\frac{Y}{b'}+\frac{Z}{c'}=1$, the axes being OX, OY, OZ.

The origin being the same for both the systems, the length of perpendicular from the origin to the plane in both the cases would be the same as only one perpendicularr can be drawn from one point to a plane, hence we have

$$\frac{1}{\sqrt{\{(1/a)^2+(1/b)^2+(1/c)^2\}}}=\frac{1}{\sqrt{\{(1/a')^2+(1/b')^2+(1/c')^2\}}}$$

or $$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}$$ Hence proved.

***Ex. 6. A point P moves on the plane $x/a+y/b+z/c=1$ which is fixed, and the plane through P perpendicular to OP meets the axes in A, B, C. If the planes through A, B, C parallel to hte co-ordinates planes meet in a point Q, show that the locus of Q is**

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}.$$ (*Kanpur* 96)

Sol. Let P be (α, β, γ), then as P lies on $x/a+y/b+z/c=1$.

so we have $$(\alpha/a)+(\beta/b)+(\gamma/c)=1. \qquad ...(i)$$

Also d. c.'s of the line OP are α, β, γ, therefore, the equation of the plane through P perpendicular to OP is

$$\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)=0$$

or $$\alpha x+\beta y+\gamma z=\alpha^2+\beta^2+\gamma^2. \qquad ...(ii)$$

The plane (ii) meets the coordinates axes in A, B, C whose co-ordinates are given by

$$\left\{\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}, 0, 0\right\};\ \left\{0, \frac{\alpha^2+\beta^2+\gamma^2}{\beta}, 0\right\};\ \left\{0, 0, \frac{\alpha^2+\beta^2+\gamma^2}{\gamma}\right\}$$

The equation of the plane through P parallel to yz-plane is $x=\dfrac{\alpha^2+\beta^2+\gamma^2}{\alpha}$. Similarly the equations of the planes through B and C parallel respectively to zx and xy-planes are

$$y=\frac{\alpha^2+\beta^2+\gamma^2}{\beta} \text{ and } z=\frac{\alpha^2+\beta^2+\gamma^2}{\gamma} \text{ respectively.}$$

The locus of Q, the point of intersection of these planes is obtained by eliminating α, β, γ between the equations of these planes and the relation (i).

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{\alpha^2}{(\alpha^2+\beta^2+\gamma^2)^2}+\frac{\beta^2}{(\alpha^2+\beta^2+\gamma^2)^2}+\frac{\gamma^2}{(\alpha^2+\beta^2+\gamma^2)^2}$$

or $$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{\alpha^2+\beta^2+\gamma^2}{(\alpha^2+\beta^2+\gamma^2)^2}=\frac{1}{\alpha^2+\beta^2+\gamma^2}$$

Also

$$\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=\frac{\alpha}{a(\alpha^2+\beta^2+\gamma^2)}+\frac{\beta}{b(\alpha^2+\beta^2+\gamma^2)}+\frac{\gamma}{c(\alpha^2+\beta^2+\gamma^2)}$$

$$= \frac{(\alpha/a)+(\beta/b)+(\gamma/c)}{(\alpha^2+\beta^2+\gamma^2)} = \frac{1}{\alpha^2+\beta^2+\gamma^2}, \text{ from (i)}$$

Hence $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}=\dfrac{1}{ax}+\dfrac{1}{by}+\dfrac{1}{cz}$ is the required locus of Q.

***Ex. 7. A triangle, the length of whose sides are a, b and c is placed so that the middle points of the sides are on the axes. Show that the lengths α, β, γ intercepted on the axes are given by**

$$8\alpha^2 = b^2+c^2-a^2,\ 8\beta^2=c^2+a^2-b^2,\ 8\gamma^2=a^2+b^2-c^2$$

and find the coordinates of its vertices. (*Kanpur* 95; *Rohilkhand* 91)

Sol. Let ABC be the given triangle which makes inttercepts α, β, γ on the axes. Therefore the equation of its plane is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1 \quad \text{...(i)}$$

The plane (i) meets the axes in $(\alpha, 0, 0)$, $(0, \beta, 0)$ and $(0, 0, \gamma)$.

Also we know that the line joining the mid-points of two sides of a triangle is parallel to the third side and half of it.

Let the mid-point of the side BC of the triangle ABC be on the x-axis, those of CA and AB on y and z- axes respectively.

Then we have $\sqrt{(\beta^2+\gamma^2)}=\frac{1}{2}a$ etc.

or $$\beta^2+\gamma^2=\tfrac{1}{4}a^2,\ \gamma^2+\alpha^2=\tfrac{1}{4}b^2,\ \alpha^2+\beta^2=\tfrac{1}{4}c^2$$

Adding, $$\alpha^2+\beta^2+\gamma^2=(1/8)(a^2+b^2+c^2).$$

$$\therefore\ \alpha^2=(\alpha^2+\beta^2+\gamma^2)-(\beta^2+\gamma^2)=(1/8)(a^2+b^2+c^2)-(1/4)a^2$$
$$=(1/8)(b^2+c^2-a^2).$$

Similarly $\beta^2=(1/8)(c^2+a^2-b^2)$ and $\gamma^2=(1/8)(a^2+b^2-c^2)$

i.e. $8\alpha^2=b^2+c^2-a^2$, $8\beta^2=c^2+a^2-b^2$ and $8\gamma^2=a^2+b^2-c^2$.

Co-ordinates of the vertices A, B, C.

Let A be (x_1, y_1, z_1). Then as mid- point of AB is $(0, 0, \gamma)$

Therefore B is $(-x_1, -y_1, 2\gamma-z_1)$. **(Note)**

Also the mid-point of AC is $(0, \beta, 0)$, so the co- ordinates of C are

$$(-x_1, 2\beta-y_1, -z_1)$$

From these, we have the mid-point of BC as

$$(-x_1, \beta-y_1, \gamma-z_1).$$

But the mid-point of BC is $(\alpha, 0, 0)$, so we have

$$-x_1=\alpha,\ 0-\beta-y_1 \quad\text{and}\quad 0=\gamma-z_1$$

i.e. $$x_1=-\alpha,\ y_1=\beta,\ z_1=\gamma.$$

Hence A, B and C are $(-\alpha, \beta, \gamma)$; $(\alpha, -\beta, \gamma)$ and $(\alpha, \beta, -\gamma)$ respectively.

Ex. 8. Find the condition that the equation

$$\phi(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

may represent a pair of planes. Prove that the product of the distance of the two planes from (α, β, γ) **is** $\dfrac{\phi(\alpha, \beta, \gamma)}{\sqrt{(\Sigma a^2 + 4\Sigma h^2 - 2\Sigma ab)}}$

Sol. For the first part (viz. the required condition) please see § 3.14 Page 30 Chapter III.

Second Part. Let the two planes represented by

$$\phi(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

be $\qquad l_1 x + m_1 y + n_1 z = 0 \quad$ and $\quad l_2 x + m_2 y + n_2 z = 0.$

Then we have $\phi(x, y, z) \equiv (l_1 x + m_1 y + n_1 z)(l_2 x + m_2 y + n_2 z)$ ...(ii)

or
$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \equiv (l_1 x + m_1 y + n_1 z)(l_2 x + m_2 y + n_2 z).$$

Comparing the coefficients on both sides, we have

$$a = l_1 l_2;\ b = m_1 m_2;\ c = n_1 n_2;\ 2f = m_1 n_2 + m_2 n_1$$
$$2g = l_1 n_2 + l_2 n_1 \quad \text{and} \quad 2h = l_1 m_2 + l_2 m_1. \qquad \text{...(iii)}$$

Let p_1 and p_2 be the distances of the two planes from (α, β, γ), then we have

$$p_1 = \frac{l_1\alpha + m_1\beta + n_1\gamma}{\sqrt{(l_1^2 + m_1^2 + n_1^2)}} \quad \text{and} \quad p_2 = \frac{l_2\alpha + m_2\beta + n_2\gamma}{\sqrt{(l_2^2 + m_2^2 + n_2^2)}}$$

$$\therefore p_1 p_2 = \frac{l_1\alpha + m_1\beta + n_1\gamma}{\sqrt{(l_1^2 + m_1^2 + n_1^2)}} \cdot \frac{l_2\alpha + m_2\beta + n_2\gamma}{\sqrt{(l_2^2 + m_2^2 + n_2^2)}}$$

$$= \frac{\phi(\alpha, \beta, \gamma)}{\sqrt{\{l_1^2 l_2^2 + m_1^2 m_2^2 + n_1^2 n_2^2 + (l_1^2 m_2^2 + l_2^2 m_1^2) + (l_1^2 n_2^2 + l_2^2 n_1^2) + (m_1^2 n_2^2 + m_2^2 n_1^2)\}}},$$

the value of the numerator has been taken from (ii)

$$= \frac{\phi(\alpha, \beta, \gamma)}{\sqrt{\{a^2 + b^2 + c^2 + \Sigma(l_1^2 m_2^2 + l_2^2 m_1^2)\}}}, \text{ from (iii)}$$

$$= \frac{\phi(\alpha, \beta, \gamma)}{\sqrt{[(a^2 + b^2 + c^2) + \Sigma\{(l_1 m_2 + l_2 m_1)^2 - 2 l_1 l_2 m_1 m_2\}]}}$$

$$= \frac{\phi(\alpha, \beta, \gamma)}{\sqrt{[a^2 + b^2 + c^2 + \Sigma\{(2h)^2 - 2ab\}]}}, \text{ from (iii)}$$

$$= \frac{\phi(\alpha, \beta, \gamma)}{\sqrt{[\Sigma a^2 + 4\Sigma h^2 - 2\Sigma ab]}}.$$

Hence proved.

Ex. 9. A plane mets a set of three mutually perpendicular planes in the sides of a triangle angles are A, B, C. Show that the first plane makes with other three planes angles whose cosine squares are cot B cot C, cot C cot A, cot A cot B.

Sol. Let the plane be $\quad x/a + y/b + z/c = 1$ ...(i)

and the three mutually perpendicular planes be co-ordinates planes.

Then the plane (i) meets the coordinates planes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

$\therefore$ The direction ratios of AB and AC are $a, b, 0$ and $a, 0, -c$ respectively, therefore, A, the angle between AB and AC, is given by

$$\tan A = \frac{\text{“}\sqrt{[\Sigma(b_1 c_2 - b_2 c_1)^2]}\text{”}}{\Sigma a_1 a_2}$$

$$= \frac{[\{(-a)(-c) - 0\}^2 + \{0 - (-c)(a)\}^2 \{0 - (a)(-b)\}^2]}{a \cdot a + (-b) \cdot 0 + 0 \cdot (-c)}$$

or $\quad \tan A = \dfrac{\sqrt{(b^2 c^2 + c^2 a^2 + a^2 b^2)}}{a^2}$ or $\cot A = \dfrac{a^2}{\sqrt{(b^2 c^2 + c^2 a^2 + a^2 b^2)}}$

Similarly we can prove that

$$\cot B = \frac{b^2}{\sqrt{(a^2 b^2 + b^2 c^2 + c^2 a^2)}} \quad \text{and} \quad \cot C = \frac{c^2}{\sqrt{(b^2 c^2 + c^2 a^2 + a^2 b^2)}}$$

Let the plane (i) make angles α, β, γ with yz, zx and xy-planes respectively. Then α is the angle between the plane (i) and the plane $x = 0$. The direction ratios of the normals to these planes are $1/a$, $1/b$, $1/c$ and $1, 0, 0$ respectively.

$$\therefore \cos\alpha = \frac{\text{“}a_1 a_2 + b_1 b_2 + c_1 c_2\text{”}}{\sqrt{(a_1^2 + b_1^2 + c_1^2)} \cdot \sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

$$= \frac{(1/a) \cdot 1 + (1/b) \cdot 0 + (1/c) \cdot 0}{\sqrt{[(1/a)^2 + (1/b)^2 + (1/c)^2]} \sqrt{(1^2 + 0^2 + 0^2)}}$$

or $\quad \cos\alpha = \dfrac{bc}{\sqrt{(b^2 c^2 + c^2 a^2 + a^2 b^2)}}$

or $\quad \cos^2\alpha = \dfrac{b^2 c^2}{(a^2 b^2 + b^2 c^2 + c^2 a^2)} = \cot B \cot C$, from above.

Similarly $\cos^2\beta = \cot C \cot A$ and $\cos^2\gamma = \cot A \cot B$.

Ex. 10. P is a given point and PM, PN are perpendiculars from P to the zx and xy-planes. If OP makes angles $\theta, \alpha, \beta, \gamma$ with the plane OMN and the co-ordinate planes, prove that

$$\mathbf{\operatorname{cosec}^2\theta = \operatorname{cosec}^2\alpha + \operatorname{cosec}^2\beta + \operatorname{cosec}^2\gamma}, \text{ **where O is the origin.**}$$

(Rohilkhand 91)

Sol. Let P be the point (a, b, c)

M and N are the feet of perpendiculars from $P(a, b, c)$ to zx and xy-planes, hence M and N are the points $(a, 0, c)$ and $(a, b, 0)$ respectively.

Also α is the angle between the line OP and the yz-plane (*i.e.* $x = 0$), then the angle between OP and the normal to the plane $x = 0$ is $90° - \alpha$ and the direction ratios of OP and the normal to the plane $x = 0$ are a, b, c and $1, 0, 0$ respectively.

$$\therefore \quad \cos(90° - \alpha) = \frac{\text{“}a_1 a_2 + b_1 b_2 + c_1 c_2\text{”}}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

or
$$\sin \alpha = \frac{a.1 + b.0 + c.0}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(1^2 + 0^2 + 0^2)}} = \frac{a}{\sqrt{(a^2 + b^2 + c^2)}}$$

or
$$\operatorname{cosec}^2 \alpha = (a^2 + b^2 + c^2)/a^2$$

Similarly $\operatorname{cosec}^2 \beta = \dfrac{a^2 + b^2 + c^2}{b^2}$ and $\operatorname{cosec}^2 \gamma = \dfrac{a^2 + b^2 + c^2}{c^2}$

$\therefore \operatorname{cosec}^2 \alpha + \operatorname{cosec}^2 \beta + \operatorname{cosec}^2 \gamma$

$$= \frac{a^2 + b^2 + c^2}{a^2} + \frac{a^2 + b^2 + c^2}{b^2} + \frac{a^2 + b^2 + c^2}{c^2}$$

$$= \frac{(a^2 + b^2 + c^2)(b^2 c^2 + c^2 a^2 + a^2 b^2)}{a^2 b^2 c^2} \qquad \text{...(i)}$$

Again the equation of any plane through O (0, 0, 0) is

$$Ax + By + Cz = 0. \qquad \text{...(ii)}$$

If (ii) passes through M $(a, 0, c)$ and N $(a, b, 0)$, then we have

$$A.a + B.0 + C.c = 0 \quad \text{and} \quad A.a + B.b + C.0 = 0$$

Solving these simultaneously, we get $\dfrac{A}{-bc} = \dfrac{B}{ca} = \dfrac{C}{ab}$

$\therefore$ From (ii) the equation of the plane OMN is

$$bcx - cay - abz = 0 \quad \text{or} \quad \frac{x}{a} - \frac{y}{b} - \frac{z}{c} = 0.$$

The direction ratios of the normal to this plane are $1/a, -1/b, -1/c$

Also θ is the angle between OP and the plane OMN, so the angle between OP whose direction ratios are (a, b, c) and the normal to the plane OMN is $90° - \theta$, therefore we get

$$\cos(90° - \theta) = \frac{a(1/a) + b(-1/b) + c(-1/c)}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{\{(1/a)^2 + (-1/b)^2 + (-1/c^2)\}}}$$

or
$$\sin \theta = \frac{-abc}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(b^2 c^2 + c^2 a^2 + a^2 b^2)}}$$

or
$$\operatorname{cosec}^2 \theta = \frac{(a^2 + b^2 + c^2)(b^2 c^2 + c^2 a^2 + a^2 b^2)}{a^2 b^2 c^2}$$

$$= \operatorname{cosec}^2 \alpha + \operatorname{cosec}^2 \beta + \operatorname{cosec}^2 \gamma, \text{ from(i)} \qquad \text{Hence proved.}$$

Ex. 11. Find the equation of the plane which bisects the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) at right angle.

Sol. Let P and Q be the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

$\therefore$ The direction ratios of the line PQ are $x_1 - x_2, y_1 - y_2, z_1 - z_2$.

Also the required plane is at right angles to the line PQ so the normal to this must be parallel to the line PQ and hence equation of this plane should be

$$(x_1 - x_2)\, x + (y_1 - y_2)\, y + (z_1 - z_2)\, z = \lambda, \qquad \text{...(i)}$$

where λ is to be determined from the condition that the plane (i) bisects the line PQ.

The coordinates of the mid-point of PQ are

$$\left[\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2), \frac{1}{2}(z_1 + z_2)\right]$$

This point lies on (i) and as such we have

$$\frac{1}{2}(x_1 - x_2)(x_1 + x_2) + \frac{1}{2}(y_1 - y_2)(y_1 + y_2) + \frac{1}{2}(z_1 - z_2)(z_1 + z_2) = \lambda$$

or
$$\frac{1}{2}(x_1^2 - x_2^2) + \frac{1}{2}(y_1^2 - y_2^2) + \frac{1}{2}(z_1^2 - z_2^2) = \lambda$$

$\therefore$ From (i) the required equation is

$$(x_1 - x_2)\, x + (y_1 - y_2)\, y + (z_1 - z_2)\, z = \frac{1}{2}[(x_1^2 - x_2^2) + (y_1^2 - y_2^2) + (z_1^2 - z_2^2)]$$

EXERCISES ON CHAPTER III

Ex. 1. Find the equations to the planes which pass through the points (2, 3, 1) and (4, – 5, 3), and are parallel to the coordinate axes. **Ans.** $y + 4z = 7$.

Ex. 2. Find the equation of the plane passing through the intersection of the planes $2x + y + 2z = 9$, $4x - 5y - 4z = 1$ and the point (3, 2, 1).

Ans. $5x - y + z = 14$.

Ex. 3. Find the equation of the plane passing through the line of intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$ and perpendicular to the plane $4x + 5y - 3z = 8$. **Ans.** $x + 7y + 13z + 96 = 0$.

Ex. 4. Find the equation of the plane through the point (2, 5, – 8) and perpendicular to each of the planes

$$2x - 3y + 4z + 1 = 0, \quad 4x + y - 2z + 6 = 0.$$

Ans. $x + 10y + 7z + 4 = 0$.

Ex. 5. Find the equation to the plane through (– 1, 3, 2) and perpendicular to the planes $x + 2y + 3z = 5$, $3x + 3y + z = 0$

Ans. $7x - 8y + 3z + 25 = 0$.

Ex. 6. Find the equation of the plane through the points (1, – 2, 2), (– 3, 1, – 2) and perpendicular to the plane $x + 2y - 3z = 5$. (*Meerut* 98)

(Hint : See Ex. 3. Page 8 Ch. III) **Ans.** $x - 16y + 11z + 9 = 0$

Ex. 6. The angle between two planes $3x - 9y + 5z = 0$ and $2x - y - 3z = 5$ is

(i) $\pi/3$; (ii) $\pi/2$; (iii) $\pi/6$; (iv) 0 **Ans.** (ii)

CHAPTER IV

The Straight Line

§ 4.01 The equation to a line.

In the last chapter we have read that every equation of the first degree represents a plane. Also as two planes intersect in a line therefore the two equations of the first degree representing these planes are satisfied by the coordinates of any point on the line intesection of these planes and as such the two linear equations in x, y, z taken together represent that line.

Thus equations $ax+by+cz+d=0$ and $a'x+b'y+c'z+d'=0$ represent the line of intersection of the planes $ax+by+cz+d=0$ and $a'x+b'y+c'z+d'=0$

§ 4.02 Symmetrical form of the equations of a line. (*Kumaun* 92)

The equations of a line are also determined by the coordinates of a fixed point on it and the direction consines or ratios of the line.

Let the line pass through a fixed point $A(\alpha, \beta, \gamma)$ and have direction–cosines l, m, n. If $P(x, y, z)$ is any point on the line at a distance r from A, then the projection of AP on the x-axis is $x-\alpha$. Also it is lr. Hence we have $x-\alpha=lr$.

Similarly we have $y-\beta=mr$ and $z-\gamma=nr$.

∴ The co-ordinates of any point (x, y, z) on the line satisfy the equations

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}\ (=r), \qquad \text{...(i)}$$

which are the equations of a straight line passing through the point $A(\alpha, \beta, \gamma)$ and l, m, n are the direction-cosines of the line.

Note 1. If instead of direction cosines of the line the direction ratios are given, the equation of the lines will remain unaltered but in this case the actual distance of P from A is not given by r.

Note 2. From the equations (i) we have

$$x=\alpha+lr, \quad y=\beta+mr \quad \text{and} \quad z=\gamma+nr$$

which are the general coordinates of any point on the line in terms of r.

Solved Examples on § 4.01 to § 4.02

Ex. 1. Choose the correct answer :

Co-ordinates of any point on the line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r, \text{ say}$$

are given by

(i) $x_1-lr,\ y_1-mr,\ z_1-nr$; (ii) lr, mr, nr;

(iii) $x_1+lr,\ y_1+mr,\ z_1+nr$; (iv) None of these.

Ans. (iii) See Note 2 of § 4.01 above.

Ex. 2. Find k so that the lines given by the following equations may be perpendicular to each other.

$$\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2} \quad \text{and} \quad \frac{x-1}{3k}=\frac{y-5}{1}=\frac{z-6}{-5}$$ (*Kumaun* 96)

Sol. The direction ratios of the given lines are $-3, 2k, 2$ and $3k, 1, -5$

$\therefore$ If these lines are perpendicular to each other, then

'$a_1a_2+b_1b_2+c_1c_2=0$'

i.e. $(-3)(3k)+(2k)(1)+2(-5)=0$

i.e. $-9k+2k-10=0$ or $7k+10=0$ or $k=-10/7$ **Ans.**

Ex. 3 (a). Find where the line $\frac{x-1}{2}=\frac{2-y}{3}=\frac{z+3}{4}$ meets the plane

$$2x+4y-z-1=0$$ (*Agra* 92)

Sol. The equations of the line are $\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z+3}{4}=r$, say ...(i)

and the plane is $2x+4y-z-1=0$ ...(ii)

Any point on the line (i) is $(1+2r, 2-3r, -3+4r)$ ...(iii)

This point lies on (ii), if $2(1+2r)+4(2-3r)-(-3+4r)-1=0$

or $-12r+12=0$ or $r=1$.

Substituting this value of r in the coordinates of the point given by (iii) we have the required point as $(1+2, 2-3, -3+4)$ or $(3, -1, 1)$ **Ans.**

Ex. 3 (b). Find the coordinates of the point of intersection of the line $x+1=\frac{1}{3}(y+3)=-\frac{1}{2}(z-2)$ with the plane $3x+4y+5z=25$.

(*Bundelkhand* 91)

Sol. Do as Ex. 2 (a) above **Ans.** $(5, 15, -10)$

Ex. 3 (c). Find the coordinates of the point of intersection of the line $x+1=\frac{1}{3}(y+3)=\frac{1}{2}(z-2)$ with the plane $3x+4y+5z=20$ (*Rohilkhand* 97)

Sol. Do as Ex. 2 (a) above. **Ans.** $(0, 0, 4)$

Ex. 4. Find the ratio in which the join of $(2, 3, 1)$ and $(-2, 1, -3)$ is cut by the plane $x-2y+3z+4=0$. Find also the coordinates of the point of intersection.

Sol. The direction ratios of the line joining the points $(2, 3, 1)$ and $(-2, 1, -3)$ are $-2-2, 1-3, -3-1$ or $-4, -2, -4$ or $2, 1, 2$.

$\therefore$ The equations of the line through $(2, 3, 1)$ and joining the given points are $\frac{x-2}{2}=\frac{y-3}{1}=\frac{z-1}{2}=r$ (say) ...(i)

Any point on this line is $(2+2r, 3+r, 1+2r)$ ...(ii)

If this point lies on the plane $x-2y+3z+4=0$, then

$(2+2r)-2(3+r)+3(1+2r)+4=0$ or $6r=-3$ or $r=-\frac{1}{2}$

Substituting this value of r in the co-ordinates of the point given by (ii), we have the required point of intersection as

$(2-1, 3-\frac{1}{2}, 1-1)$ *i.e.* $(1, \frac{5}{2}, 0)$ **Ans.**

Let the point $(1, \frac{5}{2}, 0)$ divide the join of $(2, 3, 1)$ and $(-2, 1, -3)$ in the ratio $m : n$, then

$$1 = \frac{m(-2)+n(2)}{m+n}, \frac{5}{2} = \frac{m(1)+n(3)}{m+n}; \ 0 = \frac{m(-3)+n(1)}{m+n}$$

which give $3m = n$ or $m : n = 1 : 3$. **Ans.**

Ex. 5. Prove that the distance of the point of intersection of the line $x-3 = \frac{1}{2}(y-4) = \frac{1}{2}(z-5)$ and the planes $x+y+z=17$ from the point $(3, 4, 5)$ is 3.

Sol. Any point on the given line is

$(3+1 \cdot r, 4+2r, 5+2r)$ ...See Note 2 § 4.02 P. 48 Ch. IV.

If it lies on the plane $x+y+z=17$, then we have

$$(3+r)+(4+2r)+(5+2r)=17 \text{ or } 5r=5 \text{ or } r=1.$$

Substituting this value of r in the coordinates of the point above, we have the point of intersection of the given line and plane as $(4, 6, 7)$.

$\therefore$ The required distance

$= $ the distance between $(4, 6, 7)$ and $(3, 4, 5)$

$= \sqrt{[(4-3)^2+(6-4)^2+(7-5)^2]} = \sqrt{(1^2+2^2+2^2)}$

$= 3.$ Hence proved.

Ex. 6 (a) Find the distance of the point (1, 5, 10) from the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{10}$ and plane $x-y+z=16$.

(*Kanpur* 90)

Sol. Any point on the given line is $(2+3r, -1+4r, 2+10r)$

If this point lies on the given plane $x-y+z=16$

then $(2+3r)-(-1+4r)+(2+10r)=16 \Rightarrow r=11/9.$

Substituting this value of r in the coordinates of the point, the point of intersection B of the given line and plane is

$$[2+3(11/9), -1+4(11/9), 2+10(11/9)] \text{ or } \left(\frac{17}{3}, \frac{35}{9}, \frac{128}{9}\right)$$

$\therefore$ Required distance

$= $ distance between $(1, 5, 10)$ and $(17/9, 35/9, 128/9)$

$= \sqrt{[\{1-(17/3)\}^2+\{5-(35/9)\}^2+\{10-(128/9)\}^2]}$

$= \sqrt{[3308/81]} = \sqrt{(3308)}/9.$ **Ans.**

Ex. 6 (b). Find the distance of the point $(-1, -5, -10)$ from the pointt of intersection of the line $\frac{1}{3}(x-2) = \frac{1}{4}(y+1) = (1/12)(z-2)$ and the plane $x-y+z=5$.

Sol. Do as Ex. 6 (a) above. Ans. 13

Ex. 6 (c). Find the coordinates of the foot of the perpendicular drawn from the origin to the plane $2x+3y-4z+1=0$.

Sol. The equations of the perpendicular from origin to the plane

$$2x+3y-4z+1=0 \qquad ...(i)$$

is

$$\frac{x-0}{2}=\frac{y-0}{3}=\frac{z-0}{-4} \qquad \textbf{(Note)}$$

Any point on it is $(2r, 3r, -4r)$. If this point lies on the plane (i), then

$$2(2r)+3(3r)-4(-4r)+1=0 \quad \text{or} \quad 29r=1 \quad \text{or} \quad r=1/29$$

$\therefore$ The coordinates of the required point are $(2r, 3r, -4r)$, where $r=1/29$

i.e. $$(2/29, 3/29, -4/29)$$ **Ans.**

Ex. 6 (d). Find the point in which the line $-(x+1)=\frac{1}{5}(y-12)$ $=\frac{1}{2}(z-7)$ cuts the surface $11x^2-5y^2+z^2=0$. (*Agra* 91)

Sol. Any point on the line $\frac{x+1}{-1}=\frac{y-12}{5}=\frac{z-7}{2}=r$ is

$$(-1-r,\ 12+5r,\ 7+2r). \qquad ...(i)$$

If it lies on the surface $11x^2-5y^2+z^2=0$, then we have

$$11(-1-r)^2-5(12+5r)^2+(7+2r)^2=0$$

or $$11(1+2r+r^2)-5(144+120r+25r^2)+(49+28r+4r^2)=0$$

or $$110r^2+550r^2+660=0 \quad \text{or} \quad r^2+5r+6=0 \quad \text{or} \quad r=-2,-3.$$

Substituting these values of r in (i), we have required points as $(1, 2, 3)$ and $(2, -3, 1)$. **Ans.**

****Ex. 7 (a). Find the image of the point P (3, 5, 7) in the plane**
$2x+y+z=6$. (*Avadh* 90; *Kanpur* 97, 90)

Sol. The image of the point (3, 5, 7) on the given plane is the point in which the line through (3, 5, 7) perpendicular to the given plane meets it.

Now the directin ratio of the normal to the given plane are 2, 1, 1.

$\therefore$ Equations of the line through $P(3, 5, 7)$ perpendicular to the given plane are

$$\frac{x-3}{2}=\frac{y-5}{1}=\frac{z-7}{1}$$

Any point on this line is $(3+2r, 5+r, 7+r)$

If it lies on the given plane $2x+y+z=6$, then we have

$$2(3+2r)+(5+r)+(7+r)=6 \quad \text{or} \quad 6r=-12 \quad \text{or} \quad r=-2$$

$\therefore$ Required point is $(3+2r, 5+r, 7+r)$, where $r=-2$

i.e. $$(3-4, 5-2, 7-2) \quad i.e. \quad (-1, 3, 5)$$ **Ans.**

Ex. 7 (b). Find the image of the point (1, 3, 4) in the plane $2x-y+z+3=0$. (*Bundelkhand* 90; *Gorakhpur* 92; *Purvanchal* 96, 93)

Sol. $\because$ The d.r.'s of the normal to the given plane are $2, -1, 1$, so the equations of the line through (1, 3, 4) perpendicular to the given plane are

$$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}=r \text{ (say)}$$

$\therefore$ Any point on this line is $(2r+1, -r+3, r+4)$ ...(i)

If it is lies on the given plane, then we have

$$2(2r+1)-(-r+3)+(r+4)+3=0 \quad \text{or} \quad r=-1$$

Substituting this value of r in (i), the required image is

$$(-2+1, 1+3, -1+4) \quad i.e. \quad (-1, 4, 3) \qquad \textbf{Ans.}$$

Ex. 7 (c). Find the image of the point $(2, -1, 3)$ in the plane $3x-2y+z=9$.

Sol. Do as Ex. 7 (a) above **Ans.** $(11/7, -5/7, 20/7)$

Ex. 7 (d). Find the image of the point $(2, 1, 3)$ in the plane $x+y-z+2=0$.

Sol. Do as Ex. 7 (a) above. **Ans.** $(4/3, 1/3, 11/3)$

****Ex. 7 (e). Find the imagge of the point $(-2, 1, 3)$ in the plane**

$$\mathbf{x+y-2z+1=0.}$$ *(Avadh 91)*

Sol. Do as Ex. 7 (a) above **Ans.** $(-5/4, 7/4, 3/2)$

****Ex. 8 (a). Find the distance of the point $(1, -2, 3)$ from the plane $x-y+z=5$ measured parallel to the line $\frac{1}{2}x=\frac{1}{3}y=-\frac{1}{6}z$.** *(Meerut 96P)*

Sol. In this example, we **are not required** to find the perpendicular distance of the point $(1, -2, 3)$ from the given plane, but the distance of this point from the given plane measured parallel to the given line whose direction ratios are $2, 3, -6$.

Now the equations of the line through $(1, -2, 3)$ and parallel to the line whose d.c.'s are $2, 3, -6$ are $\dfrac{x-1}{2}=\dfrac{y+2}{3}=\dfrac{z-3}{-6}$

Any point on this line is $(1+2r, -2+3r, 3-6r)$. ...(i)

If it is lies on the given plane $x-y+z=5$, then we have

$$(1+2r)-(-2+3r)+(3-6r)=5$$

or $1-7r=0$ or $r=(1/7)$. Substituting this value of r in (i), the point is $[(9/7), (-11/7), (15/7)]$ and therefore the required distance of this point from the given point $(1, -2, 3)$

$$=\sqrt{\left[\left(\frac{9}{7}-1\right)^2+\left(-\frac{11}{7}+2\right)^2+\left(\frac{15}{7}-3\right)^2\right]}=\sqrt{\left[\frac{4}{49}+\frac{9}{49}+\frac{36}{49}\right]}=1 \textbf{ Ans.}$$

***Ex. 8 (b). Find the distance of the point $(3, -4, 6)$ from the plane $2x+5y-6z=16$ measured along a line with direction cosines proportional to $(2, 1, -2)$.** *(Kumaun 94)*

Sol. Do as Ex. 8 (a) above. **Ans.** $(22/7)\sqrt{(14)}$

Ex. 8 (c). Find the equations of the straight line through (a, b, c) which are (i) parallel to z-axis and (ii) perpendicular to z-axis.

Sol. The equations of any line through (a, b, c) are

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}, \quad ...(A)$$

where l, m, n are the d.c.'s of the line.

(i) If this line is parallel to z-axis, then its d.c.'s are proportional to $(0, 0, 1)$ the d.c.'s of the z-axis.

Hence from (A) above the required equations are

$$\frac{x-a}{0}=\frac{y-b}{0}=\frac{z-c}{1}. \quad \textbf{Ans.}$$

(ii) If line given by (A) is pependicular to z-axis *i.e.* parallel to xy-plane, then we have $\quad l\,.\,0+m\,.\,0+n\,.\,1=0 \quad$ or $\quad n=0.$

From (A) above the required equations are

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{0}. \quad \textbf{Ans.}$$

***Ex. 9. Find the equations of the line through (α, β, γ) at right angls to the lines $\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$ and $\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$** *(Kumaun 91)*

Sol. Let the required line through (α, β, γ) be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}. \quad ...(i)$$

If this line is perpendicular to the given lines, then we have

$$ll_1+mm_1+nn_1=0$$

and

$$ll_2+mm_2+nn_2=0$$

Solving these, we have $\dfrac{l}{m_1n_2-m_2n_1}=\dfrac{m}{n_1l_2-n_2l_1}=\dfrac{n}{l_1m_2-l_2m_1}$

Substituting these proportionate values of l, m, n in (i) we have the required equations as $\dfrac{x-\alpha}{m_1n_2-m_2n_1}=\dfrac{y-\beta}{n_1l_2-n_2l_1}=\dfrac{z-\gamma}{l_1m_2-l_2m_1}$ **Ans.**

***Ex. 10. Show that if the axes are rectangular, the equations to the perpendicular from the point (α, β, γ) to the plane**

$$ax+by+cz+d=0 \quad \text{are} \quad \frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c}$$

and deduce the perpendicular distance of the point (α, β, γ) from the plane.

Sol. $\because$ The perpendicular from (α, β, γ) to the given plane is parallel to the normal to the given plane viz. $ax+by+cz+d=0$.

$\therefore$ The d.c.'s of this perpendicular are proportional to a, b, c.

Hence the equations of this perpendicular line are given by

$$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c}=r \text{ (say)}. \quad ...(i)$$

Any point on this line given by (i) is $(\alpha+ar,\ \beta+br,\ \gamma+cr)$. if this point lies on the plane $ax+by+cz+d=0$ we have

$$a(\alpha+ar)+b(\beta+br)+c(\gamma+cr)+d=0$$

or $$r(a^2+b^2+c^2)=-(a\alpha+b\beta+c\gamma+d)$$

or $$r=-(a\alpha+b\beta+c\gamma+d)/(a^2+b^2+c^2) \quad ...(ii)$$

Now the required distance = distance between the points (α, β, γ) and $(\alpha+ar,\ \beta+br,\ \gamma+cr)$

$$=\sqrt{[(\alpha+ar-\alpha)^2+(\beta+br-\beta)^2+(\gamma+cr-\gamma)^2]}$$
$$=\sqrt{(a^2r^2+b^2r+c^2r^2)}=r\sqrt{(a^2+b^2+c^2)}$$
$$=\frac{-(a\alpha+b\beta+c\gamma+d)\sqrt{(a^2+b^2+c^2)}}{(a^2+b^2+c^2)}, \text{ from (ii)}$$
$$=(a\alpha+b\beta+c\gamma+d)/\sqrt{(a^2+b^2+c^2)}, \text{ numerically.}$$

The foot of the perpendicular from the point (α, β, γ) on the plane $ax+by+cz+d=0$ is the point $(\alpha+ar, \beta+br, \gamma+cr)$.

Substituting value of r from (ii), we can get thte coordinates of the foot of the above perpendicular.

****Ex. 11. Find the incentre of the tetrahedron formed by the planes $x=0, y=0, z=0$ and $x+y+z=a$.** (*Kanpur* 96, 92)

Sol. Evidently the planes $x=0$, $y=0$ and $z=0$ meet in $(0, 0, 0)$. Hence the incentre lies on the perpendicular from $(0, 0, 0)$ to the plane $x+y+z=a$ and divides it in the ratio $3:1$ [3 from the vertex $(0, 0, 0)$ and 1 from the plane $x+y+z=a$].

The equations of the perpendicular from $(0, 0, 0)$ to the plane $x+y+z=a$ is

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{1}=r \text{ (say) (see Ex. 10 above)}$$

Any point on this perpendicular is (r, r, r). If it lies on the plane $x+y+z=a$, then we have $r+r+r=a$ or $r=a/3$.

$\therefore$ The perpendicular from $(0, 0, 0)$ meets the plane $x+y+z=a$ in (r, r, r) *i.e.* $(\frac{1}{3}a, \frac{1}{3}a, \frac{1}{3}a)$. Also the incentre divides the join of $(0, 0, 0)$ and $(\frac{1}{3}a, \frac{1}{3}a, \frac{1}{3}a)$ in the ratio $3:1$, therefore if (x_1, y_1, z_1) be the required incentre,

we have $x_1=\dfrac{3\cdot\frac{1}{3}a+1.0}{3+1}=\frac{1}{4}a$. Similarly $y_1=\frac{1}{4}a=z$.

$\therefore$ The required incentre is $(\frac{1}{4}a, \frac{1}{4}a, \frac{1}{4}a)$. **Ans.**

***Ex. 12. P is a point on the plane $lx+my+nz=p$. A point Q is taken on the line OP such that $OP.OQ=p^2$, prove that the locus of Q is**

$$p(lx+my+nz)=x^2+y^2+z^2.$$

Sol. Let Q be the point (α, β, γ) and $OQ=R$. Then the direction ratios of the line OQ are $\alpha-0, \beta-0, \gamma-0$, *i.e.* α, β, γ.

$\therefore$ The direction cosines of OQ are $\alpha/R, \beta/R, \gamma/R$, where

$$R = OQ = \sqrt{(\alpha^2 + \beta^2 + \gamma^2)} \qquad ...(i)$$

$\therefore$ The equations of the line OQ are $\dfrac{x-0}{\alpha/R} = \dfrac{y-0}{\beta/R} = \dfrac{z-0}{\gamma/R} = r$ (say),

where r is the distance of any point from (0, 0, 0)

Let $OP = r$, then the co-ordinates of P are $\left(\dfrac{\alpha r}{R}, \dfrac{\beta r}{R}, \dfrac{\gamma r}{R}\right)$

But it is given that P is a point on the plane $lx + my + nz = p$.

$$\therefore \quad l\frac{\alpha r}{R} + m\frac{\beta r}{R} + n\frac{\gamma r}{R} = p \quad \text{or} \quad \frac{r}{R}(l\alpha + m\beta + n\gamma) = p. \qquad ...(ii)$$

Again we are given that $OP.OQ = p^2$.

or $\qquad r \, . \, R = p^2, \quad \because \quad OP = r, \; OQ = R$

or $\qquad r = p^2/R.$

$\therefore$ from (ii) we get $(p^2/R^2)(l\alpha + m\beta + n\gamma) = p$

or $\qquad p(l\alpha + m\beta + n\gamma) = R^2 = \alpha^2 + \beta^2 + \gamma^2$, from (i).

$\therefore$ The locus of $Q(\alpha, \beta, \gamma)$ is $p(lx + my + nz) = x^2 + y^2 + z^2$.

Ex. 13. A variable plane makes intercepts on the co-ordinatte axes the sum of whose squares is constant and equal to k^2. Show that locus of the foot of the perpendicular from the origin to the plane is

$$(x^{-2} + y^{-2} + z^{-2})(x^2 + y^2 + z^2)^2 = k^2.$$

Sol. Let the varibale plane be $(x/a) + (y/b) + (z/c) = 1$, $\qquad$ ...(i)

where a, b and c are its intercepts on the co-ordinate axes

$\therefore$ according to the given condition $\quad a^2 + b^2 + c^2 = k^2$. $\qquad$ ...(ii)

Now the equations of the line through the origin and perpendicular to the plane (i) [or parallel to the normal to (i) whose direction ratios are $1/a$, $1/b$ and $1/c$] are given by $\dfrac{x-0}{1/a} = \dfrac{y-0}{(1/b)} = \dfrac{z-0}{(1/c)} = r$ (say).

Any point on this line is $(r/a, r/b, r/c.)$ $\qquad$ ...(iii)

If this point lies on the plane (i) then

$$\frac{(r/a)}{a} + \frac{(r/b)}{b} + \frac{(r/c)}{c} = 1 \quad \text{or} \quad r(a^{-2} + b^{-2} + c^{-2}) = 1.$$

Substituting this value of r in (iii) we find that the co-ordinates of the foot of the perpendicular from the origin to the plane (i) are given by

$$x = \frac{1}{a(a^{-2} + b^{-2} + c^{-2})}, \quad y = \frac{1}{b(a^{-2} + b^{-2} + c^{-2})}, \quad z = \frac{1}{c(a^{-2} + b^{-2} + c^{-2})}$$

$$\therefore x^2 + y^2 + z^2 = \frac{1}{(a^{-2} + b^{-2} + c^{-2})^2}\left[\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right]$$

$$= \frac{(a^{-2} + b^{-2} + c^{-2})}{(a^{-2} + b^{-2} + c^{-2})^2} = \frac{1}{(a^{-2} + b^{-2} + c^{-2})} \qquad ...(iv)$$

And $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=(a^{-2}+b^{-2}+c^{-2})^2\,[a^2+b^2+c^2]$

or $x^{-2}+y^{-2}+z^{-2}=(a^{-2}+b^{-2}+c^{-2})^2\,.\,k^2$, from (ii)

$=k^2/(x^2+y^2+z^2)^2$, from (iv)

or $(x^2+y^2+z^2)^2\,(x^{-2}+y^{-2}+z^{-2})=k^2$ Hence proved

***Ex. 14. Does the lines $\frac{x-1}{5}=\frac{y+2}{6}=\frac{z-3}{4}$ intersect any one of the planes (i) $2x+3y-7z+29=0$, (ii) $2x+3y-7z+25=0$, (iii) $2x+3y-6z+20=0$**

If it does, find the point of intersection.

Sol. Any point on the given line is $(5r+1, 6r-2, 4r+3)$ ...(A)

If the given line intersects the plane (i), then

$$2(5r+1)+3(6r-2)-7(4r+3)+29=0$$

or $$r(10+18-28)+(2-6-21+29)=0$$

from which r cannot be determined. Hence the given line does not meet the plane (i).

Similarly if the given line intersects the plane (ii), then proceeding as above we find that r can not be determined, hence the given line does not intersect the planee (ii).

Finally if the given line intersects the plane (iii), then

$$2(5r+1)+3(6r-2)-6(4r+3)+20=0$$

or $r(10+18-24)+(2-6-18+20)=0$ or $4r-2=0$ or $r=1/2$

Hence the given line intersects the plane (iii) and the required point of intersection from (A) is

$[5(1/2)+1,\ 6(1/2)-2,\ 4(1/2)+3]$ *i.e.* $(7/2, 1, 5)$ **Ans.**

Exercises on § 4.01-4.02

Ex. 1. Find the co-ordinates of the point of intersection of the line $(x-1)=\frac{1}{3}(y+3)=-\frac{1}{2}(z-4)$ with plane $3x+4y+5z-6=0$. **Ans.** $(0,-6,6)$

Ex. 2. Find the co-ordinates of the point where the straight line $\frac{1}{2}(x-1)=-(y+1)=\frac{1}{2}z$ interceps the plane $3x+2y-z=5$. **Ans.** $(9,-5,12)$

Ex. 3. Find the co-ordinates of the point where the line joining the points $(2,-3,1)$ and $(3,-4,-5)$ cuts the plane $2x+y+z=7$. **Ans.** $(9,-2,7)$

Ex. 4. Find the distance of the point $(3,-4,5)$ from the plane $2x+5y+6z=19$ measured along a straight line with direction cosines proportional to $2, 1, -2$. **Ans.** $83/4$

Ex. 5. Find the distance of the point $(1, 2, 3)$ from the plane $x+y+z=11$ measured parallel to the line $x+1=-\frac{1}{2}(y-12)=\frac{1}{2}(z-7)$ **Ans.** 15.

Ex. 6. Find the equations of the line through (α, β, γ) at right angles to the lines $\frac{1}{2}(x-1)=\frac{1}{3}(y-2)=\frac{1}{4}(z-4)$ and $\frac{1}{3}(x-2)=\frac{1}{4}(y-4)=\frac{1}{5}(z-5)$.

Ans. $(x-\alpha)=-\frac{1}{2}(y-\beta)=(z-\gamma)$.

Ex. 7. Find the coordinates of the image of the point $(1, 3, 4)$ in the plane $2x-y+z+3=0$. **Ans.** $(-1, 4, 3)$

Ex. 8. Find the image of the point $(1, 3, 4)$ in the plane $2x-y+z=0$

(*Bundelkhand* 96)

(Hint : See Ex. 7 (b) Page 51) **Ans.** $(1/4, 27/8, 29/8)$

Ex. 9. Find the coordinates of the image of the point (a, b, c) with respect to the coordinate planes. **Ans.** $(a, b, 0)$, $(0, b, c)$ $(a, 0, c)$

Ex. 10. Find the distance from the point $(3, 4, 5)$ to the point where the line $(x-3)=\frac{1}{2}(y-4)=\frac{1}{2}(z-5)$ meets the plane $x+y+z=2$. **Ans.** -6.

Ex. 11. Find the equations of the line through $(1, 2, -1)$ perpendicular to $3x-5y+4z=5$ and deduce the length of the perpendicular from $(1, 2, -1)$ upon the plane and also the coordinates of the foot of the perpendicular.

Ans. $(8/5)\sqrt{2}$, $[42/25, 25, 7/25]$

Ex. 12. Are the two lines

$$\frac{x-2}{3}=\frac{y-3}{2}=\frac{z+4}{4} \quad \text{and} \quad \frac{x+1}{5}=\frac{y-2}{-6}=\frac{z+3}{2}$$

perpendicular to each other? **Ans.** No.

§ 4.03. Line through two points. (*Purvanchal* 97)

Let two given points be $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Then the d.c.'s of the line PQ are x_2-x_1, y_2-y_1, z_2-z_1.

$\therefore$ The equations of the line PQ are $\dfrac{x-x_1}{x_2-x_1}=\dfrac{y-y_1}{y_2-y_1}=\dfrac{z-z_1}{z_2-z_1}$

Also we know by § 1.06 Chaptter I that if there be a point R which divides PQ internally in the ratios $\lambda : 1$, then the coordinates of R are

$$\left(\frac{\lambda x_2+x_1}{\lambda+1}, \frac{\lambda y_2+y_1}{\lambda+1}, \frac{\lambda z_2+z_1}{\lambda+1}\right). \quad ...(i)$$

Solved Examples on § 4.03.

Ex. 1. Find the equations of the line joining the points $(4, -5, -2)$ and $(-1, 5, 3)$ and show that it meets the surface $2x^2+3y^2-4z^2=1$ in coincident points.

Sol. The equations of the line joining $(4, -5, -2)$ and $(-1, 5, 3)$ are

$$\frac{x-4}{-1-4}=\frac{y-(-5)}{5-(-5)}=\frac{z-(-2)}{3-(-2)} \quad \text{or} \quad \frac{x-4}{-1}=\frac{y+5}{2}=\frac{z+2}{1}=r, \text{ say}$$

$\therefore$ Any point on it is $(4-r, -5+2r, -2+r)$

If it lies on the surface $2x^2+3y^2-4z^2=1$, then

$$2(4-r)^2+3(-5+2r)^2-4(-2+r)^2=1$$

or $\quad 2(16-8r+r^2)+3(25-20r+4r^2)-4(r^2-4r+4)=1$

or $\quad 10r^2-60r+90=0$ or $r^2-6r+9=0$ or $(r-3)^2=0$ or $r=3, 3.$

These two values of r being coincident the line (i) meets the given surface in coincident points.

***Ex. 2. Find the equations of the lines through the points (a, b, c) and (a′, b′, c′) and prove that it passes through the origin, if aa′ + bb′ + cc′ = rr′, where r and r′ are the distances of these points from the origin.**

Sol. The equations of the line through (a, b, c) and (a', b', c') are

$$\frac{x-a}{a'-a}=\frac{y-b}{b'-b}=\frac{z-c}{c'-c} \quad \text{...(i)}$$

If (i) passes through the origin *i.e.* (0, 0, 0), then

$$\frac{0-a}{a'-a}=\frac{0-b}{b'-b}=\frac{0-c}{c'-c} \quad \text{or} \quad \frac{a'-a}{a}=\frac{b'-b}{b}=\frac{c'-c}{c} \quad \textbf{(Note)}$$

or $$\frac{a'}{a}-1=\frac{b'}{b}-1=\frac{c'}{c}-1 \quad \text{or} \quad \frac{a'}{a}=\frac{b'}{b}=\frac{c'}{c}$$

From these, we can get $ab'-a'b=0,\ bc'-b'c=0,\ ca'-c'a=0$ ...(ii)

Also we are given that $r=\sqrt{(a^2+b^2+c^2)},\ r'=\sqrt{(a'^2+b'^2+c'^2)}$ ...(iii)

Now from Lagrange's Identity, we have

$$(a^2+b^2+c^2)(a'^2+b'^2+c'^2)-(aa'+bb'-cc')^2$$
$$=(ab'-a'b)^2+(bc'-b'c)^2+(ca'-c'a)^2$$

or $$r^2(r')^2-(aa'+bb'+cc')^2=0, \text{ from (ii) and (iii)}$$

or $$rr'=aa'+bb'+cc'$$ Hence proved.

**§ 4.04 Transformation from General Form to Symmetric Form.

(*Rohilkhand* 93)

Let the equations of a straight line be given by

$$\left.\begin{aligned} a_1x+b_1y+c_1z+d_1=0 \\ \text{and} \quad a_2x+b_2y+c_2z+d_2=0 \end{aligned}\right\} \quad \text{...(i)}$$

In order to transform the general form of the straight line given above by (i) we should find out the d.c.'s of the line and the co-ordinates of some point on it.

Let l, m, n be the d.c.'s of this line. Then as this line lies on both the planes given by (i), so that it is perpendicular to the normals to these planes. The direction ratios of the normals to the planes given by (i) are a_1, b_1, c_1 and a_2, b_2, c_2 respectively.

$\therefore$ We have $a_1l+b_1m+c_1n=0;\ a_2l+b_2m+c_2n=0.$

Solving these, we have $\dfrac{l}{b_1c_2-b_2c_1}=\dfrac{m}{c_1a_2-c_2a_1}=\dfrac{n}{a_1b_2-a_2b_1}$ ...(ii)

which give the direction ratios of the line.

Now the co-ordinates of any point on the line can be calculated in many ways. One of them is that we choose the point as the one where the line meets $z=0$

(*i.e.* xy-plane).

Putting $z=0$ in (i), we get

$$a_1x+b_1y+d_1=0 \quad \text{and} \quad a_2x+b_2y+d_2=0.$$

Solving these, $\dfrac{x}{b_1d_2-b_2d_1}=\dfrac{y}{d_1a_2-d_2a_1}=\dfrac{1}{a_1b_2-a_2b_1}$

or $$x=\frac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1};\ y=\frac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1}$$

$\therefore$ The co-ordinates of the point where the line meets the plane $z=0$ is

$$\left[\frac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1},\ \frac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1},\ 0\right] \quad \text{...(iii)}$$

$\therefore$ With the help of (ii) and (iii) we can write the equations of the line given by (i) in the symmetric form as

$$\frac{x-\left(\dfrac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}\right)}{b_1c_2-b_2c_1}=\frac{y-\left(\dfrac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1}\right)}{c_1a_2-c_2a_1}=\frac{z-0}{a_1b_2-a_2b_1}$$

An Important Note : If $a_1b_2-a_2b_1=0$, then we should choose the point where the line meets the plane $x=0$ or $y=0$.

Solved Exampes on § 4.04.

Ex. 1 (a). Find the symmetric equations of the line

$$\mathbf{x-y=0\ ;\ 3x-z+8=0}$$

Sol. The equations of the given planes can be written as

$$1\,.\,x-1\,.\,y+0\,.\,z=0,\ 3x+0\,.\,y-1\,.\,z+8=0 \quad \text{...(i)}$$

Let l, m, n be the d.c.'s of the line. Then as this lines lies on both the planes given by (i), so it is perpendicular to the normals of the planes given by (i).

$$\therefore\ l\,.\,1+m\,(-1)+n\,.\,0=0$$

and $$l\,.\,3+m\,(0)+n\,(-1)=0.$$

Solving these we get $l/1=m/1=n/3$ ...(ii)

Putting $x=0$ in (i) we have $y=0,\ z=8$.

$\therefore$ The line (i) meets the plane $x=0$ in $(0, 0, 8)$ ...(iii)

Hence the required form of the equations of the given line from (ii) and (iii) is $(x-0)/1=(y-0)/1=(z-8)/3$ **Ans.**

Ex. 1 (b). Find the symmetric form the equations of the line given by

$$\mathbf{x=ay+b,\ z=cy+d}$$

Sol. The equations of the given planes can be written as

$$1\,.\,x-a\,.\,y+0\,.\,z-b=0 \quad \text{and} \quad 0\,.\,x+c\,.\,y-1\,.\,z+d=0 \quad \text{...(i)}$$

Let l, m, n be the d.c.'s of the line. Then as this line lies on both the planes given by (i), so it is perpendicular to the normals to the planes given by (i).

Therefore $$l\,.\,1+m\,.\,(-a)+n\,.\,0=0$$

and $\quad l \cdot 0 + m \cdot c + n \cdot (-1) = 0.$

Solving these, we get $\quad l/a = m/1 = n/c$ ...(ii)

Putting $y = 0$ in (i), we have $x = b$ and $z = d$

$\therefore$ The line (i) meets the plane $y = 0$ at $(b, 0, d)$.

Hence the required equations are $\dfrac{x-b}{a} = \dfrac{y-0}{1} = \dfrac{z-d}{c}$. **Ans.**

Ex. 1 (c). Obtain in the symmetric form the equations of the line given by $x - 2y + 3z = 4,\ 2x - 3y + 4z = 5$.

Sol. Do as Ex. 1 (a) above. **Ans.** $x + 2 = \frac{1}{2}(y+3) = z$

Ex. 2. Find the symmetric form of the equations of the line $x + y + z + 1 = 0$ and $4x + y - 2z + 2 = 0$ and hence find the equation to the plane through (1, 1, 1) and perpendicular to the given line.

Sol. Let l, m, n be the d.c.'s of the line. Then as this line lies on both the given planes, so it is perpendicular to the normal to these planes.

$\therefore$ We have $\quad l + m + n = 0,\ 4l + m - 2n = 0.$ **(Note)**

Solving these we get $\dfrac{l}{1+2} = \dfrac{m}{-2-4} = \dfrac{n}{4-1}$ or $\dfrac{l}{1} = \dfrac{m}{-2} = \dfrac{n}{1}$ ...(i)

Putting $z = 0$ in the given equations we get

$$x + y = -1,\ 4x + y = -2$$

Solving these we get $x = -\frac{1}{3},\ y = -\frac{2}{3}$

$\therefore$ The required line meets the plane $z = 0$ at $\left(-\frac{1}{3}, -\frac{2}{3}, 0\right)$ ...(ii)

$\therefore$ From (i) and (ii) the required equations in symmetric form are

$$\frac{x+\frac{1}{3}}{1} = \frac{y+\frac{2}{3}}{-2} = \frac{z-0}{1} \quad \text{or} \quad \frac{3x+1}{3} = \frac{3y+2}{-6} = \frac{z}{1}$$

And its direction ratios are $3, -6, 1$.

Let the plane through (1, 1, 1) be $A(x-1) + B(y-1) + C(z-1) = 0$

If this plane is perpendicular to the above line, then

$A/1 = B/(-2) = C/1$ and so its equation is

$$(x-1) - 2(y-1) + (z-1) = 0 \quad \text{or} \quad x - 2y + z = 0$$ **Ans.**

Ex. 3. Find the symmetric form of the line

$$\mathbf{3x + 2y + z = 5,\ x + y - 2z = 3}$$

Sol. Let l, m, n be the d.c.'s of the line. Then as this line lies on both the given planes, so it is perpendicular to the normal to these planes.

$\therefore$ We have $3l + 2m + n = 0,\ l + m - 2n = 0$

Solving these we get $\dfrac{l}{2(-2) - 1.1} = \dfrac{m}{1.1 - 3.(-2)} = \dfrac{n}{3.1 - 2.1}$

or $\quad l/(-5) = m/7 = n/1$ ...(i)

Putting $z = 0$ in the given equation we get $3x + 2y = 5,\ x + y = 3$

Solving these we get $x = -1,\ y = 4$

$\therefore$ The required line meets the plane $z = 0$ in $(-1, 4, 0)$ ...(ii)

$\therefore$ From (i) and (ii) the required equations are

$$\frac{x+1}{-5} = \frac{y-4}{7} = \frac{z}{1}$$ **Ans.**

Ex. 4 (a). Find the equations of the line through the point $(-2, 3, 4)$ and parallel to the planes $2x + 3y + 4z = 5$ and $3x + 4y + 5z = 6$.

Sol: Let l, m, n be the d.c.'s of the line. Then as in Ex. 1 (b) Page 59 we have $$2l + 3m + 4n = 0,\ 3l + 4m + 5n = 0$$

Solving these we get

$$\frac{l}{3.5 - 4.4} = \frac{m}{4.3 - 2.5} = \frac{n}{2.4 - 3.3} \quad \text{or} \quad \frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$$

$\therefore$ The required equations of the line through $(-2, 3, 4)$ are

$$\frac{x-(-2)}{-1} = \frac{y-3}{2} = \frac{z-4}{-1} \quad \text{or} \quad \frac{x+2}{1} = \frac{y-3}{-2} = \frac{z-4}{1}$$ **Ans.**

Ex. 4 (b). Find the equations to the line through the point (1, 2, 3) parallel to the line $x - y + 2z = 5,\ 3x + y + z = 6$.

Sol. Do as Ex. 4 (a) above. **Ans.** $\dfrac{x-1}{-3} = \dfrac{y-2}{5} = \dfrac{z-3}{4}$

Ex. 4 (c). Find the equations of a straight line through the point $(3, 1, -6)$ and parallel to each of the planes.

$$x + y + 2z - 4 = 0 \quad \text{and} \quad 2x - 3y + z + 5 = 0.$$

Sol. Do as Ex. 4 (a) above. **Ans.** $\dfrac{x-3}{7} = \dfrac{y-1}{3} = \dfrac{z+6}{-5}$.

Ex. 5. Find the equations of the line through (α, β, γ) and parallel to the line $a_1x + b_1y + c_1z + d_1 = 0,\ a_2x + b_2y + c_2z + d_2 = 0$. *(Kanpur 92)*

Sol. Let l, m, n be the direction cosines of the required line. Then as this line is parallel to the line given by the planes

$$a_1x + b_1y + c_1z + d_1 = 0 \quad \text{and} \quad a_2x + b_2y + c_2z + d_2 = 0$$

$$\therefore\ a_1l + b_1m + c_1n = 0,\ a_2l + b_2m + c_2n = 0$$

Solving these we get $\dfrac{l}{b_1c_2 - b_2c_1} = \dfrac{m}{c_1a_2 - c_2a_1} = \dfrac{n}{a_1b_2 - a_2b_1}$

$\therefore$ The required equations are

$$\frac{x-\alpha}{b_1c_2 - b_2c_1} = \frac{y-\beta}{c_1a_2 - c_2a_1} = \frac{z-\gamma}{a_1b_2 - a_2b_1}$$ **Ans.**

****Ex. 6. Prove that the lines $x = ay + b,\ z = cy + d$ and $x = a'y + b',\ z = c'y + d'$ are perpendicular if $aa' + cc' = -1$.** *(Meerut 96 P; Purvanchal 94)*

Sol. As in Ex. 1 (b) Page 59 Chapter IV, if l_1, m_1, n_1 be the d.c.'s of the first line, then $l_1 . 1 + m_1(-a) + n_1 . 0 = 0$; $l_1 . 0 + m_1c + n_1(-1) = 0$.

Sovling these we get $\dfrac{l_1}{a} = \dfrac{m_1}{1} = \dfrac{n_1}{c}$ ...(i)

Similarly if l_2, m_2, n_2 be the d.c.'s of the second line then we can get

$$\frac{l_2}{a'}=\frac{m_2}{1}=\frac{n_2}{c'} \quad \text{...(ii)}$$

If the given lines are perpendicular, then we have

$$l_1 l_2+m_1 m_2+n_1 n_2=0$$

or $\quad a\,a'+1\,.\,1+c\,c'=0$, from (i) and (ii)

or $\quad aa'+cc'=-1.$ Hence proved.

Ex. 7 (a). Prove thatt the lines $x+y-z=5,\ 9x-5y+z=4$ and $6x-8y+4z=3,\ x+8y-6z+7=0$ are parallel.

(*Avadh* 95, 90; *Bundelkhand* 94)

Sol. If l_1, m_1, n_1 be the d.c.'s of the first line, then we have

$$l_1+m_1-n_1=0 \quad \text{and} \quad 9l_1-5m_1+n_1=0$$

Solving these, we get $\dfrac{l_1}{1.1-1.5}=\dfrac{m_1}{-1.9-1.1}=\dfrac{n_1}{-1.5-1.9}$

or $\quad \dfrac{l_1}{-4}=\dfrac{m_1}{-10}=\dfrac{n_1}{-14}$ or $\dfrac{l_1}{2}=\dfrac{m_1}{5}=\dfrac{n_1}{7}$ ...(i)

Similarly if l_2, m_2, n_2 be the d.c.'s of the second line, then we have

$$6l_2-8m_2+4n_2=0;\ l_2+8m_2-6n_2=0$$

Solving these, we get $\dfrac{l_2}{8.6-4.8}=\dfrac{m_2}{4.1+6.6}=\dfrac{n_2}{6.8-8.1}$

or $\quad \dfrac{l_2}{16}=\dfrac{m_2}{40}=\dfrac{n_2}{56}$ or $\dfrac{l_2}{2}=\dfrac{m_2}{5}=\dfrac{n_2}{7}$ ...(ii)

From (i) and (ii) we find that $\dfrac{l_1}{l_2}=\dfrac{m_1}{m_2}=\dfrac{n_1}{n_2}$

Hence the given lines are parallel.

Ex. 7 (b). Show that the following lines are parallel :

$$\mathbf{2x+3y-4z+2=0=3x-4y+z+1}$$

$$\mathbf{5x-y-3z+12=0=x-7y+5z-6}$$

(*Kanpur* 94)

Sol. Do as Ex. 7 (a) above.

Ex. 8 Find the equations of the line through the origin parallel to the line $x+y+z+2=0=4x+3y+2z+1$.

Sol. Let l, m, n be the direction cosines of the given line. Then as this line lies on both the given planes which constitute the given line, so it is perpendicular to the normals of these planes and thus we have

$$l\,.\,1+m\,.\,1+n\,.\,1=0 \quad \text{and} \quad 4l+3m+2n=0$$

Solving these we get $\quad \dfrac{l}{1}=\dfrac{m}{-2}=\dfrac{n}{1}$

∴ The required equations of the line through (0, 0, 1) and parallel to the line whose d.c.'s are proportional to 1, – 2, 1 are

$$\frac{x-0}{1}=\frac{y-0}{2}=\frac{z-0}{1} \quad \text{or} \quad x=-\tfrac{1}{2}y=z. \qquad \textbf{Ans.}$$

Ex. 9 (a). Find the angle between the lines $3x + 2y + z = 0 = x + y - 2z$ and $2x - y - z = 0 = 7x + 10y - 8z$. (*Agra* 92)

Sol. Let l_1, m_1, n_1 be the d.c.'s of the first line, then we have

$$3l_1 + 2m_1 + n_1 = 0,\ l_1 + m_1 - 2n_1 = 0$$

Solving these we get $\dfrac{l_1}{-4-1} = \dfrac{m_1}{1+6} = \dfrac{n_1}{3-2}$

or $$\frac{l_1}{-5} = \frac{m_1}{7} = \frac{n_1}{1} = \frac{\sqrt{(l_1^2 + m_1^2 + n_1^2)}}{\sqrt{(5^2 + 7^2 + 1^2)}} = \frac{1}{\sqrt{(75)}} = \frac{1}{5\sqrt{3}}$$ **(Note)**

$\therefore\ l_1 = -5/5\sqrt{3},\ m_1 = 7/5\sqrt{3},\ n_1 = 1/5\sqrt{3}$

Similarly if l_2, m_2, n_2 be the d.c.'s of the second line, then we have

$$2l_2 - m_2 - n_2 = 0,\ 7l_2 + 10m_2 - 8n_2 = 0$$

Solving these, $\dfrac{l_2}{8+10} = \dfrac{m_2}{-7+16} = \dfrac{n_2}{20+7}$

or $$\frac{l_2}{2} = \frac{m_2}{1} = \frac{n_2}{3} = \frac{\sqrt{(l_2^2 + m_2^2 + n_2^2)}}{\sqrt{(2^2 + 1^2 + 3^2)}} = \frac{1}{\sqrt{(14)}}$$

$\therefore\ l_2 = 2/\sqrt{(14)},\ m_2 = 1/\sqrt{(14)},\ n_2 = 3/\sqrt{(14)}$

$\therefore$ If θ be the required angle then

$$\cos\theta = \text{'}l_1l_2 + m_1m_2 + n_1n_2\text{'}$$

$$= \frac{-5}{5\sqrt{3}} \cdot \frac{2}{\sqrt{(14)}} + \frac{7}{5\sqrt{3}} \cdot \frac{1}{\sqrt{(14)}} + \frac{1}{5\sqrt{3}} \cdot \frac{3}{\sqrt{(14)}} = 0$$

or $\theta = \pi/2$ **Ans.**

Ex. 9 (b). Find the angle between the lines $x - 2y + z = x + y - z$ and $x + 2y + z = 0 = 8x + 12y + 5z$.

Sol. Do as Ex. 9 (a) above. **Ans.** $\cos^{-1}[-8/\sqrt{(406)}]$

Ex. 9 (c). Find the angle between the lines whose equations are $x + 2y - 2z = 0,\ x - 2y + z = 7$ and $x - 1 = -\frac{1}{2}(y + 2) = \frac{1}{3}z$ (*Bundelkhand* 91)

Sol. Let l_1, m_1, n_1 be the d.c.'s of the first line, then we have

$$l_1 + 2m_1 - 2n_1 = 0,\ l_1 - 2m_1 + n_1 = 0.$$

Solving these, $\dfrac{l_1}{2-4} = \dfrac{m_1}{-2-1} = \dfrac{n_1}{-2-2}$

or $$\frac{l_1}{2} = \frac{m_1}{3} = \frac{n_1}{4} = \frac{\sqrt{(l_1^2 + m_1^2 + n_1^2)}}{\sqrt{(2^2 + 3^2 + 4^2)}} = \frac{1}{\sqrt{(29)}}$$

$\therefore\ l_1 = 2/\sqrt{(29)},\ m_1 = 3/\sqrt{(29)},\ n_1 = 4/\sqrt{(29)}$

Now from the equation of second line it is obvious that its d.r.'s are $1, -2, 3$ and so its d.c.'s are $1/\sqrt{(14)}, -2/\sqrt{(14)}, 3/\sqrt{(14)}$

$\therefore$ If θ be the required angle, then

$$\cos\theta = \text{'}l_1l_2 + m_1m_2 + n_1n_2\text{'}$$

$$= \frac{2}{\sqrt{(29)}} \cdot \frac{1}{\sqrt{(14)}} + \frac{3}{\sqrt{(29)}} \cdot \frac{-2}{\sqrt{(14)}} + \frac{4}{\sqrt{(29)}} \cdot \frac{3}{\sqrt{(14)}}$$

$$= 8/[\sqrt{(29)}\sqrt{(14)}] = 8/\sqrt{(406)}$$

or $$\theta = \cos^{-1}[8/\sqrt{(406)}]$$ **Ans.**

Exercises on § 4.04

Ex. 1. Prove that the line of intersection of the planes $4x + 4y - 5z = 12$, $8x + 12y - 13z = 32$ can be written as $\frac{1}{2}(x-1) = \frac{1}{3}(y-2) = \frac{1}{4}z$.

Ex. 2. Find the equations of the line $3x - 4y + 2z + 5 = 0$, $2x + 3y - 5z - 8 = 0$ in the symmetric form. **Ans.** $\frac{x-1}{14} = \frac{y-2}{19} = \frac{z}{17}$

Ex. 3. Prove that the lines $2x + 3y - 4z = 0$, $3x - 4y + z = 7$ and $5x - y - 3z + 12 = 0$, $x - 2y + 5z - 6 = 0$ are parallel.

Ex. 4. Find the angle between the lines $3x + 2y + z - 5 = 0 = x + y - 2z - 3$; $2x - y - z - 16 = 0 = 7z + 10y - 8z - 15$. **Ans.** $90°$

Ex. 5. Find the angle between the lines $x - 2y + z = 0$, $x + y - z = 3$ and $x + 2y + z = 5$, $8x + 12y + 5z = 0$. (*Garhwal* 96, 92)

Ans. $\cos^{-1}[8/\sqrt{(29)}\sqrt{(14)}]$

Ex. 6. Find the equations of the line through (2, 3, 5) and parallel to the line given by $x + 2y - 2z = 7$, $6x + 8y - 9z = 1$.

Ans. $\frac{1}{2}(x-2) = \frac{1}{3}(y-3) = \frac{1}{4}(z-5)$

Ex. 7. Show that the condition for the lines $x = az + b$, $y = cz + d$, and $x = a_1 z + b_1$, $y = c_1 z + d_1$ to be perpendicular is $aa_1 + cc_1 = -1$.

Ex. 8. Wirte the equations of the line parallel to x-axis in symmetrical form.

Ans. $\frac{x-a}{1} = \frac{y-b}{0} = \frac{z-c}{0}$

****§ 4.05. The plane and the straight line.**

If a given line intersects a given plane to find the co-ordinates of the point of intersection and to deduce the conditions that (i) the line may be perpendicular to the plane, (ii) the line may be parallel to the plane and (iii) the line may be lying on the plane.

Let the equations of the given line and the given plane be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)} \qquad ...(i)$$

and $$ax + by + cz + d = 0 \qquad ...(ii)$$

respectively.

Any point on the line (i) is $(\alpha + lr, \beta + mr, \gamma + nr)$.

If this point lies on (ii), then we have

$$a(\alpha + lr) + b(\beta + mr) + c(\gamma + nr) + d = 0$$

or $$r(al + bm + cn) = -(a\alpha + b\beta + c\gamma + d)$$

or $$r = -(a\alpha + b\beta + c\gamma + d)/(al + bm + cn) \qquad ...(iii)$$

Therefore the point of intersection of (i) and (ii) is

$$(\alpha + lr,\ \beta + mr,\ \gamma + nr),$$

where r is given by (iii).

(i) Condition of perpendicularity.

The direction cosines of the normal to the plane (ii) are proportional a, b, c.

If the line (i) is perpendicular to the plane (ii), then it must be parallel to the normal to the plane and hence we have

$$\frac{\mathbf{l}}{\mathbf{a}} = \frac{\mathbf{m}}{\mathbf{b}} = \frac{\mathbf{n}}{\mathbf{c}},$$

which is the required condition.

(ii) Condition of parallelism.

If the line (i) is parallel to the plane (ii), then it must be perpendicular to the plane and hence we have

$$al + bm + cn = 0$$

Also (α, β, γ) should not be lie on the plane *i.e.* $a\alpha + b\beta + c\gamma + d \neq 0$ as otherwise the line will not be parallel to the plane.

$\therefore$ the conditions for parallelism of the line (i) and the plane (ii) are

$$\mathbf{al + bm + cn = 0,\ a\alpha + b\beta + c\gamma + d \neq 0.}$$

Aliter. If the line (i) is parallel to the plane (ii), then their point of intersection is at infinity, consequently the value of r given by (iii) should be infinite and the conditions for the same are

$$al + bm + cn = 0 \quad \text{and} \quad a\alpha + b\beta + c\gamma + d \neq 0$$

****(iii). Conditions for the line to lie on the plane.**

If the line (i) lies on the plane (ii) then for all values of r, the point $(\alpha + lr,\ \beta + mr,\ \gamma + nr)$ lies on the plane (ii) and so we have

$$a(\alpha + lr) + b(\beta + mr) + c(\gamma + nr) + d = 0$$

or
$$r(al + bm + cn) + (a\alpha + b\beta + c\gamma + d) = 0.$$

If this is true for all values of r, then we must have.

$$\mathbf{al + bm + cn = 0 \quad and \quad a\alpha + b\beta + c\gamma + d = 0,}$$

which are the required conditons.

Aliter. If the line (i) lies on the plane (ii), then it is perpendicular to the normal to the plane and therefore, we have

$$\mathbf{al + bm + cn = 0.}$$

Also as the line (i) lies on the plane (ii), so the point (α, β, γ) through which the line (i) passes, lies on the plane (ii) and so we hvae

$$\mathbf{a\alpha + b\beta + c\gamma + d = 0.}$$

Solved Exmples on § 4.05.

Ex. 1 (a) Find the condition that the line $\mathbf{x/l = y/m = z/n}$ may be (i) perpendicular, (ii) parallel to the plane $\mathbf{ax + by + cz = 0}$.

Sol. (i) If the given line is perpendicular to the given plane, then the given line must be parallel to the normal to the given plane and hence we have

$$l/a = m/b = n/c. \qquad \textbf{Ans.}$$

(ii) If the given line is parallel to the given plane, then the given line must be perpendicular to the normal to the given plane and hence we have

$$la + mb + nc = 0.$$ **Ans.**

Ex. 1 (b). The condition that the line $(x - \alpha)/l = (y - \beta)/m = (z - \gamma)/n$ is perpendicular to the plane $ax + by + cz + d = 0$ is

(i) $a/l = b/m = c/n$; (ii) $a/m = b/n = c/l$;

(iii) $l/a = m/b = n/c$; (iv) $al = bm = cn$

Sol. Do as Ex. 1 (a) above. **Ans.** (i), (iii)

Ex. 1 (c). Prove that the line $\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}$ is parallel to the plane $4x + 4y - 5z = 0$ (*Rohilkhand* 96)

Sol. If the given line is parallel to the given plane, then the given line must be perpendicular to the normal to the given plane and here the direction ratios of the given line are 2, 3, 4 whereas those of the normal to the given plane are 4, 4, -5.

Also "$a_1a_2 + b_1b_2 + c_1c_2$" $= 2(4) + 3(4) - 4(-5) = 8 + 12 - 20 = 0$

Hence the given line is perpendicular to the normal to the given plane and thus parallel to the given plane. Hence proved.

Ex. 2 (a). Find the direction cosines of the line whose equations are $x + y = 3$, $x + y + z = 0$ and show that it makes an angle of 30° with the plane $y - z + 2 = 0$.

Sol. Let l, m, n be the d.c.'s of the line. Then as this line is perpendicular to the normal to the planes $x + y - 3 = 0$ and $x + y + z = 0$, so we have

$$1.l + 1.m + 0.n = 0,\ 1.l + 1.m + 1.n = 0$$

Solving these, we get $\frac{l}{1} = \frac{m}{-1} = \frac{n}{0} = \frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{[1^2 + (-1)^2 + 0^2]}} = \frac{1}{\sqrt{2}}$

$\therefore\ l = 1/\sqrt{2},\ m = -1/\sqrt{2}$ and $n = 0$.

Also the direction ratios of the normal to the plane $y - z + 2 = 0$ are 0, 1 and -1.

The d.c.'s of the normal to the plane are $0, 1/\sqrt{2}, -1/\sqrt{2}$.

Hence if θ be the required angle, then $\frac{1}{2}\pi - \theta$ will be the angle between the line and the normal to the plane.

$$\therefore \cos\left(\tfrac{1}{2}\pi - \theta\right) = \text{“}l_1l_2 + m_1m_2 + n_1n_2\text{”}$$

or $$\sin\theta = 0 \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right) \cdot 0 = -\frac{1}{2}$$

or $\theta = 150°$ or $30°$, if the acute angle is taken. Hence proved.

Ex. 2 (b). Find the angle between the line given by $y + z - 5 = 0 = x + y + z$ and the plane $x - y + z = 0$ (*Kanpur* 93)

Sol. Let l, m, n be the d.c.'s of the given line, then as this is perpendicular to the normal to the planes $y + z - 5 = 0$ and $x + y + z = 0$, so we have

$$0 \,.\, l + 1 \,.\, m + 1 \,.\, n = 0 \quad \text{and} \quad 1 \,.\, l + 1 \,.\, m + 1 \,.\, n = 0$$

Solving these, we get $\dfrac{l}{0} = \dfrac{m}{1} = \dfrac{n}{-1} = \dfrac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{[0^2 + 1^2 + (-1)^2]}} = \dfrac{1}{\sqrt{2}}$

$\therefore\ l = 0,\ m = 1/\sqrt{2},\ n = -1/\sqrt{2}$

Also the direction ratios of the normal to the plane $x - y + z = 0$ are $1, -1$ and 1

$\therefore$ The d.c.'s of the normal to this plane are $\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$.

Hence if θ be the required angle, then $\frac{1}{2}\pi - \theta$ will be the angle between the line and normal to the plane.

$\therefore\ \cos\left(\frac{1}{2}\pi - \theta\right) = $ "$l_1 l_2 + m_1 m_2 + n_1 n_2$"

or $\quad \sin\theta = 0 \,.\, \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{2}} \cdot \left(-\dfrac{1}{\sqrt{2}}\right) + \left(-\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{1}{\sqrt{3}}\right) = -\dfrac{2}{\sqrt{6}}$

or $\quad \theta = \sin^{-1}(-2/\sqrt{6})$ **Ans.**

****Ex. 3 (a). Find the equation of the plane through the point (2, 1, 1), (1, – 2, 3) and parallel to the x-aixs.**

Sol. The equation of any plane through (2, 2, 1) is

$$A(x - 2) + B(y - 2) + C(z - 1) = 0 \qquad \text{...(i)}$$

If it passes through $(1, -2, 3)$, then

$$A(1 - 2) + B(-2 - 2) + C(3 - 1) = 0$$

or $\quad -A - 4B + 2C = 0 \quad$ or $\quad A + 4B - 2C = 0 \qquad$ ...(ii)

The equation of x-axis is $\quad \dfrac{x}{1} = \dfrac{y}{0} = \dfrac{z}{0} \qquad$ ...(iii)

If the line given by (iii) is parallel to the plane given by (i), then this line is perpendicular to the normal to the plane given by (i) and so we have

$$A.1 + B.0 + C.0 = 0 \quad \text{or} \quad A = 0 \qquad \text{...(iv)}$$

Sovling (ii) and (iv) we have $4B - 2C = 0$ or $C = 2B$.

Substituting these values in (i) we get the required equation as

$B(y - 2) + 2B(z - 1) = 0 \quad$ or $\quad y + 2z - 4 = 0$ **Ans.**

Ex. 3 (b). Find the equation to the plane through the points (2, – 1, 0), (3, – 4, 5) parallel to the line 2x = 3y = 4z. *(Bundelkhand* 91)

Sol. The equation of any plane through (2, – 1, 0) is

$$A(x - 2) + B(y + 1) + C(z - 0) = 0 \qquad \text{...(i)}$$

If it passes through $(3, -4, 5)$, then

$A(3 - 2) + B(-4 + 1) + C(5 - 0) = 0 \quad$ or $\quad A - 3B + 5C = 0 \qquad$ ...(ii)

Now the given line is $x/6 = y/4 = z/3$. If it is parallel to the plane (i), then this line is perpendicular to the normal to the plane given by (i) and so we have $\quad A.6 + B.4 + C.3 = 0 \quad$ or $\quad 6A + 4B + 3C = 0 \qquad$ ...(iii)

Eliminating A, B, C from (i), (ii) and (iii), we get

$$\begin{vmatrix} x-2 & y+1 & z \\ 1 & -3 & 5 \\ 6 & 4 & 3 \end{vmatrix} = 0$$

or $$(x-2)(-29)-(y+1)(-27)+z(22)=0$$

or $$29x-27y-22z-85=0$$ **Ans.**

***Ex. 3 (c). Find the equattion of the plane through the points (1, 0, – 1), (3, 2, 2) and parallel to the line $(x-1)=\frac{1}{2}(1-y)=\frac{1}{3}(z-2)$.**

Sol. Do as Ex. 3 (b) above. **Ans.** $4x-y-2z=6$

***Ex. 4 (a). Find the euquation of the plane through the point (– 1, 3, 1) and perpendicular to the line $2x+3y+4z=5$, $3x+4y+5z=6$.**

Sol. Let l, m, n be the d.c.'s of the line of intersection of the given planes, then as in Ex. 2 (a) on Page 66 Ch. IV we have

$$2l+3m+4n=0 \quad \text{and} \quad 3l+4m+5n=0$$

Solving these we get $$\frac{l}{1}=\frac{m}{-2}=\frac{n}{1} \qquad ...(i)$$

The equation of any plane through (– 1, 3, 1) is

$$A(x+1)+B(y-3)+C(z-1)=0 \qquad ...(ii)$$

If this plane is perpendicular to the line whose direction ratios are given by (i), then its normal is parallel to the line and as such we have

$$A/l=B/m=C/n= \qquad ...(iii)$$

$\therefore$ From (i) and (iii) we have $\frac{A}{1}=\frac{B}{-2}=\frac{C}{1}=k$ (say)

i.e. $$A=k,\ B=-2k,\ C=k.$$

Substituting these values in (iii), the required equation is

$$k(x+1)-2k(y-3)+k(z-1)=0 \quad \text{or} \quad x-2y+z+6=0.$$ **Ans.**

Ex. 4 (b). Show that the equation of the plane, passing through the point (α, β, γ) and perpendicular to the line x/l = y/m = z/n is

$$\mathbf{l(x-\alpha)+m(y-\beta)+n(z-\gamma)=0}$$ (*Avadh* 92, *Purvanchal* 91)

Sol. The equation of any plane through (α, β, γ) is

$$A(x-\alpha)+B(y-\beta)+C(z-\gamma)=0 \qquad ...(i)$$

The given line is $x/l=y/m=z/n$ ...(ii)

If the plane (i) is perpendicular to the line (ii), then its normal is parallel to the line (ii) and so we have $A/l=B/m=C/n=k$ (say).

Substituting these proportionate values of A, B, C in (i) we get the required equation as $l(x-\alpha)+m(y-\beta)+n(z-\gamma)=0$. Hence proved.

Ex. 5. Find the equation to the plane through (2, – 3, 4) normal to the line joining (3, 4, – 2) and (2, – 1, 6).

Sol. The equation of any plane through (2, – 3, 4) is

$$A(x-2)+B(y+3)+C(z-4)=0 \qquad ...(i)$$

Also the d.r.'s of the line joining $(3, 4, -2)$ and $(2, -1, 6)$ are $3-2, 4-(-1), (-2)-6$ *i.e.* $1, 5, -8$.

If the plane (i) is at right angles to the line joining these points *i.e.* the lines whose d.c.'s are $1, 5, -8$, then the normal to the plane (i) whose d.r.'s are A, B, C is parallel to this line and consequently we have $A/1 = B/5 = C/(-8)$

Substituting these proportionate values of A, B and C in (i) we have the required equation as

$$1(x-2)+5(y+3)-8(z-4)=0 \quad \text{or} \quad x+5y-8z+45=0 \qquad \textbf{Ans.}$$

Ex. 6. Prove the join of (2, 3, 4), (3, 4, 5) is normal to the plane through (– 2, – 3, 6), (4, 0, – 3), (0, – 1, 2) the axes being rectangular.

Sol. The equation of any plane through $(-2, -3, 6)$ is

$$A(x+2)+B(y+3)+C(z-6)=0 \qquad \text{...(i)}$$

If this plane passes through $(4, 0, -3)$ and $(0, -1, 2)$ then we have

$$A(4+2)+B(0+3)+C(-3-6)=0$$

and
$$A(0+2)+B(-1+3)+C(2-6)=0$$

or
$$2A+B-3C=0 \quad \text{and} \quad A+B-2C=0$$

Solving these we have (by simply subtracting) $A-C=0$

or
$$A=C. \quad \therefore \quad B=C$$

$\therefore$ From (i) we have the plane through three given points $(-2, -3, 6), (4, 0, -3)$ and $(0, -1, 2)$ as

$$C(x+2)+C(y+3)+C(z-6)=0 \quad \text{or} \quad x+y+z-1=0 \qquad \text{...(ii)}$$

Also the d.r.'s of the line joining $(2, 3, 4), (3, 4, 5)$ are

$$3-2, 4-3, 5-4 \quad i.e. \quad 1, 1, 1.$$

And the d.r.'s of the line normal to the plane (ii) are $1, 1, 1$ which are the same as those of the line joining the points $(2, 3, 4)$ and $(3, 4, 5)$ *i.e.* the join of the points $(2, 3, 4), (3, 4, 5)$ is perpendicular to the plane (ii).

Exercises on § 4.05

Ex. 1. Find the equation of the plane through $(1, 2, 3)$ perpendicular to the line of intersection of the planes $x+2y+3z=2$, $3x+2y+4z=0$.

Ans. $2x+5y-4z=0$

Ex. 2. Find the equation of the plane through $(3, 1, -1)$ perpendicular to the line of intersection of the planes $3x+4y+7z+4=0$ and $x-y+2z+3=0$.

Ans. $15x+y-7z=53$.

Ex. 3. Find the equation of the plane through the point $(1, 2, 3)$ and perpendicular to the line $x+2y+z=0=2x-y+z-1$.

Ans. $3x+y-5z+10=0$.

Ex. 4. Prove that the two lines in which the plane $3x-7y-5z=1$ and $5x-13y+3z+2=0$ cut the plane $8x-11y+2z=0$ include a right angle.

§ 4.06 To find the equation of a plane through a given line whose equations are given in (i) general form, (ii) symmetric form.

(i). Let the equations of the line in the general form be

$$P_1 \equiv a_1x + b_1y + c_1z + d_1 = 0 \quad \text{and} \quad P_2 \equiv a_2x + b_2y + c_2z + d_2 = 0 \quad ...(i)$$

then $P_1 + \lambda P_2 \equiv (a_1x + b_1y + c_1z + d_1) + \lambda (a_2x + b_2y + c_2z + d_2) = 0$ ...(ii)

being a first degree equation in x, y and z represents a plane, also it is satisfied by all those points which satisfy (i), hence $P_1 + \lambda P_2 = 0$, given by (ii) represents the plane through the line $P_1 = 0$ and $P_2 = 0$ given by (i).

(ii). Let the equations of the line in symmetric form be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad ...(iii)$$

As the required plane passes through this line, so it passes through (α, β, γ) which is a given point on this line.

The equation of any plane through (α, β, γ) is

$$A(x-\alpha) + B(y-\beta) + C(z-\gamma) = 0 \quad ...(iv)$$

Since the plane passes through the line (ii) whose d.c.'s are l, m, n therefore the normal to this plane whose d.r.'s are A, B, C is perpendicular the normal to the line (iii), consequently we have

$$A.l + B.m + C.n = 0$$

Hence the required equation of the plane through the line (iii) is **$A(x-\alpha) + B(y-\beta) + C(z-\gamma) = 0$, where $Al + Bm + Cn = 0$.** (*Purvanchal* 96)

§ 4.07. To find the equation of a plane through a given line and parallel to another.

In § 4.06 above we have proved that the equation of the plane through the line $\dfrac{x-\alpha_1}{l_1} = \dfrac{y-\beta_1}{m_1} = \dfrac{z-\gamma_1}{n_1}$ is

$$A(x-\alpha_1) + B(y-\beta_1) + C(z-\gamma_1) = 0 \quad ...(i)$$

where $$Al_1 + Bm_1 + Cn_1 = 0 \quad ...(ii)$$

Again if the plane given by (i) is parallel to another line

$$\frac{x-\alpha_2}{l_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2} \quad ...(iii)$$

then the normal to the plane given by (i) must be at right angles to the line given by (iii) whose d.c.'s are l_2, m_2, n_2.

$$\therefore \quad Al_2 + Bm_2 + Cn_2 = 0 \quad ...(iv)$$

Eliminating A, B, C from (i), (ii) and (iv) we have the required equattion as

$$\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or $$\Sigma [(x-\alpha_1)(m_1n_2 - m_2n_1)] = 0. \quad \textbf{(Remember)}$$

Solved Exmples on § 4.06 and § 4.07

Ex. 1 (a). Find the equation of the plane through the line

$$\mathbf{x + y - z = 0 = x - 2y + 3z - 5.}$$

Sol. The equation of the plane through the given line is

$$(x+y-z)+\lambda(x-2y+3z-5)=0, \qquad \textbf{Ans.}$$

where λ is some constant which can be evaluated if some other condition be given.

Ex. 1 (b). Find the equation of the plane through the line $ax+by+cz+d=0$, $a'x+b'y+c'z+d'=0$ and parallel to the line

$$x/l=y/m=z/n.$$

Sol. The equation of the plane through the given line is

$$(ax+by+cz+d)+\lambda(a'x+b'y+c'z+d')=0$$

...See § 4.06 Page 69 Chapter IV.

or $$(a+\lambda a')x+(b+\lambda b')y+(c+\lambda c')z+(d+\lambda d')=0 \qquad \text{...(i)}$$

If this plane is parallel to the line $x/l=y/m=z/n$, then the normal to (i) must be at right angles to the line $x/l=y/m=z/n$, whose d.c.'s are l, m, n.

$$\therefore (a+\lambda a')l+(b+\lambda b')m+(c+\lambda c')n=0$$

or $$\lambda=-(al+bm+cn)/(a'l+b'm+c'n) \qquad \text{...(ii)}$$

$\therefore$ The required plane from (i) is

$$(ax+by+cz+d)+\lambda.(a'x+b'y+c'z+d')=0,$$

where λ is given by (ii).

***Ex. 1 (c). Find the equation of the plane through the line of intersection of the planes $ax+by+cz+d=0$, $a'x+b'y+c'z+d'=0$ and parallel to x-axis.** (*Kumaun* 90)

Sol. As in Ex. 1 (b) above we can find that the equation of the plane through the line of intersection of the given planes is

$$(a+\lambda a')x+(b+\lambda b')y+(c+\lambda c')z+(d+\lambda d')=0 \qquad \text{...(i)}$$

If this plane is parallel to x-axis whose d.c.'s are 1, 0, 0 then the normal to the plane (i) must be perpendicular to x-axis, so we have

$$(a+\lambda a').1+(b+\lambda b').0+(c+\lambda c').0=0$$

or $$a+\lambda a'=0 \text{ or } \lambda=-a/a'$$

Substituting this value of λ in (i) and simplifying, we get the required equation as $$(ba'-ab')y+(ca'-c'a)z+(da'-d'a)=0 \qquad \textbf{Ans.}$$

Ex. 1 (d). Find the equation of the plane through the line of intersection of the planes $2x+y-z=3$ and $5x-3y+4z+9=0$ and parallel to the line $(x-1)/2=(y-3)/4=(z-5)/5$.

Sol. Equation of any plane through the intersection of the given planes is

$$(2x+y-z-3)+\lambda(5x-3y+4z+9)=0$$

or $$(2+5\lambda)x+(1-3\lambda)y+(4\lambda-1)z+(9\lambda-3)=0 \qquad \text{...(i)}$$

If this plane is parallel to the given line, then the normal to the plane (i) must be perpendicular to the given line whose direction cosines are 2, 4, 5.

$$\therefore \quad 2(2+5\lambda)+4(1-3\lambda)+5(4\lambda-1)=0 \quad \text{or} \quad \lambda=-1/6$$

Substituting this value of λ in (i) and simplifying we get the required equation of the plane as $$7x+9y-10z=27. \qquad \textbf{Ans.}$$

***Ex. 2. Prove that the line $\frac{1}{2}(x-3)=\frac{1}{3}(y-4)=\frac{1}{4}(z-5)$ lies on the plane $4x+4y-5z-3=0$.**

Sol. From the equations of the line it is evident that the given line passes through (3, 4, 5) and its direction ratios are 2, 3, 4.

Now we find that the point (3, 4, 5) lies on the given plane

$$4x+4y-5z-3=0 \qquad ...(i)$$

as the co-ordinates of this point satisfy (i).

Also we find from (i) that the d. ratios of the normal to the given plane (i) are 4, 4, – 5.

$\because 2.4+3.4+4.(-5)=0$, so the normal to the plane (i) is perpendicular to the given line.

Hence we find that (a) the given line passes through (3, 4, 5) and this point lies on the plane (i) and (b) the given line is perpendicular to the normal to the given plane (i).

$\therefore$ We conclude that the given line lies on the given plane. Hence proved.

Ex. 3 (a). Find the equation of the plane through the line

$\frac{1}{2}(x-2)=\frac{1}{3}(y-3)=\frac{1}{5}(z-4)$ and parallel to x-axis.

Sol. The given line is $\dfrac{x-2}{2}=\dfrac{y-3}{3}=\dfrac{z-4}{5}$

So (2, 3, 4) is a point on this line and its d.r.'s are 2, 3, 5.

$\therefore$ The equation of any plane thorugh this line is

(See § 4.06 Page 69 Chapter IV)

$$A(x-2)+B(y-3)+C(z-4)=0, \qquad ...(i)$$

where $$A.2+B.3+C.5=0. \qquad ...(ii)$$

If the plane (ii) is parallel to x-axis whose d.c.'s are 1, 0, 0, then the normal to (ii) is at right angles to x-axis.

i.e. $$A.1+B.0+C.0=0 \qquad ...(iii)$$

Eliminating A, B, C from (i), (ii) and (iii) we get the required equation as

$$\begin{vmatrix} x-2 & y-3 & z-4 \\ 2 & 3 & 5 \\ 1 & 0 & 0 \end{vmatrix}=0$$

which on simplifying reduces to $5y-3z-3=0$. **Ans.**

Ex. 3 (b). Find the equation of the plane through the line

$\frac{1}{2}(x-1)=\frac{1}{4}(y+6)=\frac{1}{2}(z+1)$ and parallel to

the line $\frac{1}{2}(x-2)=-\frac{1}{3}(y-1)=\frac{1}{5}(z+4)$.

Sol. The equation of any plane through the line

$\dfrac{x-1}{3}=\dfrac{y+6}{4}=\dfrac{z+1}{2}$ is $A(x-1)+B(y+6)+C(z+1)=0, \qquad ...(i)$

where $$A.3 + B.4 + C.2 = 0 \qquad ...(ii)$$

Also if the plane (i) is parallel to the line $\frac{x-2}{2} = \frac{y-1}{-3} = \frac{z-4}{5}$, then the normal to (i) is at right angles to this line whose d.r.'s are 2, – 3, 5

i.e. $$A \,.\, 2 + B \,.\, (-3) + C \,.\, 5 = 0. \qquad ...(iii)$$

Eliminating A, B, C from (i), (ii) and (iii) we have required equation as

$$\begin{vmatrix} x-1 & y+6 & z+1 \\ 3 & 4 & 2 \\ 2 & -3 & 5 \end{vmatrix} = 0,$$

which on simplifying reduces to $26x - 11y - 17z - 109 = 0$. **Ans.**

***Ex. 4. Find the equation of the plane containing the line $\frac{y}{b} + \frac{z}{c} = 1$, $x = 0$ and parallel to the line $\frac{x}{a} - \frac{z}{c} = 1$, $x = 0$.**

Sol. The equation of the line $x = 0$, $(y/b) + (z/c) = 1$ can be written in the symmetric form as

$$\frac{x}{0} = \frac{y - \frac{1}{2}b}{b} = \frac{z - \frac{1}{2}c}{-c} \qquad ...(i) \text{ (Note)}$$

And the equations of the line $(x/a) - (z/c) = 1$, $y = 0$ can be written in the symmetric form as

$$\frac{x - \frac{1}{2}a}{a} = \frac{y}{0} = \frac{z + \frac{1}{2}c}{c} \qquad ...(ii) \text{ (Note)}$$

The equation of any plane through the line (i) is

$$A\,(x) + B\,(y - \tfrac{1}{2}b) + C\,(z - \tfrac{1}{2}c) = 0. \qquad ...(iii)$$

where $$A.0 + B.b + C\,(-c) = 0. \qquad ...(iv)$$

Also as the plane (iii) is parallel to the line (ii), so we have

$$A \,.\, a + B \,.\, 0 + C \,.\, c = 0. \qquad ...(v)$$

Solving (iv) and (v), we have $\frac{A}{bc} = \frac{B}{-ca} = \frac{C}{-ab}$

Substituting these proportionate values of A, B and C in (iii) we have the required equation as

$$bc\,(x) - ca\,(y - \tfrac{1}{2}b) - ab\,(z - \tfrac{1}{2}c) = 0 \text{ or } bcx - cay - baz + abc = 0$$

or $$(x/a) - (y/b) - (z/c) + 1 = 0 \qquad \textbf{Ans.}$$

Ex. 5. Find the equation of the plane which contains the two parallel lines $x - 4 = -\frac{1}{4}(y-3) = \frac{1}{5}(z-2)$ and $x - 3 = -\frac{1}{4}(y+2) = \frac{1}{5}z$.

Sol. The equation of any plane through $\frac{x-4}{1} = \frac{y-3}{-4} = \frac{z-2}{5}$

is $$A\,(x-4) + B\,(y-3) + C\,(z-2) = 0. \qquad ...(i)$$

where $$A.1 + B.(-4) + C.5 = 0 \qquad ...(ii)$$

If the plane (i) contains the parallel line $\frac{x-3}{1}=\frac{y+2}{-4}=\frac{z}{5}$ also then the point (3, – 2, 0) on this line must lie on the plane (i) which gives

$$A(3-4)+B(-2-3)+C(0-2)=0$$

or $$A+5B+2C=0. \quad ...(iii)$$

Solving (ii) and (iii), we have

$$\frac{A}{-8-25}=\frac{B}{5-2}=\frac{C}{5+4} \quad \text{or} \quad \frac{A}{11}=\frac{B}{-1}=\frac{C}{-3} \quad ...(iv)$$

Substituting the proportionate values of A, B, C from (iv) in (i) we have the required equation as $11(x-4)-(y-3)-3(z-2)=0$

or $$11x-y-3z-35=0.$$ **Ans.**

***Ex. 6 (a). Find the equation of the plane passing through the line of intersection of the planes $x+y+z=6$ and $2x+3y+4z+5=0$ and perpendicular to the plane $4x+5y-3z=8$.** *(Garhwal 94)*

Sol. The equation of any plane which passes through the line of intersection of the given planes $x+y+z=6$ and $2x+3y+4z+5=0$ is

$$(x+y+z-6)+\lambda(2x+3y+4z+5)=0 \quad ...(i)$$

or $$(1+2\lambda)x+(1+3\lambda)y+(1+4\lambda)z+(5\lambda-6)=0 \quad ...(ii)$$

If this plane (ii) is perpendicular to the plane $4x+5y-3z=8$, then we have $$4(1+2\lambda)+5(1+3\lambda)-3(1+4\lambda)=0$$

or $$11\lambda+6=0 \quad \text{or} \quad \lambda=-6/11$$

$\therefore$ From (i), the required equation is

$$(x+y+z-6)-(6/11)(2x+3y+4z+5)=0$$

or $$x+7y+13z+96=0$$ **Ans.**

Ex. 6 (b). Find the equation of the plane which is perpendicular to the plane $5x+3y+6z+8=0$ and which contains the line of intersection of the planes $x+2y+3z-4=0$ and $2x+y-z+5=0$.

Sol. Do as Ex. 6 (a) above. **Ans.** $51x+15y-50z+173=0$

***Ex. 6 (c). Find the equation of the plane which contains the line $(x-1)/2=(y+1)/(-1)=(z-3)/4$ and is perpendicular to the plane $x+2y+z=12$.** *(Garhwal 95)*

Sol. The equation of any plane through the given line is

$$A(x-1)+B(y+1)+C(z-3)=0 \quad ...(i)$$

where $$A.2+B.(-1)+C.4=0 \quad ...(ii)$$

Also if the plane (i) is perpendicularr to the plane $x+2y+z=12$, we have $$A.1+B.2+C.2=0 \quad ...(iii)$$

Solving (ii) and (iii), we have

$$\frac{A}{-2-8}=\frac{B}{4-4}=\frac{C}{4+1} \quad \text{or} \quad \frac{A}{-2}=\frac{B}{0}=\frac{C}{1}$$

Substituting the proportionate values of A, B, C in (i), we have the required plane as

$$-2(x-1)+0(y+1)+(z-3)=0 \quad \text{or} \quad 2x-z+1=0$$ **Ans.**

Ex. 7 (a). Find the equation of the plane which passes through the z-axis and is perpendicular to the line

$$\frac{x-1}{\cos\theta}=\frac{y+2}{\sin\theta}=\frac{z-3}{0}$$

(*Kumaun* 91; *Lucknow* 90; *Rohilkhand* 90)

Sol. The equations of z-axis are $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$

$\therefore$ The equation of any plane through z-axis is

$$A(x)+B(y)+C(z)=0 \qquad ...(i)$$

where $\quad A.0+B.0+C.1=0 \quad i.e. \quad C=0 \qquad ...(ii)$

Also if the plane given by (i) is perpendicular to the given line, then the normal to the plane given by (i) must be parallel to the given line whose direction cosines are $\cos\theta, \sin\theta, 0$.

i.e. $$\frac{A}{\cos\theta}=\frac{B}{\sin\theta}=\frac{C}{0}. \qquad ...(iii)$$

Substituting the proportionate values of A and B from (iii) and the value of C from (ii) in (i) we get the required equation as

$$x\cos\theta+y\sin\theta=0 \quad \text{or} \quad x+y\tan\theta=0 \qquad \textbf{Ans.}$$

Ex. 7 (b). **Find the equation of the plane which passes through the x-axis and is perpendicular to the line**

$$(x-1)/\cos\theta=(y+2)/\sin\theta=(z-3)/0.$$

Sol. The equations of x-axis in the symmetric form are

$$\frac{x-0}{1}=\frac{y-0}{0}=\frac{z-0}{0} \qquad \text{(Note)}$$

$\therefore$ The equation of any palne thorugh x-axis is

$$A(x-0)+B(y-0)+C(z-0)=0 \qquad ...(i)$$

where $\quad A.1+B.0+C.0=0 \quad i.e. \quad A=0 \qquad ...(ii)$

$\therefore$ (i) reduces to $By+Cz=0 \qquad ...(iii)$

$\therefore$ The direction ratios of its normal are $0, B, C$.

If (iii) is perpendicular to the given line, then the normal to (iii) must be parallel to the given line

i.e. $$\frac{0}{\cos\theta}=\frac{B}{\sin\theta}=\frac{C}{0}=k \text{ (say)}$$

$\therefore B=k\sin\theta$; $C=0$ and $k\cos\theta=0$.

If $k\cos\theta=0$, then either $k=0$ or $\cos\theta=0$.

If $k=0$, then $B=0$ and from (iii), we do not find any plane, so

$$\cos\theta=0 \quad \text{or} \quad \theta=\pi/2.$$

$\therefore B=k\sin(\pi/2)=k$ and $C=0$.

$\therefore$ From (iii) be required equation is $ky=0$ or $y=0$. **Ans.**

*Ex. 8 (a) **Find the equation of the plane through the point** $(\alpha', \beta', \gamma')$ **and the line** $$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

Sol. The equation of the plane through the given line is

$$A(x-\alpha)+B(y-\beta)+C(z-\gamma)=0, \qquad \text{...(i)}$$

where $$A.l+B.m+C.n=0. \qquad \text{...(ii)}$$

If this plane (i) passes through (α, β, γ) also, then

$$A(\alpha'-\alpha)+B(\beta'-\beta)+C(\gamma'-\gamma)=0. \qquad \text{...(iii)}$$

Eliminating A, B, C from (i), (ii) and (iii), we have the required plane as

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l & m & n \\ a'-\alpha & \beta'-\beta & \gamma'-\gamma \end{vmatrix}=0$$ **Ans.**

Ex. 8 (b). Find the equation of the plane which passes thorugh the point (1, 2, – 1) and which contains the line

$$\tfrac{1}{2}(x+1)=\tfrac{1}{3}(y-1)=-(z+2).$$

Sol. The equation of the plane through the given line is

$$A(x+1)+B(y-1)+C(z+2)=0 \qquad \text{...(i)}$$

where $$A.2+B.3+C.(-1)=0 \qquad \text{...(ii)}$$

If this plane (i) passes through the point $(1, 2, -1)$ also, then

$$A(1+1)+B(2-1)+C(-1+2)=0 \quad \text{or} \quad 2A+B+C=0 \qquad \text{...(iii)}$$

Eliminating A, B, C from (i), (ii) and (iii) we get

$$\begin{vmatrix} x+1 & y-1 & z+2 \\ 2 & 3 & -1 \\ 2 & 1 & 1 \end{vmatrix}=0$$

or $$4(x+1)-4(y-1)-4(z+2)=0, \text{ on expanding}$$

or $$x-y-z=0.$$ **Ans.**

Ex. 8 (c). Find the equation of the plane containing the line $\frac{1}{2}(x+2)=\frac{1}{3}(y+3)=-\frac{1}{2}(z-4)$ and passing through the point (0, 6, 0).

Sol. Do as Ex. 8 (b). above **Ans.** $3x+2y+6z-12=0$.

****Ex. 9. Prove that the plane through the point (α, β, γ) and the line $x=py+q=rz+s$ is given by**

$$\begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+q & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix}=0$$ *(Garhwal 93)*

Sol. The equations of the line $x=py+q=rz+s$ can be written in the symmetric form as $$\frac{x}{1}=\frac{y+(q/p)}{1/p}=\frac{z+(s/r)}{(1/r)} \qquad \text{...(i)} \quad \textbf{(Note)}$$

$\therefore$ The equation of the plane through the line (i) is

$$A(x)+B\left(y+\frac{q}{p}\right)+C\left(z+\frac{s}{r}\right)=0 \qquad \text{...(ii)}$$

where $$A.1+B.(1/p)+C(1/r)=0. \qquad \text{...(iii)}$$

If this plane (ii) passes through the point (α, β, γ), then

$$A.\alpha+B\left(\beta+\frac{q}{p}\right)+C\left(\gamma+\frac{s}{r}\right)=0. \qquad ...(iv)$$

Eliminating A, B and C from (ii), (iii) and (iv), we have

$$\begin{vmatrix} x & y+\frac{q}{p} & z+\frac{s}{r} \\ 1 & \frac{1}{p} & \frac{1}{r} \\ \alpha & \beta+\frac{q}{p} & \gamma+\frac{s}{r} \end{vmatrix}=0.$$

Multiplying C_2 and C_3 by p and r respectively and interchanging R_2 and R_3 this reduces to

$$\begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+q & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix}=0.$$

Hence proved.

Ex. 10. Find the equation of the plane through the point (2, – 1, 1) and the line 4x – 3y + 5 = 0 = y – 2z – 5. (*Meerut* 94)

Sol. The equation of any plane through the line

$$4x-3y+5=0,\ y-2z-5=0.$$

is
$$(4x-3y+5)+\lambda(y-2z-5)=0.$$

...See § 4.06 Page 69 Chapter IV.

If it passes thorugh (2, – 1, 1), then from (i) we have

$$[4.2-3(-1)+5]+\lambda[(-1)-2(1)-5]=0 \quad \text{or} \quad \lambda=2$$

Substituting this value in (i) the required equation is

$$(4x-3y+5)+2(y-2z-5)=0 \quad \text{or} \quad 4x-y-4z-5=0. \qquad \textbf{Ans.}$$

Ex. 11. Find the equation of the plane through the lines

$$ax+by+cz=0=a'x+b'y+c'z$$

and
$$\alpha x+\beta y+\gamma z=0=\alpha' x+\beta' y+\gamma' z. \qquad (\textit{Kanpur } 92)$$

Sol. The equations of the lines (both of which pass through the origin) can be written in the symmetric form is

$$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b} \qquad ...(i)$$

and
$$\frac{x}{\beta\gamma'-\beta'\gamma}=\frac{y}{\gamma\alpha'-\gamma'\alpha}=\frac{z}{\alpha\beta'-\alpha'\beta} \qquad ...(ii)$$

The equation of any plane through the line (i) is

$$Ax+By+Cz=0, \qquad ...(iii)$$

where
$$A(bc'-b'c)+B(ca'-c'a)+C(ab'-a'b)=0 \qquad ...(iv)$$

Also if the line (ii) lies on the plane (iii), then the following conditions should be satisfied :

(a) The plane (iii) should pass through the origin [which is a point on the line (iii)] and that is true.

and (b) the normal to the plane (iii) must be at right angles to the line (ii), the condition for the same is

$$A(\beta\gamma' - \beta'\gamma) + B(\gamma\alpha' - \gamma'\alpha) + C(\alpha\beta' - \alpha'\beta) = 0 \qquad \text{...(v)}$$

Eliminating A, B, C from (iii), (iv) and (v), we have the required equation as

$$\begin{vmatrix} x & y & z \\ bc' - b'c & c'a - ca' & ab' - a'b \\ \beta\gamma' - \beta'\gamma & \gamma'\alpha - \gamma\alpha' & \alpha\beta' - \alpha'\beta \end{vmatrix} = 0$$

Ex. 12. The plane $lx + my = 0$ is rotated about the line of intersection with the plane $z = 0$ through an angle α. Prove that the equation to the plane in its new position is

$$\mathbf{lx + my \pm z\sqrt{(l^2 + m^2)}\tan\alpha = 0.}$$

Sol. The equation of any plane through the line of intersectino of the planes $lx + my = 0$ and $z = 0$ is $\quad (lx + my) + \lambda z = 0 \qquad$ ...(i)

If (i) represents the planes obtained by rotating the given plane

$$lx + my = 0 \qquad \text{...(ii)}$$

through an angle α aboutt the line of intersection of (ii) and the plane $z = 0$, then the angle between the planes (i) and (ii) is α and so we have

$$\cos\alpha = \frac{l.l + m.m + \lambda.0}{\sqrt{(l^2 + m^2 + \lambda^2)}.\sqrt{(l^2 + m^2 + 0^2)}}$$

or $\quad (l^2 + m^2 + \lambda^2)(l^2 + m^2)\cos^2\alpha = (l^2 + m^2)^2$, squaring and cross multiplying

or $\quad (l^2 + m^2 + \lambda^2)\cos^2\alpha = (l^2 + m^2) \quad$ or $\quad l^2 + m^2 + \lambda^2 = (l^2 + m^2)\sec^2\alpha$

or $\quad \lambda^2 = (l^2 + m^2)\sec^2\alpha - (l^2 + m^2) = (l^2 + m^2)\tan^2\alpha$

or $\quad \lambda = \pm\sqrt{(l^2 + m^2)}\tan\alpha$

$\therefore$ From (i) the required equation is

$$lx + my \pm z\sqrt{(l^2 + m^2)}\tan\alpha = 0. \qquad \textbf{Ans.}$$

***Ex. 13. Prove that the equation to the two planes inclined at an angle α to xy-plane and containing the line $y = 0$, $z\cos\beta = x\sin\beta$ is $(x^2 + y^2)\tan^2\beta + z^2 - 2zx\tan\beta = y^2\tan^2\alpha$.** *(Kanpur 97; Meerut 92 P)*

Sol. The equation of the plane containing the line $y = 0$, $x\sin\beta = z\cos\beta$ is

$$(x\sin\beta - z\cos\beta) + \lambda y = 0.$$

or

$$x\sin\beta + \lambda.y - z\cos\beta = 0. \qquad \text{...(i)}$$

The other plane is xy-plane *i.e.* $z = 0$.

i.e.

$$0.x + 0.y + 1.z = 0 \qquad \text{...(ii)} \quad \textbf{(Note)}$$

It is given that the angle between plane (i) and (ii) is α, so we have

$$\cos\alpha = \frac{\text{“}a_1a_2 + b_1b_2 + c_1c_2\text{”}}{\sqrt{(a_1^2 + b_1^2 + c_1^2)}\sqrt{(a_2^2 + b_2^2 + c_2^2)}},$$

$\because$ angle between planes = angle between their normals.

or

$$\cos\alpha = \frac{0.\sin\beta + 0.\lambda + 1.(-\cos\beta)}{\sqrt{(\sin^2\beta + \lambda^2 + \cos^2\beta)}\sqrt{(0^2 + 0^2 + 1^2)}}$$

Squaring and cross-multiplying, we have

$$(\lambda^2+1)\cos^2\alpha=\cos^2\beta \quad \text{or} \quad \lambda^2=(\cos^2\beta-\cos^2\alpha)/\cos^2\alpha$$

or $$\lambda=\pm[\sqrt{(\cos^2\beta-\cos^2\alpha)}]/\cos\alpha=\pm\mu \text{ (say)} \qquad ...(iii)$$

which gives two values of λ which are equal in magnitude but opposite in sign.

$\therefore$ From (i) combined equation of two required planes is

$$[(x\sin\beta-z\cos\beta)+\mu y]\,[(x\sin\beta-z\cos\beta)-\mu y]=0. \qquad \textbf{(Note)}$$

or $$(x\sin\beta-z\cos\beta)^2-\mu^2y^2=0$$

or $$(x\sin\beta-z\cos\beta)^2\cos^2\alpha=(\cos^2\beta-\cos^2\alpha)\,y^2, \text{ from (iii)}$$

or $$(x^2\sin^2\beta+z^2\cos^2\beta-2xz\sin\beta\cos\beta)\cos^2\alpha=(\cos^2\beta-\cos^2\alpha)\,y^2$$

or $$x^2\tan^2\beta+z^2-2xz\tan\beta=(\sec^2\alpha-\sec^2\beta)\,y^2,$$

dividing each term by $\cos^2\alpha\cos^2\beta$

or $$x^2\tan^2\beta+z^2-2xz\tan\beta=[(1+\tan^2\alpha)-(1+\tan^2\beta)]\,y^2$$

or $$(x^2+y^2)\tan^2\beta+z^2-2xz\tan\beta=y^2\tan^2\alpha. \qquad \text{Hence proved.}$$

***Ex. 14. Find the equation of the plane through the point $(-1, 0, 1)$ and the lines $4x-3y+1=0=y-4z+13$; $2x-y-2=0=z-5$.**

Sol. The equation of the plane through the line

$$4x-3y+1=0=y-4z+13$$

is $$(4x-3y+1)+\lambda(y-4z+13)=0 \qquad ...(i)$$

If its passes through $(-1, 0, 1)$, then

$$[4(-1)-3(0)+1]+\lambda[0-4(1)+13]=0 \quad \text{or} \quad -3+9\lambda=0 \quad \text{or} \quad \lambda=\frac{1}{3}$$

$\therefore$ From (i) the required equation is $(4x-3y+1)+\frac{1}{3}(y-4z+13)=0.$

or $$12x-9y+3+y-4z+13=0 \quad \text{or} \quad 12x-8y-4z+16=0$$

or $$3x-2y-z+4=0. \qquad \textbf{Ans.}$$

Similarly we can find the equation of the plane through the point $(-1, 0, 1)$ and the line $2x-y-2=0=z-5$ as $2x-y-z+3=0$ **Ans.**

****Ex. 15. Find the equation of a system of planes perpendicular to the line with direction ratios a, b, c.** (*Purvanchal* 97)

Sol. Let the required equation be $Ax+By+Cz+D=0$ ...(i)

If this plane is perpendicular to the line with d. ratios a, b, c, then the normal to the plane (i) must be parallel to this line.

i.e. $$\frac{A}{a}=\frac{B}{b}=\frac{C}{c}=\lambda \text{ (say)}$$

i.e. $$A=a\lambda,\ B=b\lambda \text{ and } C=c\lambda$$

$\therefore$ From (i), we have $a\lambda x+b\lambda y+c\lambda z+D=0$

or $ax+by+cz+k=0$, where $k=D/\lambda$, is the required equation of a system of planes where k is parameter *i.e.* for different values of k we shall get different members of the above system of planes. **Ans.**

Exercises on § 4.06 – § 4.07.

**Ex. 1. Find the equation to the plane through the line

$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ parallel to the line $\frac{x}{l'}=\frac{y}{m'}=\frac{z}{n'}$.

(**Hint : See § 4.07 Page 70 Chapter IV**) (*Meerut* 90 S; *Rohilkhand* 95)

Ex. 2. Show that the line $\frac{1}{2}(x+1)=-\frac{1}{3}(y-1)=\frac{1}{4}(z+5)$ lies on the plane $x+2y+z+4=0$.

Ex. 3. Obtain the equation of the plane through the line $3x-4y+5z=10$, $2x+2y-2z=4$ and parallel to the line $x=2y=3z$. **Ans.** $x-20y+27z=14$.

Ex. 4. Find the equation of the plane passing through the intersection of the planes $6x+2y+z=1$ and $2x+y+3z=4$ and which is parallel to the line $6x=3y=2z$.

Ex. 5. Find the equation of the plane determined by the parallel lines

$$\frac{1}{3}(x+1)=\frac{1}{2}(y-2)=z \quad \text{and} \quad \frac{1}{3}(x-3)=\frac{1}{2}(y+4)=z-1.$$

Ans. $8x+y-26z+6=0$.

Ex. 6. Find the equation of the plane containing the line

$\frac{1}{3}(x+3)=\frac{1}{4}(y-1)=-\frac{1}{2}(z-2)$ and the point (8, 2, 4).

Ans. $10x-28y-41z+140=0$.

Ex. 7. Find the equation to the plane containing the line $-\frac{1}{3}(x+1)=\frac{1}{2}(y-3)=(z+7)$ and the point $(0, 7, -7)$ and show that the line

$x=-\frac{1}{3}(y-7)=(z+7)$ also lies in the same plane. **Ans.** $x+y+z=0$.

****§ 4.08. To find the perpendicular distance of a point P (x_1, y_1, z_1) from a given line when its equations are given in the symmetric form.**

Let the equations of the line be $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ ...(i)

Any point on this line can be taken as $N(\alpha+lr,\ \beta+mr,\ \gamma+nr)$. ...(ii)

If N be the foot of the perpendicular from $P(x_1, y_1, z_1)$ to (i) then the line PN is perpendicular to (i).

The direction ratios of the line PN are

$$\alpha+lr-x_1,\ \beta+mr-y_1,\ \gamma+nr-z_1.$$

And as the line PN is perpendicular to line (i), so we have

$$l(\alpha+lr-x_1)+m(\beta+mr-y_1)+n(\gamma+nr-z_1)=0$$

or $$(l^2+m^2+n^2)r=(lx_1+my_1+nz_1)-(l\alpha+m\beta+n\gamma)$$

or $$r=[l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)]/(l^2+m^2+n^2).$$

Substituting this value of r in (ii) we can determine the coordinates of N, the foot of the perpendicular and then PN can be easily calculated.

§ **4.09. To find the equations of the perpendicular line from the point** P (x_1, y_1, z_1) to a given line whose equations are given (a) **in general form** and (b) in symmetric form.

(a) General form.

Let the equations of the line be

$$ax+by+cz+d=0, \quad a'x+b'y+c'z+d'=0 \qquad \text{...(i)}$$

If l, m, n be the d.c.'s of this line, then we have

$$al+bm+cn=0 \quad \text{and} \quad a'l+b'm+c'n=0.$$

Solving these we get $\dfrac{l}{bc'-b'c}=\dfrac{m}{ca'-c'a}=\dfrac{n}{ab'-a'b}$ ...(ii)

Now the equations of any plane through the line (i) is given by

$$(ax+by+cz+d)+\lambda(a'x+b'y+c'z+d')=0 \qquad \text{...(iii)}$$

If this plane passes through $P(x_1, y_1, z_1)$, then

$$(ax_1+by_1+cz_1+d)+\lambda(a'x_1+b'y_1+c'z_1+d')=0 \qquad \text{...(iv)}$$

$\therefore$ The equation of the plane through $P(x_1, y_1, z_1)$ and the line is λ-eliminant of (iii) and (iv).

Also the equation of any plane through $P(x_1, y_1, z_1)$ is

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0 \qquad \text{...(v)}$$

If this is perpendicular to the line (i), then we have

$$A/l=B/m=C/n,$$

where l, m, n are given by (ii), since the normal to the plane (v) is parallel to the line (i).

$\therefore$ From (v) the equation of the plane perpendicular to the line (i) and passing through $P(x_1, y_1, z_1)$ is $l(x-x_1)+m(y-y_1)+n(z-z_1)=0$ ...(vi)

$\therefore$ The equations of perpendicular line from $P(x_1, y_1, z_1)$ to (i) are given by (vi) and λ-eliminant of (iii) and (iv).

(b) Symmetric form.

Proceeding exactly as in § 4.08 Page 80 Chapter IV we can find the d.c.'s l_1, m_1, n_1 (say) of line PN.

Also P is (x_1, y_1, z_1). Therefore the equations of the perpendicular from $P(x_1, y_1, z_1)$ to the given line are

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}.$$

Solved Examples on § 4.08 and § 4.09.

Ex. 1 (a). Find the equations of the perpendicular from the point (3, – 1, 11) to line $\frac{1}{2}x=\frac{1}{3}(y-2)=\frac{1}{4}(z-3)$. Find also the coordinates of the foot of the perpendicular.

Sol. The given point is $P(3, -1, 11)$ and the equations of the line are

$$\frac{x}{2}=\frac{y-2}{3}=\frac{z-3}{4} \qquad \text{...(i)}$$

Any point on this line is $N(2r, 2+3r, 3+4r)$ ...(ii)

Let N be the foot of the perpendicular from $P(3, -1, 11)$ to (i).

$\therefore$ The d.r.'s of the line PN are

$$2r-3,\ 2+3r+1,\ 3+4r-11 \quad \text{or} \quad 2r-3,\ 3r+3,\ 4r-8, \qquad \text{...(iii)}$$

Also if N be the foot of the perpendicular from P to (i), then PN is perpendicular to (i) and we have

$$(2r-3).2+(3r+3).3+(4r-8).4=0$$

which gives $\quad 29r-29=0 \quad$ or $\quad r=1$.

$\therefore$ From (ii), the coordinates of N, the foot of perpendicular from P to (ii) is (2, 5, 7).

And from (iii), the d.r.'s of the perpendicular PN are

$$-1, 6, -4 \quad \text{or} \quad 1, -6, 4.$$

$\therefore$ The required equations of the perpendicular PN are given by

$$\frac{x-3}{1}=\frac{y+1}{-6}=\frac{z-11}{4}. \qquad \textbf{Ans.}$$

Ex. 1 (b). Find the equations of the perpendicular from the point $(2, 4, -1)$ to the line $(x+5)=\frac{1}{4}(y+3)=-\frac{1}{9}(z-6)$. Also obtain the foot of the perpendicular.

Sol. Do as Ex. 1 (a) above.

Ans. $(x-2)/6=(y-4)/3=(z+1)/2$; $(-4, 1, -3)$.

Ex. 1 (c). Find the distance of the point (1, 2, 3) from the line joining the points (– 1, 2, 5) and (2, 3, 4).

Sol. Equation of the line joining the points (– 1, 2, 5) and (2, 3, 4) is

$$\frac{x+1}{2+1}=\frac{y-2}{3-2}=\frac{z-5}{4-5} \quad \text{or} \quad \frac{x+1}{3}=\frac{y-2}{1}=\frac{z-5}{-1} \qquad \text{...(i)}$$

Also the given point is $P(1, 2, 3)$.

Any point on the line (i) is $N(-1+3r, 2+r, 5-r)$ ...(ii)

Let N be the foot of the perpendicular from P to (i), then the d.r.'s of PN are $\quad (-1+3r)-1, (2+r)-2, (5-r)-3 \quad$ or $\quad 3r-2, r, 2-r$ ...(iii)

Also if N be the foot of the perpendicular from P to (i), then PN is prpendicular to (i) and so we have

$$(3r-2).3+(r).1+(2-r).(-1)=0 \quad \text{or} \quad r=8/11$$

$$\therefore \text{From (ii), } N \text{ is } \left(-1+\frac{24}{11},\ 2+\frac{8}{11},\ 5-\frac{8}{11}\right) \text{ i.e. } \left(\frac{13}{11}, \frac{30}{11}, \frac{47}{11}\right)$$

$\therefore$ Required distance $= PN$

$$=\sqrt{\left[\left(\frac{13}{11}-1\right)^2+\left(\frac{30}{11}-2\right)^2+\left(\frac{47}{11}-3\right)^2\right]}$$

$$=\frac{1}{11}\sqrt{[4+64+196]}=\frac{1}{11}\sqrt{(264)}=\frac{2\sqrt{6}}{\sqrt{(11)}} \qquad \textbf{Ans.}$$

Ex. 2 (a). Find the foot and hence the length of the perpendicular from (5, 7, 3) to the line $\frac{1}{3}(x-15)=\frac{1}{8}(y-29)=-\frac{1}{5}(z-5)$. Find also the equation of the plane in which the perpendicular and the given straight line lie.

(*Avadh* 91)

Sol. The given point is $P(5, 7, 3)$ and the line is

$$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5} \quad ...(i)$$

Any point on it is $N(15+3r,\ 29+8r,\ 5-5r)$ ...(ii)

$\therefore$ The d.r.'s of line PN are $(15+3r)-5,\ (29+8r)-7,\ (5-5r)-3$

i.e. $10+3r,\ 22+8r,\ 2-5r$ ...(iii)

Let N be the foot of the perpendicular from P to (i), then PN is perpendicular to (i) and so we have

$$3(10+3r)+8(22+8r)-5(2-5r)=0 \quad \text{or} \quad r=-2$$

$\therefore$ From (ii) the foot N of tthe perpendicular PN is

$(15-6, 29-16, 5+10)$ or $(9, 13, 15)$ **Ans.**

$\therefore$ PN = distance between $P(5, 7, 3)$ and $N(9, 13, 15)$

$=\sqrt{[(9-5)^2+(13-7)^2+(15-3)^2]}=\sqrt{(16+36+144)}=14.$ **Ans.**

Again the equation of the plane contaning the given line (i) is

$$A(x-15)+B(y-29)+C(z-5)=0 \quad ...(iv)$$

where $A.3+B.8+C.(-5)=0.$ ...(v)

If the plane (iv) passes through $P(5, 7, 3)$ also then

$A(5-15)+B(7-29)+C(3-5)=0$ or $5A+11B+C=0.$ ...(vi)

Solving (v) and (vi) we have

$$\frac{A}{8+55}=\frac{B}{-25-3}=\frac{C}{33-40} \quad \text{or} \quad \frac{A}{9}=\frac{B}{-4}=\frac{C}{-1}$$

$\therefore$ From (iv) the required equation of the plane through the line (i) and the perpendicular PN [or the point P as N is a point on (i)] is $9(x-15)-4(y-29)-(z-5)=0$ or $9x-4y-z=14.$ **Ans.**

Ex. 2 (b). Find the distance of the point P (4, 3, 5) from the axis of y.

Sol. The equations of y-axis are $\frac{x}{0}=\frac{y}{1}=\frac{z}{0}$,

$\therefore$ Any point N on y-axis is $(0, r, 0)$ ...(i)

$\therefore$ The direction cosines of the line PN are $0-4, r-3, 0-5$

i.e. $-4, r-3, -5.$ ...(ii)

Let N be the foot of the perpendicular from P to y-axis, then PN is perpendicular to the y-axis whose direction cosines are 0, 1, 0 and so from (ii) we have $0.(-4)+1.(r-3)+0.(-5)=0$ or $r=3$

$\therefore$ From (i) the coordinates of N are $(0, 3, 0)$

$\therefore$ Required distance $=PN=\sqrt{[(4-0)^2+(3-3)^2+(5-0)^2]}$

$=\sqrt{(16+0+25)}=\sqrt{(41)}.$ **Ans.**

Ex. 2 (c). Find the perpendicular distance of P (1, 2, 3) from the line

$$\frac{1}{3}(x-6)=\frac{1}{2}(y-7)=-\frac{1}{2}(z-7).$$

Sol. The equations of the given line are

$$\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}$$

$\therefore$ Any point N on this line is $(3r+6,\ 2r+7,\ -2r+7)$ ...(i)

$\therefore$ d.r.'s of the line PN are

$3r+7-1,\ 2r+7-2,\ -2r+7-3$ or $3r+5,\ 2r+5,\ -2r+4$ ...(ii)

Let N be the foot of the perpendicular from P to the given line, then PN is perpendicular to the given line whose d.r.'s are $3, 2, -2$ and from (ii) we have $3(3r+5)+2(2r+5)-2(-2r+4)=0$ or $r=-1$

$\therefore$ From (i) the coordinates of N are $(3, 5, 9)$

$\therefore$ Required distance $PN = \sqrt{[(3-1)^2+(5-2)^2+(9-3)^2]}$

$= \sqrt{[4+9+36]} = 7.$ **Ans.**

Ex. 2 (d). Find the length of the perpendicular from a given point $(1, -2, 3)$ on a line through $(2, -3, 5)$ which makes equal angles with the axes.

Sol. The d.c.'s of the line which makes equal angles with the axes can be taken as l, l, l, where $l^2+l^2+l^2=1$.

$\therefore$ The equations of the given line can be written as

$$\frac{x-2}{l}=\frac{y-(-3)}{l}=\frac{z-5}{l} \quad \text{or} \quad \frac{x-2}{1}=\frac{y+3}{1}=\frac{z-5}{1} \qquad ...(i)$$

Now proceed as in Ex. 2 (c) above. **Ans.** $\sqrt{(14/3)}$.

****Ex. 3. Find the equations of the perpendicular from the origin to the line $ax+by+cz+d=0=a'x+b'y+c'z+d'$.**

Sol. The perpendicular from the origin to the given lien is the line of intersection of the following two planes :

(i) the plane through origin and countaining the given line and (ii) the plane through origin and perpendicular to the given line. **(Note)**

Now the equation of the plane through the given line is

$$(ax+by+cz+d)+\lambda(a'x+b'y+c'z+d')=0. \qquad ...(i)$$

If this plane passes through the origin (0, 0, 0), then from (i) we have

$$d+\lambda d'=0 \quad \text{or} \quad \lambda=-d/d'$$

$\therefore$ From (i) the equations of the plane through the origin and the given line is $d'(ax+by+cz)-d(a'x+b'y+c'z)=0$

or $(d'a-da')x+(d'b-db')y+(d'c-dc')z=0$ ...(ii)

Again the equation of any plane through the origin is

$$Ax+By+Cz=0 \qquad ...(iii)$$

Also if l, m, n be the d.c.'s of the given line, then we have

$$al+bm+cn=0,\ a'l+b'm+c'n=0.$$

Solving these we get $\dfrac{l}{bc'-b'c}=\dfrac{m}{ca'-c'a}=\dfrac{n}{ab'-a'b}$ ...(iv)

Now if the plane (iii) is perpendicular to the given line then the normal to the plane (iii) is parallel to the given line.

$\therefore$ We have $A/l=B/m=C/n$

or $\dfrac{A}{bc'-b'c}=\dfrac{B}{ca'-c'a}=\dfrac{C}{ab'-a'b}$, from (iv),

$\therefore$ From (iii) the equation of the plane through the origin and perpendicular to the given line is

$$(bc' - b'c)\,x + (ca' - c'a)\,y + (ab' - a'b)\,z = 0. \qquad \text{...(v)}$$

Hence planes (ii) and (v) together give the required equations of the perpendicular from the origin to the given line.

Ex. 4. Find the equations to the line through the plane (1, 2, 4) and perpendicular to the line $3x + 2y - z - 4 = 0 = x - 2y - 2z - 5$.

Sol. The equation of any plane through the given line is

$$(3x + 2y - z - 4) + \lambda\,(x - 2y - 2z - 5) = 0 \qquad \text{...(i)}$$

If it passes through the point (1, 2, 4), then

$$(3.1 + 2.2 - 4 - 4) + \lambda\,(1 - 2.2 - 2.4 - 5) = 0 \quad \text{or} \quad \lambda = -(1/16).$$

$\therefore$ From (i) the equation of the plane through (1, 2, 4) and the given line is

$$16\,(3x + 2y - z - 4) - (x - 2y - 2z - 5) = 0$$

or
$$47x + 34y - 14z - 59 = 0 \qquad \text{...(ii)}$$

Also if l, m, n be the direction cosines of the given line, then we have

$$3l + 2m - n = 0 \quad \text{and} \quad l - 2m - 2n = 0$$

Solving these we have

$$\frac{l}{-4-2} = \frac{m}{-1+6} = \frac{n}{-6-2} \quad \text{or} \quad \frac{l}{-6} = \frac{m}{5} = \frac{n}{-8} \qquad \text{...(iii)}$$

Also any plane through (1, 2, 4) is

$$A\,(x-1) + B\,(y-2) + C\,(z-4) = 0 \qquad \text{...(iv)}$$

If this plane is perpendicular to the given line then the normal to this plane must be parallel to the given line. So from (iii) we have

$$A/l = B/m = C/n \quad \text{or} \quad A/(-6) = B/5 = C/(-8)$$

Hence from (iv) the equation of the plane through (1, 2, 4) and perpendicular to the given line is

$$-6\,(x-1) + 5\,(y-2) - 8\,(z-4) = 0 \quad \text{or} \quad 6x - 5y + 8z = 28 \qquad \text{...(v)}$$

The line of intersection of the planes (ii) and (v) is the perpendicular from (1, 2, 4) to the given line, hence the required general equations of the perpendicular are $47x + 34y - 14z = 59;\ 6x - 5y + 8z = 28$ ...(vi)

Now if we want to express the equation of this line in the symmetric form, then let l_1, m_1, n_1 be d.c.'s of this perpendicular and so we have

$$47l_1 + 34m_1 - 14n_1 = 0 \quad \text{and} \quad 6l_1 - 5m_1 + 8n_1 = 0$$

Solving these, we get

$$\frac{l_1}{272-70} = \frac{m}{-84-376} = \frac{n_1}{-235-204} \quad \text{or} \quad \frac{l_1}{202} = \frac{m_1}{-460} = \frac{n_1}{-439}$$

$\therefore$ The equations (vi) of perpendicular from (1, 2, 4) to the given line can be written in the symmetric form as

$$\frac{x-1}{202} = \frac{y-2}{-460} = \frac{z-4}{-439} \qquad \textbf{Ans.}$$

***Ex. 5. Find the equations to the perpendicular from the origin to the line $x + 2y + 3z + 4 = 0,\ 2x + 3y + 4z + 5 = 0$. Find also the coordinates of the foot of the perpendicular.**

Sol. The equation of any plane thorugh the given line is

$$(x+2y+3z+4)+\lambda(2x+3y+4z+5)=0 \qquad ...(i)$$

If it passes through the origin (0, 0, 0) then $4+5\lambda=0$ or $\lambda=-4/5$.

$\therefore$ From (i), the equation of the plane through the origin and the given line is $5(x+2y+3z-4)-4(2x+3y+4z+5)=0$

or

$$3x+2y+z=0. \qquad ...(ii)$$

Also if l, m, n be the d.c.'s of the given line, then we have

$$l+2m+3n=0 \quad \text{and} \quad 2l+3m+4n=0$$

Solving, we get $\dfrac{l}{8-9}=\dfrac{m}{6-4}=\dfrac{n}{3-4}$ or $\dfrac{l}{1}=\dfrac{m}{-2}=\dfrac{n}{1}$ ...(iii)

Also any plane through the origin is $Ax+By+Cz=0$

If this plane is perpendicular to the given line, then the normal to this plane must be parallel to the given line.

$\therefore$ We have $A/l=B/m=C/n$ or $A/1=B/-2=C/1$, from (iii)

Hence the equation of the plane through the origin and perpendicular to the given line is $1.x-2.y+1.z=0$ or $x-2y+z=0$ ...(iv)

The line of intersection of the planes (ii) and (iv) is the perpendicular from the origin to the given line, hence the required equation of the perpendicular are $3x+2y+z=0, x-2y+z=0.$ ...(v) **Ans.**

$\therefore$ The d.r.'s l_1, m_1, n_1 of this perpendicular are given by

$$3l_1+2m_1+n_1=0 \text{ and } l_1-2m_1+n_1=0$$

Solving these we have $\dfrac{l_1}{2+2}=\dfrac{m_1}{1-3}=\dfrac{n_1}{-6-2}$ or $\dfrac{l_1}{2}=\dfrac{m_2}{-1}=\dfrac{n_1}{-4}$

$\therefore$ The equations (v) of the perpendicular from the origin to the given line can be written in the symmetric form as

$$\frac{x-0}{2}=\frac{y-0}{-1}=\frac{z-0}{-4}=r \text{ (say)} \qquad ...(vi)$$

****Foot of the perpendicular :**

Any point of this line (*i.e.* the perpendicular) is $(2r, -r, -4r)$. ...(vii)

If it is the foot of the perpendicular from the origin to the given line then it must satisfy each of the equations of the given line *i.e.* must lie on each of the planes whose line of intersection is the given line. **(Note)**

$\therefore$ From $x+2y+3z+4=0$, we have

$$(2r)+2(-r)+3(-4r)+4=0 \quad \text{or} \quad r=1/3.$$

$\therefore$ From (vii), the coordinates of the required foot of the perpendicular are $(2/3, -1/3, -4/3)$. **Ans.**

***Ex. 6. The equations of a given line AB are $x/2=y/-3=z/5$. Through a point P (1, 2, 3), PN is drawn prpendicular to the line AB and PQ is drawn parallel to the plane $3x+4y+5z=0$ to meet AB in Q. Find the coordinates of N and Q and the equations of PN and PQ.**

Sol. Any point on the line AB is $N(2r, -3r, 5r)$ ...(i)

$\therefore$ The d.r.'s of the line PN are proportional to

$$2r-1, -3r-2, 5r-3. \quad \text{...(ii)}$$

$\therefore$ If PN is perpendicular to AB (whose d.r.'s are $2, -3, 5$), then we have

$$2(2r-1)-3(-3r-2)+5(5r-3)=0 \quad \text{or} \quad r=11/38$$

$\therefore$ From (i) the point N is $(11/19, -33/38, 55/38)$. **Ans.**

And from (ii) the d.r.'s of PN are $-8/19, -109/38, -59/38$ or $16, 109, 59$.

The equations of the line PN which passes through $P(1, 2, 3)$ and whose direction ratios are $16, 109, 59$ is $\dfrac{x-1}{16}=\dfrac{y-2}{109}=\dfrac{z-3}{59}$ **Ans.**

Again as before any point Q on the line AB is $(2r, -3r, 5r)$.

$\therefore$ The d.r.'s of the line PQ are $2r-1, -3r-2, 5r-3$.

If PQ is parallel to the plane $3x+4y+5z=0$ then PQ is perpendicular to the normal to this plane and so we have

$$3(2r-1)+4(-3r-2)+5(5r-3) \text{ or } r=26/19$$

$\therefore$ The point Q is $(52/19, -78/19, 130/19)$ and d.r.'s of the line PQ are

$$\frac{52}{19}-1, -\frac{78}{19}-2, \frac{130}{19}-3 \quad \text{or} \quad 33, -116, 73.$$

$\therefore$ The equations of PQ the line passing thorugh $P(1, 2, 3)$ are

$$(x-1)/(33)=(y-2)/(-116)=(z-3)/73. \quad \textbf{Ans.}$$

***Ex. 7 (a). Find the distance of the point A (1, – 2, 3) from the line PQ drawn through P (2, – 3, 5) making equal angles with the axes.**

Sol. Since the line PQ makes equal angles with the axes, so its direction ratios are $1, 1, 1$.

$\therefore$ The equations of the line PQ are $\dfrac{x-2}{1}=\dfrac{y+3}{1}=\dfrac{z-5}{1}$

$\therefore$ Any point N on this line is $(2+r, -3+r, 5+r)$ ...(i)

$\therefore$ The direction ratios of the line AN are

$$(2+r)-1, (-3+r)-(-2), (5+r)-3.$$

i.e. $$1+r, \ -1+r, \ 2+r$$

Now if N be the foot of the perpendicular drawn from A to the line PQ, then AN is perpendicular to PQ and so we have

$$(1+r).1+(-1+r).1+(2+r).1=0 \quad \textbf{(Note)}$$

or $$3r+2=0 \quad \text{or} \quad r=-2/3.$$

$\therefore$ From (i), the co-ordinates of N are

$$\left(2-\frac{2}{3}, -3-\frac{2}{3}, 5-\frac{2}{3}\right) \ i.e. \ \left(\frac{4}{3}, -\frac{11}{3}, \frac{13}{3}\right)$$

$\therefore$ The required distance $= AN$

$$=\sqrt{\left[\left(1-\frac{4}{3}\right)^2+\left(-2+\frac{11}{3}\right)^2+\left(3-\frac{13}{2}\right)^2\right]}=\sqrt{\left(\frac{14}{3}\right)} \quad \textbf{Ans.}$$

***Ex. 7 (b). Find the distance of the point (3, 8, 2) from the line**

$$\frac{1}{2}(x-1)=\frac{1}{4}(y-3)=\frac{1}{3}(z-2)$$

measured parallel to the plane $3x+2y-2z+15=0$.

Sol. Let the given point (3, 8, 2) be P.

Any point N on the given line is $(1+2r, 3+4r, 2+3r)$ ...(i)

$\therefore$ The direction ratios of the line PN are

$(1+2r)-3, (3+4r)-8, (2+3r)-2$ *i.e.* $2r-2, 4r-5, 3r$. ...(ii)

If PQ is parallel to the plane $3x+2y-2z+15=0$, then PQ is perpendicular to the normal to this plane, consequently

$$3(2r-2)+2(4r-5)-2(3r)=0 \quad \text{or} \quad r=2.$$

$\therefore$ From (i) the coordinates of N are

$$[1+2(2), 3+4(2), 2+3(2)] \text{ i.e. } (5, 11, 8)$$

$\therefore$ Required distance = distance between $P(3, 8, 2)$ and $N(5, 11, 8)$

$$=\sqrt{[(5-3)^2+(11-8)^2+(8-2)^2]}=7.$$ **Ans.**

Ex. 8. Find the locus of points equidistant from the lines

$$y-mx=0=z-c;\ y+mx=0=z+c. \qquad (\textit{Rohilkhand } 91)$$

Sol. The given lines are

$$\frac{x}{1/m}=\frac{y}{1}=\frac{z-c}{0} \quad ...(i); \qquad \frac{x}{1/m}=\frac{y}{-1}=\frac{z+c}{0} \quad ...(ii)$$

Let $P(x_1, y_1, z_1)$ be a point equidistant from these lines. Let PN_1 and PN_2 be the perpendiculars from P to these lines (i) and (ii) respectively.

Any point N_1 on the line (i) is $[(r/m), r, c]$ ...(iii)

$\therefore$ The d.r.'s of the line PN_1 are $x_1-(r/m),\ y_1-r,\ z_1-c$.

If N_1 is the foot of the perpendicular from P on the line (i), then PN_1 is perpendicular to the line (i) and so we have

$$(1/m)[x_1-(r/m)]+1.(y_1-r)+0.(z_1-c)=0$$

or $$(x_1/m)+y_1=r[1+(1/m^2)] \text{ or } mx_1+m^2y_1=r(m^2+1)$$

or $$r=(mx_1+m^2y_1)/(m^2+1)$$

$\therefore$ From (iii), N_1 is $\left[\dfrac{x_1+my_1}{m^2+1}, \dfrac{mx_1+m^2y_1}{m^2+1}, c\right]$

$$\therefore PN_1^2=\left(x_1-\frac{x_1+my_1}{m^2+1}\right)^2+\left(y_1-\frac{mx_1+m^2y_1}{m^2+1}\right)^2+(z_1-c)^2$$

$$=\frac{m^2(y_1-mx_1)^2}{(m^2+1)^2}+\frac{(y_1-mx_1)^2}{(m^2+1)^2}+(z_1-c)^2$$

or $$PN_1^2=\frac{(y_1-mx_1)^2}{(m^2+1)}+(z_1-c)^2, \text{ on simplifying} \qquad ...(iv)$$

Similarly we can prove that $PN_2^2 = \dfrac{(y_1 + mx_1)^2}{(m^2+1)} + (z_1 + c)^2$ ...(v)

$\therefore$ The point $P(x_1, y_1, z_1)$ is equidistant from the given lines,

so $$PN_1 = PN_2 \quad i.e. \quad PN_1^2 = PN_2^2$$

or $$\frac{(y_1 - mx_1)^2}{m^2+1} + (z_1 - c)^2 = \frac{(y_1 + mx_1)^2}{m^2+1} + (z_1 + c)^2, \text{ from (iv), (v)}$$

or $$[(z_1 - c)^2 - (z_1 + c)^2](m^2+1) = (y_1 + mx_1)^2 - (y_1 - mx_1)^2$$

or $$[(2z_1)(-2c)](m^2+1) = (2y_1)(2mx_1) \quad \text{or} \quad mx_1y_1 + cz_1(m^2+1) = 0$$

$\therefore$ The locus of P is $mxy + c(m^2+1)z = 0$. **Ans.**

Ex. 9 (a). Given a point P and a straight line AB, show how to find the equations of the perpendicular PN drawn from P on AB as the line of intersection of two suitable planes.

(b) If P is (1, 2, 3) and AB is $\frac{1}{3}(x-2) = \frac{1}{4}(y-3) = \frac{1}{5}(z-4)$, find the line PN in this manner and obtain the coordinates of N.

Sol. (a) Let P be (x_1, y_1, z_1) and the equations of the straight line AB be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

Now PN can be looked upon as the line of intersection of the plane through the line AB and the point P and the plane perpendicular to AB and the point P.

Now the equation of any plane through the line AB is

$$A(x-\alpha) + B(y-\beta) + C(z-\gamma) = 0 \quad \text{...(i)}$$

where $$Al + Bm + Cn = 0 \quad \text{...(ii)}$$

If (i) passes through $P(x_1, y_1, z_1)$, then

$$A(x_1-\alpha) + B(y_1-\beta) + C(z_1-\gamma) = 0 \quad \text{...(iii)}$$

Solving (ii) and (iii) simultaneously we get

$$\frac{A}{m(z_1-\gamma) - n(y_1-\beta)} = \frac{B}{n(x_1-\alpha) - l(z_1-\gamma)} = \frac{C}{l(y_1-\beta) - m(x_1-\alpha)}$$

Substituting these proportionate values of A, B, C in (i) we get the equation of the plane through the line AB and the point P as

$$\Sigma[\{m(z_1-\gamma) - n(y_1-\beta)\}(x-\alpha)] = 0 \quad \text{...(iv)}$$

...See Ex. 8 (a) Page 75 Ch. IV also

Also equation of any plane through $P(x_1, y_1, z_1)$ is

$$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0 \quad \text{...(v)}$$

If it is perpendicular to the line AB whose d.c.'s are perportional to l, m, n then we must have $a/l = b/m = c/n$.

$\therefore$ The equation of the plane through the point P and perpendicular to the line AB is $l(x-x_1) + m(y-y_1) + n(z-z_1) = 0$ ...(vi)

Hence the equations of PN, the line of intersection of the planes through AB and P and perp. to AB and P, are given by (iv) and (vi).

(b) Here P is (1, 2, 3) and the line AB is given by

$$\frac{1}{3}(x-2)=\frac{1}{4}(y-3)=\frac{1}{5}(z-4) \quad \text{...}(\alpha)$$

And plane through the line AB is given by

$$A(x-2)+B(y-3)+C(z-4)=0, \quad \text{...(I)}$$

where
$$A.3+B.4+C.5=0 \quad \text{...(II)}$$

Also if (I) passes through P (1, 2, 3), then

$$A(1-2)+B(2-3)+C(3-4)=0 \quad \text{or} \quad A+B+C=0 \quad \text{...(III)}$$

Solving (II) and (III) we get

$$\frac{A}{4-5}=\frac{B}{5-3}=\frac{C}{3-4} \quad \text{or} \quad \frac{A}{1}=\frac{B}{-2}=\frac{C}{1}=k \text{ (say)}$$

$$\therefore A=k,\ B=-2k,\ C=k.$$

Substituting these values of A, B, C in (I) we get the equation of the plane through the line AB and the point P as

$$(x-2)-2(y-3)-(z+4)=0 \quad \text{or} \quad x-2y+z=0 \quad \text{...(IV)}$$

Also equation of any plane through P (1, 2, 3) is

$$a(x-1)+b(y-2)+c(z-3)=0 \quad \text{...(V)}$$

If it is perpendicular to the line AB, then

$$a/3=b/4=c/5=\lambda \text{ (say)} \Rightarrow a=3\lambda,\ b=4\lambda,\ c=5\lambda.$$

$\therefore$ From (V) the equation of the plane through $P(1, 2, 3)$ and perpendicular to the line AB is $3(x-1)+4(y-2)+5(z-3)=0$

or
$$3x+4y+5z=26 \quad \text{...(VI)}$$

Hence the equations of the line PN are $x-2y+z=0$, $3x+4y+5z=26$

Also the foot N of the perpendicular PN is the point of intersection of the plane (VI) and the line AB given by (α).

Any point on the line AB given by (α) is

$$(2+3r,\ 3+4r,\ 4+5r) \quad \text{...(VII)}$$

If it lies on the plane given by (VI), we get

$$3(2+3r)+4(3+4r)+5(4+5r)=26$$

or
$$50r=26-38=-12 \quad \text{or} \quad r=-(6/25)$$

$\therefore$ The coordinates of N from (VII) are

$$\left(2-\frac{18}{25}, 3-\frac{24}{25}, 4-\frac{30}{25}\right) \quad \text{or} \quad \left(\frac{32}{25}, \frac{51}{25}, \frac{70}{25}\right) \quad \textbf{Ans.}$$

Exercises on § 4.08 — § 4.09

Ex. 1. Find the distance of the point (1, 2, 3) from the line

$$(x-2)=\frac{1}{2}(y-3)=\frac{1}{3}(z-4). \quad \textbf{Ans. } \sqrt{(3/7)}.$$

Ex. 2. Find the equations of the perpendicular drawn from the point (5, 9, 3) to the line $\frac{1}{2}(x-1)=\frac{1}{3}(y-2)=\frac{1}{4}(z-3)$.

Ans. $(x-5)=\frac{1}{2}(y-9)=-\frac{1}{2}(z-3)$.

Ex. 3. Find the equations to the perpendicular from the origin to the line $x+4y+4z-27=0$, $2x+2y+3z-21=0$. Also find the coordinates of the foot of the perpendicular. **Ans.** $\frac{x-0}{1}=\frac{y-0}{30}=\frac{z-0}{17}$, $\left(\frac{1}{7}, \frac{30}{7}, \frac{17}{7}\right)$

Ex. 4. Find the coordinates of the foot of the perpendicular drawn from the point $P(5, 9, 3)$ to the line $\frac{1}{2}(x-1)=\frac{1}{3}(y-2)=\frac{1}{4}(z-3)$. *(Avadh 90)*

Ans. (3,5,7)

Ex. 5. Find the equations of the perpendicular from the point (1, 6, 3) to the line $x=\frac{1}{2}(y-1)=\frac{1}{3}(z-1)$. **Ans.** $\frac{x-1}{0}=\frac{y-6}{-5}=\frac{z-3}{2}$ *i.e.* $x=1, 2y+3z=21$

§ 4.10. Projection of a line on a given plane.

Definition 1. If A be the point of intersection of the given line and the given plane and B be the foot of the perpendicular from any point on this line to this plane, then AB is called the projection of the given line on the given plane.

Definition 2. The line of intersection of the given plane with another plane through the given line and perpendicular to the given plane is called the projection of the given line on the given plane.

Solved Examples on § 4.10.

***Ex. 1. Find the projection (or 'image') of the line**

$$(x-1)/3=(y-2)/4=(z-3)/5$$

on (or 'in' if image is used in place of 'projection' above) the plane

$$x-y+z+2=0$$

Sol. The given line is $\frac{x-1}{3}=\frac{y-2}{4}=\frac{z-3}{5}$. ...(i)

Any point on this line A is $(1+3r, 2+4r, 3+5r)$ ...(ii)

If A lies on the given plane $x-y+z+2=0$, we have

$$(1+3r)-(2-4r)+(3+5r)+2=0, \quad \text{or} \quad r=-1$$

$\therefore$ From (ii) the point A is $(-2,-2,-2)$, where A is the point of ntersection of the given plane and the given line.

Now from (ii) it is evident that a point on the given line is $C(1, 2, 3)$ (for $r=0$). Let B be the foot of tthe perpendicular from C on the given plane.

Now BC is a line perpendicular to the given plane, *i.e.* it is normal to the given plane and as such the direction ratios of BC are $1, -1, 1$ (the coefficients of x, y and z in the equation of the given plane).

$\therefore$ The equations of BC a line passing through $C(1, 2, 3)$ and having $1, -1, 1$ as direction ratios are $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-3}{1}$.

Any point on this lie BC is $(1+r, 2-r, 3+r)$. If the point is B *i.e.* if this point lies on the given plane, then we have

$(1+r)-(2-r)+(3-r)+2=0$ or $3r+4=0$ or $r=-(4/3)$

$\therefore$ The point B is $\left(1-\frac{4}{3}, 2+\frac{4}{3}, 3-\frac{4}{3}\right)$ *i.e.* $\left(-\frac{1}{3}, \frac{10}{3}, \frac{5}{3}\right)$

$\therefore$ The directoin ratios of the projection AB of the given line are

$$-\frac{1}{3}+2, \frac{10}{3}+2, \frac{5}{3}+2 \quad i.e. \quad -5, 16, 11$$

$\therefore$ The required equations of the projection AB are

$$\frac{x+2}{-5}=\frac{y+2}{6}=\frac{z+2}{11}$$ **Ans.**

Ex. 2. If L is the line $\frac{1}{2}(x-1)=-y=(z+2)$, find the direction cosines of the projection of L on the plane $2x+y-3z=4$ and the equation of the plane through L parallel to the line $2x+5y+3z=4$. $x-y-5z=6$.

Sol. Any point on the line L is $A(1+2r, -r, 2+r)$. ...(i)

If A lies on the plane $2x+y-3z=4$, ...(ii)

$2(1+2r)+(-r)-3(-2+r)=4$, which does not give any value of r as the coefficient of r is zero.

This shows that the line L is parallel to the plane (ii).

In such case we should adopt the following method —

Any plane through the line L is $A(x-1)+By+C(z+2)=0$, ...(iii)

where $A(2)+B(-1)+C(1)=0$. ...(iv)

If the plane (iii) is perpendicular to the given plane (ii), then

$$2.A+1.B-3.C=0 \quad \text{...(v)}$$

Solving (iv) and (v), we get $\frac{A}{3-1}=\frac{B}{2+6}=\frac{C}{2+2}$ or $\frac{A}{1}=\frac{B}{4}=\frac{C}{2}$.

$\therefore$ From (iii) the equation of the plane through L and perpendicular to the given plane (ii) is $1.(x-1)+4(y)+2(z+2)=3$

or $x+4y+2z+3=0$ ...(vi)

The plane (ii) and (vi) together give the projection of L on the plane (ii). If l, m, n be the d.c.'s of the projection, then from (iii) and (iv) we have

$$2l+m-3n=0 \quad \text{and} \quad l+4m+2n=0.$$

Solving, $\frac{l}{14}=\frac{m}{-7}=\frac{n}{7}$ or $\frac{l}{2}=\frac{m}{-1}=\frac{n}{1}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(2^2+1^2+1^2)}}=\frac{1}{\sqrt{6}}$

or $l=2/\sqrt{6}, \ m=-1/\sqrt{6}, \ n=1/\sqrt{6}$ **Ans.**

Let l_1, m_1, n_1 be the d.c.'s of the line given by

$$2x+5y+3z=4, \quad x-y-5z=6. \quad \text{...(vii)}$$

Then $2l_1+5m_1+3n_1=0, \ l_1-m_1-5n_1=0.$

Solving, $\dfrac{l_1}{-25+3}=\dfrac{m_1}{3+10}=\dfrac{n_1}{-2-5}$ or $\dfrac{l_1}{22}=\dfrac{m_1}{-13}=\dfrac{n_1}{7}$ ...(viii)

Now the equation of any plane through L is (iii) where (iv) holds.

If this plane is parallel to the line (vii) whose d.c.'s are given by (viii), then the normal to the plane (iii), must be perpendicular to the line (vii) and so we have $Al_1+Bm_1+Cn_1=0$ or $22A-13B-7C=0.$ ...(ix)

Solving (iv) and (ix) we get $\dfrac{A}{6}=\dfrac{B}{8}=\dfrac{C}{-4}$ or $\dfrac{A}{3}=\dfrac{B}{4}=\dfrac{C}{-2}$.

$\therefore$ From (iii), the equation of the plane through L parallel to (vii) is

$3(x-1)+4(y)-2(z+2)=0$ or $3x+4y-2z=7.$ **Ans.**

Ex. 3. Find the projection of the line $3x-y+2z=1,\ x+2y-z=2$ on the plane $3x+2y+z=0$ in the symmetric form. (*Kanpur* 97)

Sol. Any plane through the given line is

$$(3x-y+2z-1)+\lambda(x+2y-z-2)=0$$

or $$(3+\lambda)x+(2\lambda-1)y+(2-\lambda)z-(1+2\lambda)=0. \quad ...(i)$$

If this plane is perpendicular to the given plane $3x+2y+z=0.$ ...(ii)

Then we have $3(3+\lambda)+2(2\lambda-1)+1(2-\lambda)=0$ or $\lambda=-3/2.$

$\therefore$ From (i) the equation of the plane through the given line and perpendicular to the given plane (ii) is

$$(3-\tfrac{3}{2})x+(-3-1)y+(2+\tfrac{3}{2})z-(1-3)=0$$

or $$3x-8y+7z+4=0 \quad ...(iii)$$

The projection of the given line on the given plane is the line of intersection of the planes (ii) and (iii). (See def. § 4.10 Page 91 Ch. IV)

$\therefore$ The equations of the projection of the given line on the given plane are $3x+2y+z=0,\ 3x-8y+7z+4=0.$ ...(iv)

Let l, m, n be the d.c.'s of this line, then as this line lies on both the planes given by (iv), so it is perpendicular to the normal to these planes.

$\therefore$ We have $3l+2m+n=0,\ 3l-8m+7n=0$

Solving these, we get $\dfrac{l}{14+8}=\dfrac{m}{3-21}=\dfrac{n}{-24-6}$ or $\dfrac{l}{11}=\dfrac{m}{-9}=\dfrac{n}{-15}$...(v)

Again putting $z=0$ in (iv), we get $3x+2y=0,\ 3x-8y+4=0$

Solving these, we get $x=-4/15,\ y=2/5$

$\therefore$ The line (iv) meets the plane $z=0$ at $(-4/15,\ 2/5,\ 0)$

$\therefore$ From (v) and (vi), the required equations are

$$\frac{x+(4/15)}{11}=\frac{y-(2/5)}{-9}=\frac{z-0}{-15}=0 \quad i.e.\quad \frac{15x+4}{11}=\frac{5y-2}{-3}=\frac{z}{-1} \quad \textbf{Ans.}$$

Exercises § 4.10

Ex. 1. Find the equations of the projection of the line

$\frac{1}{2}(x-1)=-(y+1)=\frac{1}{4}(z-3)$ on the plane $x+2y+z=6$

(*Rohilkhand* 92)

Ans. $(1/4)(x-3)=-(1/7)(y+2)=-(1/10)(z-7)$

Ex. 2. Prove that the image of the line

$(x-1)=-9(y-2)=-3(z+3)$ in the plane $3x-3y+10z=26$ is the line

$$(1/9).(x-4)=+(y+1)=-(1/3)(z-7).$$

Ex. 3. Find the equation of the orthogonal projection of the line

$\frac{1}{2}(x-1)=\frac{1}{3}(y-2)=\frac{1}{4}(z-4)$ on the plane $x+3y+z+5=0.$

Ex. 4. Find the image of the line $\frac{1}{2}(x+1)=\frac{1}{3}(y+2)=\frac{1}{4}(z+3)$ in the plane $x-2y+3z-4=0$

Ans. $\dfrac{2x-3}{20}=\dfrac{4y-7}{116}=\dfrac{z-2}{16}$

****§ 4.11 Coplanar Lines.** (*Agra* 91; *Avadh* 92; *Kanpur* 95)

To find the condition that two lines whose equations are given may intersect (i.e. are coplanar) and equation of the plane in which they lie.

The equation of the given lines may be given in three ways :—

(a) both in symmetric form, (b) one in symmetric form the other in general form and (c) both in general form.

****(a) Both lines given in symmetric form.** (*Kumaun* 94)

Let the given lines be

$$\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1} \quad \text{...(i)}, \qquad \frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2} \quad \text{...(ii)}$$

The equation of any plane through the line (i) is

$$A(x-\alpha_1)+B(y-\beta_1)+C(z-\gamma_1)=0, \quad \text{...(iii)}$$

where

$$Al_1+Bm_1+Cn_1=0. \quad \text{...(iv)}$$

If the two lines given by (i) and (ii) intersect, then the line (ii) must lie on the plane (iii) *i.e.* the normal to the plane (iii) must be at right angles to (ii), consequently we get

$$Al_2+Bm_2+Cn_2=0. \quad \text{...(v)}$$

Also from (ii) it is evident that $(\alpha_2, \beta_2, \gamma_2)$ is a point on (ii).

If the line (ii) lies on the plane (iii), then this point $(\alpha_2, \beta_2, \gamma_2)$ must lie on (iii) and so we have $A(\alpha_2-\alpha_1)+B(\beta_2-\beta_1)+C(\gamma_2-\gamma_1)=0.$ ...(vi)

Hence the given line (i) and (ii) intersect *i.e.* they are coplanar, then eliminating A, B, C from (iv), (v) and (vi) we have the required condition as

$$\begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Also the equation of the plane in which the given line (i) and (ii) lie is obtained by eliminating A, B, C from (iii), (iv) and (v) and is

$$\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0.$$

To find the point of intersection of the lines (i) and (ii).

Any point on the line (i) is $(\alpha_1 + l_1 r_1, \beta_1 + m_1 r_1, \gamma_1 + n_1 r_1)$ and any point on the line (ii) is $(\alpha_2 + l_2 r_2, \beta_2 + m_2 r_2, \gamma_2 + n_2 r_2)$.

If the two lines intersect then for some values of r_1, and r_2 these points must coincide *i.e.* we have

$$\alpha_1 + l_1 r_1 = \alpha_2 + l_2 r_2, \beta_1 + m_1 r_1 = \beta_2 + m_2 r_2, \gamma_1 + n_1 r_1 = \gamma_2 + n_2 r_2$$

or $$l_1 r_1 - l_2 r_2 + (\alpha_1 - \alpha_2) = 0 \quad ...(i)$$

$$m_1 r_1 - m_2 r_2 + (\beta_1 - \beta_2) = 0 \quad ...(ii)$$

and $$n_1 r_1 - n_2 r_2 + (\gamma_1 - \gamma_2) = 0 \quad ...(iii)$$

Solving (i) and (ii) we can obtain tthe values of r_1 and r_2 and if they satisfy (iii) also then two lines intersectt and we can find the point of intersection by substituting the value of r_1 in $(\alpha_1 + l_1 r_1, \beta_1 + m_1 r_1, \gamma_1 + n_1 r_1)$ or the value of r_1 in $(\alpha_2 + l_2 r_2, \beta_2 + m_2 r_2, \gamma_2 + n_2 r_2)$.

***(b) One line in symmetric form, the other in general form.**

Let the equations of the lines be $$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad ...(i)$$

and $$ax + by + cz + d = 0 = a'x + b'y + c'z + d' \quad ...(ii)$$

The equation of the plane through the line (ii) is

$$(ax + by + cz + d) + \lambda (a'x + b'y + c'z + d') = 0 \quad ...(iii)$$

or $$(a + \lambda a')x + (b + \lambda b')y + (c + \lambda c')z + (d + \lambda d') = 0 \quad ...(iv)$$

If this plane is parallel to the line (i), then we have

$$l(a + \lambda a') + m(b + \lambda b') + n(c + \lambda c') = 0$$

or $$\lambda (a'l + b'm + c'n) = -(al + bm + cn)$$

or $$\lambda = -(al + bm + cn)/(a'l + b'm + c'n)$$

Putting this value of λ in (iii), the equation of the plane through the line (ii) and parallel to the line (i) is

$$(a'l + b'm + c'n)(ax + by + cz + d)$$
$$= (al + bm + cn)(a'x + b'y + c'z + d') \quad ...(iv)$$

If the line (i) lies in this plane then the point (α, β, γ) on the line (i) must satisfy (iv) and so condition for the lines (i) and (ii) to be coplanar is

$$(a'l + b'm + c'n)(a\alpha + b\beta + c\gamma + d) = (al + bm + cn)(a'\alpha + b'\beta + c'\gamma + d')$$

or $$\frac{a\alpha + b\beta + c\gamma + d}{al + bm + cn} = \frac{a'\alpha + b'\beta + c'\gamma + d'}{a'l + b'm + c'n}$$

Also from (iv) putting $\dfrac{a'l + b'm + c'n}{al + bm + cn} = \dfrac{a'\alpha + b'\beta + c'\gamma + d'}{a\alpha + b\beta + c\gamma + d}$ the equation of the plane in which (i) and (ii) lie is

$$\frac{ax + by + cz + d}{a\alpha + b\beta + c\gamma + d} = \frac{a'x + b'y + c'z + d'}{a'\alpha + b'\beta + c'\gamma + d'}$$

(c) Both the lines in general form.

Let the equations of the lines be

$$a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2 \quad \text{...(i)}$$
$$a_3x + b_3y + c_3z + d_3 = 0 = a_4x + b_4y + c_4z + d_4$$

If these two lines are coplanar, then they intersect and let (x_1, y_1, z_1) be their point of intersection.

If (x_1, y_1, z_1) is their point of intersection, then it satisfies all the four planes given by (i) and which intersect in the two lines.

$\therefore$ we have
$$a_1x_1 + b_1y_1 + c_1z_1 + d_1 = 0$$
$$a_2x_1 + b_2y_1 + c_2z_1 + d_2 = 0$$
$$a_3x_1 + b_3y_1 + c_3z_1 + d_3 = 0$$
$$a_4x_1 + b_4y_1 + c_4z_1 + d_4 = 0$$

Eliminating x_1, y_1 and z_1 from these we have the required conditon as

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$$

In numerical examples however, the solution of the above determinant is a tedious job and so the students are advised to transform the given equations (in the general form) into the symmetric form.

Solved Examples on § 4.11.

Ex. 1 (a). Prove that the lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad \textbf{and} \quad \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} \textbf{ are coplanar.}$$

Also find their point of intersection. (*Bundelkhand 95, 92; Garhwal 90*)

Sol. Any point on the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is

$$(1 + 2r,\ 2 + 3r,\ 3 + 4r) \quad \text{...(i)}$$

Similarly any point on the line $\frac{x-2}{3} = \frac{y-3}{3} = \frac{z-4}{5}$ is

$$(2 + 3r', 3 + 4r', 4 + 5r') \quad \text{...(ii)}$$

If the two given lines are coplanar, then they intersect and so for some values of r and r', the points (i) and (ii) must coincide.

i.e. $1 + 2r = 2 + 3r'$; $2 + 4r = 3 + 4r'$; $3 + 4r = 4 + 5r'$

Solving first and third of these we get $r = -1$, $r' = -1$.

Also these values of r and r' satisfy the second equation hence the lines intersect *i.e.* the given lines are coplanar.

Putting $r = -1$ in (i), required point of intersection is $(-1, -1, -1)$ **Ans.**

Ex. 1 (b). Prove that the lines

$$(x-1) = \frac{1}{2}(y-1) = \frac{1}{3}(z-1) \ ; \ \frac{1}{2}(x-4) = \frac{1}{3}(y-6) = \frac{1}{3}(z-7)$$

are coplanar and find the coordinates of the point of intersection.

(*Avadh 94*)

Sol. Any point on the line $\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3}$ is

$$(1+r,\ 1+2r,\ 1+3r) \qquad ...(ii)$$

Similarly any point on the line $\frac{x-4}{2}=\frac{y-6}{3}=\frac{z-7}{3}$ is

$$(4+2r',\ 6+3r',\ 7+3r') \qquad ...(ii)$$

If the two given lines are coplanar, then they intersect and so for some values of r and r', the points (i) and (ii) must coincide.

i.e. $\qquad 1+r=4+2r',\ 1+2r=6+3r',\ 1+3r=7+3r$

Solving second and third of these, we get $r=1, r'=-1$ which satisfy first equation also. Hence the lines intersect *i.e.* given lines are coplanar.

Putting $r=1$ in (i), the required point is

$$(1+1,\ 1+2,\ 1+3) \ \ i.e.\ \ (2, 3, 4). \qquad \textbf{Ans.}$$

***Ex. 2 (a). Show that the lines**

$$\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1} \quad \text{and} \quad \frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$$

intersect. Find the coordinates of the point of intersection and the equation of the plane containing them. (*Avadh* 91; *Kanpur* 97; *Meerut* 98; *Rohilkhand* 96)

Sol. Any point on the line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$ is

$$(-1-3r, 3+2r, -2+r) \qquad ...(i)$$

Similarly any point on the line $\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$ is

$$(r', 7-3r', -7+2r') \qquad ...(ii)$$

If the two given lines intersect then for some value of r and r' the two above points (i) and (ii) must coincide *i.e.*

$$-1-3r=r';\ 3+2r=7-3r';\ -2+r=-7+2r'$$

Solving the first two of these equations we get $r=-1,\ r'=2$.

These values of r and r' satisfy the third equation also, hence the given lines intersect.

Substituting these values r and r' in (i) or (ii) we get the required coordinates of the point of intersection as $(2, 1, -3)$. **Ans.**

Also the equation of the plane containing the given lines is

$$\begin{vmatrix} x+1 & y-3 & z+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix}=0, \text{ see § 4.11 (a) Page 94 Ch. IV}$$

or $\qquad (x+1)(4+3)-(y-3)(-6-1)+(z+2)(9-2)=0$

or $\qquad x+y+z=0.$ **Ans.**

Ex. 2 (b). Prove that the two lines

$$x-3=-\tfrac{1}{3}(y+4)=\tfrac{1}{3}(z-5) \text{ and } x-4=\tfrac{1}{3}(y-5)=-\tfrac{1}{4}(z+6)$$

intersect, and find the coordinates of the point of intersection.

Sol. Any point on the first line is $(3+r, -4-3r, 5+3r)$ and any point on the second line is $(4+r', 5+3r', -6-4r')$.

If these two lines intersect then for some values of r and r', these points must concide *i.e.* $3+r=4+r', -4-3r=5+3r', 5+3r=-6-4r'$

All these three equations are satisfied by $r=-1$ and $r'=-2$.

Hence the given lines intersect and putting $r=-1$ or $r'=-2$ in the above coordinates, we have the required point of intersection as

$$(3-1, -4+3, 5-3) \quad \text{or} \quad (2, -1, 2). \qquad \textbf{Ans.}$$

****Ex. 3. Prove that the lines**

$$\frac{x-a}{a'}=\frac{y-b}{b'}=\frac{z-c}{c'} \text{ and } \frac{x-a'}{a}=\frac{y-b'}{b}=\frac{z-c'}{c}$$

intersect and find the coordinates of the point of intersection and the equation of the plane in which they lie.

Sol. Any point on the first line is $(a+a'r, b+b'r, c+c'r)$ and any point on the second line is $(a'+ar', b'+br', c'+cr')$.

If these two lines intersect then for some values of r' and r these points must coincide *i.e.* $a+a'r=a'+ar'; b+b'r=b'+br'; c+c'r=c'+cr'$

Evidently all these equations are satisfied by $r=1=r'$.

Hence the given lines intersect and putting $r=1$ or $r'=1$ in the above coordinates we have the required point of intersection as

$$(a+a', b+b', c+c') \qquad \textbf{Ans.}$$

Also the equation of the plane in which the given lines lie is

$$\begin{vmatrix} x-a & y-b & z-c \\ a & b & c \\ a' & b' & c' \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} x & y & z \\ a & b & c \\ a' & b' & c' \end{vmatrix}=0,$$

adding the second row to the first.

****Ex. 4. Prove that the straight lines**

$$\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma}, \frac{x}{l}=\frac{y}{m}=\frac{z}{n} \text{ and } \frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma} \textbf{ will lie in one plane if}$$

$$\frac{l}{\alpha}(b-c)+\frac{m}{\beta}(c-a)+\frac{n}{\gamma}(a-b)=0.$$

(*Avadh* 93; *Kanpur* 91; *Meerut* 97, 95)

Sol. Here we observe that all the three given lines pass through the origin O and hence if they are coplanar they must be perpendicular to some line through O. Let d.c.'s of this line through O be l_1, m_1, n_1.

Then as this line is perpendicular to the given lines, so we have

$$l_1\alpha+m_1\beta+n_1\gamma=0 \qquad \text{...(i)}$$

$$l_1 l+m_1 m+n_1 n=0 \qquad \text{...(ii)}$$

$$l_1 a\alpha+m_1 b\beta+n_1 c\gamma=0. \qquad \text{...(iii)}$$

Eliminating l_1, m_1 and n_1 from (i), (ii) and (iii), we have the required condition as

$$\begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ a\alpha & b\beta & c\gamma \end{vmatrix} = 0 \text{ or } \begin{vmatrix} 1 & 1 & 1 \\ l/\alpha & m/\beta & n/\gamma \\ a & b & c \end{vmatrix} = 0,$$

taking α, β and γ common from first, second and third columns.

or $\quad -(l/\alpha)(c-b)+(m/\beta)(c-a)-(n/\gamma)(b-a)=0$, expanding the det. with respect to second row.

or $\quad (l/\alpha)(b-c)+(m/\beta)(c-a)+(n/\gamma)(a-b)=0$. Hence proved.

Ex. 5. Prove that the three lines drawn from the origin O with d.c.'s l_1, m_1, n_1 ; l_2, m_2, n_2 and l_3, m_3, n_3 i.e. $x/l_1 = y/m_1 = z/n_1$, $x/l_2 = y/m_2 = z/n_2$ and $x/l_3 = y/m_3 = z/n_3$ are coplanar, if

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$$

(*Agra* 90)

Sol. Since the three lines whose d.c.'s are given pass through the origin, so if they are coplanar then they must be perpendicular to the same line through the origin. Let the d.c.'s of this line through the origin be l, m, n.

Then as this line is perpendicular to the given lines, so we have

$$ll_1 + mm_1 + nn_1 = 0$$

$$ll_2 + mm_2 + nn_2 = 0$$

and
$$ll_3 + mm_3 + nn_3 = 0$$

Eliminating l, m, n between these we have the required condition as

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$$

Hence proved.

Ex. 6. Show that the lines

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}, \ \frac{x}{a\alpha} = \frac{y}{b\beta} = \frac{z}{c\gamma}, \ \frac{x}{\alpha/a} = \frac{y}{\beta/b} = \frac{z}{\gamma/c}$$

are coplanar if $a = b$ or $b = c$ or $c = a$. (*Kanpur* 92; *Meerut* 96)

Sol. As in last example, we can find that given lines are coplanar if

$$\begin{vmatrix} \alpha & \beta & \gamma \\ a\alpha & b\beta & c\gamma \\ \alpha/a & \beta/b & \gamma/c \end{vmatrix} = 0 \text{ or } \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1/a & 1/b & 1/c \end{vmatrix} = 0$$

or $\quad \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{vmatrix} = 0$, multiplying 1st, 2nd and 3rd columns by a, b and c respectively.

or $\begin{vmatrix} a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \\ 1 & 0 & 0 \end{vmatrix}=0$, substracting first column from second and third columns.

or $\begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix}=0$ or $(b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}=0$

or $(b-a)(c-a)(c-b)=0$ or $a=b; c=a; c=b.$ **Ans.**

***Ex. 7. Prove that the lines $\dfrac{x-a+d}{\alpha-\delta}=\dfrac{y-a}{\alpha}=\dfrac{z-a-d}{\alpha+\delta}$ and $\dfrac{x-b+c}{\beta-\gamma}=\dfrac{y-b}{\beta}=\dfrac{z-b-c}{\beta+\gamma}$ are coplanar and find the equation to the plane in which they lie.**

Sol. Given lines are coplanar, if

$$\begin{vmatrix} (a-d)-(b-c) & a-b & (a+d)-(b+c) \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}=0,$$

...See § 4.11 Pages 94 - 96 Ch. IV

Adding third column to first we get

$$\begin{vmatrix} 2(a-b) & a-b & (a+d)-(b+c) \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix}=0$$

The first column being twice the second column, the determinant on the left vanishes, hence the given lines are coplanar.

Also the equation of the plane in which the two given lines lie is

$$\begin{vmatrix} x-a+d & y-a & z-a-d \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix}=0$$

or $\begin{vmatrix} x+z-2a & y-a & z-a-d \\ 2\alpha & \alpha & \alpha+\delta \\ 2\beta & \beta & \beta+\gamma \end{vmatrix}=0$, adding third column to the first.

or $\begin{vmatrix} (x+z-2a)-2(y-a) & y-a & z-a-d \\ 2\alpha-2(\alpha) & \alpha & \alpha+\delta \\ 2\beta-2(\beta) & \beta & \beta+\gamma \end{vmatrix}=0$, subtracting twice second column from first.

or $$\begin{vmatrix} x+z-2y & y-a & z-a-d \\ 0 & \alpha & \alpha+\delta \\ 0 & \beta & \beta+\gamma \end{vmatrix}=0$$

or $(x+z-2y)[\alpha(\beta+\gamma)-\beta(\alpha+\delta)]=0$ or $x+z-2y=0.$ **Ans.**

Ex. 8. Show that the lines $x-4=-\frac{1}{2}(y+1)=z$ and $4x-y+5z-7=0=2x-5y-z-3$ are coplanar. Find the equation of the plane containing them. (*Rohilkhand* 95)

Sol. Any plane through the second line is

$$(4x - y + 5z - 7) + \lambda (2x - 5y - z - 3) = 0$$

or $$(4 + 2\lambda) x - (1 + 5\lambda) y + (5 - \lambda) z - (7 + 3\lambda) = 0 \quad ...(i)$$

If it is parallel to the line $\frac{x-4}{1} = \frac{y+1}{-2} = \frac{z}{1}$, then we have

$1.(4 + 2\lambda) - 2.\{-(1 + 5\lambda)\} + 1.(5 - \lambda) = 0$ or $11\lambda + 11 = 0$ or $\lambda = -1$

Hence from (i) the equation of the plane through the second line and parallel to the first is $(4 - 2) x - (1 - 5) y + (5 + 1) z - (7 - 3) = 0$

or $$2x + 4y + 6z - 4 = 0 \quad \text{or} \quad x + 2y + 3z - 2 = 0 \quad ...(ii)$$

Also from the equation of the first line it is evident that $(4, -1, 0)$ is a point on this line. And from (ii) we find that the point $(4, -1, 0)$ lies on the plane given by (ii). Hence the given lines are coplanar and the equation of the plane containing them is given by (ii).

Ex. 9. Prove that the line $(1/2)(x - 9) = -(y + 4) = (z - 5)$ and $6x + 4y - 5z = 4, x - 5y + 2z = 12$ are coplanar. Find also their point of intersection. (*Gorakhpur* 91)

Sol. Any plane through the second line is

$$(6x + 4y - 5z - 4) + \lambda (x - 5y + 2z - 12) = 0$$

or $$(6 + \lambda) x + (4 - 5\lambda) y + (2\lambda - 5) z - (4 + 12\lambda) = 0 \quad ...(i)$$

If it is parallel to the first line viz. $\frac{x-9}{2} = \frac{y+4}{-1} = \frac{z-5}{1}$, then we have

$$2.(6 + \lambda) - 1.(4 - 5\lambda) + 1.(2\lambda - 5) = 0 \quad \text{or} \quad \lambda = -1/3$$

Hence from (i) the equation of the plane through the second line and parallel to the first is

$$(6 - \tfrac{1}{3}) x + (4 + \tfrac{5}{3}) y + (-\tfrac{2}{3} - 5) z - (4 - 4) = 0$$

or $$17x + 17y - 17z = 0 \quad \text{or} \quad x + y - z = 0 \quad ...(ii)$$

Also from the equation of the first line it is evident that $(9, -4, 5)$ is a point on this line. And from (ii) we find that the point $(9, -4, 5)$ lies on the plane given by (ii). Hence the given lines are coplanar *i.e.* they intersect.

Now any point on the first line is $(9 + 2r, -4 - r, 5 + r)$ ...(iii)

As the two given lines intersect, therefore for some value of r, the point given by (iii) lies on the second line and so satisfies the equations of the two planes which constitute the second line and so we have

$$6(9 + 2r) + 4(-4 - r) - 5(5 + r) = 4$$

and $$(9 + 2r) - 5(-4 - r) + 2(5 + r) = 12.$$

Both of these give $r = -3$ and so from (iii) the required point of intersection is $(9 - 6, -4 + 3, 5 - 3)$ *i.e.* $(3, -1, 2)$. **Ans.**

***Ex. 10. Show that the lines**

$$\tfrac{1}{2}(x-1)=\tfrac{1}{3}(y-2)=\tfrac{1}{4}(z-3) \text{ and } 4x-3y+1=0=5x-3z+2$$

are coplanar. Also find their point of intersection.

(*Gorakhpur* 92; *Rohilkhand* 97, 94, 90)

Sol. Any plane through the second line is

$$(4x-3y+1)+\lambda(5x-3z+2)=0$$

or
$$(4+5\lambda)x-3y-3\lambda z+(1+2\lambda)=0. \quad \text{...(i)}$$

If it is parallel to the line $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}$, then we have

$$2.(4+5\lambda)+3.(-3)+4.(-3\lambda)=0 \text{ or } -2\lambda-1=0 \quad \text{or} \quad \lambda=-\tfrac{1}{2}.$$

Hence from (i) the equation of the plane through the second line and parallel to the first line is

$$[4-(5/2)]\,x-3y+(3/2)\,z+[1-2.(1/2)]=0$$

or
$$(3/2)\,x-3y+(3/2)\,z=0 \quad \text{or} \quad x-2y+z=0. \quad \text{...(ii)}$$

Also from the equations of the first line it is evident that (1, 2, 3) is a point on this line. And we find from (ii) that the point (1, 2, 3) lies on this plane (ii). Hence the given lines are coplanar *i.e.* they intersect.

To find the point of intersection : Any point on the first line is

$$(1+2r, 2+3r, 3+4r). \quad \text{...(iii)}$$

As the two given lines intersect therefore for some value of r the point (iii) lies on the second line and so we have

$$4(1+2r)-3(2+3r)+1=0$$

and
$$5(1+2r)-3(3+4r)+2=0$$

Both of these give $r=-1$ and therefore from (iii) the required point of intersection is $(1-2, 2-3, 3-4)$ or $(-1, -1, -1)$. **Ans.**

***Ex. 11. Prove that the lines $3x-5=4y-9=3z$ and $x-1=2y-4=3z$ meet in a point and the equation of the plane in which they lie is $3x-8y+13=0$.**

Sol. The equations of the line $3x-5=4y-9=3z$ can be rewritten as

$$\frac{x-(5/3)}{(\frac{1}{3})}=\frac{y-(9/4)}{(\frac{1}{4})}=\frac{z}{(\frac{1}{3})} \quad \text{or} \quad \frac{x-(5/3)}{4}=\frac{y-(9/4)}{3}=\frac{z}{4} \quad \text{...(i)}$$

Similarly the equation of the second line can be rewritten as

$$\frac{x-1}{1}=\frac{y-2}{(\frac{1}{2})}=\frac{z}{(\frac{1}{3})} \quad \text{or} \quad \frac{x-1}{6}=\frac{y-2}{3}=\frac{z}{2} \quad \text{...(ii)}$$

Any point on the line (i) is $[(5/3)+4r, (9/4)+3r, 4r]$ ...(iii)

and any point on the line (ii) is $(1+6r', 2+3r', 2r')$ ...(iv)

$\therefore$ If the two lines meet in a point, then for some values of r and r' the points given by (iii) and (iv) must be identical.

i.e. $\quad (5/2)+4r=1+6r'\,;\ (9/4)+3r=2+3r';\ 4r=2r'.$

Solving first and third of these we get $r = 1/12$, $r' = 1/6$ which satisfy the second viz. $(9/4) + 3r = 2 + 3r'$ also. Hence the two lines intersect *i.e.* are coplanar.

[Putting $r = (1/12)$ in (iii) we can get the point of intersection as $(2, \frac{5}{2}, \frac{1}{3})$.

Also the equation of the plane through the line given by (i) and (ii) is

$$\begin{vmatrix} x-1 & y-2 & z \\ 6 & 3 & 2 \\ 4 & 3 & 4 \end{vmatrix} = 0$$ ...See § 4.11 (a) Pages 94–95 Chapter IV

or $\quad 6(x-1) - 16(y-2) + 6(z) = 0 \quad$ or $\quad 3x - 8y + 3z + 13 = 0.$

Hence proved.

***Ex. 12 (a). Show that the lines $x + y + z - 3 = 0 = 2x + 3y + 4z - 5$ and $4x - y + 5z - 7 = 0 = 2x - 5y - z - 3$ are coplanar and find the plane in which they lie.** (*Avadh* 95; *Lucknow* 90; *Purvanchal* 90)

Sol. Let l, m, n be the d.c.'s of the line $x + y + z = 3$, $2x + 3y + 4z = 5$

Then we must have $l + m + n = 0$, $2l + 3m + 4n = 0$

...See § 4.04 P. 58 Ch IV

or $$\frac{l}{4-3} = \frac{m}{2-4} = \frac{n}{3-2} \quad \text{or} \quad \frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$$

which gives the direction ratios of the first line.

Let $(x_1, y_1, 0)$ be any point on this line, then we have

$$x_1 + y_1 = 3 \quad \text{and} \quad 2x_1 + 3y_1 = 5 \qquad \textbf{(Note)}$$

Solving these we get $x_1 = 4$, $y_1 = -1$

$\therefore$ Any point on the first line is $(4, -1, 0)$, so the equation of the first line in the symmetric form can be written as

$$\frac{x-4}{1} = \frac{y+1}{-2} = \frac{z-0}{1} \qquad ...(i)$$

Similarly the equations of the other given line in the symmetric form can be written as $$\frac{x_1 - (11/7)}{13} = \frac{y_1 - 0}{7} = \frac{z_1 - (1/7)}{-9} \qquad ...(ii)$$

Any point on the line (i) is $(4 + r, -2r - 1, r)$ ...(iii)

and on the line (ii) is $\left(13r' + \frac{11}{7}, 7r', -9r' + \frac{1}{7}\right)$ ...(iv)

If these two lines meet in a point, then for some values of r and r', the points given by (iii) and (iv) must be identical.

i.e. $\quad 4 + r = 13r' + (11/7), -2r - 2 = 7r', r = -9r' + (1/7)$

Solving last two of these we get $r = -10/11$, $r' = 9/77$ which satisfy first viz. $4 + r = 13r' + (11/7)$ also.

Hence the two given lines intersect *i.e.* are coplanar.

[Putting $r' = 9/77$ in (iv), the point of intersection of the given lines can be found to be (34/11, 9/11, 10/11).]

The equation of the plane in which these lines lie is

$$\begin{vmatrix} x-4 & y+1 & z-0 \\ 1 & -2 & 1 \\ 13 & 7 & -9 \end{vmatrix} = 0, \qquad \text{...See § 4.11 (a) Pages 94–95 Chapter IV}$$

or $$(x-4)[18-7]-(y+1)[-9-13]+z[7+26]=0$$

or $$(x-4)+(y+1)(2)+z(3)=0 \quad \text{or} \quad x+2y+3z=2 \qquad \textbf{Ans.}$$

***Ex. 12 (b). Show that the line $x + 2y - z = 3$, $3x - y + 2z = 1$ is coplanar with the line $2x - 2y + 3z = 2$, $x - y + z + 1 = 0$ and find the plane in which these two lines lie.**

Sol. Do as Ex. 12 (a) above. **Ans.** $7x - 7y + 8z + 3 = 0$

***Ex. 13. Prove that the lines $x = ay + b = cz + d$ and $x = \alpha y + \beta = \gamma z + \delta$ are coplanar if $(\gamma - c)(a\beta - b\alpha) - (\alpha - a)(c\delta - d\gamma) = 0$.** *(Kanpur 91)*

Sol. The equations of the given lines can be written in the symmetric form as $\dfrac{x-0}{1}=\dfrac{y+(b/a)}{(1/a)}=\dfrac{z+(d/c)}{(1/c)}$; $\dfrac{x-0}{1}=\dfrac{y+(\beta/\alpha)}{(1/\alpha)}=\dfrac{z+(\delta/\gamma)}{(1/\gamma)}$

These lines will be coplanar if

$$\begin{vmatrix} 0-0 & \dfrac{b}{a}-\dfrac{\beta}{\alpha} & \dfrac{d}{c}-\dfrac{\delta}{\gamma} \\ 1 & \dfrac{1}{a} & \dfrac{1}{c} \\ 1 & \dfrac{1}{\alpha} & \dfrac{1}{\gamma} \end{vmatrix} = 0, \qquad \text{See § 4.11 (a) Pages 94-95, Chapter IV}$$

or $$\begin{vmatrix} 0 & (b\alpha - a\beta)/a\alpha & (d\gamma - c\delta)/c\gamma \\ 0 & \dfrac{1}{a}-\dfrac{1}{\alpha} & \dfrac{1}{c}-\dfrac{1}{\gamma} \\ 1 & \dfrac{1}{\alpha} & \dfrac{1}{\beta} \end{vmatrix} = 0, \text{ substituting third row from second}$$

or $$\begin{vmatrix} (b\alpha - a\beta)/a\alpha & (d\gamma - c\delta)/c\gamma \\ (\alpha - a)/a\alpha & (\gamma - c)/c\gamma \end{vmatrix} = 0,$$

or $$\begin{vmatrix} b\alpha - a\beta & d\gamma - c\delta \\ \alpha - a & \gamma - c \end{vmatrix} = 0$$

or $$(b\alpha - a\beta)(\gamma - c) - (\alpha - a)(d\gamma - c\delta) = 0$$

or $$(\gamma - c)(a\beta - b\alpha) - (\alpha - a)(c\delta - d\gamma) = 0. \qquad \text{Hence proved.}$$

Ex. 14. Show that the line of intersection of the planes $7x - 4y + 7z + 16 = 0$, $4x + 3y + 3y - 2z + 3 = 0$ is coplanar with the line of intersection of $x - 3y + 4z + 6 = 0$, $x - y + z + 1 = 0$.

Sol. The given lines will be coplanar if

$$\begin{vmatrix} 7 & -4 & 7 & 16 \\ 4 & 3 & -2 & 3 \\ 1 & -3 & 4 & 6 \\ 1 & -1 & 1 & 1 \end{vmatrix} = 0 \qquad \text{...See § 4.11 (c) Page 96 Chapter IV}$$

or $\begin{vmatrix} 3 & -4 & 3 & 12 \\ 7 & 3 & 1 & 6 \\ -2 & -3 & 1 & 3 \\ 0 & -1 & 0 & 0 \end{vmatrix} = 0$, adding 2nd column to the rest.

or $\begin{vmatrix} 3 & 3 & 12 \\ 7 & 1 & 6 \\ -2 & 1 & 3 \end{vmatrix} = 0$, expanding with respect to last row

or $\begin{vmatrix} 9 & 0 & 3 \\ 9 & 0 & 3 \\ -2 & 1 & 3 \end{vmatrix} = 0,$

The determinant on the left vanishes as two rows are ideintical and hence the given lines are coplanar.

***Ex. 15. A, A′ ; B, B′ ; C, C′ are points on the coordinates axes. Prove that the lines of intersection of the planes A′BC, AB′ C′ ; B′ CA, BC′ A′ and C′ AB, CA′ B′ are coplanar.** (*Kanpur* 96; *Meerut* 90)

Sol. Let A and A' be $(a, 0, 0)$ and $(a', 0, 0)$; B and B' be $(0, b, 0)$ and $(0, b', 0)$; C and C' be $(0, 0, c)$ and $(0, 0, c')$.

Then the equations (intercept form) of the planes $A'BC$ and $AB'C'$ are

$$\frac{x}{a'}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{and} \quad \frac{x}{a}+\frac{y}{b'}+\frac{z}{c'}=1$$

∴ The equation of the plane through the line of intersection of these two planes is $\left(\frac{x}{a'}+\frac{y}{b}+\frac{z}{c}-1\right)+\lambda\left(\frac{x}{a}+\frac{y}{b'}+\frac{z}{c'}-1\right)=0$, for some value of λ.

If we take $\lambda = 1$, the line of intersection lies in the plane

$$\left(\frac{x}{a'}+\frac{y}{b}+\frac{z}{c}-1\right)+\left(\frac{x}{a}+\frac{y}{b'}+\frac{z}{c'}-1\right)=0$$

or $$\left(\frac{1}{a}+\frac{1}{a'}\right)x+\left(\frac{1}{b}+\frac{1}{b'}\right)y+\left(\frac{1}{c}+\frac{1}{c'}\right)z=2.$$

The symmetry in this equation indicates that the lines of intersection of the other two pairs of planes also lie in this plane and hence the lines of intersection of the given pairs of planes are coplanar.

****Ex. 16. Find the equations of the plane through the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ and perpendicular to the plane containing the lines.**

$$\frac{x}{m}=\frac{y}{n}=\frac{z}{l} \quad \text{and} \quad \frac{x}{n}=\frac{y}{l}=\frac{z}{m}$$

(*Agra* 91; *Kanpur* 95; *Purvanchal* 93; *Rohilkhand* 91)

Sol. The equation of the plane containing the lines

$\frac{x}{m}=\frac{y}{n}=\frac{z}{l}$ and $\frac{x}{n}=\frac{y}{l}=\frac{z}{m}$, both of which pass through the origin is

$$\begin{vmatrix} x & y & z \\ m & n & l \\ n & l & m \end{vmatrix}=0$$

or
$$(mn-l^2)\,x+(ln-m^2)\,y+(ml-n^2)\,z=0. \qquad \text{...(i)}$$

Also the equation of the plane through the line $x/l=y/m=z/n$ is

$$ax+by+cz=0 \qquad \text{...(ii)}$$

where
$$al+bm+cn=0, \qquad \text{...(iii)}$$

(See § 4.06 Page 69 Ch. IV)

If the planes (i) and (ii) are perpendicular, the we have

$$a\,(mn-l^2)+b\,(nl-m^2)+c\,(ml-n^2)=0 \qquad \text{...(iv)}$$

Solving (iii) and (iv), we get

$$\frac{a}{m\,(ml-n^2)\;-n\,(nl-m^2)}=\frac{b}{n\,(nm-l^2)-l\,(ml-n^2)}$$
$$=\frac{c}{l\,(nl-m^2)-m\,(nm-l^2)}$$

or
$$\frac{a}{(m^2-n^2)\,l+mn\,(n-m)}=\frac{b}{(n^2-l^2)\,m+nl\,(n-l)}=\frac{c}{(l^2-m^2)\,n+lm\,(l-m)}$$

or
$$\frac{a}{(m-n)\,(lm+mn+nl)}=\frac{b}{(n-l)\,(nm+lm+nl)}=\frac{c}{(l-m)(ln+nm+lm)}$$

or
$$\frac{a}{m-n}=\frac{b}{n-l}=\frac{c}{l-m}$$

$\therefore$ from (ii), equation of the required plane is

$$(m-n)\,x+(n-l)\,y+(l-m)\,z=0$$ **Ans.**

Exercises on § 4.11

Ex. 1. Show that the line $\frac{1}{2}(x-1)=\frac{1}{3}(y-1)=\frac{1}{4}(z-1)$ and $\frac{1}{3}(x-5)=\frac{1}{2}(y-7)=(z-9)$ are coplanar and find the equation of the plane containing them.

Ans. $x-2y+z=0$

Ex. 2. Prove that the two lines $2\,(x-1)=\frac{1}{2}(y-1)=\frac{1}{3}(z-1)$ and $\frac{1}{2}(x-2)=\frac{1}{3}(y-5)=\frac{1}{3}(z-7)$ are coplanar and find their point of intersection.

Ans. (2, 5, 7)

Ex. 3. Prove that the lines $\frac{1}{2}(x+1)=\frac{1}{3}(y-2)=\frac{1}{4}(z-3)$ and $\frac{1}{3}(x+2)=\frac{1}{4}(y-3)=\frac{1}{5}(z-4)$ are coplanar.

Ex. 4. Prove that the lines $\frac{1}{3}(x-2)=\frac{1}{4}(y-3)=\frac{1}{5}(z-4)$ and $2x-3y+z=0=x+y+2z+4$ are coplanar. Also find their point of intersection. **Ans.** $(-1,-1,-1)$

Ex. 5. Show that the lines $\frac{1}{3}(x+5)=(y+4)=-\frac{1}{2}(z-7)$ and $3x+2y+z-2=0=x-3y+2z-13$ are coplanar and find the equation to the plane in which they lie. **Ans.** $21x-19y+22z=125$. (*Purvanchal* 97)

Ex. 6. Prove that the line $\frac{1}{2}(x+1)=\frac{1}{3}(y+2)=\frac{1}{4}(z+3)$ and $\frac{1}{3}(x-3)=\frac{1}{4}(y-2)=\frac{1}{5}(z-1)$ are coplanar and determine the plane containing them.

Ex. 7. Show that the lines $\frac{1}{2}(x+3)=\frac{1}{3}(y+5)=-\frac{1}{3}(z-7)$ and $\frac{1}{4}(x+1)=\frac{1}{5}(y+1)=-(z+1)$ are coplanar. Find the equation of the plane containing them. **Ans.** $6x-5y-z=0$ (*Meerut* 92)

Ex. 8. Show that the lines $\frac{1}{4}(x-5)=\frac{1}{7}(y-7)=-\frac{1}{5}(z+3)$ and $\frac{1}{7}(x-8)=(y-4)=\frac{1}{3}(z-5)$ are coplanar. Find their common point and equation of the plane in which they lie. **Ans.** $(1,3,2)$; $17x-7y-24z+172=0$

***Ex. 9.** Prove that the lines $x=\frac{1}{2}(y-2)=\frac{1}{3}(z+3)$ and $\frac{1}{2}(x-2)=\frac{1}{3}(y-6)=\frac{1}{4}(z-3)$ are coplanar and find the equation of the plane in which they lie and the point of intersection. **Ans.** $x-2y+z+7=0$; $(2,6,3)$

Ex. 10. Show that the lines $(x-4)=-\frac{1}{4}(y+3)=\frac{1}{7}(z+1)$ and $\frac{1}{2}(x-1)=-\frac{1}{3}(y+1)=\frac{1}{8}(z+10)$ intersect and find the coordinates of the point of intersection. **Ans.** $(5,-7,6)$ (*Bundelkhand* 93; *Kumaun* 90)

Ex. 11. Show that the lines $x+2y-5z+9=0=3x-y+2z-5$ and $2x+3y-z-3=0=4x-5y+z+3$ are coplanar.

***Ex. 12.** Show that the two straight lines $x=mz+a$, $y=nz+b$ and $x=m'z+a'$, $y=n'z+b'$ would intersect only if

$$(a-a')(n-n')=(b-b')(m-m')$$

§ 4.12. To obtain the equations of a straight line intersecting two given lines.

Case I. Given Lines in Symmetric Form.

Let the given lines be $$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \text{ (say)} \qquad \text{...(i)}$$

and $$\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}=r' \text{ (say)} \qquad \text{...(ii)}$$

Any point on the line (i) is $(\alpha + lr, \beta + mr, \gamma + ar)$ and any point on line (ii) is $(\alpha' + l'r', \beta' + m'r', \gamma' + n'r')$.

The required line is the line which joins these two points for some values of r and r' which will be obtained from other given conditions.

Case II. Given lines in General form.

Let the given lines be $u = 0, v = 0,$ and $u' = 0, v' = 0$

Then the equations of the required line are

$$u + kv = 0 \quad \text{and} \quad u' + k'v' = 0,$$

where k and k' will be obtained by another condition.

Solved Examples on § 4.12.

Ex. 1 (a). Find the equations to the line which intersect the lines $2x + y - 4 = 0 = y + 2z$ and $x + 3z = 4$, $2x + 5z = 8$ and passes through the point $(2, -1, 1)$.

Sol. The equations of the required line are given by

$$(2x + y - 4) + \lambda(y + 2z) = 0 \quad \text{...(i)}$$

and

$$(x + 3z - 4) + \mu(2x + 5z - 8) = 0 \quad \text{...(ii)}$$

If the above lines pass through $(2, -1, 1)$, then

$$(4 - 1 - 4) + \lambda(-1 + 2) = 0 \quad \text{and} \quad (2 + 3 - 4) + \mu(4 + 5 - 8) = 0$$

i.e. $\quad -1 + \lambda = 0 \quad$ and $\quad 1 + \mu = 0 \quad$ *i.e.* $\quad \lambda = 1, \ \mu = -1$

$\therefore$ From (i) and (ii), the required equations are

$$(2x + y - 4) + (y + 2z) = 0, \ (x + 3z - 4) - (2x + 5z - 8) = 0$$

or

$$2x + 2y + 2z = 4, \ -x - 2z + 4 = 0$$

or

$$x + y + z = 2, \ x + 2z = 4. \qquad \textbf{Ans.}$$

Ex. 1 (b). Find the equations of the straight line drawn from the origin to intersect the lines $3x + 2y + 4z - 5 = 0 = 2x - 3y + 4z + 1$ and $2x - 4y + z + 6 = 0 = 3x - 4y + z - 3$ (*Garhwal* 95)

Sol. The equation of the required line are given by

$$(3x + 2y + 4z - 5) + \lambda(2x - 3y + 4z + 1) = 0 \quad \text{...(i)}$$

and

$$(2x - 4y + z + 6) + \mu(3x - 4y + z - 3) = 0 \quad \text{...(ii)}$$

If the above line passes through the origin (0, 0, 0), then

$$-5 + \lambda = 0 \quad \text{and} \quad 6 - 3\mu = 0 \quad i.e. \quad \lambda = 5, \ \mu = 2$$

$\therefore$ From (i) and (ii) the required equations are

$$(3x + 2y + 4z - 5) + 5(2x - 3y + 4z + 1) = 0$$

and

$$(2x - 4y + z + 6) + 2(3x - 4y + z - 3) = 0$$

or

$$13x - 13y + 24z = 0 \quad \text{and} \quad 8x - 12y + 3z = 0 \qquad \textbf{Ans.}$$

Ex. 2 (a). Find the equation to the line drawn parallel to $\frac{1}{4}x = y = z$ so as to meet the lines $5x - 6 = 4y + 3 = z$ and $2x - 4 = 3y + 5 = z$.

Sol. The equations of the lines intersecting the given lines are

$$\{(5x - 6) - (4y + 3)\} + k\{(4y + 3) - z\} = 0$$

and

$$\{(2x - 4) - (3y + 5)\} + k'\{(3y + 5) - z\} = 0$$

$$\left.\begin{array}{ll}\text{or} & 5x+4(k-1)y-kz+(3k-9)=0 \\ \text{and} & 2x+3(k'-1)y-k'z+(5k'-9)=0\end{array}\right\}\quad ...(i)$$

If the line given by (i) is parallel to the line $x/4=y/1=z/1$ ...(ii)
then the line (ii) is perpendicular to the normals to each of the planes given by (i) and so we have

$$4.5+1.4(k-1)+1(-k)=0 \quad \text{or} \quad k=-16/3$$

and $$4.2+1.3(k'-1)+1(-k')=0 \quad \text{or} \quad k'=-5/2.$$

Substituting these values of k and k' in (i) we have the required equations as $15x-76y+16z-75=0$, $4x-21y+5z-43=0$. **Ans.**

Ex. 2 (b). Find the equations to the line that intersects the lines $2x+y-1=0=x-2y+3z$; $3x-y+z+2=0=6x+5y-2z-3$ and is parallel to the line $x=\frac{1}{2}y=\frac{1}{3}z$. *(Purvanchal 96)*

Sol. The equations of the lines intersecting the given lines are

$$(2x+y-1)+\lambda(x-2y+3z)=0$$

and $$(3x-y+z+2)+\mu(6x+5y-2z-3)=0$$

$$\left.\begin{array}{ll}\text{or} & (2+\lambda)x+(1-2\lambda)y+3\lambda z+(3\lambda-1)=0 \\ \text{and} & (3+6\mu)x+(5\mu-1)y+(1-2\mu)z+(2-3\mu)=0\end{array}\right\}\quad ...(i)$$

If the line given by (i) is parallel to the line $x/1=y/2=z/3$ ...(ii)
the line (ii) is perpendicular to the normals to each of the planes given by (i) and so we have

$$1.(2+\lambda)+2(1-2\lambda)+3(3\lambda)=0 \quad \text{or} \quad \lambda=-2/3$$

and $$1.(3+6\mu)+2(5\mu-1)+3(1-2\mu)=0 \quad \text{or} \quad \mu=-2/5$$

Substituting these values of λ and μ in (i), the required equations are

$$4x+7y-6z-9=0,\ 3x-15y+9z+16=0.$$ **Ans.**

Ex. 3 (a). A line with direction ratios 7, – 5, 2 is drawn to intersect the lines $\frac{x-7}{-1}=\frac{y+2}{1}=\frac{z-5}{3}$, $\frac{x-3}{2}=\frac{y-5}{4}=\frac{z+3}{-3}$.

Find the coordinates of the points of intersection and the length intercepted on it.

Sol. Given lines are $$\frac{x-7}{-1}=\frac{y+2}{1}=\frac{z-5}{3}=r\ (\text{say}) \quad ...(i)$$

and $$\frac{x-3}{2}=\frac{y-5}{4}=\frac{z+3}{-3}=r'\ (\text{say}) \quad ...(ii)$$

Any point on (i) is $P(7-r,-2+r,5+3r)$ and any point on (ii) is

$$P'(3+2r',5+4r',-3-3r'). \quad ...(iii)$$

$\therefore$ The direction ratios of the line PP' are

$$[(3+2r')-(7-r),(5+4r')-(-2+r),(-3-3r')-(5+3r)]$$

or $$(r+2r'-4,-r+4r'+7,-3r-3r'-8)$$

But the direction ratios of PP' are given to be $7,-5,2$ so we get

$$\frac{r+2r'-4}{7}=\frac{-r+4r'+7}{-5}=\frac{-3r-3r'-8}{2}$$

which gives $7(-r+4r'+7)=-5(r+2r'-4)$

and $2(r+2r'-4)=7(-3r-3r'-8)$

or $2r-38r'-29=0$ and $23r+25r'+48=0$

Solve these for r and r'. Substituting the values so obtained in (iii) we get the coordinates of the required points of intersection P and P' and then find the length of PP'

Ex. 3 (b). A line with d.c.'s proportional to (2, 7, −5) is drawn to intersect the lines $\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1}$ **and** $\frac{x-3}{3}=\frac{y-3}{2}=\frac{z-6}{4}$

Find the coordinates of the point of intersection and the length intercepted on it.

Hint : Do as Ex. 3 (a) above. **Ans.** (2, 8, −3), (0, 1, 2) and $\sqrt{(78)}$

***Ex. 4. Find the equations of the line intersecting the lines** $x-a=y=z-a$, $x+a=y=\frac{1}{2}(z+a)$ **and parallel to the line**

$$\frac{1}{2}(x-a)=y-a=\frac{1}{3}(z-2a)$$

Sol. Any point on the line $x-a=y=z-a=r$ (say)

is $$P(a+r, r, a+r) \quad ...(i)$$

and any point on the line $x+a=y=\frac{1}{2}(z+a)=r'$ (say)

is $$P'(-a+r', r', -a+2r') \text{ (say)} \quad ...(ii)$$

∴ The direction ratios of PP' are

$$\{(a+r)-(-a+r'), r-r', (a+r)-(-a-2r')\}$$

or $$\{r-r'+2a, r-r', r-2r'+2a\} \quad ...(iii)$$

But the line PP' is parallel to the line $\frac{x-a}{2}=\frac{y-a}{1}=\frac{z-2a}{3}$

i.e. the direction ratios of PP' are 2, 1, 3.

∴ From (iii) we get $\frac{r-r'+2a}{2}=\frac{r-r'}{1}=\frac{r-2r'+2a}{3}$

which gives $r-r'+2a=2(r-r')$; $3(r-r')=r-2r'+2a$

or $r-r'-2a=0$; $2r-r'-2a=0$

Solving these we get $r=0$ and $r'=-2a$.

Substituting the values of r and r' in (i) and (ii) we find that P is $(a, 0, a)$ and P' is $(-3a, -2a, -5a)$.

∴ The required equation of PP' is $\frac{x-a}{2}=\frac{y-0}{1}=\frac{z-a}{3}$ **Ans.**

Ex. 5. Find the equations of the straight line drawn through the origin which will intersect both the lines

$$\frac{1}{2}(x-1)=\frac{1}{3}(y-2)=\frac{1}{4}(z-3),\ \frac{1}{4}(x+2)=\frac{1}{3}(y-3)=\frac{1}{2}(z-4).$$

Sol. The equation of the plane containing the first line is

$$A(x-1)+B(y-2)+C(z-3)=0, \quad ...(i)$$

where $$2A+3B+4C=0 \qquad ...(ii)$$

If this plane passes through the origin, then we have

$$A(-1)+B(-2)+C(-3)=0 \qquad ...(iii)$$

Solving (ii) and (iii) we get $\dfrac{A}{-1}=\dfrac{B}{2}=\dfrac{C}{-1}$

$\therefore$ From (i) the plane through the first line and the origin is given by

$$-(x-1)+2(y-2)-(z-3)=0 \quad \text{or} \quad x-2y+z=0 \qquad ...(iv)$$

Also the equation of the plane through the given second line is

$$a(x+2)+b(y-3)+c.(z-4)=0, \qquad ...(v)$$

where $$a.4+b.3+c.2=0 \qquad ...(vi)$$

If this plane given by (v) passes through the origin, then

$$a(2)+b(-3)+c(-4)=0 \qquad ...(vii)$$

Solving (vi) and (vii) we get $\dfrac{a}{-6}=\dfrac{b}{20}=\dfrac{c}{-18}$

$\therefore$ From (v), equation of the plane through the second line and origin is

$$-6(x+2)+20(y-3)-18(z-4)=0 \quad \text{or} \quad 3x-10y+9z=0 \qquad ...(viii)$$

$\therefore$ The required line is given by (iv) and (viii).

Ex. 6. Show that the equations of the line through (a, b, c) which is parallel to the plane lx + my + nz = 0 and intersects the line

$$A_1x+B_1y+C_1z+D_1=0=A_2x+B_2y+C_2z+D_2$$

are $$l(x-a)+m(y-b)+n(z-c)=0$$

and $$\frac{A_1x+B_1y+C_1z+D_1}{A_1a+B_1b+C_1c+D_1}=\frac{A_2x+B_2y+C_2z+D_2}{A_2a+B_2b+C_2c+D_2} \qquad \textit{(Avadh 90)}$$

Sol. The required line is the line of intersection of the two planes as stated below :—

(i) The palne through the point (a, b, c) and parallel to the given plane $lx+my+nz=0$ is given by $l(x-a)+m(y-b)+n(z-c)=0$. ...(i)

(ii) The plane through the point (a, b, c) and the line

$$(A_1x+B_1y+C_1z+D_1)+\lambda(A_2x+B_2y+C_2z+D_2)=0, \qquad ...(ii)$$

where λ is calculated from the condition that the plaen (ii) passes through (a, b, c).

$\therefore$ From (ii), $(A_1a+B_1b+C_1c+D_1)+\lambda(A_2a+B_2b+C_2c+D_2)=0,$

or $$\lambda=-(A_1a+B_1b+C_1c+D_1)/(A_2a+B_2b+C_2c+D_2)$$

$\therefore$ From (ii) putting this value of λ the equation of the plane through (a, b, c) and the given line is

$$\frac{A_1x+B_1y+C_1z+D_1}{A_1a+B_1b+C_1c+D_1}=\frac{A_2x+B_2y+C_2z+D_2}{A_2a+B_2b+C_2c+D_2} \qquad ...(iii)$$

Here the equations of the required line are given by (i) and (iii).

Ex. 7. Find the equations of the line through the point (– 4, 3, 1) parallel to the plane x + 2y – z = 5 so as to intersect the line $-\frac{1}{3}(x+1)=\frac{1}{2}(y-3)=-(z-2)$. Find also the point of intersection.

Sol. Let P be the point $(-4, 3, 1)$.

Let the line through $P(-4, 3, 1)$ parallel to the given plane meet the given line $\frac{x+1}{-3}=\frac{y-3}{2}=\frac{z-2}{-1}=r$ (say) in Q.

Then the coordinates of Q may be taken as $(-1-3r, 3+2r, 2-r)$. ...(i)

$\therefore$ Direction ratios of PQ are $[-4-(-1-3r), 3-(3+2r), 1-(2-r)]$

or $$(3r-3, -2r, r-1) \quad ...(ii)$$

But PQ is parallel to the plane $x+2y-z=5$. ...(iii)

the direction ratios of whose normal are $1, 2, -1$.

Since PQ is parallel to the plane (iii), so PQ is perp. to the normal to the plane (iii) and so we have $(3r-3).1+(-2r).2+(r-1)(-1)=0$

or $$-2r-2=0 \quad \text{or} \quad r=-1$$

Hence from (i) the point of intersection Q is $(2, 1, 3)$. **Ans.**

Also from (ii) the direction ratios of PQ are $-6, 2, -2$ or $3, -1, 1$.

$\therefore$ The equation of the line PQ are

$$\frac{x-(-4)}{3}=\frac{y-3}{-1}=\frac{z-1}{1} \quad \text{or} \quad \frac{x+4}{3}=\frac{y-3}{-1}=\frac{z-1}{1}$$ **Ans.**

****Ex. 8. Show that the equation of the straight line through the origin cutting each of the lines**

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \quad \text{and} \quad \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

are $$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ l_1 & m_1 & n_1 \end{vmatrix} = 0 = \begin{vmatrix} x & y & z \\ x_2 & y_2 & z_2 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

Sol. Equation of the plane through the first line is

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0, \quad ...(i)$$

where $$Al_1+Bm_1+Cn_1=0 \quad ...(ii)$$

If this plane passes through the origin then from (i) we have

$$A(-x_1)+B(-y_1)+C(-z_1)=0 \quad \text{or} \quad Ax_1+By_1+Cz_1=0 \quad ...(iii)$$

Eliminating A, B, C from (i), (ii) and (iii) we get

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ x_1 & y_1 & z_1 \end{vmatrix} = 0$$

or $\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$, adding third row to first and interchanging second and third rows. ...(iv)

Similarly the equation of the plane through the origin and second line is

$$\begin{vmatrix} x & y & z \\ x_2 & y_2 & z_2 \\ l_1 & m_2 & n_2 \end{vmatrix} = 0 \quad ...(v)$$

The planes (iv) and (v) taken together give the required line.

Hence proved.

Exercises on § 4.12.

Ex. 1. Find the equations to the line intersecting the lines $x-1=y=z-1, 2x+2=2y=z+1$ and parallel to the line
$\frac{1}{2}(x-1)=(y-1)=\frac{1}{3}(z-2)$. **Ans.** $\frac{1}{2}(x-1)=y=\frac{1}{3}(z-1)$

Ex. 2. Find the equations to the straight line drawn through the origin which will intersect both the lines.

$$(x-1)=\frac{1}{4}(y+3)=\frac{1}{3}(z-5); \frac{1}{2}(x-4)=\frac{1}{2}(y+3)=\frac{1}{4}(z-14).$$

Ans. $29x-2y-7z=0=20x-6y-7z.$

Ex. 3. A line with direction cosines proportional to (2, 1, 1) meet each of the lines given by the equations $x=y+a=z; x+a=2y=2z$.

Find the coordinates of each of the points of intersection.

Ans. $(3a, 2a, 3a), (a, a, a)$

Ex. 4. Find the equations to the line which can be drawn from the point $(2, -1, 3)$ to intersect the lines

$$\frac{1}{2}(x+1)=\frac{1}{3}(y-2)=\frac{1}{3}(z-3) \quad \text{and} \quad \frac{1}{4}(x-4)=\frac{1}{2}y=\frac{1}{3}(z+3).$$

Ans. $12x+4y-9z+7=0=11x-10y+2z-38.$

Ex. 5. Find the equations of the line through the point (3, 1, 2) intersecting the line $x+3=y+1=2(z-2)$ and parallel to the plane

$4x+y+5z=0.$ **Ans.** $\frac{x-3}{10}=\frac{y-1}{-11}=\frac{z-2}{-13}$

Ex. 6. Find the equations to the line through the point (1, 2, 3) and intersecting the lines.

$$x+2y-3z=0=3x-2y-z \quad \text{and} \quad x-y-5=0=z-x+5.$$

**§ 4.13. Intersection of three planes. *(Meerut 90)*

Let the planes be $a_ix+b_iy+c_iz+d_i=0, i=1, 2, 3.$

Now if we take two equations at a time, we get three lines of intersection of the above three planes and following three cases arise :—

(I) Three lines of intersection of these planes may coincide and then the three given **planes intersect in a common line.**

(II). Three lines of intersection of three planes may be **parallel** and then the three given **planes form a triangular prism** and

(III) Three lines of intersection of three planes may **intersect in a point** and then the three given **planes intersect in a point.**

Now we are to **find the condition** that the planes

$$u_1 \equiv a_1x+b_1y+c_1z+d_1=0 \qquad \text{...(i)}$$
$$u_2 \equiv a_2x+b_2y+c_2z+d_2=0 \qquad \text{...(ii)}$$
$$u_3 \equiv a_3x+b_3y+c_3z+d_3=0 \qquad \text{...(iii)}$$

(I) may intersect in a common line,

(II) may form a triangular prism, and

(III) may intersect in a point.

Case I. *Planes intersecting in a common line.*

The equation of any plane through the intersection of the planes (i) and (ii) is

$$u_1 + \lambda u_2 = 0$$

i.e.
$$(a_1x + b_1y + c_1z + d_1) + \lambda (a_2x + b_2y + c_2z + d_2) = 0$$

or
$$(a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2)z + (d_1 + \lambda d_2) = 0 \quad \text{...(iv)}$$

If the planes (i), (ii) and (iii) intersect in a common line, then by properly choosing the value of λ the plane (iv) can be made to represent the plane (iii).

Hence comparing the coefficients of x, y, z and the constant terms in (iii) and (iv) we have

$$\frac{a_1 + \lambda a_2}{a_3} = \frac{b_1 + \lambda b_2}{b_3} = \frac{c_1 + \lambda c_2}{c_3} = \frac{d_1 + \lambda d_2}{d_3} = k \text{ (say)}$$

i.e.
$$a_1 + \lambda a_2 - ka_3 = 0$$
$$b_1 + \lambda b_2 - kb_3 = 0$$
$$c_1 + \lambda c_2 - kc_3 = 0$$
$$d_1 + \lambda d_2 - kd_3 = 0.$$

Now we are to eliminate two variables λ and k and for this we require only three equations whereas there are four equations. Therefore we take any three equations and obtain one condition by eliminating λ and k between them. Now we can choose a set of three equations from the above four equations in four ways and so it appears that we shall get four different conditions. But we can easily find that any two sets of three equations are equivalent to the remaining two sets and consequently we shall get only two independent conditions.

The necessary conditions are expressed by the rectangular array

$$\left\|\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{matrix}\right\| = 0 \quad \text{...(v)}$$

The notation signifies that any of the four determinants of third order obtained by omitting one vertical column are zero and thus we get two independent conditions.

Case II. *The planes forming a triangular prism.* (*Kumaun* 95)

The planes will form a triangular prism provided the three lines of intersection of the planes taken two at a time are parallel.

Let l, m, n be the direction ratios of the line of intersection of the planes (i) and (ii), then we have-

$a_1l + b_1m + c_1n = 0$ and $a_2l + b_2m + c_2n = 0$, as this line is prependicular to the normals to these planes.

Solving these we have $\dfrac{l}{b_1c_2 - b_2c_1} = \dfrac{m}{c_1a_2 - c_2a_1} = \dfrac{n}{a_1b_2 - a_2b_1}$

If this line is parallel to the plane (iii), then this must be at right angles to the normal to the plane (iii) and so we get

$$a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$$

or $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \quad \text{...(vi)}$$

The symmetry of this result shows that this is the condition for the line of intersection of any two planes to be parallel to the third plane.

Hence the planes (i), (ii) and (iii) form a triangular prism if (vi) is satisfied and other determinants of (iv) **do not vanish.**

Case III. *The planes intersecting in a point.* (*Kumaun* 95)

Solving the equations (i), (ii) and (iii) by Crammer's Rule **(See Author's Algebra or Matrices)** we get

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

or $$\frac{x_1}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_4}$$

where $$\Delta_1 = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \quad \text{and} \quad \Delta_4 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

or $$x = -\frac{\Delta_1}{\Delta_4}, \; y = \frac{\Delta_2}{\Delta_4}, \; z = -\frac{\Delta_3}{\Delta_2}$$

Hence if the planes intersect in a point at a finite distance then we must have $\Delta_4 \neq 0$, which is the required condition.

Working Rule. Let the three planes be given by equations (i), (ii) and (iii) on Page 113 Ch. IV and let us denote $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 as in case III above.

Now we proceed as follows —

(a) Evaluate Δ_4 and if $\Delta_4 \neq 0$, then the planes intersect in a point and the coordinaes of this point can be obtained by solving the given equations.

(b) If $\Delta_4 = 0$, then evaluate any one of Δ_1, Δ_2 and Δ_3 and we have :—

(i) If none of Δ_1, Δ_2 and Δ_3 is zero, then the planes form triangular prism and

(ii) If any one of Δ_1, Δ_2 and Δ_3 is zero, then the planes intersect in a line.

Note : If $\Delta_4 = 0$ and $\Delta_1 \neq 0$, then $\Delta_2 \neq 0, \Delta_3 \neq 0$.

And If $\Delta_4 = 0$ and $\Delta_1 = 0$, then $\Delta_2 = 0, \Delta_3 = 0$.

Hence evaluation of Δ_4 and only one of the determinats out of $\Delta_1, \Delta_2, \Delta_3$ is sufficient. **(Remember)**

Solved Examples on § 4.13.

Ex. 1. Examine the nature of intersection of planes.

$$\mathbf{2x - y + z = 4,\ 5x + 7y + 2z = 0,\ 3x + 4y - 2z + 3 = 0.}$$

Sol. The given planes are

$$2x - y + z - 4 = 0 \quad ...(i)$$
$$5x + 7y + 2z + 0 = 0 \quad ...(ii)$$
$$3x + 4y - 2z + 3 = 0 \quad ...(iii)$$

$\therefore$ The 'rectangular array' is

$$\left\|\begin{matrix} 2 & -1 & 1 & -4 \\ 5 & 7 & 2 & 0 \\ 3 & 4 & -2 & 3 \end{matrix}\right\|$$

$$\therefore \quad \text{"}\Delta_4\text{"} = \begin{vmatrix} 2 & -1 & 1 \\ 5 & 7 & 2 \\ 3 & 4 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 9 & 2 \\ 7 & 2 & -2 \end{vmatrix}$$

subtractingg 2 times third column from first and adding third column to second.

or $\Delta_4 = -61 \neq 0$. (See § 4.13 Case III Page 115 Ch. IV)

Hence the given planes intersect in a point.

Solving (i), (ii) and (iii) we get

$$\frac{x}{\begin{vmatrix} -1 & 1 & -4 \\ 7 & 2 & 0 \\ 4 & -2 & 3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 2 & 1 & -4 \\ 3 & 2 & 0 \\ 3 & -2 & 3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & -1 & -4 \\ 5 & 7 & 0 \\ 3 & 4 & 3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} 2 & -1 & 1 \\ 5 & 7 & 2 \\ 3 & 4 & -2 \end{vmatrix}}$$

which give $x = 1, y = -1, z = 1$.

Hence the planes meet in the point $(1, -1, 1)$.

Ex. 2. Show that the planes $\mathbf{2x + 4y + 2z = 7 ; 5x + y - z = 9}$, $\mathbf{x - y - z = 6}$ form a triangular prism.

Sol. Here the rectangular array is

$$\left\|\begin{matrix} 2 & 4 & 2 & -7 \\ 5 & 1 & -1 & -9 \\ 1 & -1 & -1 & -6 \end{matrix}\right\|$$

$$\therefore \Delta_4 = \begin{vmatrix} 2 & 4 & 2 \\ 5 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 6 & 4 \\ 5 & 6 & 4 \\ 1 & 0 & 0 \end{vmatrix}$$, adding 1st column to 2nd and 3rd.

$$= 6 \times 4 \begin{vmatrix} 2 & 1 & 1 \\ 5 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0, \text{ two columns being identical}$$

Also $\Delta_2 = \begin{vmatrix} 4 & 2 & -7 \\ 1 & -1 & -9 \\ -1 & -1 & -6 \end{vmatrix} = \begin{vmatrix} 6 & 0 & -25 \\ 1 & -1 & -9 \\ -2 & 0 & 3 \end{vmatrix}$, adding twice 2nd row to 1st and subtracting 2nd row from 3rd.

or $\Delta_1 = -\begin{vmatrix} 6 & -25 \\ -2 & 3 \end{vmatrix}$, expanding w.r. to 2nd column

$$= -[18 - 50] = 32 \neq 0$$

Thus we have $\Delta_4 = 0, \Delta_1 \neq 0$ and so the given planes form a triangular prism. (See working rule and note on Pages 115-116 Ch. IV).

Ex. 3. Prove that the planes $2x - 3y - 7z = 0$, $2x - 14y - 13z = 0$, $8x - 31y - 33z = 0$ pass through one line and find its equation.

Sol. The rectangular array is

$$\left\| \begin{matrix} 2 & -3 & -7 & 0 \\ 3 & -14 & -13 & 0 \\ 8 & -31 & -33 & 0 \end{matrix} \right\|$$

$$\therefore \Delta_4 = \begin{vmatrix} 2 & -3 & -7 \\ 3 & -14 & -13 \\ 8 & -31 & -33 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 7 \\ 3 & 14 & 13 \\ 8 & 31 & 33 \end{vmatrix},$$

taking -1 common from 2nd and 3rd columns.

$= \begin{vmatrix} 2 & 1 & 1 \\ 3 & 11 & 4 \\ 8 & 23 & 9 \end{vmatrix}$, subtracting 1st column from 2nd and 3 times 1st column from 3rd respectively.

$= \begin{vmatrix} 0 & 0 & 1 \\ -5 & 7 & 1 \\ -10 & 14 & 9 \end{vmatrix}$, subtracting 3rd column once from 2nd and twice from first,

$= 0$, expanding with respect to first row and evaluating.

Also $\Delta_1 = \begin{vmatrix} -3 & -7 & 0 \\ -14 & -13 & 0 \\ -31 & -33 & 0 \end{vmatrix} = 0$

Since $\Delta_4 = 0$ and $\Delta_1 = 0$, therefore the given planes meet in a line.

(See working rule and note on Pages 115-116 Ch. IV.)

The equations of the line of intersectin are given by

$$2x - 3y - 7z = 0,\ 3x - 14y - 13z = 0$$

(taking any two of the given planes)

These may be written in the symmetric form as

$$\frac{x}{39 - 94} = \frac{y}{-21 + 26} = \frac{z}{-28 + 9} \quad \text{or} \quad \frac{x}{59} = \frac{y}{-5} = \frac{z}{19}. \qquad \textbf{Ans.}$$

Ex. 4. Prove that the three planes $x+y+z+6=0$, $x+2y+2z+6=0$, $x+3y+3z+6=0$ intersect in a common line.

Sol. The rectangular array is

$$\left\|\begin{matrix}1 & 1 & 1 & 6\\ 1 & 2 & 2 & 6\\ 1 & 3 & 3 & 6\end{matrix}\right\| = 0$$

$$\Delta_4 = \begin{vmatrix}1 & 1 & 1\\ 1 & 2 & 2\\ 1 & 3 & 3\end{vmatrix} = 0 \because \text{two columns are identical}$$

Also $\Delta_1 = \begin{vmatrix}1 & 1 & 6\\ 2 & 2 & 6\\ 3 & 3 & 6\end{vmatrix} = 0$, since two columns are identical.

$\therefore$ As $\Delta_4 = 0$ and $\Delta_1 = 0$, so the given planes meet in a line.

(See working rule and Note on Pages 115-116 Ch. IV)

Ex. 5. Show that the planes $x+ay+(b+c)z+d=0$, $x+by+(c+a)z+d=0$, $x+cy+(a+b)z+d=0$ pass through one line.

Sol. The rectangular array is

$$\left\|\begin{matrix}1 & a & b+c & d\\ 1 & b & c+a & d\\ 1 & c & a+d & d\end{matrix}\right\|$$

$$\therefore\ \Delta_4 = \begin{vmatrix}1 & a & b+c\\ 1 & b & c+a\\ 1 & c & a+b\end{vmatrix} = \begin{vmatrix}1 & a & a+b+c\\ 1 & b & b+c+a\\ 1 & c & c+a+b\end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix}1 & a & 1\\ 1 & b & 1\\ 1 & c & 1\end{vmatrix},$$

$= 0$, two columns being identical.

Also $\Delta_2 = \begin{vmatrix}1 & b+c & d\\ 1 & c+a & d\\ 1 & a+b & d\end{vmatrix} = d\begin{vmatrix}1 & b+c & 1\\ 1 & c+a & 1\\ 1 & a+b & i\end{vmatrix} = 0$

Since $\Delta_4 = 0$, $\Delta_2 = 0$, therefore the given lines intersect in a line.

(See Working Rule and Note on Pages 115-116 Ch. IV)

****Ex. 6 (a). Prove that the planes $x = cy + bz$, $y = az + cx$, $z = bx + ay$ pass through one line if $a^2 + b^2 + c^2 + 2abc = 1$, and find its equations.**

(*Avadh* 94, 93; *Garhwal* 93, 90; *Kanpur* 95, 93; *Rohilkhand* 91)

Sol. The given planes are $x - cy - bz = 0$, $cx - y + az = 0$, $bx + ay - z = 0$

$\therefore$ The rectangular array is $\left\|\begin{matrix}1 & -c & -b & 0\\ c & -1 & a & 0\\ b & a & -1 & 0\end{matrix}\right\|$ (Note)

$$\therefore \Delta_4 = \begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = \begin{vmatrix} 1 & -c & -b \\ 0 & c^2-1 & bc+a \\ 0 & a+bc & b^2-1 \end{vmatrix}$$

substracting c times 1st row from 2nd and b times from third.

$$= \begin{vmatrix} c^2-1 & bc+a \\ a+bc & b^2-1 \end{vmatrix}, \text{ expanding with respect to first column}$$

$$= (c^2-1)(b^2-1) - (bc+a)(bc+a)$$

$$= (c^2b^2 - c^2 - b^2 + 1) - (b^2c^2 + a^2 + 2abc)$$

$$= 1 - a^2 - b^2 - c^2 - 2abc$$

and $\Delta_1 = \begin{vmatrix} -c & -b & 0 \\ -1 & a & 0 \\ a & -a & 0 \end{vmatrix} = 0.$

Now if the given planes intersect in a line then Δ_4 must be zero

i.e. $1 - a^2 - b^2 - c^2 - 2abc = 0$ *i.e.* $a^2 + b^2 + c^2 + 2abc = 1$. Hence proved.

If l, m, n be the d.c.'s of this line then this line being perpendicular to normals to the given three planes we have $1.l - c.m - b.n = 0$, $c.l - 1.m + a.n = 0$

and $$b \,.\, l + a \,.\, m - 1 \,.\, n = 0$$

Solving the above three equations in pairs, we get

$$\frac{l}{-ac-b} = \frac{m}{-bc-a} = \frac{n}{-1+c^2} \qquad \text{...(i)}$$

$$\frac{l}{(1-a^2)} = \frac{m}{ab+c} = \frac{n}{ca+b} \qquad \text{...(ii)}$$

and $$\frac{l}{c+ab} = \frac{m}{-b^2+1} = \frac{n}{a+bc} \qquad \text{...(iii)}$$

From (ii) and (iii) we get

$$\frac{l^2}{(ab+c)(1-a^2)} = \frac{m^2}{(1-b^2)(ab+c)} \qquad \textbf{(Note)}$$

or $$l^2/(1-a^2) = m^2/(1-b^2)$$

Similarly from (i) and (ii) we find that $l^2/(1-a^2) = n^2/(1-c^2)$

$\therefore$ We have $l^2/(1-a^2) = m^2/(1-b^2) = n^2/(1-c^2)$

or $$\frac{l}{\sqrt{(1-a^2)}} = \frac{m}{\sqrt{(1-b^2)}} = \frac{n}{\sqrt{(1-c^2)}}$$

Also the three given planes pass through the origin, so the equations of their line of intersection are given by

$$\frac{x}{\sqrt{(1-a^2)}} = \frac{y}{\sqrt{(1-b^2)}} = \frac{z}{\sqrt{(1-c^2)}} \qquad \textbf{Ans.}$$

*Ex. 6 (b). **Prove that the planes $ny - mz = \lambda$, $lz - nx = \mu$, $mx - ny = \nu$ have a common line if and only if $l\lambda + m\mu + n\nu = 0$** (*Kanpur* 96)

Sol. Give planes are $0.x+ny-mz-\lambda=0,\quad -nx+0.y+lz-\mu=0$

and $$mx-ny+0\,.\,z-\nu=0$$

$\therefore$ The rectangular array is $\left\|\begin{matrix} 0 & n & -m & -\lambda \\ -n & 0 & l & -\mu \\ m & -n & 0 & -\nu \end{matrix}\right\|=0$

$$\therefore \Delta_4=\begin{vmatrix} 0 & n & -m \\ -n & 0 & l \\ m & -n & 0 \end{vmatrix}=n\begin{vmatrix} n & -m \\ -n & 0 \end{vmatrix}+m\begin{vmatrix} n & -m \\ 0 & l \end{vmatrix}$$

$$=n(-mn)+m(nl)=mn(l-n) \qquad \text{...(i)}$$

And $$\Delta_1=\begin{vmatrix} n & -m & -\lambda \\ 0 & l & -\mu \\ -n & 0 & -\nu \end{vmatrix}=\begin{vmatrix} n & -m & -\lambda \\ 0 & l & -\mu \\ 0 & -m & -\nu-\lambda \end{vmatrix},\ \text{adding } R_1 \text{ to } R_3$$

$$=n\begin{vmatrix} l & -\mu \\ -m & -\nu-\lambda \end{vmatrix}=n(-l\nu-l\lambda-m\mu). \qquad \text{...(ii)}$$

Now if the given plane intersects in a line, then Δ_4 must be zero

i.e. $mn(l-n)=0$ and $\Delta_1=0$ (or $\Delta_2=0$ or $\Delta_3=0$)

i.e. $l-n=0$ and $l\nu+l\lambda+m\mu=0,\ \because m\neq 0,\ n\neq 0$

i.e. $l=n$ and $l\nu+l\lambda+m\mu=0$

i.e. $n\nu+l\lambda+m\mu=0$ *i.e.* $l\lambda+m\mu+n\nu=0$ Hence proved.

****Ex. 7. For what values of λ do the planes**

$$\mathbf{x-y+z+1=0,\ \lambda x+3y+2z-3=0,\ 3x+\lambda y+z-2=0}$$

(i) intersect in a point; (ii) intersect along a line; (iii) form a traingular prism?

Sol. The equations of the given planes can be written as

$$x-y+z+1=0$$
$$\lambda x+3y+2z-3=0$$
$$3x+\lambda y+z-2=0$$

$\therefore$ The rectangular array is $\left\|\begin{matrix} 1 & -1 & 1 & 1 \\ \lambda & 3 & 2 & -3 \\ 3 & \lambda & 1 & -2 \end{matrix}\right\|=0$

$$\therefore \Delta_4=\begin{vmatrix} 1 & -1 & 1 \\ \lambda & 3 & 2 \\ 3 & \lambda & 1 \end{vmatrix}=\begin{vmatrix} 0 & 0 & 1 \\ \lambda-2 & 5 & 2 \\ 2 & \lambda+1 & 1 \end{vmatrix},\ \text{replacing } C_1,\ C_2 \text{ by } C_1-C_3,\ C_2+C_3$$

$$=\begin{vmatrix} \lambda-2 & 5 \\ 2 & \lambda+1 \end{vmatrix},\ \text{expanding with respect to first row.}$$

$$=(\lambda-2)(\lambda+1)-(5\times 2)=\lambda^2-\lambda-12$$
$$=(\lambda-4)(\lambda+3) \qquad \text{...(I)}$$

Also $$\Delta_1=\begin{vmatrix} -1 & 1 & 1 \\ 3 & 2 & -3 \\ \lambda & 1 & -2 \end{vmatrix}=\begin{vmatrix} 0 & 1 & 1 \\ 0 & 2 & -3 \\ \lambda-2 & 1 & -2 \end{vmatrix},\ \text{replacing } C_1 \text{ by } C_1+C_2$$

$= (\lambda - 2)\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}$, expanding with respect to first column.

$= (\lambda - 2)[-3 - 2] = -5(\lambda - 2);$ ...(II)

$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ \lambda & 2 & -3 \\ 3 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ \lambda - 2 & 2 & -5 \\ 2 & 1 & -3 \end{vmatrix}$, replacing C_1, C_3 by $C_1 - C_2$, $C_3 - C_2$.

$= -\begin{vmatrix} \lambda - 2 & -5 \\ 1 & -3 \end{vmatrix} = -[3(\lambda - 2) + 10] = 3\lambda - 16$ ...(III)

Also $\Delta_3 = \begin{vmatrix} 1 & -1 & 1 \\ \lambda & 3 & -3 \\ 3 & \lambda & -2 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ \lambda + 3 & 3 & 0 \\ 3 + \lambda & \lambda & \lambda - 2 \end{vmatrix}$, replacing C_1, C_3 by $C_1 + C_2$ and $C_3 + C_2$

$= (\lambda - 2)(\lambda + 3)$ ...(IV)

(i) If the given planes intersect in a point, then $\Delta_4 \neq 0$ and so from (I) we must have $\lambda \neq 4, \lambda \neq -3$.

(ii) If the given planes intersect in a line, then we know (See § 4.13 Working Rule Page 115 Ch. IV) that $\Delta_4 = 0$ and any one of Δ_1, Δ_2 and Δ_3 is zero.

Here from (I), (II), (III) and (IV) we find that if $\lambda = -3$, then $\Delta_4 = 0$ and $\Delta_3 = 0$.

Consequently for $\lambda = -3$, the given planes intersect in a line.

(iii) If the given planes form a triangular prism, then we know (See § 4.13 Working Rule Page 115 Ch. IV) that $\Delta_4 = 0$ and none of $\Delta_1, \Delta_2, \Delta_3$ is zero.

Here from (I), (II), (III) and (IV) we find that if $\lambda = 4$, then $\Delta_4 = 0$ and none of $\Delta_1, \Delta_2, \Delta_3$ is zero.

Consequently for $\lambda = 4$, the given planes form a triangular prism.

Ex. 8. Show that the planes $cy - bz = l$, $az - cx = m$, $bx - ay = n$ intersect in a line if $al + bm + cn = 0$, and the direction ratios of the line are a, b, c.

Sol. The equations of given planes can be written as

$$0 . x + c . y - b . z - l = 0$$
$$c . x + 0 . y - a . z + m = 0$$
$$b . x - a . y + 0 . z - n = 0$$

$\therefore$ The rectangular array is $\left\| \begin{matrix} 0 & c & -b & -l \\ c & 0 & -a & m \\ b & -a & 0 & -n \end{matrix} \right\|$

$\therefore \Delta_4 = \begin{vmatrix} 0 & c & -b \\ c & 0 & -a \\ b & -a & 0 \end{vmatrix} = -c\begin{vmatrix} c & -a \\ b & 0 \end{vmatrix} - b\begin{vmatrix} c & 0 \\ b & -a \end{vmatrix} = 0$

and $\Delta_1 = \begin{vmatrix} c & -b & -l \\ 0 & -a & m \\ -a & 0 & -n \end{vmatrix} = c\begin{vmatrix} -a & m \\ 0 & -n \end{vmatrix} - a\begin{vmatrix} -b & -l \\ -a & m \end{vmatrix}$

$= acn - a(-bm - al) = a(al + bm + cn)$

If the given planes meet in a line then we must have $\Delta_4 = 0$ and $\Delta_1 = 0$

$\therefore$ If $\Delta_1 = 0$ we get $a(al + bm + cn) = 0$

or $$al + bm + cn = 0, \quad \because \quad a \neq 0.$$ Hence proved.

Also if l_1, m_1, n_1 be the d.c.'s of the line of intersection of the given planes then as this line is perpendicular to the normal to the given planes, so we have $0.l_1 + c.m_1 - b.n_1 = 0$, $c.l_1 + 0.m_1 - a.n_1 = 0$

and $$b.l_1 - a.m_1 + 0.n_1 = 0$$

Solving these taking two at a time, we have

$$\frac{l_1}{-ac} = \frac{m_1}{-bc} = \frac{n_1}{-c^2}, \frac{l}{-a^2} = \frac{m_1}{-ab} = \frac{n_1}{-ac} \text{ and } \frac{l_1}{-ab} = \frac{m_1}{-b^2} = \frac{n_1}{-bc}$$

All of these reduce to $\frac{l_1}{a} = \frac{m_1}{b} = \frac{n_1}{c}$, which shows that the direction ratios of the line of intersection of the given planes are a, b, c.

Ex. 9. Prove that the planes $x = y \sin\psi + z \sin\phi$, $y = z \sin\theta + x \sin\psi$ and $z = x \sin\phi + y \sin\theta$ intersect in the line $\frac{x}{\cos\theta} = \frac{y}{\cos\phi} = \frac{z}{\cos\psi}$ if $\theta + \phi + \psi = \frac{1}{2}\pi$.

Sol. The equation of the given planes can be written as

$$x - y\sin\psi - z\sin\phi = 0 \quad \text{...(i)}$$

$$x\sin\psi - y + z\sin\theta = 0 \quad \text{...(ii)}$$

and $$x\sin\phi + y\sin\theta - z = 0 \quad \text{...(iii)}$$

Let l, m, n be the d.c.'s of the line of intersection of the planes (i) and (ii), then as this line is perpendicular to the normal to the planes (i) and (ii), we have $l.1 - m.\sin\psi - n.\sin\phi = 0$, $l\sin\psi - m.1 + n\sin\theta = 0$

Solving these we have

$$\frac{l}{-\sin\theta\sin\psi - \sin\phi} = \frac{m}{-\sin\phi\sin\psi - \sin\theta} = \frac{n}{-1 + \sin^2\psi} \quad \text{...(iv)}$$

Now if $\theta + \phi + \psi = \frac{1}{2}\pi$, then $\phi = \frac{1}{2}\pi - (\theta + \psi)$

$\therefore \sin\phi = \sin[\frac{1}{2}\pi - (\theta + \psi)] = \cos(\theta + \psi)$

$= \cos\theta\cos\psi - \sin\theta\sin\psi$

or $$\sin\phi + \sin\theta\sin\psi = \cos\theta\cos\psi$$

Similarly $\sin\theta + \sin\phi\sin\psi = \cos\phi\cos\psi$

Substituting these values in (iv) we have

$$\frac{l}{\cos\theta\cos\psi}=\frac{m}{\cos\phi\cos\psi}=\frac{n}{\cos^2\psi} \quad \text{or} \quad \frac{l}{\cos\theta}=\frac{m}{\cos\phi}=\frac{n}{\cos\psi} \quad \text{...(v)}$$

Also the planes (i) and (ii) pass through the origin, so the equations of the line of intersection of the planes (i) and (ii) is

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad \text{or} \quad \frac{x}{\cos\theta}=\frac{y}{\cos\phi}=\frac{z}{\cos\psi}. \quad \text{...(vi)}$$

If this line (vi) lies on the plane (iii), then the point (0, 0, 0) on this line must be on the plane (iii) which is true, also the normal to the plane (iii) must be at right angles to the line (vi), the condition for the same is

$$\sin\phi\cos\theta+\sin\theta\cos\phi-1.\cos\psi=0.$$

or $\sin(\theta+\phi)-\cos\psi=0$ or $\sin[\frac{1}{2}-\psi]-\cos\psi=0$ $\because \theta+\phi+\psi=\frac{1}{2}\pi$

or $\cos\psi-\cos\psi=0$, which being true the line (vi) lies on the plane (iii).

****Ex. 10. The plane $(x/a)+(y/b)+(z/c)=1$ meets the axes in A, B and C. Prove that the planes through the axes and the internal bisector of the angles of the triangle ABC pass through the line**

$$\frac{x}{a\sqrt{(b^2+c^2)}}=\frac{y}{b\sqrt{(c^2+a^2)}}=\frac{z}{c\sqrt{(a^2+b^2)}}$$

Sol. The coordinates of the point A, B and C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

The d.c.'s of the sides AB and AC of the ΔABC are

$$\frac{a}{\sqrt{(a^2+b^2)}}, \frac{-b}{\sqrt{(a^2+b^2)}}, 0 \quad \text{and} \quad \frac{a}{\sqrt{(a^2+c^2)}}, 0, \frac{-c}{\sqrt{(a^2+c^2)}} \text{ respectively}$$

Also we know that the d.r.'s of internal bisector of the two lines whose d.c.'s are l_1, m_1, n_1 and l_1, m_2, n_2 are given by

$$\frac{1}{2}(l_1+l_2), \frac{1}{2}(m_1+m_2), \frac{1}{2}(n_1+n_2)$$

$\therefore$ The d.c.'s of the internal bisector of AB and AC are

$$\frac{1}{2}\left[\frac{a}{\sqrt{(a^2+b^2)}}-\frac{a}{\sqrt{(a^2+c^2)}}\right], \frac{1}{2}\left[-\frac{b}{\sqrt{(a^2+b^2)}}-0\right], \frac{1}{2}\left[0+\frac{c}{\sqrt{(a^2+c^2)}}\right]$$

or $$\frac{1}{2}a\left[\frac{1}{\sqrt{(a^2+b^2)}}-\frac{1}{\sqrt{(a^2+c^2)}}\right], -\frac{1}{2}\frac{b}{\sqrt{(a^2+b^2)}}, \frac{1}{2}\frac{c}{\sqrt{(a^2+c^2)}}$$

or l, m, n (say). ...(i)

Any plane through x-axis *i.e.* $y=0, z=0$ is $y+\lambda z=0$ ...(ii)

If the internal bisector of AB and AC whos d.c.'s l, m, n are given by (i) above lies on the plane (ii) then we have $l.0+m.1+n.\lambda=0$ or $\lambda=-m/n$

$\therefore$ From (ii), the equation of the plane through x-axis and the internal bisector of AB and AC is $y-(m/n)z=0$ or $y/m=z/n$.

or $\dfrac{2y\sqrt{(a^2+b^2)}}{b}=\dfrac{2z\sqrt{(a^2+c^2)}}{c}$, putting the values of m and n

or $$\frac{y}{b\sqrt{(c^2+a^2)}}=\frac{z}{c\sqrt{(a^2+b^2)}} \quad \text{...(iii)}$$

Similarly the equations of the other planes are

$$\frac{z}{c\sqrt{(a^2+b^2)}}=\frac{x}{a\sqrt{(b^2+c^2)}} \quad \text{...(iv)}$$

and $$\frac{x}{a\sqrt{(b^2+c^2)}}=\frac{y}{b\sqrt{(c^2+a^2)}} \quad \text{...(v)}$$

Evidently the line of intersection of the planes (iii), (iv) and (v) is

$$\frac{x}{a\sqrt{(b^2+c^2)}}=\frac{y}{b\sqrt{(c^2+a^2)}}=\frac{z}{c\sqrt{(a^2+b^2)}} \quad \text{Hence proved.}$$

Ex. 11. The plane x/a + y/b + z/c = 1 meets the axes OX, OY and OZ in A, B, C respectively. Prove that the planes through the axes perpendicular to the sides of the triangle ABC pass through the line ax = by = cz.

Find also the coordinates of the orthocentre of the triangle ABC.

Sol. The coordinates of A, B and C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

The equations of the side BC of the ΔABC are

$$\frac{x-0}{0}=\frac{y-b}{b}=\frac{z-0}{-c} \quad \textbf{(Note)} \quad \text{...(i)}$$

The equation of the plane through OX *i.e.* $y=0=z$ is $y+\lambda z=0$.

If this plane is perpendicular to the line (i), then the normal to this plane must be parallel to (i), the condition for the same is

$$\frac{0}{0}=\frac{b}{1}=\frac{-c}{\lambda} \quad \text{or} \quad \lambda=-\frac{c}{b}.$$

$\therefore$ The equation of the plane through OX and perpendicular to the side BC of the ΔABC is $y-(c/b)z=0$ or $by=cz$ ...(iii)

Similarly the equations of the other planes are

$$cz=ax \quad \text{...(iv)} \quad \text{and} \quad ax=by \quad \text{...(v)}$$

Evidently the line of intersection of the planes (iii), (iv) and (v) is

$$ax=by=cz \quad \text{or} \quad \frac{x}{1/a}=\frac{y}{1/b}=\frac{z}{1/c}=r \text{ (say)} \quad \text{...(vi)}$$

Also the orthocentre of the triangle ABC lies on the three planes given by (iii), (iv) and (v) and it also lies on the plane of ΔABC. **(Note)**

i.e. the orthocentre of ΔABC lies on the line (vi) and the plane of ΔABC, viz. $$x/a+y/b+z/c=1.$$

i.e. the orthocentre of ΔABC is the point of intersection of the line (vi) and the plane $x/a+y/b+z/c=1$.

Now from (vi), any point on the line (vi) is $(r/a, r/b, r/c)$.

If this point lies on the plane $x/a+y/b+z/c=1$ we have

$$\frac{r}{a^2}+\frac{r}{b^2}+\frac{r}{c^2}=1 \quad \text{or} \quad r=1/(a^{-2}+b^{-2}+c^{-2}).$$

$\therefore$ The required coordinates of the orthocentre of $\Delta\, ABC$ are

$$\left(\frac{r}{a},\frac{r}{b},\frac{r}{c}\right), \text{ where } r=\frac{1}{a^{-2}+b^{-2}+c^{-2}} \qquad \textbf{Ans.}$$

Exercises on § 4.13.

Ex. 1. Determine the values of k such that the following system of planes may (i) intersect in a point, (ii) intersect in a line, (iii) form a triangular prism :—

$3x+y+kz=2,\ 2x+3y+4z=3,\ x-2y-3z=-k.$ **Ans.** (i) $k\neq 1$, (ii) $k=1$

Ex. 2. Find the condition that three planes

$$a_r x+b_r y+c_r z+d_r=0,\ r=1,2,3.$$

may intersect in a common line.

[Hint : See § 4.13 Case I Page 114 Ch. IV]

Ex. 3. Examine the nature of intersection of the planes

$$2x+3y-z=2,\ 3x+3y+z=4 \text{ and } x-y+2z=5.$$

Ans. Planes intersect in $(24/5,-3,-7/5)$

Ex. 4. Examine the nature of intersection of the planes

$$x+2y+3z=6,\ 3x+4y+5z=2 \quad \text{and} \quad 5x+4y+3z+18=0$$

Ans. Planes intersect in a line.

Ex. 5. Show that the planes $x+y-z=2,\ 2x-y-z+2=0,$ $x-5y+z+4=0$ form a triangular prism and calculate the breadth of each face of the prism. **Ans.** $(2/7)\sqrt{(21)}, (3/4)\sqrt{(42)}, (3/14)\sqrt{(42)}$.

Ex. 6. Show that the planes $5x+2y-4z+2=0,\ 4x-2y-5z-2=0$ and $2x+8y-2z+1=0$ form a prism.

Ex. 7. Interpret the following types of solutions of the set of simultaneous equations $a_1x+b_1y+c_1z+d_1=0,\quad a_2x+b_2y+c_2z+d_2=0,$ $a_3x+b_3y+c_3z+d_3=0.$

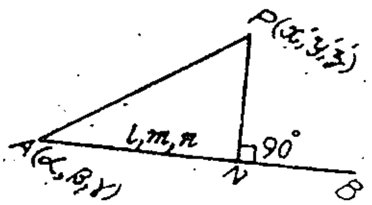
(i) unique solution, (ii) no solution, that is the equations are inconsistent, (iii) infinitely many solutions.

****§ 4.14. To find the perpendicular distance of a point from a line and the coordinates of the foot of the perpendicular.** (*Kumaun* 93)

Let the equations of the line AB (say) in the symmetric form be $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ and let the point be $P(x',y',z')$.

From P draw PN perpendicular to AB. Join AP.

Then in right angled triangle APN, we have $\qquad PN^2=AP^2-AN^2 \qquad$ ...(i)



(Fig. 2)

Now $AP^2 = (x' - \alpha)^2 + (y' - \beta)^2 + (z' - \gamma)^2$ ...(ii)

And AN = the projection of AP on the line whose d.c.'s are l, m, n

or $$AN = l(x' - \alpha) + m(y' - \beta) + n(z' - \gamma) \quad \text{...(iii)}$$

$\therefore$ From (i), (ii) and (iii) we have PN^2

$$= [(x' - \alpha)^2 + (y' - \beta)^2 + (z' - \gamma)^2] - [l(x' - \alpha) + m(y' - \beta) + n(z' - \gamma)^2]$$

$$= [(x' - \alpha)^2 + (y' - \beta)^2 + (z' - \gamma)^2](l^2 + m^2 + n^2)$$
$$- [l(x' - \alpha) + m(y' - \beta) + n(z' - \gamma)]^2, \quad \because l^2 + m^2 + n^2 = 1$$

$$= \{m(z' - \gamma) - n(y' - \beta)\}^2 + \{n(x' - \alpha) - l(z' - \gamma)\}^2$$
$$+ \{l(y' - \beta) - m(x' - \alpha)\}^2,$$

using Lagrange's Identity See Ch. II

$$= \begin{vmatrix} m & n \\ y' - \beta & z' - \gamma \end{vmatrix}^2 + \begin{vmatrix} n & l \\ z' - \gamma & x' - \alpha \end{vmatrix}^2 + \begin{vmatrix} l & m \\ x' - \alpha & y' - \beta \end{vmatrix}^2$$

Note. Students should remember that l, m, n are the direction cosines of the line AB. In case direction ratios are given we should find the direction cosines of the line.

Coordinates of N, the foot of perpendicular.

Let $AN = r$, then the coordinates of N, which is a point on AD at a distance r from A are $(\alpha + lr, \beta + mr, \gamma + nr)$. ...(iv)

$\therefore$ The direction ratios of PN are $\alpha + lr - x', \beta + mr - y', \gamma + nr - z'$.

As PN is perpendicular to AB, whose d.c.'s are l, m, n so we have

$$l(\alpha + lr - x') + m(\beta + mr - y') + n(\gamma + nr - z') = 0$$

or $$(l^2 + m^2 + n^2) r = l(x' - \alpha) + m(y' - \beta) + n(z' - \gamma)$$

or $$r = l(x' - \alpha) + m(y' - \beta) + n(z' - \gamma), \text{ since } l^2 + m^2 + n^2 = 1$$

Substituting this value of r in (iv) we can find the coordinates of N, the foot of the perpendicular from P to the line AB.

Solved Examples on § 4.14.

Ex. 1. Find the distance of P (x', y', z') from the line through A (a, b, c) whose direction cosines are $\cos \alpha, \cos \beta, \cos \gamma$.

Sol. Proceed exactly as in § 4.14 P. 125 Ch. IV. Here the co-ordinates of A are (a, b, c) instead of (α, β, γ) and the d.c.'s of the line AB are $\cos \alpha, \cos \beta, \cos \gamma$ instead of l, m, n in § 4.14 P. 125 Ch. IV.

***Ex. 2. From the point P (1, 2, 3), PN is drawn perpendicular to the straight line $(x - 2)/3 = (y - 3)/4 = (z - 4)/5$. Find the distance PN, the equations to PN and coordinates of N.**

Sol. The equations of the given line AB (say) are given as

$$\frac{x - 2}{3} = \frac{y - 3}{4} = \frac{z - 4}{5} \quad \text{...(i)}$$

where A is (2, 3, 4) say.

Let N, the foot of the perpendicular from P to AB be at a distance r from A. Then from (i) are coordinates of N are $(2+3r, 3+4r, 4+5r)$. ...(ii)

$\therefore$ The direction ratios of the line PN are

$(2+3r)-1, (3+4r)-2, (4+5r)-3$ *i.e.* $1+3r, 1+4r, 1+5r$

Also direction ratios of the line AB given by (i) are 3, 4, 5.

As PN is perpendicular to AB, so we have

$(1+3r).3+(1+4r).4+(1+5r).5=0$ or $50r+12=0$ or $r=-6/25$.

$\therefore$ From (ii) the coordinates of N are $\left(\frac{32}{25}, \frac{51}{25}, \frac{70}{25}\right)$ **Ans.**

$\therefore$ The distance PN = distance between P and N

$$=\sqrt{\left[\left(\frac{32}{25}-1\right)^2+\left(\frac{51}{25}-2\right)^2+\left(\frac{70}{25}-3\right)^2\right]} \quad \textbf{(Note)}$$

$$=\frac{1}{25}\sqrt{\left[(7)^2+(1)^2+(-5)^2\right]}=\frac{5\sqrt{3}}{25}=\frac{\sqrt{3}}{5} \quad \textbf{Ans.}$$

And the direction ratios of PN are $1+3r, 1+4r, 1+5r$

where $r=-6/25$

i.e. d. ratios of PN are $7/25, 1/25, -5/25$ *i.e.* $7, 1, -5$.

$\therefore$ The equations of the perpendicular PN which passes through (1, 2, 3) and whose d. ratios are 7, 1, −5 are

$$\frac{x-1}{7}=\frac{y-2}{1}=\frac{z-3}{-5}. \quad \textbf{Ans.}$$

Ex. 3 (a). Prove that the equations of the perpendicular from the point (1, 6, 3) to the line $x=\frac{y-1}{2}=\frac{z-2}{3}$ are $\frac{x-1}{0}=\frac{y-6}{-3}=\frac{z-3}{2}$ and the coordinates of the foot of the perpendicular are (1, 3, 5).

(*Kumaun* 93; *Meerut* 91)

Sol. Refer Fig. 2 Page 125 Ch. IV.

Here the point P is (1, 6, 3) and the equations of line AB are

$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ $\therefore$ The d.r.'s of the line AB are 1, 2, 3.

Let N be the foot of the perpendicular from P to AB be

$(r, 1+2r, 2+3r)$ ...(i)

The d.r.'s of the line PN are

$r-1, (1+2r)-6, (2+3r)-3$ *i.e.* $r-1, 2r-5, 3r-1$ ...(ii)

As PN is perpendicular to AB whose d.r.'s are 1, 2, 3 so we get

$1.(r-1)+2(2r-5)+3(3r-1)=0$ or $r=1$

$\therefore$ From (i), the coordinates of N are (1, 3, 5). Hence proved.

And from (ii), the d.r.'s of the line PN are $0, -3, 2$. Also perpendicular PN passes through $P(1, 6, 3)$. Hence the equations of the perpendicular PN are

$$\frac{x-1}{0}=\frac{y-6}{-3}=\frac{z-3}{2}.$$ Hence proved.

Ex. 3 (b). Find the equations of the perpendicular and coordinates of the foot of perpendicular drawn from the point (5, 9, 3) to the line

$$\frac{1}{2}(x-1)=\frac{1}{3}(y-2)=\frac{1}{4}(z-3)$$ (*Purvanchal* 93)

Sol. Do as Ex. 3 (a) above. **Ans.** $x-5=\frac{1}{2}(y-9)=-\frac{1}{2}(z-3)$; (3, 5, 7)

****Ex. 3 (c). Find the equations of the perpendicular from (1, 3, 7) on the line $x=3-5t, y=2+5t, z=-7+2t$.**

Sol. The equations of the given line AB (say) are given by

$$x=3-5t,\ y=2+5t,\ z=-7+2t.$$

or $$\frac{x-3}{-5}=\frac{y-2}{5}=\frac{z+7}{2}=t \quad \text{...(i)} \quad \textbf{(Note)}$$

Let the given point (1, 3, 7) be P and N be the foot of the perpendicular from $P(1,3,7)$ on the line AB given by (i). Let A be $(3,2,-7)$, which evidently lies on (i). Let $AN=t$ then from (i) the coordinates of N are

$$(3-5t,\ 2+5t,\ -7+2t). \quad \text{(ii)}$$

$\therefore$ d ratios of the line PN are

$(3-5t)-1, (2+5t)-3, (-7+2t)-7$ *i.e.* $2-5t, -1+5t, -14+2t$

Also d. ratios of the line AB given by (i) are $-5, 5, 2$.

As PN is perpendicular to AB, so we have

$$(2-5t).(-5)+(-1+5t).5+(-14+2t).2=0$$

or $$54t-43=0 \quad \text{or} \quad t=43/54.$$

$\therefore$ From (ii) the coordinates of N are $\left(-\frac{53}{54}, \frac{323}{54}, -\frac{292}{54}\right)$

Also d. ratios of PN are

$$2-5(43/54), -1+5(43/54), -14+2(43/54)$$

i.e. $$2-\frac{215}{54}, -1+\frac{215}{54}, -14+\frac{86}{54} \quad \textit{i.e.} \quad -107, 161, -670$$

$\therefore$ The equations of PN are $\frac{x-1}{-107}=\frac{y-3}{161}=\frac{z-7}{-670}$ **Ans**

Ex. 4 (a). How far is the point (4, 1, 1) from the line of intersection o $x+y+z-4=0=x-2y-z-4$?

Sol. Refer Fig 2 Page 125 Ch. IV.

The equations of the line AB (say) are

$$(x-4)+y+z=0 \quad \text{and} \quad (x-4)-2y-z=0$$

$\therefore$ The equations of the line in the symmetric form are

$$\frac{x-4}{-1+2}=\frac{y}{1+1}=\frac{z}{-2-1} \quad \text{or} \quad \frac{x-4}{1}=\frac{y}{2}=\frac{z}{-3} \quad \text{...(}$$

$A(4,0,0)$ is any point on this line given by (i).

Here P is (4, 1, 1) and let N, the foot of the perpendicular from P on the line (i), be at a distance r from A.

Then the coordinates of N are $(4+r, 2r, -3r)$ ...(ii)

$\therefore$ The d.r.'s of the line PN are

$$(4+r)-4, 2r-1, -3r-1 \quad \text{or} \quad r, 2r-1, -3r-1$$

Also from (i) we find that the d.c.'s of the line AB are $1, 2, -3$

Since PN is perpendicular to AB, so we have

$1 \cdot r + 2(2r-1) - 3(-3r-1) = 0$ or $14r + 1 = 0$ or $r = -1/14$

$\therefore$ From (ii) the coordinates of N are $\left(\frac{55}{14}, -\frac{1}{7}, \frac{3}{14}\right)$.

$\therefore$ The required distance $= PN$

$$= \sqrt{\left[\left\{4-\frac{55}{14}\right\}^2 + \left\{1+\frac{1}{7}\right\}^2 + \left\{1-\frac{3}{14}\right\}^2\right]}$$

$$= \sqrt{\left[\left(\frac{1}{14}\right)^2 + \left(\frac{16}{14}\right)^2 + \left(\frac{11}{14}\right)^2\right]} = \frac{1}{14}\sqrt{\left[1+256+121\right]}$$

$= (1/14)\sqrt{(378)} = (3/14)\sqrt{(42)}$. **Ans.**

****Ex. 4 (b). In Ex. 6 (b) Page 119, show that the distance of the line from the origin is $\sqrt{[(\lambda^2+\mu^2+\nu^2)]/(l^2+m^2+n^2)}$.** *(Kanpur 96)*

Sol. In Ex. 6 (b) Page 119, if l_1, m_1, n_1 be the d.c.'s of the common line, then this line being perpendicular the given three planes, we have

$$0 \cdot l_1 + n \cdot m_1 - m \cdot n_1 = 0, \ -n \cdot l_1 + 0 \cdot m_1 + l \cdot n_1 = 0$$

and $$m \cdot l_1 - n \cdot m_1 + 0 \cdot n_1 = 0$$

Solving the above three equations in pairs, we get

$$\frac{l_1}{nl} = \frac{m_1}{mn} = \frac{n_1}{n^2} \quad \text{or} \quad \frac{l_1}{l} = \frac{m_1}{m} = \frac{n_1}{n} \quad \text{...(i)}$$

$$\frac{l_1}{nl} = \frac{m_1}{lm} = \frac{n_1}{n^2} \quad \text{...(ii)} \quad \text{and} \quad \frac{l_1}{-mn} = \frac{m_1}{-m^2} = \frac{n_1}{-nm} \quad \text{or} \quad \frac{l_1}{n} = \frac{m_1}{m} = \frac{n_1}{n} \quad \text{...(iii)}$$

Also from Ex. 6 (b) Page 119 we have $l = n$ as a condition that the plane intersects in a line. Substituting n for l, all the above three equations (i), (ii) and (iii) reduce to $\frac{l_1}{n} = \frac{m_1}{m} = \frac{n_1}{n}$ or $\frac{l_1}{l} = \frac{m_1}{m} = \frac{n_1}{n}$ **(Note)** ...(iv)

Also putting $x = 0$ in the equations of the given planes of Ex. 6 (b) Page 119, we have $ny - mz = \lambda$, $lz = \mu$, $-ny = \nu$

These give $x = 0, y = -\frac{\nu}{n}, z = \frac{\mu}{l}$

and $l\nu + m\mu + l\lambda = 0$ or $l\lambda + m\mu + n\nu = 0$, $\because l = n$ ...(v)

which is true from Ex. 6 (b) Page 119.

$\therefore$ Any point on the line common to three given planes is $(0, -\nu/n, \mu/l)$

$\therefore$ The equations of this common line are

$$\frac{x-0}{l/\sqrt{(\Sigma l^2)}}=\frac{y+(\nu/n)}{m/(\Sigma l^2)}=\frac{z-(\mu/l)}{n/\sqrt{(\Sigma l^2)}} \quad \text{...(v)}$$

$\therefore$ Square of the length of the perpendicular from the origin (0, 0, 0) to the line (v)

$$=\begin{vmatrix} m/(\sqrt{\Sigma l^2}) & n/\sqrt{(\Sigma l^2)} \\ 0+(\nu/n) & 0-(\mu/l)\end{vmatrix}^2+\begin{vmatrix} n/(\sqrt{\Sigma l^2}) & l/\sqrt{(\Sigma l^2)} \\ 0-(\mu/l) & 0-0\end{vmatrix}^2$$

$$+\begin{vmatrix} l/(\sqrt{\Sigma l^2}) & m/\sqrt{(\Sigma l^2)} \\ 0-0 & 0+(\nu/n)\end{vmatrix}^2, \text{ See § 4.14 P. 125}$$

$$=\frac{1}{\Sigma l^2}\left[\begin{vmatrix} m & n \\ \nu/n & -\mu/l\end{vmatrix}^2+\begin{vmatrix} n & l \\ -\mu/l & 0\end{vmatrix}^2+\begin{vmatrix} l & m \\ 0 & \nu/n\end{vmatrix}^2\right]$$

$$=\frac{1}{\Sigma l^2}\left[\left(\frac{-m\mu}{l}-\nu\right)^2+\mu^2+\left(\frac{l\nu}{n}\right)^2\right]$$

$$=\frac{1}{\Sigma l^2}\left[\left(\frac{-m\mu-l\nu}{l}\right)^2+\mu^2+\left(\frac{l\nu}{n}\right)^2\right]$$

$$=\frac{1}{\Sigma l^2}\left[\left(\frac{m\mu+n\nu}{l}\right)^2+\mu^2+\left(\frac{n\nu}{n}\right)^2\right], \quad \because \ l=n$$

$$=\frac{1}{\Sigma l^2}\left[\left(\frac{-l\lambda}{l}\right)+\mu^2+\nu^2\right], \quad \because \ l\lambda+m\mu+n\nu=0$$

$$=\frac{1}{\Sigma l^2}\left[\lambda^2+\mu^2+\nu^2\right]=\frac{\lambda^2+\mu^2+\nu^2}{(l^2+m^2+n^2)}$$

$\therefore$ Required perpendicular distance

$$=\sqrt{[(\lambda^2+\mu^2+\nu^2)/(l^2+m^2+n^2)]} \qquad \text{Hence proved.}$$

***Ex. 5. Find the equation of the two planes through the origin which are parallel to the line $\frac{x-1}{2}=\frac{y+3}{-1}=\frac{z-1}{-2}$ and distance $\frac{5}{3}$ from it.**

(*Meerut* 91 *S*)

Sol. The equation of any plane through origin is

$$Ax+By+Cz=0. \quad \text{...(i)}$$

If this plane is parallel to the line, whose direction ratios are 2, – 1, – 2, then the normal to this plane (i) must be perpendicular to the given line

i.e. $A.2+B.(-1)+C(-2)=0$ or $2A-B-2C=0$ ...(ii)

Also the plane (i) is at a distance (5/3) from the given line *i.e.* at a distance 5/3 from the point (1, –3, – 1) on this line.

$$\therefore \frac{A(1)+B(-3)+C(-1)}{\sqrt{(A^2+B^2+C^2)}}=\frac{5}{3}$$

or $$9(A-3B-C)^2=25(A^2+B^2+C^2)$$

or $$9(A^2+9B^2+C^2-6AB+6BC-2AC)=25(A^2+B^2+C^2)$$

or $$8A^2-28B^2+8C^2+27AB-27BC+9CA=0 \quad ...(iii)$$

From (ii) we have $B=2(A-C)$. Substituting this in (iii) we get

$$8A^2-28\{4(A-C)^2\}+8C^2+54A(A-C)-54C(A-C)+9CA=0$$

or $$-50A^2-50C^2+125AC=0 \quad \text{or} \quad 2A^2+2C^2-5AC=0$$

or $$(2A-C)(A-2C)=0 \quad \text{or} \quad A=\tfrac{1}{2}C, 2C$$

$\therefore$ From (ii) we have $B=2(A-C)=2(\frac{1}{2}C-C)$, if $A=\frac{1}{2}C$

or $2(2C-C)$ if $A=2C$.

i.e. $$B=-C, 2C$$

Thus we have two cases $A=\frac{1}{2}C, B=-C$ and $A=2C=B$

$\therefore$ From (i) the required equations are

$$\tfrac{1}{2}Cx-Cy+Cz=0 \quad \text{and} \quad 2Cx+2Cy+Cz=0$$

or $$x-2y+2z=0 \quad \text{and} \quad 2x+2y+z=0.$$ **Ans.**

Ex. 6. Find the length and equations of the perpendicular from the origin to the line $x+2y+3z+4=0=2x+3y+4z+5$. Also find the coordinates of the foot of the perpendicular.

Sol. Refer Fig. 2 Page 125 Ch. IV.

The equations of the given line AB (say) in the symmetric form can be found as $$\frac{x-2}{1}=\frac{y+3}{-2}=\frac{z-0}{1}. \quad ...(i)$$

$A(2, -3, 0)$ is any point on this line. Also here P is $(0, 0, 0)$ and let N, the foot of the perpendicular from P to the line (i), be at a distance r from A.

Then the coordinates of N are $(2+r, -3-2r, r)$ ...(ii)

$\therefore$ The d.r.'s of the line PN are $(2+r, -3-2r, r)$ ...(iii)

Also the d.r.'s of the line AB from (i) are $1, -2, 1$.

$\therefore$ As PN is perpendicular to AB, so we have

$1.(2+r)-2(-3-2r)+1.r=0$ or $6r+8=0$ or $r=-(4/3)$

$\therefore$ From (ii) coordinates of N are $\left(\frac{2}{3}, -\frac{1}{3}, -\frac{4}{3}\right)$. **Ans.**

and from (iii) the d.r.'s of the line PN are

$$(2/3), -(1/3), -(4/3) \quad \text{or} \quad 2, -1, -4, \quad ...(iv)$$

The required length of the perpendicular

$$=PN=\sqrt{[(2/3)^2+(-1/3)^2+(-4/3)^2]}=\tfrac{1}{3}\sqrt{(21)}$$ **Ans.**

Also the equations of the line PN passing through $P(0, 0, 0)$ and d.r.'s $2, -1, -4$ [See results (iv) above] are

$$\frac{x-0}{2}=\frac{y-0}{-1}=\frac{z-0}{-4} \quad \text{or} \quad \frac{x}{2}=\frac{y}{-1}=\frac{z}{-4}.$$ **Ans.**

***Ex. 7. Find the locus of a point which moves so that its distance from the line $x = y = -z$ is twice its distance from the plane**

$$x + y + z = 0$$ (*Kanpur* 90)

Sol. Let $P(x_1, y_1, z_1)$ be the point whose locus is required under the given condition.

Its distance from the given plane

$$=\frac{x_1+y_1+z_1}{\sqrt{(1^2+1^2+1^2)}}=\frac{x_1+y_1+z_1}{\sqrt{3}}=p \text{ (say)}, \quad \text{...(i)}$$

Also any point on the given line $\frac{x}{1}=\frac{y}{1}=\frac{z}{-1}$ is $A(0, 0, 0)$.

Let the foot of the perpendicular from P to this line be N and let it be at distance r from $A(0, 0, 0)$, then the coordinates of N are $(r, r, -r)$ and the d.r.'s of PN are $x_1 - r, y_1 - r, z_1 + r$.

Also the d.r.'s of the given line AB (say) are $1, 1, -1$.

Since PN is perpendicular to AB, so we have

$$1.(x_1-r)+1.(y_1-r)-1.(z_1+r)=0 \quad \text{or} \quad r=\tfrac{1}{3}(x_1+y_1-z_1)$$

$\therefore$ The coordinates of N are $(r, r, -r)$, where $r=\frac{1}{3}(x_1+y_1-z_1)$

$$\therefore PN^2=(x_1-r)^2+(y_1-r)^2+(z_1+r)^2$$
$$=x_1^2+y_1^2+z_1^2+3r^2-2r(x_1+y_1-z_1)$$
$$=x_1^2+y_1^2+z_1^2+3.(1/9)(x_1+y_1-z_1)^2-\tfrac{2}{3}.(x_1+y_1-z_1)^2$$
$$=x_1^2+y_1^2+z_1^2-\tfrac{1}{3}(x_1+y_1-z_1)^2$$
$$=(2/3)[x_1^2+y_1^2+z_1^2-x_1y_1+y_1z_1+z_1x_1] \quad \text{...(ii)}$$

Also according to the problem $PN = 2p$ or $PN^2 = 4p^2$

or $\quad (2/3)[x_1^2+y_1^2+z_1^2-x_1y_1+y_1z_1+z_1x_1]=(4/3)(x_1+y_1+z_1)^2$, from (i) and (ii)

or $\quad x_1^2+y_1^2+z_1^2+5x_1y_1+3y_1z_1+3z_1x_1=0$, on simplifying.

The required locus of $P(x_1, y_1, z_1)$ is

$$x^2+y^2+z^2+5xy+3yz+3zx=0$$ **Ans.**

Ex. 8. Find the locus of a point whose distance from x-axis is twice its distance from the yz-plane.

Sol. Let $P(\alpha, \beta, \gamma)$ be the point whose locus is to be found according to the given problem.

Distance of P from yz-plane $= x$-coordinate of the point $F = \alpha$ **(Note)**

Again square of distance of $P(\alpha, \beta, \gamma)$ from x-axis *i.e.*

These two sets of relations can also be deduced algebracilly from the two sets (i) and (ii).

Again from the relations (ii), we have

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0,\ l_1 l_3 + m_1 m_3 + n_1 n_3 = 0$$

Solving these simultaneouly, we get

$$\frac{l_1}{m_2 n_3 - m_3 n_2} = \frac{m_1}{n_2 l_3 - n_3 l_2} = \frac{n_1}{l_2 m_3 - l_3 m_2}$$

$$= \frac{\sqrt{[(l_1^2 + m_1^2 + n_1^2)]}}{\sqrt{[\Sigma(m_2 n_3 - m_3 n_2)^2]}} = \frac{\pm\sqrt{(1)}}{\sqrt{(\sin 90°)}} \qquad \textbf{(Note)}$$

$$= \pm 1. \qquad \textbf{(Note)}$$

These express the d.c.'s of one line in terms of those of the other two.

Also let $\Delta = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$

Then $\Delta^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$

$$= \begin{vmatrix} \Sigma l_1^2 & \Sigma l_1 l_2 & \Sigma l_1 l_2 \\ \Sigma l_1 l_2 & \Sigma l_2^2 & \Sigma l_2 l_3 \\ \Sigma l_3 l_1 & \Sigma l_3 l_2 & \Sigma l_3^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

from relations (i) and (ii)

or $\Delta^2 = 1$ or $\Delta = \pm 1$ *i.e.* $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \pm 1.$

****§ 6.06 Invariants.**

Show that if by any change of rectangular axes, the expression $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ be transformed, then the expressions $a + b + c$, $A + B + C$ and Δ are invariants i.e. remain unchanged in value and A, B, C are the cofactors of a, b, c in the determinant

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Proof. Here $A = bc - f^2,\ B = ca - g^2,\ C = ab - h^2$ and the value of

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$\therefore \quad A + B + C = bc + ca + ab - f^2 - g^2 - h^2$$

Now two cases arise :

Case I. If only origin is shifted to (x_1, y_1, z_1) axes remainingg parallel.

In this case the given expression transforms into

$$a(x+x_1)^2 + b(y+y_1)^2 + c(z+z_1)^2 + 2f(y+y_1)(z+z_1) + 2g(z+z_1)(x+x_1) + 2h(x+x_1)(y+y_1)$$

or $$[ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy] + 2x(ax_1 + hy_1 + gz_1) + 2y(hx_1 + by_1 + fz_1) + 2z(gx_1 + fy_1 + cz_1) + \dots$$

The second degree terms remain unaltered after the transformation and as $a+b+c$, $A+B+C$ and Δ contain only a, b, c, f, g, h *i.e.* the coefficiens of second degree terms, so these are invariants *i.e.* remain unchanged.

Case II. If only the (rectangular) axes are rotated, origin remaining same.

In this case let the given expression $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ transform into $\quad a_1x^2 + b_1y^2 + c_1z^2 + 2f_1yz + 2g_1zx + 2h_1xy$

Also as origin remains unchanged, so the distance of any point $P(x, y, z)$ from the origin remains unchanged and so $x^2 + y^2 + z^2$ remains the same.

$\therefore$ The expression

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + \lambda(x^2 + y^2 + z^2) \qquad \dots(i)$$

transforms into

$$a_1x^2 + b_1y^2 + c_1z^2 + 2f_1yz + 2g_1zx + 2h_1xy + \lambda(x^2 + y^2 + z^2) \qquad \dots(ii)$$

If for some value of λ, the expression (i) is the product of two linear factors, then for the same value of λ, the expression (ii) can also be written as the product of two linear factors.

Now (i) can be written as

$$(a+\lambda)x^2 + (b+\lambda)y^2 + (c+\lambda)z^2 + 2fyz + 2gzx + 2hxy$$

and this can be written as the product of two linear factors if

$$\text{“}abc + 2fgh - af^2 - bg^2 - ch^2 = 0\text{”}$$

i.e. if $(a+\lambda)(b+\lambda)(c+\lambda) + 2fgh - (a+\lambda)f^2 - (b+\lambda)g^2 - (c+\lambda)h^2 = 0$

i.e. if $\lambda^3 + (a+b+c)\lambda^2 + (ab+bc+ca - f^2 - g^2 - h^2)\lambda + (abc + 2fgh - af^2 - bg^2 - ch^2) = 0$

i.e. if $\lambda^3 + (a+b+c)\lambda^2 + (A+B+C)\lambda + \Delta = 0, \qquad \dots(iii)$

where the symbols have their usual meanings.

Similarly (ii) can be written as the product of two linear factors if

$$\lambda^3 + (a_1 + b_1 + c_1)\lambda^2 + (A_1 + B_1 + C_1)\lambda + \Delta_1 = 0, \qquad \dots(iv)$$

where $A = b_1c_1 - f_1^2$ etc. and $\Delta_1 = a_1b_1c_1 + 2f_1g_1h_1 - a_1f_1^2 - b_1g_1^2 - c_1h_1^2$

$\because$ The equations (iii) and (iv) have the same roots, so comparing the coefficients, we get

$$\frac{x-0}{1}=\frac{y-0}{0}=\frac{z-0}{0} \text{ is}$$

$$\begin{vmatrix} \alpha-0 & \beta-0 \\ 1 & 0 \end{vmatrix}^2+\begin{vmatrix} \beta-0 & \gamma-0 \\ 0 & 0 \end{vmatrix}^2+\begin{vmatrix} \gamma-0 & \alpha-0 \\ 0 & 1 \end{vmatrix}^2$$

i.e. $$(-\beta)^2+(0)^2+(\gamma)^2 \quad i.e \quad \beta^2+\gamma^2$$

$\therefore$ According to the problem we have

$$\sqrt{(\beta^2+\gamma^2)}=2\alpha \quad \text{or} \quad 4\alpha^2=\beta^2+\gamma^2$$

$\therefore$ Required locus of $P(\alpha, \beta, \gamma)$ is $4x^2=y^2+z^2$. Ans

***Ex. 9. Find the equaton of the right circular cylinder of ra**
whose axis passes through (1, 2, 3) and has direction cosines prop
to (2, 3, 6).

Sol. The equatons of the axis of the cylinder are

$$\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z-3}{6}.$$

If $P(x, y, z)$ by any point on the cylinder, then the l
perpendicular from P to the line (i) is equal to radius 2 of the cy

i.e. $$(2)^2=\frac{1}{2^2+3^2+6^2}\left[\begin{vmatrix} x-1 & y-2 \\ 2 & -3 \end{vmatrix}^2+\begin{vmatrix} y-2 & z \\ -3 & \end{vmatrix}\right.$$

See § 4.14 and Note on P.

or $$4=(1/49)\left[(-3x-2y+7)^2+(6y-3z-21)^2+(\right.$$

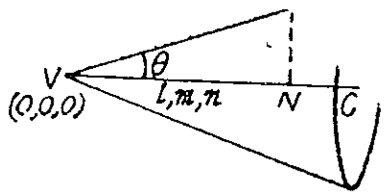
or $$196=(3x+2y-7)^2+9(2y-z-7)^2+4(3x-z)^2$$

Ex. 10. Show that the equation to a right circular
is the origin, the semi-vertical angle θ and whose axis has
l, m, n is $\Sigma[yn-zm]^2=(x^2+y^2+z^2)\sin^2\theta$.

Sol. Let $P(x, y, z)$ be a point on the one whose vertex V is (0, 0, 0) and semi-vertical angle is θ.

From P draw PN perpendicular to its axis VC, whose d.c.'s are l, m, n.

The equation of VC is $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.



Then PN^2

$$=\begin{vmatrix} x & y \\ l & m \end{vmatrix}^2+\begin{vmatrix} y & z \\ m & n \end{vmatrix}^2+\begin{vmatrix} z & x \\ n & l \end{vmatrix}^2$$

(Fig. 3)
....See § 4.14 Page 125 Ch. IV

$$=(xm-yl)^2+(yn-zm)^2+(zl-nx)^2=\Sigma(yn-zm)^2 \quad \text{...(i)}$$

And VP = distance between $V(0, 0, 0)$ and $P(x, y, z)$

or $\quad VP = \sqrt{(x^2 + y^2 + z^2)}$ or $VP^2 = x^2 + y^2 + z^2$ ...(ii)

Also from $\Delta\, VPN$ it is evident that

$$PN = VP \sin\theta \quad \text{or} \quad PN^2 = VP^2 \sin^2\theta$$

or $\quad \Sigma\,(yn - zm)^2 = (x^2 + y^2 + z^2)\sin^2\theta$, from (i) and (ii). Hence proved.

***Ex. 11. Find the equation to the right circular cone whose vertex is (2, 3, 5), the semi-vertical angle is 30°, and the axis is a line equally inclined to the coordinate axes.**

Sol. Refer Fig. 3 Page 133 Ch. IV.

If the axis VC of the cone [Here V is $(2, -3, 5)$] is equally inclined to the ...nate axes, then its direction cosines are l, l, l, where

$$\sqrt{(l^2 + l^2 + l^2)} = 1 \quad \text{or} \quad l = 1/\sqrt{3}$$

...direction cosines of the axis VC of the cone are

$$1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3} \text{ and } V \text{ is } (2, -3, 5).$$

...equations of the axis VC are $\dfrac{x-2}{1/\sqrt{3}} = \dfrac{y+3}{1/\sqrt{3}} = \dfrac{z-5}{1/\sqrt{3}}$

...$x, y, z)$ be any point on the cone, then the length of the ...from P to the axis VC is given by

$$\begin{vmatrix} y+3 \\ 1/\sqrt{3} \end{vmatrix}^2 + \begin{vmatrix} y+3 & z-5 \\ 1/\sqrt{3} & 1/\sqrt{3} \end{vmatrix}^2 + \begin{vmatrix} z-5 & x-2 \\ 1/\sqrt{3} & 1/\sqrt{3} \end{vmatrix}^2$$

$$\begin{vmatrix} y+3 \\ 1 \end{vmatrix}^2 + \begin{vmatrix} y+3 & z-5 \\ 1 & 1 \end{vmatrix}^2 + \begin{vmatrix} z-5 & x-2 \\ 1 & 1 \end{vmatrix}^2 \Bigg]$$

$$...-5)^2 + (y - z + 8)^2 + (z - x - 3)^2]$$

$$+ 2y^2 + 2z^2 - 2xy - 2yz - 2xz - 4x + 26y - 22z + 98] \qquad ...(i)$$

... Square of the distance between $V(2, -3, 5)$ and $P(x, y, z)$

$$= (x-2)^2 + (y+3)^2 + (z-5)^2$$

$$= x^2 + y^2 + z^2 - 4x + 6y - 10z + 38 \qquad ...(ii)$$

...o from $\Delta\, VPN$ we get $PN = VP \sin\theta = VP \sin 30°$

$$PN^2 = VP^2 \left(\tfrac{1}{2}\right)^2$$

or $\quad (2/3)\,[x^2 + y^2 + z^2 - xy - yz - zx - 2x + 13y - 11z + 49]$

or $\quad = (1/4)\,[x^2 + y^2 + z^2 - 4x + 6y - 10z + 38]$, from (i) and (ii)

or $\quad 5x^2 + 5y^2 + 5z^2 - 8xy - 8yz - 8zx - 4x + 86y - 58z + 278 = 0.$ **Ans.**

Exercises on § 4.14.

Ex. 1. Find the equation, foot and length of the perpendicular from (2, 3, 5) to the line $(x - 15)/3 = (y - 29)/8 = (z - 5)/(-3)$.

Ex. 2. Find the distance of $(-2, 1, 5)$ from the line through (5, 7, 3) whose direction cosines are proportional to 2, −3, 6. **Ans.** $4\sqrt{(61)}/7$.

Ex. 3. Find the equation of a right circular cone whose vertex is '(2, – 3, 5) and axis the line PQ, which is equally inclined to the axis and 'hich passes through the point $A\,(1, -2, 5)$

Ans. $x^2+y^2+z^2+6yz+6zx+6xy=0$

Ex. 4. Find the perpendicular distance of the point (2, 4, – 1) from the ne $x+5=\frac{1}{4}(y+3)=-\frac{1}{9}(z-6)$. Also find its equations.

****§ 4.15. To find the shortest distance (or S. D.) between two given nes and to obtain the equation of this shortest distance.** (*Kumaun* 95, 93)

Definition. Two lines are said to be *skew lines or non-intersecting lines* if ey do not lie in the same plane and the straight line which is perpendicular to ich of these two non-intersecting lines is called the line of shortest distance or D. The length of this line intercepted between the given lines is called the ngth of the shortest distance.

****Lengths and equations of S. D.** (*Kanpur 97*)

Method I. *Length of S.D.*

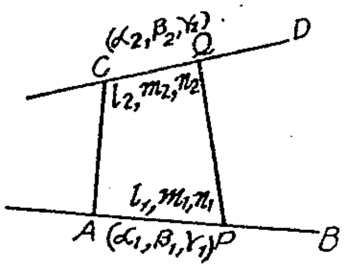
Let the equations of the given lines be

$$\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1} \qquad \text{...(i)}$$

d

$$\frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2} \qquad \text{...(ii)}$$

Let PQ be the line which is perpendicular both the given lines AB and CD. Let l, m, n be direction ratios of the line PQ.



Then as PQ is perpendicular to the lines and (ii), so we have $ll_1+mm_1+nn_1=0$

l $\quad ll_2+mm_2+nn_2=0.$

Solving these, we get

(Fig. 4)

$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1} \qquad \text{...(iii)}$$

$\therefore$ If $\lambda,\ \mu,\ \nu$ be the actual d.c.'s of the line PQ, then we have

$$\lambda=\frac{(m_1n_2-m_2n_1)}{\sqrt{[(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2]}}$$

$$\lambda=\frac{m_1n_2-m_2n_1}{\sqrt{[\Sigma\,(m_1n_2-m_2n_1)^2]}}$$

Similarly $\mu=\dfrac{n_1l_2-n_2l_1}{\sqrt{[\Sigma\,(m_1n_2-m_2n_1)^2]}}$ and $\nu=\dfrac{l_1m_2-l_2m_1}{\sqrt{[\Sigma\,(m_1n_2-m_2n_1)^2]}}$ ...(iv)

From figure 4 above it is evident that PQ is the projection of AC [where nd C are points $(\alpha_1,\beta_1,\gamma_1)$ and $(\alpha_2,\beta_2,\gamma_2)$ on the line (i) and (ii) ectively] on the line PQ whose d.c.'s are (λ, μ, ν) given by (iv).

$\therefore\ PQ = (\alpha_1 - \alpha_2)\lambda + (\beta_1 - \beta_2)\mu + (\gamma_1 - \gamma_2)\nu.$ ...See § 2.09 Ch.

$$= \frac{(\alpha_1 - \alpha_2)(m_1n_2 - m_2n_1) + (\beta_1 - \beta_2)(n_1l_2 - n_2l_1) + (\gamma_1 - \gamma_2)(l_1m_2 - l_2m_1)}{\sqrt{[\Sigma (m_1n_2 - m_2n_1)^2]}}$$

$$= \begin{vmatrix} \alpha_1 - \alpha_2 & \beta_1 - \beta_2 & \gamma_1 - \gamma_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{[\Sigma (m_1n_2 - m_2n_1)^2]}$$ **(Note)** ...(v

An important Note : If the given lines are coplanar, then

$$\text{S.D.} = 0 \quad i.e. \quad \Sigma (\alpha_1 - \alpha_2)(m_1n_2 - m_2n_1) = 0$$

Equations of S.D. (*Kanpur* 9

From the figure 4 above it is evident that the line PQ (which represen S.D.) is the line of intersection of the plane containing the lines AB and PQ a the plane containing the lines CD and PQ. **(Not**

Now the equation of the plane containing lines AB and PQ is

$$\begin{vmatrix} x - \alpha_1 & y - \beta_1 & z - \gamma_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0$$ ...(v

and the equations of the plane containing the lines CD and PQ is

$$\begin{vmatrix} x - \alpha_2 & y - \beta_2 & z - \gamma_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix} = 0$$ ...(v

The line of shortest distance PQ is the line of intersection of these t planes given by (vi) and (vii), hence the equations of the line of the S.D. given by (vi) and (vii).

Method II. The general coordinates of points on the two lines given (i) and (ii) above are $(\alpha_1 + l_1r_1, \beta_1 + m_1r_1, \gamma_1 + n_1r_1)$ say point P

and $(\alpha_2 + l_2r_2, \beta_2 + m_2r_2, \gamma_2 + n_2r_2)$ say point Q.

If these are the points where the line of S.D. meets the given lines (i) (ii) respectively, then the line PQ must be at right angles to both the lines and (ii). Find the direction ratios of the lines PQ and apply the conditions PQ is perpendicular to the lines (i) and (ii).

Solving the two equations so obtained, we can find the values of r_1 r_2. Hence the coordinates of P and Q and also the d.c.'s of the line PQ known, which enable us to find the distance between P and Q and the equa of the line PQ.

Note. This method is useful when the coordinates of P and Q required.

Method III. The shortest distance can also be obtained if we use the that it is equal to the length of the perpendicular from any point on one of lines to the plane through the other line parallel to the first.

Note : This method is generally used when the equations of one lir given in general form while those of the other are in the symmetric form.

Method IV. If the equations of both the lines are given in general f as $u_1 = 0 = v_1$ and $u_2 = 0 = v_2$.

Then the equations of any plane through the first line is

$$u_1 + k_1 v_1 = 0. \quad \text{...(A)}$$

And the equation of any plane through the second line is

$$u_2 + k_2 v_2 = 0. \quad \text{...(B)}$$

Choose k_1 and k_2 in such a manner that planes (A) and (B) are parallel then the required S. D. is the distance between these two parallel planes and the equations of S.D. will be given by planes through each of the given planes and perpendicular to these parallel planes.

An Important Note : For convenience we generally reduce the given equations to the symmetric form (if they are not so) and use method I as given on Pages 135-36 Ch. IV.

Solved Examples on § 4.15

****Ex. 1 (a). Find the S. D. between the lines**

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-2}{1} \quad \text{...(i)} \quad \text{and} \quad \frac{x-1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \quad \text{...(ii)}$$

(*Meerut* 91, 90)

Sol. Let l, m, n be the d.c.'s of the S. D. of the given lines.

Then we have $\quad l - 2m + n = 0, \; 7l - 6m + n = 0$

Solving these, we get $\dfrac{l}{-2+6} = \dfrac{m}{7-1} = \dfrac{n}{-6+14}$

or
$$\frac{l}{2} = \frac{m}{3} = \frac{n}{4} = \frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(2^2+3^2+4^2)}} = \frac{1}{\sqrt{(29)}}$$

$\therefore$ The direction cosines of S. D. are $\dfrac{2}{\sqrt{(29)}}, \dfrac{3}{\sqrt{(29)}}, \dfrac{4}{\sqrt{(29)}}$ ...(iii)

Also as $A\,(3, 5, 2)$ is a point on the line (i) and $B\,(1, -1, -1)$ is a point on the line (ii).

$\therefore$ The length of S. D. = projection of join of A and B on the line whose d.c.'s are given by (iii)

$$= \frac{2}{\sqrt{(29)}}[3-1] + \frac{3}{\sqrt{(29)}}[5-(-1)] + \frac{4}{\sqrt{(29)}}[2-(-1)]$$

$$= \frac{2\,(2) + 3\,(6) + 4\,(3)}{\sqrt{(29)}} = \frac{34}{\sqrt{(29)}} \qquad \textbf{Ans.}$$

Ex. 1 (b). Find the S. D. between the lines

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \quad \text{and} \quad \frac{x+1}{7} = \frac{y+6}{-1} = \frac{z+1}{1}$$ (*Meerut* 92)

Ans. $2\sqrt{(29)}$

Hint. Do as Ex. 1 (a) above.

***Ex. 2 (a). Find the length and the equations of the common perpendicular to the two lines**

$$\frac{x+3}{-4} = \frac{y-6}{3} = \frac{z}{2} \quad \text{...(i)} \quad \text{and} \quad \frac{x+2}{-4} = \frac{y}{1} = \frac{z-7}{1} \quad \text{...(ii)}$$

(*Avadh* 95, 91, 90; *Gorakhpur* 91; *Purvanchal* 95)

Sol. Let l, m, n be the d.c.'s of the line of the common perpendicular (or S. D.) to the two given lines. Then we have

$$-4.l+3m+2n=0,\ -4l+m+n=0$$

Solving these we get $\dfrac{l}{3-2}=\dfrac{m}{-8+4}=\dfrac{n}{-4+12}$

or $$\frac{l}{1}=\frac{m}{-4}=\frac{n}{8}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{[1^2+(-4)^2+8^2]}}=\frac{1}{9}$$

$\therefore$ The d.c.'s of the S. D are $(1/9), -(4/9), (8/9)$

and the d.r.'s of the S. D. are $1, -4, 8$. ...(iii)

Also $A(-3, 6, 0)$ is a point on the line (i) and $B(-2, 0, 7)$ is a point on the line (ii).

$\therefore$ The length of S. D. = projection of join of A and B on the line whose d.c.'s are given by (iii)

$$=(1/9)[(-3)-(-2)]-(4/9)[(6)-(0)]+(8/9)[(0)-(7)]$$
$$=-(1/9)-(24/9)-(56/9)=9 \text{ (numerically).}$$

Equations of S. D.

The equation of the plane through the line (ii) and S. D. is

$$\begin{vmatrix} x+3 & y-6 & z \\ -4 & 3 & 2 \\ 1 & -4 & 8 \end{vmatrix}=0 \text{ or } 32x+34y+13z=108 \quad ...(iv)$$

And the equation of the plane through the line (ii) and S. D. is

$$\begin{vmatrix} x+2 & y & z-7 \\ -4 & 1 & 1 \\ 1 & -4 & 8 \end{vmatrix}=0 \text{ or } 4x+11y+5z-27=0 \quad ...(v)$$

$\therefore$ From (iv) and (v) the equations of the S. D. are

$$32x+34y+13z-108=0 \quad \text{and} \quad 4x+11y+5z-27=0. \quad \textbf{Ans.}$$

Ex. 2 (b). Show that the shortest distance between the lines $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}$; $\dfrac{x-2}{3}=\dfrac{y-4}{4}=\dfrac{z-5}{5}$ is $\dfrac{1}{\sqrt{6}}$ and that its equations are $11x+2y-7z+6=0,\ 7x+y-5z+7=0$.

(*Bundelkhand* 95, 94; *Meerut* 94, 91 *S*; *Rohilkhand* 92)

Sol. The given lines are

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \quad ...(i) \quad \text{and} \quad \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5} \quad ...(ii)$$

Any point P on the line (i) is $(1+2r_1, 2+3r_1, 3+4r_1)$

and any point Q on the line (ii) is $(2+3r_2, 4+4r_2, 5+5r_2)$ ...(iii)

$\therefore$ The direction ratios of the line PQ are ...(iv)

$$(1+2r_1)-(2+3r_2), (2+3r_1)-(4+4r_2), (3+4r_1)-(5+5r_2)$$

or $$2r_1-3r_2-1, 3r_1-4r_2-2, 4r_1-5r_2-2 \quad ...(v)$$

If PQ is the required S. D., then PQ is perpendicular to both the given lines and as such we have

$$2(2r_1-3r_2-1)+3(3r_1-4r_2-2)+4(4r_1-5r_2-2)=0$$

and $\quad 3(2r_1-3r_2-1)+4(3r_1-4r_2-2)-5(4r_1-5r_2-2)=0$

or $\quad 29r_1-38r_2-16=0 \quad$ and $\quad 32r_1-50r_2-21=0$

Solving these, we get $r_1=1/3$ and $r_2=-1/6$.

Substituting these values of r_1 and r_2 in (iii), (iv) and (v), we have the coordinates of P and Q as $\left(\frac{5}{3}, 3, \frac{13}{3}\right)$ and $\left(\frac{3}{2}, \frac{10}{3}, \frac{25}{6}\right)$

And the d.r.'s of the line PQ are $\frac{1}{6}, \frac{-1}{3}, \frac{1}{6}$ or $1, -2, 1$.

$\therefore$ The length of S. D.

$$=PQ=\sqrt{\left[\left(\frac{5}{3}-\frac{3}{2}\right)^2+\left(3-\frac{10}{3}\right)^2+\left(\frac{13}{3}-\frac{25}{6}\right)^2\right]}$$

$$=\sqrt{\left[\frac{1}{36}+\frac{1}{9}+\frac{1}{36}\right]}=\frac{1}{\sqrt{6}}.$$ Hence proved.

Also the line PQ is the line of intersection of the plane containing the line (i) and PQ and the plane containing the line (ii) and PQ.

The equation of the plane containing the line (i) and PQ is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{vmatrix}=0$$

or $$\begin{vmatrix} x-z+2 & y+2z-8 & z-3 \\ -2 & 11 & 4 \\ 0 & 0 & 1 \end{vmatrix}=0,$$

adding twice third column to second and subtracting third column from first.

or $\quad 11(x-z+2)+2(y+2z-8)=3 \quad$ or $\quad 11x+2y-7z+6=0 \quad$ ...(vi)

And the equation of the plane containing the line (ii) and PQ is

$$\begin{vmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ 1 & -2 & 1 \end{vmatrix}=0$$

or $$\begin{vmatrix} x-z+3 & y+2z-14 & z-5 \\ -2 & 14 & 5 \\ 0 & 0 & 1 \end{vmatrix}=0,$$

adding twice third column to second and subtracting third column from first.

or $\quad 14(x-z+3)+2(y+2z-14)=0 \quad$ or $\quad 7x+y+5z+7=0 \quad$ ...(vii)

$\therefore$ The equations of S. D. are given by (vi) and (vii).

Another method for equations of S.D.

The line PQ is the line through $P(5/3, 3, 13/3)$ and having d.r.'s as $1, -2, 1$

$\therefore$ The equations of the S. D. are

$$\frac{x-(5/3)}{1}=\frac{y-3}{-2}=\frac{z-(13/3)}{1} \quad \text{or} \quad \frac{3x-5}{1}=\frac{3y-9}{-2}=\frac{3z-13}{1} \qquad \textbf{Ans.}$$

***Ex. 2 (c). Find the length of the S. D. between the lines**

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}, \frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$$

Find also its equations. (*Meerut* 95, 92 *P*)

Sol. Do as Ex. 2 (a) above. **Ans.** $4\sqrt{(29)}; \frac{1}{2}(x-1)=\frac{1}{3}(y-2)=\frac{1}{4}(z-3)$

Ex. 2 (d). Find the points on the lines

$$\frac{1}{3}(x-6)=-(y-7)=z-4 \quad \text{and} \quad -\frac{1}{3}x=\frac{1}{2}(y+9)=\frac{1}{4}(z-2)$$

which are nearest to each other. Hence find the S. D. between the lines and also its equations.

[**Hint :** Do as Ex. 2 (b) above.] **Ans.** $3\sqrt{(30)}; \frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}$

****Ex. 3 (a). Find the S. D. between lines**

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1} \quad \text{...(i)} \quad \text{and} \quad \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4} \quad \text{...(ii)}$$

Find also its equations and the points in which it meets the given lines. (*Agra* 92, 90; *Garhwal* 94, 92, 91; *Kanpur* 97; *Meerut* 96 *P*, 93, 90 *S*; *Purvanchal* 92, 91;

Sol. Any point on the line (i) is $(3-3r, 8-r, 3+r)$, say point P

And any point on the line (ii) is $(-3-3r', -7+2r', 6+4r')$ say point Q

Then the direction ratios of the line PQ are

$$(3+3r)-(-3-3r'), (8-r)-(-7+2r'), (3+r)-(6+4r')$$

or
$$3r+3r'+6, \ -r-2r'+15, \ r-4r'-3 \qquad \text{...(iii)}$$

Now if PQ is the S. D. between the given lines then PQ is perpendicular to both (i) and (ii), the conditions for the same are

$$3(3r+3r'+6)-1(-r-2r'+15)+1.(r-4r'-3)=0$$

and
$$-3(3r+3r'+6)+2(-r-2r'+15)+4(r-4r'-3)=0$$

or
$$11r+7r'=0 \quad \text{and} \quad 7r+29r'=0$$

Solving these we find $r=0$ and $r'=0$

Substituting these values of r and r', we find that the coordinates of P and Q are $(3, 8, 3)$ and $(-3, -7, 6)$ respectively. **Ans.**

And the d.r.'s of the line PQ from (iii) are $6, 15, -3$ or $2, 5, -1$

Now the required S. D.

$$= PQ = \sqrt{[\{3-(-3)\}^2+\{8-(-7)\}^2+(3-6)^2]}$$
$$= \sqrt{(36+225+9)} = 3\sqrt{(30)}. \qquad \textbf{Ans.}$$

Also PQ is a line through $P(3, 8, 3)$ and of direction ratios $2, 5, -1$ so its equations are

$$\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{1} \qquad \textbf{Ans.}$$

***Ex. 3 (b). Find the S. D. between the lines**

$$\frac{x-3}{1}=\frac{y-8}{-1}=\frac{z-3}{1} \quad \text{and} \quad \frac{x+3}{1}=\frac{y+7}{2}=\frac{z-6}{4}$$ (*Rohilkhand* 95)

Hint : Do as Ex. 3 (a). above. **Ans.** $5\sqrt{6}$

***Ex. 3 (c). Find the magnitude and the equations of the line of S. D. between the lines**

$$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7} \quad \text{and} \quad \frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$$

Hint : Do as Ex. 3 (a) above. (*Kanpur* 91; *Purvanchal* 97)

Ans. 14; $\dfrac{x-9}{2}=\dfrac{y-13}{3}=\dfrac{z-15}{6}$

****Ex. 4. Find the S. D. between the lines**

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \quad \text{...(i)} \quad \text{and} \quad \frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5} \quad \text{...(ii)}$$

(*Kumaun* 96)

Sol. Let l, m, n be the d.c.'s of the S. D. Since the lines of S. D. is perpendicular to both the given lines so we have

$$2.l+3.m+4.n=0 \quad \text{and} \quad 3l+4m+5n=0$$

Solving these we get $\dfrac{l}{-1}=\dfrac{m}{2}=\dfrac{n}{-1}=\dfrac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{[(-1)^2+2^2+(-1)^2]}}=\dfrac{1}{\sqrt{6}}$

or $$l=-(1/\sqrt{6}),\ m=(2/\sqrt{6}),\ n=-(1/\sqrt{6}) \quad \text{...(iii)}$$

Also from the given equations it is evident that $A(1, 2, 3)$ is a point on (i) and $B(2, 3, 4)$ is a point on (ii).

Now S. D. is the projection of AB on the line whose d.c.'s are given by (iii) and so we have

S.D. $=l(1-2)+m(2-3)+n(3-4)$, where l, m, n are given by (iii)

$$=-\frac{1}{\sqrt{6}}(-1)+\frac{2}{\sqrt{6}}(-1)-\frac{1}{\sqrt{6}}(-1)=(1/\sqrt{6})(1-2+1)=0.$$

(**Note.** If S. D. $=0$, then the lines intersect *i.e.* are coplanar).

****Ex. 5. Show that the S. D. between any two opposite edges of the tetrahedron formed by the planes $y+z=0$, $z+x=0$, $x+y=0$, $x+y+z=a$ is $2a/\sqrt{6}$ and the three lines of S. D. intersect at the point $x=y=z=-a$.**

(*Avadh* 94; *Kanpur* 95)

Sol. The equations of the line (or edge) of intersection of the planes

$y+z=0$ and $z+x=0$ are $\dfrac{x}{1}=\dfrac{y}{1}=\dfrac{z}{-1}$. ...(i)

Similarly the equations of the edge of intersection of the planes

$x+y=0$ and $x+y+z=a$ are $\dfrac{x}{1}=\dfrac{y}{-1}=\dfrac{z-a}{0}$. ...(ii)

Let l, m, n be the d.c.'s of the S. D. between the lines (i) and (ii). Then as it is perpendicular to both (i) and (ii), so we have

$$1.l+1.m-1.n=0 \quad \text{and} \quad 1.l-1.m+0.n=0.$$

Solving these we get

$$\frac{l}{-1}=\frac{m}{-1}=\frac{n}{-2}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{[(-1)^2+(-1)^2+(-2)^2]}}=\frac{1}{\sqrt{6}}.$$

$$\therefore\ l=-\frac{1}{\sqrt{6}},\ m=-\frac{1}{\sqrt{6}},\ n=-\frac{1}{\sqrt{6}} \qquad \text{...(iii)}$$

From (i) and (ii) it is evident that A (0, 0, 0) is a point on line (i) and B (0, 0, a) is a point on the line (ii). Also required S. D. is the projection of AB on the line whose d.c.'s are given by (iii).

$\therefore$ Required S. D. $= l\,(0-0)+m\,(0-0)+n\,(0-a)$,

where l, m, n are given by (iii)

$=-(2/\sqrt{6})\,(-a)=(2/\sqrt{6})\,a$ Hence proved.

Now the equation of the plane through (i) and the S. D. is

$$\begin{vmatrix} x & y & z \\ 1 & 1 & -1 \\ -1 & -1 & -2 \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} x & y & z \\ 1 & 1 & -1 \\ 0 & 0 & -3 \end{vmatrix}=0, \text{ adding 2nd row to 3rd.}$$

or

$$x-y=0 \qquad \text{...(iv)}$$

Similarly the plane through the line (ii) and S. D. is

$$\begin{vmatrix} x & y & z-a \\ 1 & -1 & 0 \\ -1 & -1 & -2 \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} x & y+x & z-a \\ 1 & 0 & 0 \\ -1 & -2 & -2 \end{vmatrix}=0, \text{ adding first column to 2nd}$$

or

$$(y+x)-(z-a)=0 \quad \text{or} \quad x+y-z+a=0 \qquad \text{...(v)}$$

$\therefore$ The equations of S. D. of (i) and (ii) are given by (iv) and (v)

i.e. $x-y=0$ and $x+y-z+a=0$.

Both these equations are satisfied by $x=y=z=-a$.

Similarly we can show that other S.D.'s between other pairs of opposite edges of the tetrahedron are also satisfied by $x=y=z=-a$

Hence these S. D.'s meet in the point $x=y=z=-a$. Hence proved.

****Ex. 6 (a). Find the S. D. between the z-axis and the line**

$$\mathbf{ax+by+cz+d=0=a'x+b'y+c'z+d'.}$$

Show that it meets the z-axis at a point whose distance from the origin is $\dfrac{\mathbf{(bc'-b'c)(db'-d'b)+(ca'-c'a)(ad'-a'd)}}{\mathbf{(bc'-b'c)^2+(ca'-c'a)^2}}$

(*Avadh* 92, 90; *Kanpur* 93; *Kumaun* 91; *Purvanchal* 96; *Rohilkhand* 94)

Sol. The plane through the given line is

$$(ax+by+cz+d)+\lambda\,(a'x+b'y+c'z+d')=0$$

or

$$(a+\lambda a')\,x+(b+\lambda b')\,y+(c+\lambda c')\,z+(d+\lambda d')=0 \qquad \text{...(i)}$$

If this plane is parallel to z-axis, whose d.c.'s are 0, 0, 1, then the normal to the plane (i) is perpendicular to z-axis and so we get

$$(a+\lambda a')\,.\,0+(b+\lambda b')\,.\,0+(c+\lambda c')\,.\,1=0 \quad \text{or} \quad \lambda=-c/c'.$$

$\therefore$ From (i) the equation of the plane through the given line and parallel to z-axis is $(ax+by+cz+d)-(c/c')\,(a'x+b'y+c'z+d')=0$

or

$$(c'a-ca')\,x+(c'b-b'c)\,y+(d'c-c'd)=0 \qquad \text{...(ii)}$$

Also any point on the z-axis can be taken as origin *i.e.* (0, 0, 0).

$\therefore$ Required S. D. = length of perpendicular from (0, 0, 0) to the plane (ii)

$$=\frac{(dc'-cd')}{\sqrt{[(c'a-ca')^2+(c'b-b'c)^2]}} \quad \textbf{Ans.}$$

Let the S. D. meet the z-axis and the given line at A and B and let A be $(0, 0, z_1)$ and $B(x_2, y_2, z_2)$. Then the direction ratios of the line AB are x_2, y_2, z_2-z_1 respectively.

Also the direction ratios of the given line are

$$bc'-b'c,\ ca'-c'a,\ ab'-a'b. \qquad ...(iii)$$

Now AB is perpendicular to z-axis, whose d.c.'s are 0, 0, 1, so we have

$$0.x_2+0.y_2+1.(z_2-z_1)=0 \quad \text{or} \quad z_1=z_2 \qquad ...(iv)$$

And as AB is perpendicular to the given line, whose d.c.'s are given by (iii) above, so we have

$$x_2(bc'-b'c)+y_2(ca'-c'a)+(z_2-z_1)(ab'-a'b)=0$$

or $$x_2(bc'-b'c)+y_2(ca'-c'a)=0, \qquad ...(v)$$

with the help of (iv).

Also $B(x_2, y_2, z_2)$ being a point on the given line we have

$$ax_2+by_2+(cz_2+d)=0 \qquad ...(vi)$$

and $$a'x_2+b'y_2+(c'z_2+d')=0. \qquad \textbf{(Note)} \qquad ...(vii)$$

Eliminating x_2, y_2 between (v), (vi) and (vii) we get

$$\begin{vmatrix} bc'-b'c & ca'-c'a & 0 \\ a & b & cz_2+d \\ a' & b' & c'z_2+d' \end{vmatrix}=0$$

or $$z_2\begin{vmatrix} bc'-b'c & ca'-c'a & 0 \\ a & b & c \\ a' & b' & c' \end{vmatrix}+\begin{vmatrix} bc'-b'c & ca'-c'a & 0 \\ a & b & d \\ a' & b' & d' \end{vmatrix}=0$$

or $$z_2[(bc'-b'c)^2+(ca'-c'a)^2]+[(bc'-b'c)(bd'-b'd)$$
$$+(ca'-c'a)(da'-d'a)]=0,$$

expanding the determinantts

or $$z_2=\frac{[(bc'-b'c)(bd'-b'd)+(ca'-c'a)(da'-d'a)]}{(bc'-b'c)^2(ca'-c'a)^2} \qquad ...(viii)$$

$\therefore$ The required distance $=z_1=z_2$, from (iv) and the value of z_2 is given by (viii) above. Hence proved.

Ex. 6 (b). Find the S. D. between axis of x and the line

$$\mathbf{ax+by+cz+d=0,\ a'x+b'y+c'z+d'=0} \qquad (Avadh\ 94)$$

Sol. The plane through the given line is

$$(ax+by+cz+d)+\lambda(a'x+b'y+c'z+d')=0 \qquad ...(i)$$

or $$(a+\lambda a')x+(b+\lambda b')y+(c+\lambda c')z+(d+\lambda d')=0$$

If this plane is parallel to x-axis, whose d.c.'s are 1, 0, 0 then the normal to the plane is perpendicular to x-axis and so we get

$$(a+\lambda a').1+(b+\lambda b').0+(c+\lambda c').0=0 \quad \text{or} \quad \lambda=-a/a'$$

$\therefore$ From (i), the equation of the plane through the given line and parallel to x-axis is $(ax+by+cz+d)-(a/a')(a'x+b'y+c'z+d')=0$

or $$(ba'-ab')y+(ca'-c'a)z+(da'+d'a)=0 \qquad \text{...(ii)}$$

Also any point on x-axis can be taken as origin (0, 0, 0)

$\therefore$ Required S. D. = length of perpendicular from (0, 0, 0) to the plane (ii)

$$=\frac{(da'-d'a)}{\sqrt{[(ba'-b'a)^2+(ca'-c'a)^2]}} \qquad \textbf{Ans.}$$

Ex. 6 (c). Find the S. D. between the z-axis and the line

$$\mathbf{x+y+2z=3,\ 2x+3y+4z=4.}$$

Sol. The plane through the given line is

$$(x+y+2z-3)+\lambda(2x+3y+4z-4)=0 \qquad \text{...(i)}$$

or $$(1+2\lambda)x+(1+3\lambda)y+(2+4\lambda)z-(3+4\lambda)=0$$

If this plane is parallel to z-axis whose d.c.'s are 0, 0, 1, then the normal to this plane must be perpendicular to z-axis and so we have

$$(1+2\lambda).0+(1+3\lambda).0+(2+4\lambda).1=0 \quad \text{or} \quad 2+4\lambda=0 \quad \text{or} \quad \lambda=-\tfrac{1}{2}.$$

$\therefore$ From (i), the equation of the plane through the given line and parallel to z-axis is $(x+y+2z-3)-(1/2)(2x+3y+4z-4)=0$

or $$(2x+2y+4z-6)-(2x+3y+4z-4)=0 \quad \text{or} \quad y+2=0 \qquad \text{...(ii)}$$

Also any point on z-axis is (0, 0, 0).

$\therefore$ Required S. D. = length of perpendicular from (0, 0, 0) to the plane (ii) **(Note)**

$$=\frac{2}{\sqrt{(1)^2}}=2 \qquad \textbf{Ans.}$$

***Ex. 7. Prove that the S. D. between the diagonals of rectangular parallelopiped and the edges not meeting it are**

$$\frac{bc}{\sqrt{(b^2+c^2)}},\ \frac{ca}{\sqrt{(c^2+a^2)}},\ \frac{ab}{\sqrt{(a^2+b^2)}},$$

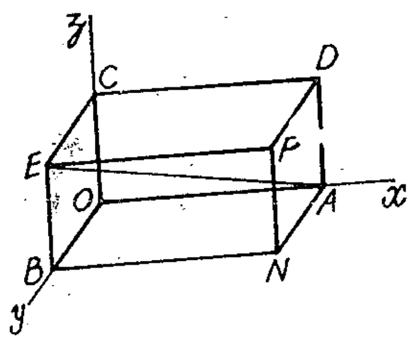
where a, b, c are the lengths of the edges.

Sol. In the parallelopiped (as shown in the adjoining Fig. 5) let the sides $OA=a$, $OB=b$ and $OC=c$.

Let the edges OA, OB and OC be taken as co-ordinates axes as shown in adjoining fig. 5.

Consider the diagonal AE and the edge OB not intersecting this diagonal.

From the figure it is evident that the co-ordinates of A, B and E are $(a, 0, 0)$, $(0, b, 0)$ and $(0, b, c)$ respectively.



(Fig. 5)

The direction ratios of AE are $a-0, 0-b, 0-c$.
i.e. $a, -b, -c$ respectively.

$\therefore$ The equations of AE and OB are $\frac{x-a}{a}=\frac{y-0}{-b}=\frac{z-0}{-c}$ ...(i)

and $$\frac{x}{0}=\frac{y}{b}=\frac{z}{0}. \quad \text{...(ii)}$$

If l, m, n be the d.c.'s of the S. D. between AE and OB, then as the line of S.D. is perpendicular to both AE and OB, therefore, we have
$$l.a-m.b-n.c=0 \quad \text{and} \quad l.0+m.b+n.0=0.$$

Solving these we get $\frac{l}{bc}=\frac{m}{0}=\frac{n}{ab}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(b^2c^2+a^2b^2)}}$

or $$\frac{l}{bc}=\frac{m}{0}=\frac{n}{ab}=\frac{1}{b\sqrt{(c^2+a^2)}}$$

or $$l=\frac{c}{\sqrt{(c^2+a^2)}},\ m=0,\ n=\frac{a}{\sqrt{(c^2+a^2)}} \quad \text{...(iii)}$$

Now the S.D. between AE and OB.
= the projection of the join $A(a, 0, 0)$ and $O(0, 0, 0)$ on the line whose d.c.'s are given by (iii).
$= l(a-0)+m(0-0)+n(0-0)$, where l, m, n are given by (iii)
$= ac/\sqrt{(a^2+c^2)}$ ...(iv)

Similarly we can find other S.D.'s

Ex. 8. Show that the shortest distance between an edge of cube and a diagonal which does not meet it is the join of their mid-points.

Sol. Proceed exactly as in Ex. 7 above.

Here $a=b=c$ and so we can find that S. D. between AE and $OB=\frac{a.a}{\sqrt{(a^2+a^2)}}$, from (iv) of last example.

i.e. S.D. between AE and $OB=a/\sqrt{2}$

Also here A, B and E are the points $(a, 0, 0), (0, a, 0)$ and $(0, a, a)$.

$\therefore$ The mid-points of AE and OB are $(\frac{1}{2}a, \frac{1}{2}a, \frac{1}{2}a)$ and $(0, \frac{1}{2}a, 0)$

So the length of their join
$=\sqrt{[(\frac{1}{2}a-0)^2+(\frac{1}{2}a-\frac{1}{2}a)^2+(\frac{1}{2}a-0)^2]}=\sqrt{(\frac{1}{2}a^2)}=a/\sqrt{2}$. Hence proved.

****Ex. 9. Show that the equation of the plane containing the line $\frac{y}{b}+\frac{z}{c}=1, x=0$ and parallel to the line $\frac{x}{a}-\frac{z}{c}=1, y=0$ is $\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$ and if 2d is the S.D. show that $d^{-2}=a^{-2}+b^{-2}+c^{-2}$.** *(Kanpur 90)*

Sol. The equation of the plane containing the line
$$\frac{y}{b}+\frac{z}{c}=1,\ x=0 \quad \text{is} \quad \left(\frac{y}{b}+\frac{z}{c}-1\right)+\lambda x=0$$

or $$\lambda x + (1/b) y + (1/c) z - 1 = 0 \qquad \text{...(i)}$$

If it is parallel to the line $\frac{x}{a} - \frac{z}{c} = 1,\ y = 0$ *i.e.* $\frac{x-a}{a} = \frac{y}{0} = \frac{z}{c}$, then the normal to the plane (i) must be perpendicular to the line and so we have

$$\lambda . a + (1/b).0 + (1/c).c = 0 \quad \text{or} \quad \lambda = -1/a.$$

$\therefore$ From (i), the equation of the required plane is

$$\left(\frac{y}{b} + \frac{z}{c} - 1\right) - \frac{1}{a} x = 0 \quad \text{or} \quad \frac{x}{a} - \frac{y}{b} - \frac{z}{c} + 1 = 0 \qquad \text{...(ii)}$$

Hence proved.

Now any point on the line $\frac{x-a}{a} = \frac{y}{0} = \frac{z}{c}$ is $(a, 0, 0)$ therefore

$$2d = \text{S.D. between the given lines.}$$

or $2d$ = perpendicular distance of the point $(a, 0, 0)$ from the plane (ii)

$$= \frac{a.(1/a) - 0.(1/b) - 0.(1/c) + 1}{\sqrt{[(1/a)^2 + (-1/b)^2 + (-1/c)^2]}} = \frac{2}{\sqrt{[a^{-2} + b^{-2} + c^{-2}]}}$$

or $$d^{-2} = a^{-2} + b^{-2} + c^{-2}.$$ Hence proved.

Ex. 10 (a). Find the length and equation of the line of S.D. between the two lines $\frac{1}{2}(x-1) = \frac{1}{4}(y-3) = z + 2$ and

$$\mathbf{3x - y - 2z + 4 = 0 = 2x + y + z + 1}$$

Sol. Any plane through the second line is

$$(3x - y - 2z + 4) + \lambda (2x + y + z + 1) = 0 \qquad \text{...(i)}$$

or $$(3 + 2\lambda) x + (\lambda - 1) y + (\lambda - 2) z + (4 + \lambda) = 0$$

If this plane is parallel to the first line then its normal must be at right angles to first line and as such, we have

$$(3 + 2\lambda) 2 + (\lambda - 1).4 + (\lambda - 2).1 = 0 \quad \text{or} \quad \lambda = 0.$$

$\therefore$ From (i) the plane through the second line and parallel to the first line is $$3x - y - 2z + 4 = 0 \qquad \text{...(ii)}$$

Now the required S. D. between the given lines

= perpendicular distance of a point $(1, 3, -2)$ on the first line to the plane (ii)

$$= \frac{3.1 - 1.3 - 2.(-2) + 4}{\sqrt{[3^2 + (-1)^2 + (-2)^2]}} = \frac{8}{\sqrt{(14)}} \qquad \textbf{Ans.}$$

Now the S.D. is the line of intersection of the plane through the given lines and perpendicular to the plane (ii) found above.

The equation of the plane through the first line is

$$A (x - 1) + B (y - 3) + C (z + 2) = 0 \qquad \text{...(iii)}$$

where $$A.2 + B.4 + C.1 = 0 \qquad \text{...(iv)}$$

Also as the plane (iii) is perpendicular to plane (ii), so we get

$$A.3 + B.(-1) + C(-2) = 0 \qquad \text{...(v)}$$

Eliminating A, B and C between (iii), (iv) and (v) we get the equation of the plane through the first line and perpendicular to plane (ii) as

$$\begin{vmatrix} x-1 & y-3 & z+2 \\ 2 & 4 & 1 \\ 3 & -1 & -2 \end{vmatrix} = 0$$

or $$x - y + 2z + 6 = 0 \qquad ...(vi)$$

Also as in (i) the equation of any plane through the second line is

$$(3+2\lambda)\,x + (\lambda-1)\,y + (\lambda-2)\,z + (4+\lambda) = 0 \qquad ...(vii)$$

If it is perpendicular to the plane (ii), then

$$(3+2\lambda).3 + (\lambda-1)(-1) + (\lambda-2)(-2) = 0 \quad \text{or} \quad \lambda = -14/3.$$

$\therefore$ From (vii), the equation of the plane through the second line and perpendicular to the plane (ii) is $19x + 17y + 20z + 2 = 0$ ...(viii)

The required equations of S.D. between the given lines are (vi) and (viii).

Ans.

Ex. 10 (d). Find the length of S.D. between the lines

$\frac{1}{2}(x-2) = \frac{1}{3}(y+1) = \frac{1}{4}z$ **and** $2x + 3y - 5z - 6 = 0 = 3x - 2y - z + 3.$

Hint : Do as Ex. 10 (a) above. **Ans.** $97/(13\sqrt{6})$.

Ex. 11 (a). Find the S. D. between the line $x = 0, (y/2) + (z/3) = 1$ and $y = 0, (x/4) - (z/3) = 1$.

Sol. Equations of the plane through the given lines are

$$x + \lambda\left[\tfrac{1}{2}y + \tfrac{1}{3}z - 1\right] = 0 \quad \text{or} \quad 6x + \lambda(3y + 2z - 6) = 0$$

and $$y + \mu\left[\tfrac{1}{4}x - \tfrac{1}{3}z - 1\right] = 0 \qquad 12y + \mu(3x - 4z - 12) = 0$$

i.e. $$6x + 3\lambda\,y + 2\lambda\,z - 6\lambda = 0 \qquad ...(i)$$

and $$3\mu\,x + 12y - 4\mu\,z - 12\mu = 0 \qquad ...(ii)$$

If the plane (i) and (ii) are parallel, then we get

$$\frac{6}{3\mu} = \frac{3\lambda}{12} = \frac{2\lambda}{-4\mu}, \text{ comparing coefficients of } x, y, z$$

From $$\frac{6}{3\mu} = \frac{2\lambda}{-4\mu} \text{ we get } \lambda = -4$$

And form $$\frac{6}{3\mu} = \frac{3\lambda}{12} \text{ we get } 8 = \lambda\,\mu$$

or $$\mu = -2, \text{ putting the value of } \lambda.$$

$\therefore$ From (i) and (ii) substituting values of λ and μ we get the parallel planes as $3x - 6y - 4z + 12 = 0$ and $3x - 6y - 4z - 12 = 0$

Now any point on the plane $3x - 6y - 4z + 12 = 0$ is $(0, 0, 3)$ **(Note)**

$\therefore$ The required length of S.D.

= length of perpendicular from $(0, 0, 3)$ on the plane $3x - 6y - 4z = 12$

$$= \frac{3.0 - 6.0 - 4.3 - 12}{\sqrt{[3^2 + (-6)^2 + (-4)^2]}} = \frac{-24}{\sqrt{(61)}} = \frac{24}{\sqrt{(61)}}, \text{ numerically.}$$ **Ans.**

***Ex. 11 (b). Find the length and equations of the S.D. between**

$$3x - 9y + 5z = 0 = x + y - z \qquad ...(i)$$

and $$6x + 8y + 3z - 13 = 0 = x + 2y + z - 3 \quad ...(ii)$$

(*Meerut* 96)

Sol. The equations of the plane through the given lines are

$$(3x - 9y + 5z) + \lambda (x + y - z) = 0$$

and $$(6x + 8y + 3z - 13) + \mu (x + 2y + z - 3) = 0$$

i.e. $$(3 + \lambda) x + (\lambda - 9) y + (5 - \lambda) z = 0 \quad ...(iii)$$

and $$(6 + \mu) x + (8 + 2\mu) y + (3 + \mu) z - (13 + 3\mu) = 0 \quad ...(iv)$$

If the plane (iii) and (iv) are parallel, then we have

$\dfrac{3+\lambda}{6+\mu} = \dfrac{\lambda - 9}{8 + 2\mu} = \dfrac{5 - \lambda}{3 + \mu}$, comparing coeff. of x, y, z.

From $\dfrac{3+\lambda}{6+\mu} = \dfrac{\lambda - 9}{8 + 2\mu}$ we get $(3 + \lambda)(8 + 2\mu) = (\lambda - 9)(6 + \mu)$

or $$24 + 6\mu + 8\lambda + 2\lambda\mu = 6\lambda + \lambda\mu - 54 - 9\mu$$

or $$\lambda\mu + 2\lambda + 15\mu + 78 = 0 \quad ...(v)$$

From $\dfrac{3+\lambda}{6+\mu} = \dfrac{5 - \lambda}{3 + \mu}$ we get $(3 + \lambda)(3 + \mu) = (5 - \lambda)(6 + \mu)$

or $$9 + 3\lambda + 3\mu + \lambda\mu = 30 - 6\lambda + 5\mu - \lambda\mu$$

or $$2\lambda\mu + 9\lambda - 2\mu - 21 = 0 \quad ...(vi)$$

From (v) and (vi) on eliminating $\lambda\mu$, we get

$$5\lambda - 32\mu - 177 = 0 \quad \text{or} \quad 5\lambda = 32\mu + 177 \quad ...(vii)$$

From (v) we get $5\lambda\mu + 10\lambda + 75\mu + 390 = 0$

or $$\mu(32\mu + 177) + 2(32\mu + 177) + 75\mu + 390 = 0, \text{ from (vii)}$$

or $$32\mu^2 + 316\mu + 744 = 0 \quad \text{or} \quad 8\mu^2 + 79\mu + 186 = 0$$

or $$(8\mu + 31)(\mu + 6) = 0 \quad \text{or} \quad \mu = -6, \ -31/8.$$

$\therefore$ From (vii), when $\mu = -6$, $\lambda = -3$ and when

$$\mu = -\frac{31}{8}, \quad \lambda = \frac{53}{5}$$

Putting values $\lambda = \dfrac{53}{5}$, $\mu = -\dfrac{31}{8}$ in (iii) and (iv), we get the parallel planes as $17x + 2y - 7z = 0$, $17x + 2y - 7z - 11 = 0$

[The other pair of values of λ and μ does not give parallel planes].

Any point on the plane $17x + 2y - 7z = 0$ is $(0, 0, 0)$

$\therefore$ The required length of S.D.

= length of the perpendicular from $(0, 0, 0)$ to $-17x - 2y + 7z + 11 = 0$

$$= \frac{11}{\sqrt{[(-17)^2 + (-2)^2 + (7)^2]}} = \frac{11}{\sqrt{(342)}} = \frac{11}{3\sqrt{(38)}} \qquad \textbf{Ans.}$$

Also we know the equation of any plane through (i) is

$$(3 + \lambda) x + (\lambda - 9) y + (5 - \lambda) z = 0.$$

If it is perpendicular to the plane $17x + 2y - 7z = 0$,

then we have $(3 + \lambda).17 + (\lambda - 9).2 + (5 - \lambda)(-7) = 0$ or $\lambda = 1/(13)$.

$\therefore$ The equation of the planes through (i) and perpendicular to the plane

$17x+2y-7z=0$ is $\left(3+\frac{1}{13}\right)x+\left(\frac{1}{13}-9\right)y+\left(5-\frac{1}{13}\right)z=0$

or $40x-116y+64z=0$ or $10x-29y+16z=0$ ...(viii)

And the equation of any plane through the line (ii) is

$$(6+\mu)\,x+(8+2\mu)\,y+(3+\mu).z-(13+3\mu)=0$$

If it is perpendicular to the plane $17x+2y-7z=0$,

then we get $(6+\mu)\,17+(8+2\mu)\,(2)+(3+\mu)\,(-7)=0$

or $14\mu+97=0$ or $\mu=-97/14$.

$\therefore$ The equation of the plane through the line (ii) and perpendicular to the plane $17x+2y-7z=0$ is

$$[6-(97/14)]\,x+[8-(97/7)]\,y+[3-(97/14)]\,z-[13-(291/14)]=0$$

or $-(13/14)\,x-(41/7)\,y-(55/14)\,z+(109/14)=0$

or $13x+82y+55z-109=0$. ...(ix)

The required equations of the S.D. between the given lines are (viii) and (ix). **Ans.**

Ex. 11 (c). Find the length and equation of the S.D. between $x-2y+z=0=x+y+z$ and $6x+8y+3z-13=0=x+2y+z-3$

(*Kanpur* 94)

Sol. Do as Ex. 11 (b) above.

****Ex. 12. Two straight lines**

$$\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1},\quad \frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}$$

are cut by a third line whose d.c.'s are λ, μ, ν. Show that d the length intercepted on the third line is given by

$$d\begin{vmatrix} l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\\ \lambda & \mu & \nu\end{vmatrix}=\begin{vmatrix}\alpha_1-\alpha_2 & \beta_1-\beta_2 & \gamma_1-\gamma_2\\ l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\end{vmatrix}$$

Deduce the length of S.D. between the first two lines.

Sol. Let the third line with d.c.'s $\lambda, \mu\ \nu$ meet the first line

$$\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1}, \qquad ...(i)$$

at $P\,(\alpha_1+l_1r_1,\ \beta_1+m_1r_1,\ \gamma_1+n_1r_1)$.

Then the equations of third line can be written as

$$\frac{x-(\alpha_1+l_1r_1)}{\lambda}=\frac{y-(\beta_1+m_1r_1)}{\mu}=\frac{z-(\gamma_1+n_1r_1)}{\nu}=d\text{ (say)}, \qquad ...(ii)$$

$\therefore$ The coordinates of any point Q at a distance d from the point P on the line (ii) are $\alpha_1+l_1r_1+d\lambda,\ \beta_1+m_1r_1+d\mu,\ \gamma_1+n_1r_1+d\nu$. ...(iii)

According to the given problem this point Q given by (iii) is a point on the second line $\frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}$ at a distance d from P.

Hence the point (iii) is the same as $(\alpha_2+l_2r_2,\ \beta_2+m_2r_2,\ \gamma_2+n_2r_2)$....(iv)

Comparing (iii) and (iv) we get $d\lambda+\alpha_1+l_1r_1=\alpha_2+l_2r_2$ etc.

or
$$d\lambda+(\alpha_1-\alpha_2)+l_1r_1-l_2r_2=0;$$

Similarly $d\mu+(\beta_1-\beta_2)+m_1r_1-m_2r_2=0.$
$$d\nu+(\gamma_1-\gamma_2)+n_1r_1-n_2r_2=0.$$

Eliminating r_1 and r_2 from these we get

$$\begin{vmatrix} d\lambda+(\alpha_1-\alpha_2) & l_1 & l_2 \\ d\mu+(\beta_1-\beta_2) & m_1 & m_2 \\ d\nu+(\gamma_1-\gamma_2) & n_1 & n_2 \end{vmatrix}=0$$

or
$$d\begin{vmatrix} \lambda & l_1 & l_2 \\ \mu & m_1 & m_2 \\ \nu & n_1 & n_2 \end{vmatrix}+\begin{vmatrix} \alpha_1-\alpha_2 & l_1 & l_2 \\ \beta_1-\beta_2 & m_1 & m_2 \\ \gamma_1-\gamma_2 & n_1 & n_2 \end{vmatrix}=0$$

or
$$d\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ \lambda & \mu & \nu \end{vmatrix}=-\begin{vmatrix} \alpha_1-\alpha_2 & \beta_1-\beta_2 & \gamma_1-\gamma_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \quad ...(v)$$

d being the distance between two points, neglecting the negative sign we have the required result.

If d stands for the S.D. between the given lines, then the lines with d.c.'s $\lambda, \mu\ \nu$ is perpendicular to both the given lines and as such we have $\lambda.l_1+\mu.m_1+\nu.n_1=0$ and $\lambda.l_2+\mu.m_2+\nu.n_2=0.$

Solving these we get

$$\frac{\lambda}{m_1n_2-m_2n_1}=\frac{\mu}{n_1l_2-n_2l_1}=\frac{\nu}{l_1m_2-l_2m_1}=\frac{1}{\sqrt{[\Sigma\,(m_1n_2-m_2n_1)^2]}} \quad ...(vi)$$

Also from (v) the coefficient of d

$$=\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ \lambda & \mu & \nu \end{vmatrix}$$

$$=\lambda\,(m_1n_2-m_2n_1)+\mu\,(n_1l_2-n_2l_1)+\nu\,(l_1m_2-l_2m_1)$$

$$=\frac{\Sigma\,(m_1n_2-m_2n_1)}{\sqrt{[\Sigma\,(m_1n_2-m_2n_1)^2]}}, \text{ putting the values of } \lambda, \mu, \nu \text{ from (vi)}$$

$$=\sqrt{[\Sigma\,(m_1n_2-m_2n_1)^2]}.$$

$\therefore$ From (v), d the S.D. is given by

$$d=\begin{vmatrix} \alpha_1-\alpha_2 & \beta_1-\beta_2 & \gamma_1-\gamma_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}\div\sqrt{[\Sigma\,(m_1n_2-m_2n_1)^2]}$$

Exercises on § 4.15

Ex. 1. Find the shortest distance between the lines

$$\frac{x-2}{0}=\frac{y-1}{1}=\frac{z}{1} \text{ and } \frac{x-3}{2}=\frac{y-5}{2}=\frac{z-1}{1}$$

and also the points where it intersects the given lines. (*Bundelkhand* 92)

Ans. $\frac{5}{3}, \left(2, \frac{7}{2}, \frac{4}{3}\right), \left(\frac{13}{9}, \frac{31}{9}, \frac{2}{9}\right)$

Ex. 2. Find the length and equation of the common perpendicular to the lines $-\frac{1}{4}(x+2)=\frac{1}{5}(y-5)=\frac{1}{2}z$ and $-\frac{1}{4}(x+3)=\frac{1}{3}y=(z-6)$

Ans. $\frac{1}{27}\sqrt{(45129)}$; $\frac{x-5}{-53}=\frac{y+24}{92}=\frac{z-46}{-184}$

Ex. 3. Find the S.D. between the lines

$\frac{1}{2}(x-3)=-\frac{1}{7}(y+15)=\frac{1}{5}(z-9)$ and $\frac{1}{2}(x+1)=y-1=-\frac{1}{3}(z-9)$. Also find the points where this S.D. meets these lines. (*Bundelkhand* 93)

Ex. 4. Obtain the coordinates of the point where the shortest distance line between the lines $\frac{x-23}{-6}=\frac{y-19}{-4}=\frac{z-25}{3}$ and $\frac{x-12}{-9}=\frac{y-1}{4}=\frac{z-5}{2}$ meets them. (*Kumaun* 95)

Ex. 5. Find the S.D. between the lines given below :

$x=-\frac{1}{2}(y-2)=z+1$ and $\frac{1}{2}(x-3)=-\frac{1}{3}(y+1)=z+1$. **Ans.** 0

Ex. 6. Find the S.D. between the lines $-\frac{1}{2}(x+3)=\frac{1}{2}(y+7)=\frac{1}{4}(z-6)$ and $\frac{1}{3}(x-3)=-(y-8)=(z-3)$. Also find the equations of the line S.D.

Ans. $3\sqrt{(30)}, \frac{1}{2}(x-3)=(y-8)/5=-(z-3)$

Ex. 7. Find the magnitude and the equations of the line of S.D. between the lines $(x-3)/2=-(y+15)/7=(z-9)/5$ and $\frac{1}{3}(x+1)=y-1=-\frac{1}{3}(z-9)$.

Ans. $4\sqrt{3}, x+1=y+1=z+1$

Ex. 8. Find the length of the shortest distance between the lines $x=y=z$ and $x+y=2,\ x+z=2$. **Ans.** 0

Ex. 9. Find the S.D. between the lines

$$\frac{x-5}{3}=\frac{y-7}{-6}=\frac{z-3}{7};\ \frac{x-9}{3}=\frac{y-13}{8}=\frac{z-15}{-5}.$$

Ex. 10. Find the magnitude and the equations of the line of S.D. between the lines $\frac{1}{2}(x-3)=-(1/7)(y+12)=(1/5)(z-9)$ and

$\frac{1}{2}(x+1)=y-1=-\frac{1}{3}(z-9)$. (*Kumaun* 92)

Ex. 11. Find the length of the S.D. between the lines

$\frac{x+3}{-4}=\frac{y-6}{3}=\frac{z}{2},\ \frac{x+2}{-4}=\frac{y}{1}=\frac{z-7}{1}$ (*Lucknow* 90)

Ex. 12. Find the S.D. between the straight lines

$$\frac{1}{2}(x-2)=\frac{1}{3}(y+1)=\frac{1}{4}z \quad \text{and} \quad 4x+y+z=0=2x-3y-5z.$$

Ans. $28/[3\sqrt{(22)}]$

Ex. 13. Find the magnitude and the equations of S.D. between the lines

$$\frac{1}{4}x=\frac{1}{3}(y+1)=\frac{1}{2}(z-2) \quad \text{and} \quad 5x-2y-3z+6=0=x-3y+2z-3$$

(*Meerut* 98)

Ans. $8/\sqrt{(14)}$; $x-y+2z+6=0=19x+17y+20z+2$

Ex. 14. Find the S.D. between z-axis and the line

$$\frac{1}{2}(x-3)=\frac{1}{2}(y-5)=-(z+1)/5. \qquad \textbf{Ans. } \sqrt{(962)}/9$$

Ex. 15. Find the S.D. between the lines $y+z=0=z+x$ and $x+y=0=x+y+z-a$. (See Ex. 5 Page 141 Ch. IV) **Ans.** $2a/\sqrt{6}$

Ex. 16. Find the length and equations of the S.D. between

$$x-2y+z=0=x+y+z \quad \text{and} \quad 6x+8y+3z-13=0=x+2y+z-3$$

Ans. $3/\sqrt{(38)}$; $10x-29y+16z=0=13x+82y+55z-109$

MISCELLANEOUS SOLVED EXAMPLES

***Ex. 1. Find the equations of the straight lines which bisects the angles between the lines** $\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$; $\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$.

Sol. We can show as in Ex. 6 P. 46 Ch. II that the d.c.'s of the lines bisecting the angles between the given lines are proportional to

$$l_1 \pm l_2,\ m_1 \pm m_2,\ n_1 \pm n_2.$$

Also both the given lines pass through (0, 0, 0) and hence their bisectors also pass through (0, 0, 0).

$\therefore$ The equations of the required bisectors are

$$\frac{x}{l_1 \pm l_2}=\frac{y}{m_1 \pm m_2}=\frac{z}{n_1 \pm n_2} \qquad \textbf{Ans.}$$

Ex. 2. Prove that equation of plane through the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

is $(x-\alpha)\frac{\lambda}{l}+(y+\beta)\frac{\mu}{m}+(z-\gamma)\frac{\nu}{n}=0$, **where** $\lambda+\mu+\nu=0$

Hence find the equation of the plane containing one given line and parallel to another given straight line.

Sol. The given plane evidently passes through (α, β, γ), a point on the given line.

Also the given line is perpendicular to the normal to the plane if

$$l\frac{\lambda}{\lambda}+m\frac{\mu}{m}+n\frac{\nu}{n}=0 \quad i.e. \quad \lambda+\mu+\nu=0. \qquad \text{...(i)}$$

Hence the planes $\Sigma (x-\alpha)\dfrac{\lambda}{l}=0$ passes through the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

Again if the plane $\Sigma (x-\alpha)\dfrac{\lambda}{l}=0$ is parallel to another line

$$\frac{x}{l'}=\frac{y}{m'}=\frac{z}{n'} \text{ then } \frac{\lambda}{l}\,.\,l'+\frac{\mu}{m}\,.\,m'+\frac{\nu}{n}\,.\,n'=0$$

or $$\lambda\,(l'/l)+\mu\,(m'/m)+\nu\,(n'/n)=0. \qquad \text{...(ii)}$$

Solving (i) and (ii) we get

$$\frac{\lambda}{(n'/n)-(m'/m)}=\frac{\mu}{(l'/l)-(n'/n)}=\frac{\nu}{(m'/m)-(l'/l)}$$

or $$\frac{\lambda\, mn}{mn'-m'n}=\frac{\mu\, nl}{nl'-n'l}=\frac{\nu\, ml}{lm'-l'm}$$

or $$\frac{(\lambda/l)}{mn'-m'n}=\frac{(\mu/m)}{nl'-n'l}=\frac{(\nu/n)}{lm'-l'm}$$

Substituting these values of $\lambda/l, \mu/m, \nu/n$ in the equation

$$\Sigma (x-\alpha)\frac{\lambda}{l}=0.$$

we have the required equation as $\Sigma (x-\alpha)(mn'-m'n)=0.$ **Ans.**

Ex. 3. Prove that if the straight line $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ intersect the curve $ax^2+by^2=1,\ z=0$ then $a\,(\alpha n-\gamma l)^2+b\,(\beta n-\gamma m)^2=n^2$

Sol. Any point on the given line is $(\alpha+lr, \beta+mr, \gamma+nr)$.

If this point lies on the curve $ax^2+by^2=1,\ z=0$, then

$$a\,(\alpha+lr)^2+b\,(\beta+mr)^2=1 \qquad \text{...(i)}$$

and $$\gamma+nr=0 \quad \text{or} \quad r=-\gamma/n \qquad \text{...(ii)}$$

Substituting the value of r from (ii) in (i) we have the required condition

as $$a\left[\alpha+l\left(-\frac{\gamma}{n}\right)\right]^2+b\left[\beta+m\left(-\frac{\gamma}{n}\right)\right]^2=1$$

or $$a\,(\alpha n-l\gamma)^2+b\,(\beta n-\gamma m)^2=n^2$$ Hence proved.

***Ex. 4. Prove that a line which passes through (α, β, γ) and intersects the parabola $y=0, z^2=4ax$ lies on the surface**

$$(\beta z-y\gamma)^2=4a\,(\beta-\gamma)\,(\beta x-\alpha y) \qquad (\textit{Kanpur } 91)$$

Sol. Let the equation of the line be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{...(i)}$$

$\therefore$ Any point on this line is $(\alpha+lr, \beta+mr, \gamma+nr)$

If this point lies on the parabola $y=0,\ z^2=4ax$.

then $$\beta + mr = 0 \quad \text{or} \quad r = -\beta/m \qquad ...(ii)$$

and $$(\gamma + nr)^2 = 4a(\alpha + lr) \qquad ...(iii)$$

Eliminating r between (ii) and (iii) we get

$$\left(\gamma - \frac{n\beta}{m}\right)^2 = 4a\left(\alpha - \frac{l\beta}{m}\right)$$

or $$(m\gamma - n\beta)^2 = 4am(\alpha m - l\beta) \qquad ...(iv)$$

Eliminating l, m, n between (i) and (iv) we get the required locus as

$$[(y-\beta)\gamma - (z-\gamma)\beta]^2 = 4a(y-\beta)[\alpha(y-\beta) - (x-\alpha)\beta]$$

or $$(y\gamma - z\beta)^2 = 4a(y-\beta)(\alpha y - \beta x)$$

or $$(z\beta - \gamma y)^2 = 4a(\beta - y)(\beta x - \alpha y). \qquad \text{Hence proved.}$$

Ex. 5. Find the equations to the two planes through the origin which are parallel to the line $\frac{1}{2}(x-1) = -(y+3) = -\frac{1}{2}(z+1)$ and distant 5/3 from it.

Sol. Any plane through the origin is $ax + by + cz = 0$ ...(i)

The d. ratios of its normal are a, b, c

If this plane (i) is parallel to the given line, then the normal to (i) must be perpendicular to the given line whose d.c.'s are $2, -1, -2$

$$\therefore \quad a.2 + b(-1) + c(-2) = 0 \qquad ...(ii)$$

Also the plane (i) is at a distance 5/3 from the given line. Now any point on the given line is $(1, -3, -1)$, so from above we have

$$\frac{a(1) + b(-3) + c(-1)}{\sqrt{(a^2+b^2+c^2)}} = \pm\frac{5}{3} \qquad \textbf{(Note)}$$

or $$3(a - 3b - c) = \pm 5\sqrt{(a^2+b^2+c^2)}$$

or $$9(a-3b-c)^2 = 25(a^2+b^2+c^2)$$

or $$9\left[\tfrac{1}{2}(b+2c) - 3b - c\right]^2 = 25\left[\tfrac{1}{4}(b+2c)^2 + b^2 + c^2\right],$$

$\because$ from (ii), $2a = b + 2c$

or $$9[-5b]^2 = 25[b^2 + 4c^2 + 4bc + 4b^2 + 4c^2]$$

or $$b^2 - bc - 2c^2 = 0 \text{ or } (b+c)(b-2c) = 0$$

Either $b = -c$ or $b = 2c$

If $b = -c$, from (ii) we get $2a = c$ or $a = \frac{1}{2}c$

and the cooresponding plane from (i) is $x - 2y + 2z = 0$

If $b = 2c$, from (ii) we get $2a = 2c + 2c$ or $a = 2c$

and from (i), the corresponding plane is $2x + 2y + z = 0.$

Ans. $x - 2y + 2z = 0,\ 2x + 2y + z = 0.$

***Ex. 6. Prove that the three lines drawn from origin with d.r.'s 2, 1, 5; 2, –1 1; 6, – 4, 1 are coplanar.** (*Purvanchal* 91)

Sol. Equations of these lines in the symmetric form can be written as

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{5};\ \frac{x}{2} = \frac{y}{-1} = \frac{z}{1};\ \frac{x}{6} = \frac{y}{-4} = \frac{z}{1}$$

The equation of the plane passing through first two lines is

$$\begin{vmatrix} x & y & z \\ 2 & 1 & 5 \\ 2 & -1 & 1 \end{vmatrix} = 0 \quad \text{... See § 4.11 (a) Page 94}$$

or $\quad x(1+5)+y(10-2)+z(-2-2)=0 \quad$ or $\quad 3x+4y-2z=0 \quad$ ...(i)

And the equation of the plane through last two lines is

$$\begin{vmatrix} x & y & z \\ 2 & -1 & 1 \\ 6 & -4 & 1 \end{vmatrix} = 0, \text{ as above}$$

or $\quad x(-1+4)+y(6-2)+z(-8+6)=0 \quad$ or $\quad 3x+4y-2z=0,$

which is the same as given by the equation (i).

Hence the given lines are coplanar.

Ex. 7 (a). Find the equations of the straight line through the origin and cutting each of the lines

$$\frac{x-\alpha_1}{l_1}=\frac{y-\beta_1}{m_1}=\frac{z-\gamma_1}{n_1} \quad \textbf{and} \quad \frac{x-\alpha_2}{l_2}=\frac{y-\beta_2}{m_2}=\frac{z-\gamma_2}{n_2}$$

Sol. The equation of the plane containing the first line is

$$A(x-\alpha_1)+B(y-\beta_1)+C(z-\gamma_1)=0, \quad \text{...(i)}$$

where $\quad Al_1+Bm_1+Cn_1=0. \quad$ ...(ii)

If (i) passes through the origin, then we have

$$A(0-\alpha_1)+B(0-\beta_1)+C(0-\gamma_1)=0$$

or $\quad A\alpha_1+B\beta_1+C\gamma_1=0 \quad$ ...(iii)

Solving (ii) and (iii) we get

$$\frac{A}{m_1\gamma_1-n_1\beta_1}=\frac{B}{n_1\alpha_1-l_1\gamma_1}=\frac{C}{l_1\beta_1-m_1\alpha_1} \quad \text{...(iv)}$$

$\therefore$ From (i) the equation of the plane through the origin and the first line is

$$(m_1\gamma_1-n_1\beta_1)(x-\alpha_1)+(n_1\alpha_1-l_1\gamma_1)(y-\beta_1)+(l_1\beta_1-m_1\alpha_1)(z-\gamma_1)=0$$

or $\quad (m_1\gamma_1-n_1\beta_1)x+(n_1\alpha_1-l_1\gamma_1)y+(l_1\beta_1-m_1\alpha_1)z=0 \quad$ ...(v)

Similarly the equations of the plane through the origin and the second line is $\quad (m_2\gamma_2-n_2\beta_2)x+(n_2\alpha_2-l_2\gamma_2)y+(l_2\beta_2-m_2\alpha_2)z=0 \quad$ ...(vi)

The equations (v) and (vi) together represent the required line.

Ex. 7 (b). Find the foot of the perpendicular from the origin to the plane $2x+3y-4z+1=0$. Also find the image of the origin in the plane.

Sol. Do yourself.

****Ex. 8. If the lines**

$$\frac{x-5}{4}=\frac{y-3}{6}=\frac{z-7}{5} \quad \textbf{and} \quad \frac{x+1}{2}=\frac{y+4}{5}=\frac{z-9}{10}$$

are in a plane then find the equation of plane containing them. If the lines are not in a plane then find the S.D. between them. (*Bundelkhand* 96)

Sol. Any point on the line $\frac{x-5}{4}=\frac{y-3}{6}=\frac{z-7}{5}$ is

$$(5+4r_1, 3+6r_1, 7+5r_1) \quad \text{...(i)}$$

Similarly any point on the line $\dfrac{x+1}{2}=\dfrac{y+4}{5}=\dfrac{z-9}{10}$ is

$$(-1+2r_2, -4+5r_2, 9+10r_2) \quad \text{...(ii)}$$

If the two given lines intersect, then for some values of r_1 and r_2 the two above points (i) and (ii) must coincide *i.e.* $5+4r_1=-1+2r_2$, $3+6r_1=-4+5r_2$ and $7+5r_1=9+10r_2$

Solving the first two of these equations we get $r_1=-2$, $r_2=-1$

But these values of r_1 and r_2 do not satisfy the above third equation viz. $7+5r_1=9+10r_2$. Hence the given lines do not lie on a plane *i.e.* are not coplanar.

S.D. between the given lines.

Let l, m, n be the d.c.'s of the S.D. between the lines. Then we have

$$4l+6m+5n=0 \quad \text{and} \quad 2l+5m+10n=0$$

Solving these, we get $\dfrac{l}{60-25}=\dfrac{m}{10-40}=\dfrac{n}{20-12}$

or $$\frac{l}{35}=\frac{m}{-30}=\frac{n}{8}=\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{[(35)^2+(-30)^2+(8)^2]}}=\frac{1}{\sqrt{(2189)}}$$

$\therefore$ The d.c.'s of S.D. are $\dfrac{35}{\sqrt{(2189)}}, -\dfrac{30}{\sqrt{(2189)}}, \dfrac{8}{\sqrt{(2189)}}$ ...(iii)

Also $A(5, 3, 7)$ is a point on the line $\dfrac{x-5}{4}=\dfrac{y-3}{6}=\dfrac{z-7}{5}$ and $B(-1, -4, 9)$ is a point on the line $\dfrac{x+1}{2}=\dfrac{y+4}{5}=\dfrac{z-9}{10}$

$\therefore$ Required length of S.D. = projection of join of A and B on the line whose d.c.'s are given by (iii)

$$=\frac{35}{\sqrt{(2189)}}[5+1]-\frac{30}{\sqrt{(2189)}}[4+4]+\frac{8}{\sqrt{(2189)}}[7-9]$$

$$=\frac{210-210-16}{\sqrt{(2189)}}=\frac{16}{\sqrt{(2189)}}, \text{ numerically}$$ **Ans.**

****Ex. 9. If the plane $ax+hy+gz=0$, $hx+by+fz=0$, $gx+fy+cz=0$ have a common line of intersection, then**

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

and the direction ratios of the line satisfy the equations

$$\frac{l^2}{\frac{\partial \Delta}{\partial a}}=\frac{m^2}{\frac{\partial \Delta}{\partial b}}=\frac{n^2}{\frac{\partial \Delta}{\partial c}} \quad \textit{(Garhwal 91)}$$

Sol. The given planes are $ax+hy+gz=0$. ...(i)

$hx+by+fz=0$ ...(ii) and $gx+fy+cz=0$ ...(iii)

If l, m, n be the d.c.'s of the common line of intersection, then from (i) and (ii) we have $al+hm+gn=0,\ hl+bm+fn=0.$

Solving these we get $\dfrac{l}{hf-bg}=\dfrac{m}{gh-af}=\dfrac{n}{ab-h^2}$ ...(iv)

Hence equations of the line of intersection of the planes (i) anad (ii) are

$$\frac{x-0}{hf-bg}=\frac{y-0}{gh-af}=\frac{z-0}{ab-h^2} \quad ...(v)$$

If this line lies on the plane (iii), then we have

$$g(hf-gb)+f(gh-af)+c(ab-h^2)=0$$

or $$abc+2fgh-af^2-bg^2-ch^2=0 \quad ...(vi)$$

or $$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0,$$ Hence proved

Also $\dfrac{\partial \Delta}{\partial a}=bc-f^2,\ \dfrac{\partial \Delta}{\partial b}=ca-g^2,\ \dfrac{\partial \Delta}{\partial c}=ab-h^2.$ ...(viii)

Now $(hf-bg)^2=h^2f^2+b^2g^2-2hfgb$

$=h^2f^2+b^2g^2-b(af^2+bg^2+ch^2-abc)$, from (vi)

$=h^2f^2-abf^2-bch^2+ab^2c$

$=f^2(h^2-ab)-bc(h^2-ab)$

or $$(hf-bg)^2=(h^2-ab)(f^2-bc)=(ab-h^2)(bc-f^2) \quad ...(viii)$$

Similarly $$(gh-af)^2=(ab-h^2)(ca-g^2) \quad ...(ix)$$

$\therefore$ From (iv) with the help of (viii) and (ix) we have

$$\frac{l^2}{(ab-h^2)(bc-f^2)}=\frac{m^2}{(ab-h^2)(ca-g^2)}=\frac{n^2}{(ab-h^2)^2}$$

or $$\frac{l^2}{(bc-f^2)}=\frac{m^2}{(ca-g^2)}=\frac{n^2}{(ab-h^2)}$$

or $$\frac{l^2}{\left(\dfrac{\partial \Delta}{\partial a}\right)}=\frac{m^2}{\left(\dfrac{\partial \Delta}{\partial b}\right)}=\frac{n^2}{\left(\dfrac{\partial \Delta}{\partial c}\right)}, \text{ from (vii)},$$

Hence proved.

Ex. 10. The direction cosines of OA, OB, OC are l_r, m_r, n_r, r = 1, 2 and 3 respectively and OA′, OB′, OC′ bisect the angles BOC, COA, AOB. Show that the planes AOA′, BOB′, COC′ pass through the line

$$\frac{x}{l_1+l_2+l_3}=\frac{y}{m_1+m+m_3}=\frac{z}{n_1+n_2+n_3}$$

Sol. The d.c.'s of the line OA' which bisects the angle BOC are

$$\frac{1}{2}(l_2+l_3),\ \frac{1}{2}(m_2+m_3),\ \frac{1}{2}(n_2+n_3) \qquad \textbf{(Note)}$$

$\therefore$ The equations of OA' are $\dfrac{x}{l_2+l_3}=\dfrac{y}{m_2+m_3}=\dfrac{z}{n_2+n_3}$ ...(i)

Also equations of OA are $\dfrac{x}{l_1}=\dfrac{y}{m_1}=\dfrac{z}{n_1}$. ...(ii)

$\therefore$ Equation of the plane AOA', which contains the line (i) and (ii) is

$$\begin{vmatrix} x & y & z \\ l_1 & m_1 & n_1 \\ l_2+l_3 & m_2+m_3 & n_2+n_3 \end{vmatrix}=0$$

or

$$\begin{vmatrix} x & y & z \\ l_1 & m_1 & n_1 \\ l_1+l_2+l_3 & m_1+m_2+m_3 & n_1+n_2+n_3 \end{vmatrix}=0,$$

adding second row to the third.

Similarly the equations to the planes BOB' and COC' are

$$\begin{vmatrix} x & y & z \\ l_2 & m_2 & n_2 \\ \Sigma l_1 & \Sigma m_1 & \Sigma n_1 \end{vmatrix}=0 \quad \text{and} \quad \begin{vmatrix} x & y & z \\ l_3 & m_3 & n_3 \\ \Sigma l_1 & \Sigma m_1 & \Sigma n_1 \end{vmatrix}=0$$

where $\Sigma l_1 = l_1+l_2+l_3$ etc.

Now the plane AOA' evidently passes through the line

$$\frac{x}{l_1+l_2+l_3}=\frac{y}{m_1+m_2+m_3}=\frac{z}{n_1+n_2+n_3} \qquad \text{...(iv)}$$

as any point on (iv) viz. $r(l_1+l_2+l_3), r(m_1+m_2+m_3), r(n_1+n_2+n_3)$ satisfies (iii), which can be verified by putting these values of x, y, z in (iii) resulting in first and third rows being identical.

Similarly we can prove that the planes BOB' and COC' also pass though line (iv).

Hence proved.

***Ex. 11. Show that the distance of the point (x_0, y_0, z_0) from $u \equiv ax+by+cz+d=0$, $v \equiv a'x+b'y+c'z+d'=0$ is**

$$\left[\frac{(a'u_0-av_0)^2+(b'u_0-bv_0)^2+(c'u_0-cv_0)^2}{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}\right]^{1/2}$$

Sol. The equations of the given lines in the symmetric form can be written (see § 4.04 Page 58 Ch. IV) as

$$\frac{x-\dfrac{bd'-b'd}{ab'-a'b}}{bc'-b'c}=\frac{y-\dfrac{da'-d'a}{ab'-a'b}}{ca'-c'a}=\frac{z-0}{ab'-ab'} \qquad \text{...(i)}$$

$\therefore$ If p be the required perpendicular distance, then with the help of § 4.14 Page 125 Ch. IV we get

$$p^2 = \frac{1}{A^2}\left[\begin{vmatrix} x_0 - \dfrac{bd' - b'd}{ab' - a'b} & y_0 - \dfrac{da' - d'a}{ab' - a'b} \\ bc' - b'c & ca' - c'a \end{vmatrix}^2 + \begin{vmatrix} y_0 - \dfrac{da' - d'a}{ab' - a'b} & z_0 - 0 \\ ca' - c'a & ab' - a'b \end{vmatrix}^2 + \begin{vmatrix} z_0 - 0 & x_0 - \dfrac{bd' - b'd}{ab' - a'b} \\ ab' - a'b & bc' - b'c \end{vmatrix}^2\right] \quad ...(ii)$$

where $$A^2 = (bc' - b'c)^2 + (ca' - c'a)^2 + (ab' - a'b)^2 \quad ...(iii)$$

Now the first determinant in (ii)

$$= \left(x_0 - \frac{bd' - b'd}{ab' - a'b}\right)(ca' - c'a) - \left(y_0 - \frac{da' - d'a}{ab' - a'b}\right)(bc' - b'c)$$

$$= x_0 (ca' - c'a) - y_0 (bc' - b'c) - \frac{(da' - d'a)(bc' - b'c) - (bd' - b'd)(ca' - c'a)}{ab' - a'b}$$

$$= x_0 (ca' - c'a) - y_0 (bc' - b'c) - \frac{(dc' - d'c)(ab' - a'b)}{ab' - a'b}, \text{ on simplifying}$$

$$= x_0 (ca' - c'a) - y_0 (bc' - b'c) - (dc' - d'c)$$

$$= c(a'x_0 + b'y_0 + c'z_0 + d') - c'(ax_0 + by_0 + cz_0 + d) \quad \textbf{(Note)}$$

$$= cv_0 - c'u_0$$

Similarly we can evaluate other determinants of (ii).

$\therefore$ From (ii) we have

$$p^2 = \frac{1}{A^2}\left[(a'u_0 - av_0)^2 + (b'u_0 - bv_0)^2 + (c'u_0 - cv_0)^2\right],$$

where A^2 is given by (iii). Hence proved.

***Ex. 12. Find the S.D. between z-axis and the line $x + y + z + 3 = 0 = 3x + y + 2z + 2$** (*Rohilkhand* 93)

Sol. Equations of z-axis in symmetric form are

$$\frac{x - 0}{0} = \frac{y - 0}{0} = \frac{z - 0}{1}. \quad ...(i)$$

Also let l, m, n be the d.c.'s of the line

$$x + y + z + 3 = 0 = 3x + y + 2z + 2 \quad ...(ii)$$

Then as this line is perpendicular to the normals to both the planes $x + y + z + 3 = 0$ and $3x + y + 2z + 2 = 0$, so we have

$$l + m + n = 0 \quad \text{and} \quad 3l + m + 2n = 0$$

Solving these, we get $\dfrac{l}{2-1} = \dfrac{m}{3-2} = \dfrac{n}{1-3}$ or $\dfrac{l}{1} = \dfrac{m}{1} = \dfrac{n}{-2}$ ...(iii)

Putting $z = 0$ in the given equations (iii) we get

$$x + y + z = 0 \quad \text{and} \quad 3x + y + 2 = 0$$

Solving these we get $x = 1/2$, $y = -7/2$

$\therefore$ The line (ii) meets the plane $z=0$ in $(1/2, -7/2, 0)$ ...(iv)

$\therefore$ From (iii) and (iv), the equations of the line (ii) in this symmetric form are

$$\frac{x-(1/2)}{1}=\frac{y+(7/2)}{1}=\frac{z-0}{-2} \quad ...(v)$$

Now we are to find S.D. between the lines (i) and (v)

Let l_1, m_1, n_1 be the d.c.'s of the line of S.D. to the lines (i) and (v), then S.D. being perpedicular to both these lines, we have

$$l.0+m.0+n.1=0 \text{ and } l.1+m.1+n.(-2)=0$$

Solving these we get $\dfrac{l}{1}=\dfrac{m}{-1}=\dfrac{n}{0}=\dfrac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{[1^2+(-1)^2+0^2]}}=\dfrac{1}{\sqrt{2}}$

$\therefore$ d.c.'s of S.D. are $1/\sqrt{2}, -1/\sqrt{2}, 0$.

Also $O(0, 0, 0)$ is a point on the line (i) and $B(1/2, -7/2, 0)$ is a point on the line (v).

$\therefore$ Required S.D. = Projection OB on a line whose d.c.s' are

$1/\sqrt{2}, -1/\sqrt{2}, 0$

$$=\frac{1}{\sqrt{2}}\left(\frac{1}{2}-0\right)-\frac{1}{\sqrt{2}}\left(-\frac{7}{2}-0\right)+0\,(0-0) \quad \textbf{(Note)}$$

$$=\frac{1}{\sqrt{2}}\left(\frac{1}{2}+\frac{7}{2}\right)=2\sqrt{2} \quad \textbf{Ans.}$$

Ex. 13 (a). Show that the S.D. between the line $x+a=2y=-12z$ and $x=y+2a=6z-6a$ is $2a$.

Sol. The given lines are

$$\frac{x+a}{2}=\frac{y}{1}=\frac{z}{-1/6} \quad \text{and} \quad \frac{x}{1}=\frac{y+2a}{1}=\frac{z-a}{1/6}$$

or $$\frac{x+a}{12}=\frac{y}{6}=\frac{z}{-1} \quad ...(i)$$

and $$\frac{x}{6}=\frac{y+2a}{6}=\frac{z-a}{1} \quad ...(ii)$$

Let l, m, n be the d.c.'s of the line S.D. to the given lines, then S.D. being perpendicular to both the lines (i) and (ii) we have

$$12l+6m-n=0 \quad \text{and} \quad 6l+6m+n=0.$$

Solving these we get $\dfrac{l}{2}=\dfrac{m}{-3}=\dfrac{n}{6}=\dfrac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(2^2+3^2+6^2)}}=\dfrac{1}{7}$

$\therefore$ d.c.'s of S.D. are $2/7, -3/7, 6/7$. ...(iii)

Also $A(-a, 0, 0)$ is a point on the line (i) and $B(0, -2a, a)$ is a point on the line (ii).

$\therefore$ Required S. D. = projection of AB on a line whose d.c.'s are given by (iii)

$$=\frac{2}{7}[0+a]-\frac{3}{7}[-2a-0]+\frac{6}{7}[a-0]=2a$$

Hence proved.

***Ex. 13 (b). If the axes are rectangular, find the S.D. between the lines $y = az + b$, $z = \alpha x + \beta$ and $y = a'z + b'$, $z = \alpha' x + \beta'$.**

Also deduce the condition for the lines to intersect. (*Rohilkhand* 90)

Sol. The equations of the given lines can be written in the symmetric form as $\dfrac{x + (\beta/\alpha)}{(1/\alpha)} = \dfrac{y-b}{a} = \dfrac{z}{1}$ and $\dfrac{x + (\beta'/\alpha')}{(1/\alpha')} = \dfrac{y-b'}{a'} = \dfrac{z}{1}$

$\therefore$ If l, m, n be the d.c.'s of the required S.D., then we have

$$l(1/\alpha) + m.a + n.1 = 0 \quad \text{and} \quad l(1/\alpha') + ma' + n.1 = 0$$

Solving these we get $\dfrac{l}{(a-a')} = \dfrac{m}{(1/\alpha') - (1/\alpha)} = \dfrac{n}{(1/\alpha)a' - (1/\alpha').a}$

or
$$\frac{l}{\alpha\alpha'(a-a')} = \frac{m}{(\alpha - \alpha')} = \frac{n}{(a'\alpha' - a\alpha)} \qquad \text{...(i)}$$

Also any point A on the first given line is $\left(-\dfrac{\beta}{\alpha}, b, 0\right)$ and a point on the second given line is $B\left(-\dfrac{\beta'}{\alpha'}, b', 0\right)$

$\therefore$ The required S.D. = the projection of AB on the line whose d.c.'s are l, m, n given by (i)

$$= \left[l\left\{\left(-\frac{\beta}{\alpha} + \frac{\beta'}{\alpha'}\right)\right\} + m\{b - b'\} + n\{0 - 0\}\right] \div \sqrt{(l^2 + m^2 + n^2)}$$

$$= \frac{\alpha\alpha'(a-a')(\alpha\beta' - \alpha'\beta)}{\alpha\alpha'\sqrt{(\Sigma l^2)}} + \frac{(\alpha - \alpha')(b-b')}{\sqrt{(\Sigma l^2)}}$$

$$= \frac{(a-a')(\alpha\beta' - \alpha'\beta) + (\alpha - \alpha')(b-b')}{\sqrt{[\alpha^2\alpha'^2(a-a')^2 + (\alpha-\alpha')^2 + (a'\alpha' - a\alpha)^2]}} \qquad \textbf{Ans.}$$

If the given lines intersect, then this S.D. $= 0$ and we have the required condition as $(a-a')(\alpha\beta' - \alpha'\beta) + (\alpha - \alpha')(b - b') = 0$. **Ans.**

Ex. 14. If the lines $\frac{1}{4}(x-5) = (1/6)(y-3) = \frac{1}{4}(z-7)$ and $\frac{1}{2}(x+1) = (1/5)(y+4) = (1/10)(z-9)$ are in a plane then find the equation of the plane containing them. If the lines are not in a plane then find the shortest distance between them. (***Bundelkhand*** **90**)

Sol. Do yourself.

***Ex. 15. Show that the S.D. between the lines.**

$$\frac{x - x_1}{\cos\alpha_1} = \frac{y - y_1}{\cos\beta_1} = \frac{z - z_1}{\cos\gamma_1}, \quad \frac{x - x_2}{\cos\alpha_2} = \frac{y - y_2}{\cos\beta_2} = \frac{z - z_2}{\cos\gamma_2}$$

meets the first line in a point whose distance from (x_1, y_1, z_1) is $[\Sigma(x_1 - x_2)(\cos\alpha_1 - \cos\theta\cos\alpha_2)]/\sin^2\theta$, where θ is the angle between the lines. (*Kanpur* 96)

Sol. Let the S.D. meet the first line at

$P\{x_1 + r_1\cos\alpha_1, y_1 + r_1\cos\beta_1, z + r_1\cos\gamma_1\}$ and the second line at

$$Q\,[x_2 + r_2 \cos \alpha_2, y_2 + r_2 \cos \beta_2, z_2 + r_2 \cos \gamma_2].$$

Then the distance of P from (x_1, y_1, z_1) is evidently r_1 which we are to evaluate.

The direction ratios of the line PQ are

$$[x_1 + r_1 \cos \alpha_1 - x_2 - r_2 \cos \alpha_2, y_1 + r_1 \cos \beta_1 - y_2 - r_2 \cos \beta_2, \\ z_1 + r_1 \cos \gamma_1 - z_2 - r_2 \cos \gamma_2]$$

Also the line PQ is perp. to both the given lines and hence

$$\cos \alpha_1 \{x_1 + r_1 \cos \alpha_1 - x_2 - r_2 \cos \alpha_2\} \\ + \cos \beta_1 \{y_1 + r_1 \cos \beta_1 - y_2 - r_2 \cos \beta_2\} \\ + \cos \gamma_1 \{z_1 + r_1 \cos \gamma_1 - z_2 - r_2 \cos \gamma_2\} = 0 \quad \text{...(i)}$$

and

$$\cos \alpha_2 \{x_1 + r_1 \cos \alpha_1 - x_2 - r_2 \cos \alpha_2\} \\ + \cos \beta_2 \{y_1 + r_1 \cos \beta_1 - y_2 - r_2 \cos \beta_2\} \\ + \cos \gamma_2 \{z_1 + r_1 \cos \gamma_1 - z_2 - r_2 \cos \gamma_2\} = 0 \quad \text{...(ii)}$$

From (i) we get

$$\{(x_1 - x_2) \cos \alpha_1 + (y_1 - y_2) \cos \beta_1 + (z_1 - z_2) \cos \gamma_1\} \\ + r_1 \{\cos^2 \alpha_1 + \cos^2 \beta_1 + \cos^2 \gamma_1\} \\ - r_2 \{\cos \alpha_1 \cos \alpha_2 + \cos \beta_1 \cos \beta_2 + \cos \gamma_1 \cos \gamma_2\} = 0 \quad \textbf{(Note)}$$

or $\{\Sigma (x_1 - x_2) \cos \alpha_1\} + r_1 \{1\} - r_2 \{\cos \theta\} = 0,$

since $\Sigma \cos^2 \alpha_1 = 1$ and $\Sigma \cos \alpha_1 \cos \alpha_2 = \cos \theta$

or $$r_1 - r_2 \cos \theta + \{\Sigma (x_1 - x_2) \cos \alpha_1\} = 0 \quad \text{...(iii)}$$

Similarly from (ii) we can find that

$$-r_2 + r_1 \cos \theta + \{\Sigma (x_1 - x_2) \cos \alpha_2\} = 0 \quad \text{...(iv)}$$

Eliminating r_2 from (iii) and (iv) we get

$$r_1 (1 - \cos^2 \theta) + \{\Sigma (x_1 - x_2) \cos \alpha_1\} - \Sigma (x_1 - x_2) \cos \alpha_2 \cos \theta = 0$$

or $$r_1 = \frac{\Sigma (x_1 - x_2)(\cos \alpha_1 - \cos \alpha_2 \cos \theta)}{\sin^2 \theta}$$

Now r_1 being the distance of P from (x_1, y_1, z_1), neglecting its sign we have the required result.

***Ex. 16. Find the image of the line $x = 3 - 6t, y = 2t, z = 3 + 2t$ in the plane $3x + 4y - 5z + 26 = 0$.** (*Kanpur* 91)

Sol. The given line is $x - 3 = -6t, y = 2t, z - 3 = 2t$.

or $$\frac{x-3}{-6} = \frac{y-0}{2} = \frac{z-3}{2} = t \quad \text{...(i)} \quad \textbf{(Note)}$$

Any point on this line is $A\,(3 - 6t, 2t, 3 + 2t)$. ...(ii)

If A lies on the plane $3x + 4y - 5z + 26 = 0$, then we have

$$3\,(3 - 6t) + 4\,(2t) - 5\,(3 + 2t) + 26 = 0 \quad \text{or} \quad t = -1$$

$\therefore$ From (ii), the point A is $(9, -2, 1)$, where A is the point of intersection of the given plane and the given line.

Also from (i) it is evident that any point on the given line is $C(3, 0, 3)$. Let B be the foot of the perpendicular from C on the given plane.

Now BC is a line perpendicular to the given plane *i.e.* it is normal to the given plane and as such the direction ratios of BC are 3, 4, – 5 (the coefficients of x, y, z in the equation of the given plane).

$\therefore$ Equations of BC are $\dfrac{x-3}{3}=\dfrac{y-0}{4}=\dfrac{z-3}{-5}$.

Any point on this line is $(3+3r, 4r, 3-5r)$. If this point is B *i.e.* if this point lies on the given plane then we have

$$3(3+3r)+4(4r)-5(3-5r)+26=0 \quad \text{or} \quad r=-2/5$$

$\therefore$ The point B is $\left(3-\dfrac{6}{5}, -\dfrac{8}{5}, 3+2\right)$ *i.e.* $\left(\dfrac{9}{5}, \dfrac{-8}{5}, 5\right)$

$\therefore$ The direction ratios of the projection AB of the given line are

$$9-\frac{9}{5}, -2+\frac{8}{5}, 1-5 \text{ i.e. } \frac{36}{5}, \frac{-2}{5}, -4$$

i.e. $36, -2, -20$ *i.e.* $18, -1, -10.$

$\therefore$ The required equations of the projection AB are

$$\frac{x-9}{18}=\frac{y+2}{-1}=\frac{z-1}{-10}$$ **Ans.**

****Ex. 17. A square ABCD of diagonal 2a is folded along the diagonal AC so that the planes DAC, BAC are at right angles. Find the S.D. between DC and AB.**

(*Meerut* 90)

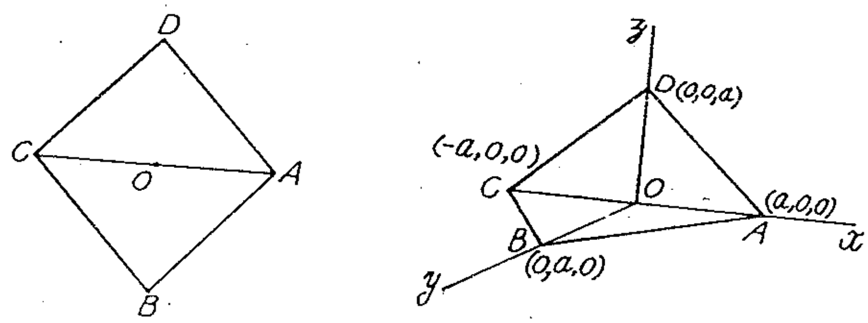
Sol. Let O, the centre of the square, be taken as the origin and OA, OB and OD, be taken as x, y and z-axes respectively.

Then the co-ordinates of A, B, C and D are $(a, 0, 0)$, $(0, a, 0)$ $(-a, 0, 0)$ and $(0, 0, a)$ respectively.

Therefore equations of AB are $\dfrac{x-a}{a}=\dfrac{y-0}{-a}=\dfrac{z-0}{0}$ ...(i)

Equations of DC are $\dfrac{x-0}{a}=\dfrac{y-0}{0}=\dfrac{z-a}{a}$ ...(ii)

Now any point on the line DC is $(0, 0, a)$.



(Fig. 6)

And the equaton of the plane through the line AB and parallel to DC *i.e.* through (i) and parallel to (ii) is

$$\begin{vmatrix} x-a & y-0 & z-0 \\ a & -a & 0 \\ a & 0 & a \end{vmatrix}=0$$

or $(x-a)(-a.a)-y(a.a)+z(a.a)=0$ or $x+y-z-a=0$ ...(iii)

$\therefore$ Required S.D. = length of perpendicular from $(0, 0, a)$ to the plane (iii)

$$=\frac{0+0-a-a}{\sqrt{[1^2+1^2+(-1)^2]}}=\frac{2a}{\sqrt{3}} \text{ (numerically)}$$ **Ans.**

Ex. 18. Evaluate 'a' so that the lines $ax-4y+7z+16=0$ $=4x+3y-2z+3$ and $x-3y+4z+6=0=x-y+z+1$ are coplanar

(*Kanpur* 94)

Sol. Given lines will be coplanar if

$$\begin{vmatrix} a & -4 & 7 & 16 \\ 4 & 3 & -2 & 3 \\ 1 & -3 & 4 & 6 \\ 1 & -1 & 1 & 1 \end{vmatrix}=0$$ ...See § 4.11 (c) Page 96 Chapter IV

or $$\begin{vmatrix} a-4 & -4 & 3 & 12 \\ 7 & 3 & 1 & 6 \\ -2 & -3 & 1 & 3 \\ 0 & -1 & 0 & 0 \end{vmatrix}=0,$$ adding second column to first, third and fourth

or $$(-1)\begin{vmatrix} a-4 & 3 & 12 \\ 7 & 1 & 6 \\ -2 & 1 & 3 \end{vmatrix}=0,$$ expanding w.r. to R_4

or $$\begin{vmatrix} a+2 & 0 & 3 \\ 9 & 0 & 3 \\ -2 & 1 & 3 \end{vmatrix}=0,$$ applying R_1-3R_3 and R_2-R_3

or $$\begin{vmatrix} a+2 & 2 \\ 9 & 3 \end{vmatrix}=0,$$ expanding w.r. to C_2

or $3(a+2)-(9\times 3)=0$ or $3a=27-6=21$ or $a=7$ **Ans.**

***Ex. 19. If the three planes through P and the three given lines $y=1, z=-1$; $z=1, x=-1$; $x=1, y=-1$ all pass through one line, P must lie on the surface $yz+zx+xy+1=0$.**

Sol. Let P be the point (x_1, y_1, z_1).

The equation of any plane through the line $y=1, z=-1$

i.e. $y-1=0, z+1=0$ is $(y-1)+\lambda(z+1)=0$ ...(i)

$\because$ This plane passes through $P(x_1, y_1, z_1)$,

$\therefore (y_1 - 1) + \lambda (z_1 + 1) = 0 \quad$ or $\quad \lambda = \dfrac{-(y_1 - 1)}{z_1 + 1}$

$\therefore$ From (i), the equation of the plane through the line

$y = 1, z = -1 \quad$ is $\quad (y - 1) - \dfrac{(y_1 - 1)}{(z_1 + 1)} (z + 1) = 0$

or $\qquad (y - 1)(z_1 + 1) - (y_1 - 1)(z + 1) = 0$

or $\qquad 0.x + (z_1 + 1) y - (y_1 - 1) z - (y_1 + z_1) = 0 \qquad$ ...(ii)

Similarly the equations of other two planes through P and the other two lines are $\qquad -(z_1 - 1) x + 0 . y + (x_1 + 1) z - (x_1 + z_1) = 0 \qquad$ ...(iii)

and $\qquad (y_1 + 1) x - (x_1 - 1) y + 0.z - (x_1 + y_1) = 0 \qquad$ ...(iv)

If these three planes given by (ii), (iii) and (iv) pass through one line, then we must have $\Delta_4 = 0$. ...See § 4.14 Page 125 Ch. IV

i.e. $\begin{vmatrix} 0 & z_1 + 1 & -(y_1 - 1) \\ -(z_1 - 1) & 0 & (x_1 + 1) \\ (y_1 + 1) & -(x_1 - 1) & 0 \end{vmatrix} = 0$

or $\qquad (z_1 - 1) [-(x_1 - 1)(y_1 - 1)] + (y_1 + 1) [(z_1 + 1)(x_1 + 1)] = 0,$

expanding the determinant.

or $\qquad -(z_1 - 1)(x_1y_1 - x_1 - y_1 + 1) + (y_1 + 1)(x_1z_1 + x_1 + z_1 + 1) = 0$

or $\qquad x_1y_1 + y_1z_1 + z_1x_1 + 1 = 0$, on simplifying.

$\therefore$ The locus of $P (x_1, y_1, z_1)$ is $xy + yz + zx + 1 = 0$. Hence proved.

EXERCISES ON STRAIGHT LINE

Ex. 1. Prove that the planes $3x + y + z = 7$ and $x + 2y + z = 0$ pass through a line lying on the plane $12x - y + 2z = 35$.

***Ex. 2.** Prove that every straight line can be represented by two linear equations in x, y and z; conversely two linear equations in x, y and z represent a straight line.

Ex. 3. Find the co-ordinates of the point where the line given by $x + 3y - z = 6,\ y - z = 4$ cuts the plane $2x + 2y + z = 0$. **Ans.** $(2, 0, -4)$

Ex. 4. A line is given by $x + y + z = 0, x = y$. Prove that it is perpendicular to the plane $x + y = 2z$.

Ex. 5. Find the equations of the line through the point (1, 2, 4) and perpendicular to the line $3x + 2y - z = 4, x - 2y - 2z = 5$.

Ex. 6. The plane $(x/a) + (y/b) + (z/c) = 1$ cuts the coordinate axes at A, B, C. Find the equations to the line BC. **Ans.** $x/0 = -(y - b)/b = z/c$

Ex. 7. Find the equation of the plane through the point $(1, -2, 3)$ and the line $2x - y + 4z + 4 = 0 = x + 2y - 4z + 8$. **Ans.** $34x + 33y - 52z + 188 = 0$

Ex. 8. Find the equation of the plane passing through the line $\frac{1}{2}(x - 1) = -(y + 1) = \frac{1}{4}(z - 3)$ and

(i) perpendicular to the plane $x + 2y + z = 12$

(ii) parallel to the line $x = \frac{1}{2} y = \frac{1}{3} z$.

Ans. (i) $9x - 2y - 5z + 4 = 0$; (ii) $11x + 2y - 5z + 6 = 0$

Ex. 9. Find the equations of the line drawn perpendicular from the origin to the line of intersection of the planes.

$$x + 2y + 3z + 4 = 0 \quad \text{and} \quad 2x + 3y + 4z + 5 = 0.$$

Ans. $3x + 2y + z = 0, x - 2y + z = 0$

Ex. 10. Show that any two linear equations in x, y, z whose coefficients of first degree terms are not proportional represent a straight line.

Ex. 11. Show that every linear equation $ax + by + cz + d = 0$ represents a plane. The triplet (a, b, c) are the direction numbers (D.N.'S) of any normal to the planes and two linear equations (in x, y, z) whose coefficients of the first degree terms are not proportional, represent a straight line.

Ex. 12. Show that the distance of the point (α, β, γ) from the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ measured parallel to the plane $ax + by + cz + d = 0$ is given by $d^2 = \frac{(a^2 + b^2 + c^2) \Sigma \{m (z_1 - \gamma) - n (y_1 - \beta)\}^2 - [\Sigma (x_1 - a)(bn - cm)]^2}{(al + bm + cn)^2}$

***Ex. 13.** Find the length of the perpendicular from the point (1, 2, 3) on the straight line $x = 7 + 3t, y = 6 + 2t, z = 7 - 2t$. **Ans.** $4 \sqrt{(52/17)}$

Ex. 14. Find the point of intersection to the lines

$$\frac{1}{3}(x + 1) = -\frac{1}{2}(y - 3) = -(z + 2); \; (x - 3) = \frac{1}{2}(y - 3) = \frac{1}{3} z.$$

Also find the equation of the plane passing through two lines.

Ans. $(2, 1, -3)$, $2x + 5y - 4z = 21$

Ex. 15. Prove that the lines $\frac{1}{3}(x + 1) = \frac{1}{5}(y + 3) = -\frac{1}{7}(z - 5)$ and $x - 2 = \frac{1}{3}(y - 4) = \frac{1}{5}(z - 6)$ intersect. Find the point of intersection and the plane in which they lie. **Ans.** $(\frac{1}{2}, -\frac{1}{2}, \frac{3}{2})$; $x - 2y + z = 0$.

Ex. 16. Find the equation of the plane which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ and which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$.

Ex. 17 (a). Show that the lines $3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$ and $\frac{x + 4}{3} = \frac{y + 6}{5} = \frac{z - 1}{-2}$ are coplanar. Find also the coordinates of their point of intersection and the equation of the plane in which they lie.

Ans. $(2, 4, -3)$, $45x - 17y + 25z = -53$

Ex. 17 (b). Find the distance of the point $(1, -2, 3)$ from the plane $x - y + z = 5$ measured parallel to the line $x/2 = y/3 = -(z - 3)/4$. Find also the coordinates of the foot of the perpendicular. *(Kanpur 91)*

Ex. 18. Show that if the equation $x+y+z=0$ and $ayz+bzx+cxy=0$ represent two lines and $abc=0, a+b+c=0$, then the lines are perpendicular to each other.

Ex. 19. Find the equations of the line which intersects the line $x=18+4t, y=3t, z=10+t$ and $x=-11+4t, y=-10+3t, z=-7t$ perpendicularly.

Ex. 20. Find the distance of the point $(-1, 2, 3)$ from the line through $(3, 4, 5)$ whose direction cosines are proportional to $2, -3, 6$. **Ans.** $2\sqrt{5}$

Ex. 21. Find the equations of the line drawn through the point $(3, -4, 1)$ parallel to the plane $2x+y-z=5$ so as to cut the line $\frac{x-3}{2}=\frac{y+1}{-5}=\frac{z-2}{-1}$. Find also the co-ordinates of the point of intersection and the equation of the plane through given line and the required line.

Ex. 22. A line through the origin makes angles α, β, γ with its projection on the co-ordinate planes, which are rectangular. The distance of any point (x, y, z) from the line and its projections are d, a, b, c respectively. Prove that

$$d^2=(a^2-x^2)\cos^2\alpha+(b^2-y^2)\cos^2\beta+(c^2-z^2)\cos^2\gamma.$$

Ex. 23. Find the image of the straight line $x=1+t, y=2+3t, z=3+4t$ in the plane $z=0$.

Ex. 24. Find the image in the plane $x-y+3z+2=0$ of the line $x=2+t, y=3-4t, z=4-2t$.

(Hint : See Ex. 16 Page 162 Ch. IV.)

***Ex. 25.** Let λ be the shortest distance between the lines $by+cz=1$, $x=0$ and $ax-cz=1, y=0$ and let μ be the length of a diagonal of the cuboid whose three concurrent edges are of lengths a, b, c respectively, then show that $\lambda\mu=2$.

****Ex. 26.** Prove that the equation of the plane through the line $x=x_0+lt, y=y_0+mt, z=z_0+nt$ and through the point (x_1, y_1, z_1) can be put in the form

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_1-x_0 & y_1-y_0 & z_1-z_0 \\ l & m & n \end{vmatrix}=0$$

***Ex. 27.** Find the length of S. D. length the lines

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{7}, \quad \frac{x+1}{7}=\frac{y-1}{-6}=\frac{z+1}{1}$$

Also find the equations and the points, where it intersects the lines.

Ans. $2\sqrt{(29)}; \frac{1}{2}(x-1)=\frac{1}{3}(y-2)=\frac{1}{4}(z-3)$

Ex. 28. Prove that the locus of a line which intersects the three lines $y-z=1, x=0$; $z-x=1, y=0$; $x-y=1, z=0$ is

$$x^2+y^2+z^2-2yz-2zx-8xy=1.$$

Ex. 29. If OA, OB, OC are three mutually perpendicular lines each of length a, show that the S.D. between OA and BC is $\frac{1}{3}a\sqrt{2}$ and its bisects BC.

Ex. 30. Find the length and the equations of S.D. between the lines $5x-y-z=0=x-2y+z+3$ and $7x-4y-2z=0=x-y+z-3$.

Ans. $13/\sqrt{(75)}$; $17x+20y-19z-39=0=8x+5y-31z+67$.

Ex. 31. Find the magnitude and the equations of the line of S.D. between the lines $\dfrac{x-8}{3}=\dfrac{y+9}{-16}=\dfrac{z-10}{7}$ and $\dfrac{x-6}{3}=\dfrac{y-5}{8}=\dfrac{z-20}{-5}$

Ex. 32. Prove that non-parallel lines $x=\alpha+tl,\ y=\beta+tm,\ z=\gamma+tn$ and $x=\alpha'+t'l', y=\beta'+t'm', z'=\gamma'+t'n'$ are coplanar if

$$\begin{vmatrix} \alpha-\alpha' & \beta-\beta' & \gamma-\gamma' \\ l' & m' & n' \\ l & m & n \end{vmatrix}=0$$

Ex. 33. A straight line with direction numbers $<1,-1,1>$ is drawn to intersect the lines $x=5+3t, y=1+t, z=-3-5t$ and $x=4+2t,\ z=7-t$ $y=1-5t$,. Find the point of intersection and the length intercepted on it. Also find its equations.

Ex. 34. The planes $3x-y+z+1=0, 5x+y+3z=0$ intersect in the line PQ. Find the equation to the plane through the point (2, 1, 4) and perpendicular to PQ.

(*Meerut* 93)

Ex. 35. Find the shortest distance between the lines AB and CD, where the points A, B, C and D are respectively (1, 0, 0), (0, 1, 0), (0, 0, 1) and (1, 1, 1).

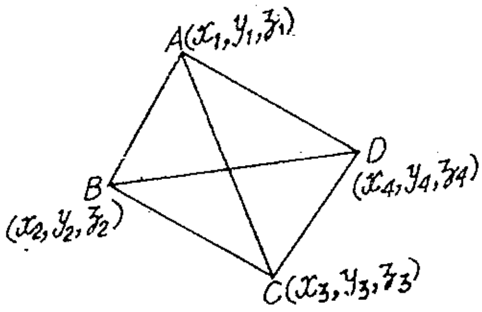
(*Meerut* 97)

CHAPTER V

Volume of Tetrahedron

****§ 5.01 Volume of tetrahedron in terms of the co-ordinates of the vertices.** (*Kanpur* 93; *Kumaun* 94, 92)

Let V be the volume of the tetrahedron $ABCD$, where the co-ordinates of the vertices A, B, C and D are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4).



(Fig. 1)

Let Δ be the area of ΔBCD and p be the length of perpendicular from the vertex A to opposite face BCD.

Then $V = \frac{1}{3} p . \Delta$ ...(i)

Now if Δ_x, Δ_y and Δ_z, be the projections of the triangle BCD on the co-ordinates planes, then we have

$$\Delta_x = \frac{1}{2} \begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} ; \ \Delta_y = \frac{1}{2} \begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix}$$

and $$\Delta_z = \frac{1}{2} \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} \quad ...(ii)$$

...See § 3.16 in Chapter III

Also $$\Delta = \sqrt{(\Delta_x^2 + \Delta_y^2 + \Delta_z^2)} \quad ...(iii)$$

Again the equation of the plane BCD is

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

or $$x \begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} - y \begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix} + z \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} - \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} = 0$$

or $$x\,(2\Delta_x) - y\,(2\Delta_y) + z\,(2\Delta_z) - \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} = 0, \text{ from (ii)}$$

$\therefore p =$ length of perp. from $A\,(x_1, y_1, z_1)$ to the plane BCD

$$= [x_1\,(2\,\Delta_x) - y_1\,(2\Delta_y) + z_1\,(2\Delta_z) - \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}]$$

$$\div \sqrt{[(2\Delta_x)^2 + (2\Delta_y)^2] + (2\Delta_z)^2]} \quad \text{...See § 3.11 Ch. III}$$

$$= [x_1 \begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} - y_1 \begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix} + z_1 \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$

$$- \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}] \div 2\sqrt{(\Delta_x^2 + \Delta_y^2 + \Delta_z^2)}$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \div 2\Delta, \text{ from (iii)} \quad \text{...(iv)}$$

$\therefore$ From (i), $V = \frac{1}{3} p . \Delta$.

or $$V = (1/6) \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}, \text{ substituting value of } p \text{ from (iv)} \quad \text{...(v)}$$

An Important Note. We know that the interchange of two rows of a determinant changes its sign **(See Author's Algebra or Matrices)** and thus we shall often get a negative value for the volume of a tetrahedron. We define the volume of $OABC$ to be positive if the rotation determined by ABC is anti-clockwise with respect to origin O. So we have

$$\text{vol. } DABC = -\text{ vol. } ADBC = -\text{ vol. } ABDC = +\text{ vol. } ABCD.$$

Another Form

$$V = (1/6) \begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 & 0 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 & 0 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 & 0 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

, substracting 2nd row from 1st, 3rd from 2nd and 4th from 3rd row respectively.

$$= (1/6) \begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ x_2 - x_3 & y_2 - y_3 & z_2 - z_3 \\ x_3 - x_4 & y_3 - y_4 & z_3 - z_4 \end{vmatrix} \quad \text{...(vi)}$$

Particular case (One vertex at origin).

Let $x_4 = 0 = y_4 = z_4$ *i.e.* D is the origin

Then from (v), we have $$V = (1/6) \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \quad \text{...(vii)}$$

§ 5.02 Volume of tetrahedron when equations of its four faces are given.

Let the equations of four faces of the tetrahedron be

$$a_1x + b_1y + c_1z + d_1 = 0 \quad \text{...(i)}$$
$$a_2x + b_2y + c_2z + d_2 = 0 \quad \text{...(ii)}$$
$$a_3x + b_3y + c_3z + d_3 = 0 \quad \text{...(iii)}$$
$$a_4x + b_4y + c_4z + d_4 = 0 \quad \text{...(iv)}$$

Solving (ii), (iii) and (iv), we have

$$\frac{x}{\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix}}$$

(See Author's Algebra or Matrices)

or
$$\frac{x}{A_1} = \frac{-y}{-B_1} = \frac{z}{C_1} = \frac{-1}{-D_1},$$

where A_1, B_1, C_1, D_1 are cofactors of a_1, b_1, c_1 and d_1 respectively in the determinant
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

$\therefore$ The point of intersection of the planes (ii), (iii) and (iv) above is
$$\left(\frac{A_1}{D_1}, \frac{B_1}{D_1}, \frac{C_1}{D_1}\right).$$

Similarly solving other sets of three planes from the given planes (i), (ii), (iii) and (iv) the other points of intersection *i.e.* other three vertices of the tetrahedron are
$$\left(\frac{A_2}{D_2}, \frac{B_2}{D_2}, \frac{C_2}{D_2}\right); \quad \left(\frac{A_3}{D_3}, \frac{B_3}{D_3}, \frac{C_3}{D_3}\right) \quad \text{and} \quad \left(\frac{A_4}{D_4}, \frac{B_4}{D_4}, \frac{C_4}{D_4}\right)$$

$\therefore$ The required volume of tetrahedron

$$= \text{“}(1/6) \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}\text{”}$$ ...See § 5.01 (v) Page 1 of this Chapter.

$$= (1/6) \begin{vmatrix} A_1/D_1 & B_1/D_1 & C_1/D_1 & 1 \\ A_2/D_2 & B_2/D_2 & C_2/D_2 & 1 \\ A_3/D_3 & B_3/D_3 & C_3/D_3 & 1 \\ A_4/D_4 & B_4/D_4 & C_4/D_4 & 1 \end{vmatrix}$$

$$= \frac{1}{6D_1D_2D_3D_4} \begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}$$

$= \dfrac{1}{6D_1D_2D_3D_4}\Delta^{4-1}$. since we know that if Δ' be the determinant formed by cofactors of a determinant Δ of nth order then $\Delta' = \Delta^{n-1}$ (See **Author's Matrices or Algebra**)

or $\quad V = \Delta^3/(6D_1D_2D_3D_4)$.

§ 5.03 Volume of a tetrahedron in terms of three conterminus edges and the angles which they make with each other. (*Kanpur* 96)

Let the lengths of the edges OA, OB and OC of the tetrahedron $OABC$ be a, b and c and the angles BOC, COA and OAB be $\lambda\,\mu\,\nu$ respectively. Let the point O be taken as origin and any three mutually perpendicular lines through O be taken as co-ordinate axes.

Let the directin cosines of the lines OA, OB and OC be l_1, m_1, n_1; l_2, m_2, n_2 and l_3, m_3, n_3 respectively. Then the co-ordinates of A, B and C are (l_1a, m_1a, n_1a), (l_2b, m_2b, n_2b) and (l_3c, m_3c, n_3c) respectively. Then we have

$$\cos\angle BOC = \cos\lambda = l_2l_3 + m_2m_3 + n_2n_3, \qquad \text{...(i)}$$

$$\cos\angle COA = \cos\mu = l_3l_1 + m_3m_1 + n_3n_1, \qquad \text{...(ii)}$$

and

$$\cos\angle AOB = \cos\nu = l_1l_2 + m_1m_2 + n_1n_2, \qquad \text{...(iii)}$$

Also $\quad l_1^2 + m_1^2 + n_1^2 = 1 = l_2^2 + m_2^2 + n_2^2 = l_3^2 + m_3^2 + n_3^2 \qquad$ ...(iv)

Therefore volume V of tetrahedron $OABC$ is given by

$$V = \text{“}(1/6)\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\text{”}$$ ...See § 5.01 (vii) Page 2 of this chapter

$$= (1/6)\begin{vmatrix} l_1a & m_1a & n_1a \\ l_2b & m_2b & n_2b \\ l_3c & m_3c & n_3c \end{vmatrix}, \qquad \because A \text{ is } (l_1a, m_1a, n_1a) \text{ etc.}$$

or

$$V = (1/6)\,abc\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \qquad \text{...(v)}$$

Now $$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} l_1^2 + m_1^2 + n_1^2 & l_1l_2 + m_1m_2 + n_1n_2 & l_1l_3 + m_1m_3 + n_1n_3 \\ l_2l_1 + m_2m_1 + n_2n_1 & l_2^2 + m_2^2 + n_2^2 & l_2l_3 + m_2m_3 + n_2n_3 \\ l_3l_1 + m_3m_1 + n_3n_1 & l_3l_2 + m_3m_2 + n_3n_2 & l_3^2 + m_3^2 + n_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}, \text{ from (i), (ii), (iii), and (iv)}$$

$\therefore$ From (v), we have $V = \pm(1/6)\, abc \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}^{1/2}$

Here the negative sign may be neglected while calculating magnitude of the volume.

Solved Examples on Volume of Tetrahedron.

Ex. 1. A, B, C are (3, 2, 1), (– 2, 0, – 3), (0, 0, – 2). Find the locus of P if the volume of the tetrahedron PABC is 5.

Sol. Let P be (x, y, z). Then if V be the volume of the tetrahedron $PABC$ from § 5.01 (Important Note) Page 2 of this chapter, we have

$$V = \text{“} -(1/6) \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \text{”} = -\frac{1}{6} \begin{vmatrix} x & y & z & 1 \\ 3 & 2 & 1 & 1 \\ -2 & 0 & -3 & 1 \\ 0 & 0 & -2 & 1 \end{vmatrix} \quad \text{(Note)}$$

or $5 = -(1/6) \begin{vmatrix} x & y & z+2 & 0 \\ 3 & 2 & 3 & 0 \\ -2 & 0 & -1 & 0 \\ 0 & 0 & -2 & 1 \end{vmatrix}$, subtracting 4th row from rest remembering V is given as 5

or $-30 = \begin{vmatrix} x & y & z+2 \\ 3 & 2 & 3 \\ -2 & 0 & -1 \end{vmatrix}$, expanding with respect to 4th column

or $-30 = \begin{vmatrix} x-2z-4 & y & z+2 \\ -3 & 2 & 3 \\ 0 & 0 & -1 \end{vmatrix}$, subtracting twice third column from first.

or $-30 = -\begin{vmatrix} x-2z-4 & y \\ -3 & 2 \end{vmatrix}$, expanding with respect to 3rd row

or $-30 = -2(x-2z-4) - 3y$ or $2x + 3y - 4z = 38$. **Ans.**

Ex. 2. A, B and C are three fixed points and a variable point P moves so that the volume of the tetrahedron PABC is constant. Find the locus of P.

Sol. Let A, B and C be the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively and P be (x_1, y_1, z_1).

Now the volume of tetrahedron $PABC$ = constant (given)

so we have $\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ a & 0 & 0 & 1 \\ 0 & b & 0 & 1 \\ 0 & 0 & c & 1 \end{vmatrix} = \text{constant}$

or $$\begin{vmatrix} x_1 & y_1 & z_1-c & 0 \\ a & 0 & -c & 0 \\ 0 & b & -c & 0 \\ 0 & 0 & c & 1 \end{vmatrix} = \text{constant, substracting 4th row from the rest.}$$

or $$\begin{vmatrix} x_1 & y_1 & z_1-c \\ a & 0 & -c \\ 0 & b & -c \end{vmatrix} = \text{constant}$$

or $$x_1\,(bc) - y_1\,(-ac) + (z_1 - c)\,(ab) = \text{constant}$$

or $$bc\,x_1 + ca\,y_1 + ab\,z_1 = \text{constant}$$

or $$\frac{x_1}{a} + \frac{y_1}{b} + \frac{z_1}{c} = \text{constant}$$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \text{constant}$, which is a plane parallel to the plane ABC whose equation is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 .$$

Ex. 3 (a). Find the volume of the tetrahedron whose vertices are the points (2, – 1, – 3), (4, 1, 3), (3, 2, – 1) and (1, 4, 2).

Sol. We know that if (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) are the vertices of a tetrahedron, then its volume is given by

$$V = (1/6) \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

$\therefore$ Required volume $= (1/6) \begin{vmatrix} 2 & -1 & -3 & 1 \\ 4 & 1 & 3 & 1 \\ 3 & 2 & -1 & 1 \\ 1 & 4 & 2 & 1 \end{vmatrix}$

$$= (1/6) \begin{vmatrix} 6 & 0 & 0 & 2 \\ 4 & 1 & 3 & 1 \\ -1 & 1 & -4 & 0 \\ -3 & 3 & -1 & 0 \end{vmatrix},$$ adding 2nd row to 1st and subtracting from the rest.

$$= (1/6) \begin{vmatrix} 0 & 0 & 0 & 2 \\ 1 & 1 & 3 & 1 \\ -1 & 1 & -4 & 0 \\ -3 & 3 & -1 & 0 \end{vmatrix},$$ subtracting 3 times 4th column from 1st.

$$= (1/6)\,(-2) \begin{vmatrix} 1 & 1 & 3 \\ -1 & 1 & -4 \\ -3 & 3 & -1 \end{vmatrix},$$ expanding with respect to 1st row.

$= -(1/3)\begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 6 & 8 \end{vmatrix}$, adding 1st row to 2nd and 3 times 1st row to 3rd.

$= -(1/3)\begin{vmatrix} 2 & -1 \\ 6 & 8 \end{vmatrix}$, expanding with respect to 1st column.

$= -\frac{1}{3}[16+6] = (22/3)$, numerically. **Ans.**

****Ex. 3 (b). Show that the volume of the tetrahedron formed by the planes $my+nz=0$, $nz+lx=0$, $lx+my=0$ and $lx+my+nz=p$ is**

$$\frac{2}{3}p^3/(lmn).$$ (*Agra* 90; *Kanpur* 95)

Hence deduce the area of the triangle formed on the plane

$$lx+my+nz=p.$$

Sol. The planes are

$my+nz=0$, ...(i) $nz+lx=0$, ...(ii)

$lx+my=0$, ...(iii) and $lx+my+nz=p$. ...(iv)

The point of intersection of the planes (i), (ii) and (iii) is obviously (0, 0, 0) *i.e.* the origin.

Solving (ii), (iii) and (iv), we can easily find

$$x=-p/l,\ y=p/m \quad \text{and} \quad z=p/n$$

i.e. these planes intersect in the point $(-p/l, p/m, p/n)$,

Similarly the other two vertices of tetrahedron are

$$(p/l, -p/m, p/n) \quad \text{and} \quad (p/l, p/m, -p/n),$$

$\therefore$ The volume of the tetrahderon

$= \text{“}(1/6)\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\text{”}$ ...See § 5.01 (vii) Page 2 of this chapter

$$= (1/6)\begin{vmatrix} -p/l & p/m & p/n \\ p/l & -p/m & p/n \\ p/l & p/m & -p/n \end{vmatrix} = \frac{p^3}{6\,lmn}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= \frac{p^3}{6\,lmn}\begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix}$, adding 1st column to 2nd and 3rd

$= \frac{p^3}{6\,lmn}(4) = \frac{2}{3}\cdot\frac{p^3}{lmn}$ Hence proved.

Let Δ be the area of the triangle formed by the plane (iv) and d be the length of perpendicular from the opposite vertex (0, 0, 0) to the plane (iv).

i.e. d = length of perp. from (0, 0, 0) to $lx+my+nz=p$

$$= \frac{p}{\sqrt{(l^2+m^2+n^2)}}$$

Also volume of tetrahedron $=\frac{1}{3}\,.\,\Delta\,.\,d$

i.e. $\dfrac{2p^3}{3\,l\,m\,n}=\dfrac{1}{3}\Delta\,.\,\dfrac{p}{\sqrt{(l^2+m^2+n^2)}}$ or $\Delta=\dfrac{2p^2\sqrt{(l^2+m^2+n^2)}}{l\,m\,n}$ **Ans.**

***Ex. 3 (c). Find the volume of the tetrahedron formed by the coordinate planes and the plane lx + my + nz = p.** *(Kanpur 92)*

Sol. The planes are $x=0$ ...(i), $y=0$ ...(ii)

$z=0$ ...(iii) and $lx+my+nz=p$ ...(iv)

The point of intersection of the planes (i), (ii) and (iii) is obviously (0, 0, 0) *i.e.* the origin.

Solving (ii), (iii) and (iv) we find $x=p/l, y=0, z=0$ *i.e.* these planes intersect in the point $(p/l, 0, 0)$.

Similarly the other two vertices of the tetrahedron are $(0, p/m, 0)$ and $(0, 0, p/n)$.

$\therefore$ Required volume [See § 5.01 (vii) Page 2 this ch.]

$$=(1/6)\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=(1/6)\begin{vmatrix} p/l & 0 & 0 \\ 0 & p/m & 0 \\ 0 & 0 & p/m \end{vmatrix}=p^2/(6\,lmn)$$ **Ans.**

***Ex. 3 (d). Find the volume of the tetrahedron formed by planes whose equations are y + z = 0, z + x = 0, x + y = 0 and x + y + z = 1.** *(Kumaun 91)*

Sol. The planes are $y+z=0$ ...(i), $z+x=0$ ...(ii)

$x+y=0$ ...(iii) and $x+y+z=1$ ...(iv)

The point of intersection of the planes (i), (ii) and (iii) is obviously the origin *i.e.* (0, 0, 0).

Solving (ii), (iii) and (iv) we get $y=1, z=1$ and therefore from (iv) we have $x=-1$.

i.e. these planes intersect in $(-1, 1, 1)$.

Similarly the other two vertices of the tetrahedron are $(1, -1, 1)$ and $(1, 1, -1)$

$\therefore$ Required volume [See § 5.01 (vii) Page 2 of this ch.]

$$=(1/6)\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=(1/6)\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$=(1/6)\begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix}$$, adding first column to the rest.

$$=(1/6)\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}=-(1/6)(-4)=4/6=2/3$$ **Ans.**

****Ex. 4. Find the locus of the centroid of the tetrahedron of constant volume $64k^3$, formed by the three co-cordinate planes and a variable plane.**

(*Bundelkhand* 96; *Kanpur* 97, 91, 90 ; *Kumaun* 93)

Sol. Let the equation of the variable plane be $lx + my + nz = p$. ...(i)

Also the equations of the co-ordinate planes are $x = 0$, $y = 0$, $z = 0$.

Solving these four equations taking three at a time, we get co-ordinates of the vertices of the tetrahedron as

$(0, 0, 0)$; $(0, 0, p/n)$; $(p/l, 0, 0)$ and $(0, p/m, 0)$.

Let the centroid of this tetrahedron be (x_1, y_1, z_1), then

$$x_1 = \frac{1}{4}\left(0 + 0 + \frac{p}{l} + 0\right) = \frac{p}{4l}$$

Similarly, $y_1 = \dfrac{p}{4m}$ and $z_1 = \dfrac{p}{4n}$

or $$\frac{p}{l} = 4x_1, \quad \frac{p}{m} = 4y_1 \quad \text{and} \quad \frac{p}{n} = 4z_1. \qquad \text{...(ii)}$$

Also the volume of tetrahedron $= 64k^3$. (given)

i.e. $(1/6)\begin{vmatrix} p/l & 0 & 0 \\ 0 & p/m & 0 \\ 0 & 0 & p/n \end{vmatrix} = 64k^3$, ...See § 5.01 (vii) Page 2 of this chapter

or $$\frac{p^3}{6\,lmn} = 64k^3 \quad \text{or} \quad \frac{1}{6}\left(\frac{p}{l}\right)\left(\frac{p}{m}\right)\left(\frac{p}{n}\right) = 64k^3 \qquad \text{...(iii)}$$

$\therefore$ From (ii) $(1/6)(4x_1)(4y_1)(4z_1) = 64k^3$ or $x_1 y_1 z_1 = 6k^3$,

$\therefore$ Required locus of the centroid (x_1, y_1, z_1) is $xyz = 6k^3$. **Ans.**

***Ex. 5. In the above example find the locus of the foot of the perpendicular from the origin to the variable plane.**

Sol. Let (x_2, y_2, z_2) be the foot of the perpendicular from the origin to the plane (i) of last example.

Then the direction cosines of this perpendicular *i.e.* the line joining $(0, 0, 0)$ and (x_2, y_2, z_2) are proportional to x_2, y_2, z_2.

But the direction cosines of the perpendicular which is also the normal to the plane (i) are proportional to l, m, n.

Hence $$\frac{x_2}{l} = \frac{y_2}{m} = \frac{z_2}{n}. \qquad \text{...(iv)}$$

Also (x_2, y_2, z_2) lies on (i), so we have $lx_2 + my_2 + nz_2 = p$. ...(v)

From (iv), we have $$\frac{x_2^2}{lx_2} = \frac{y_2^2}{my_2} = \frac{z_2^2}{nz_2} \qquad \text{(Note)}$$

(multiplying num, and denom. of each fraction by the same quantity) or $\dfrac{x_2^2}{lx_2}=\dfrac{y_2^2}{my_2}=\dfrac{z_2^2}{nz_2}=\dfrac{x_2^2+y_2^2+z_2^2}{lx_2+my_2+nz_2}=\dfrac{x_2^2+y_2^2+z_2^2}{p}$, from (v)

or $\dfrac{x_2}{l}=\dfrac{x_2^2+y_2^2+z_2^2}{p}$ etc. (Note)

or $\dfrac{p}{l}=\dfrac{x_2^2+y_2^2+z_2^2}{x_2}$, $\dfrac{p}{m}=\dfrac{x_2^2+y_2^2+z_2^2}{y_2}$, $\dfrac{p}{n}=\dfrac{x_2^2+y_2^2+z_2^2}{z_2}$

Substituting these value in (iii) of last example, we have

$$\frac{1}{6}\cdot\frac{(x_2^2+y_2^2+z_2^2)^3}{x_2y_2z_2}=64k^3$$

or

$$(x_2^2+y_2^2+z_2^2)^3=384k^3\,x_2y_2z_2$$

$\therefore$ The required locus of (x_2, y_2, z_2) is

$$(x^2+y^2+z^2)^3=384k^3\,xyz.$$ Ans.

****Ex. 6. If the lengths of two opposite edges of a tetrahedron are a, b, their shortest distance is equal to d and the angle between them is θ, then prove that its volume is (1/6) abd sin θ.** *(Garhwal 91)*

Sol. Refer Fig 1 Page 1 of this chapter.

Let the vertex A be taken as origin *i.e.* (0, 0, 0) and let $AB=a$. Let the direction cosines of AB be taken as l_1, m_1, n_1. Then the co-ordinates of B are (l_1a, m_1a, z_1a), and the equations of the line $A\,B$ are

$$\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1} \quad ...(i)$$

The edge opposite to AB is CD. Let the vertex C be (α, β, γ) and the direction cosines of the line CD be l_2, m_2, n_2. Given $CD=b$.

$\therefore$ The equations of the line CD are

$$\frac{x-\alpha}{l_2}=\frac{y-\beta}{m_2}=\frac{z-\gamma}{n_2} \quad ...(ii)$$

$$=b \text{ for } D.$$

$\therefore$ The co-ordinattes of D are $(\alpha+l_2b, \beta+m_2b, \gamma+n_2b)$.

Also if θ be the angle between the lines AB and CD whose d.c.'s are l_1, m_1, n_1 and l_2, m_2, n_2 respectively, then

$$\sin\theta=\sqrt{[\Sigma\,(m_1n_2-m_2n_1)^2]} \quad ...(iii)$$

Now $d=$ the shortest distance between lines (i) and (ii)

$$=\begin{vmatrix}\alpha-0 & \beta-0 & \gamma-0\\ l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\end{vmatrix}\div\sqrt{[\Sigma\,(m_1n_2-m_2n_1)^2]},$$ ...See § 4.15 on Straight Line

$$=\begin{vmatrix}\alpha & \beta & \gamma\\ l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\end{vmatrix}\div\sin\theta \text{, from (iii)}$$

or $$d\sin\theta = \begin{vmatrix} \alpha & \beta & \gamma \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \quad \text{...(iv)}$$

Now the volume of the tetrahedron ABC.

$= \text{“}(1/6) \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\text{”}$, since one of the vertices *viz.* A is taken as origin.

$= (1/6) \begin{vmatrix} \alpha & \beta & \gamma \\ al_1 & am_1 & an_1 \\ \alpha + l_2 b & \beta + m_2 b & \gamma + n_2 b \end{vmatrix}$, substituting the co-ordinates of B, C, D **(Note)**

$= (1/6)\, a \begin{vmatrix} \alpha & \beta & \gamma \\ l_1 & m_1 & n_1 \\ l_2 b & m_2 b & n_2 b \end{vmatrix}$, substracting 1st row from 3rd.

$= (1/6)\, ab \begin{vmatrix} \alpha & \beta & \gamma \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = (1/6)\, ab \,.\, d\sin\theta$, from (iv) Hence proved.

Ex. 7. A, B, C, D are coplanar and A', B', C', D' are their projections on any plane, prove that

$$\textbf{vol. } AB'C'D' = -\textbf{vol. } A'BCD.$$

Sol. Let the coordinates of A, B, C and D be (x_1, y_1, z_1); (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) respectively.

Then the coordinates of their projection A', B', C', D' on yz-plane (say) are $(0, y_1, z_1)$ $(0, y_2, z_2)$, $(0, y_3, z_3)$ and $(0, y_4, z_4)$

$$\therefore \text{ vol. } AB'C'D' = (1/6) \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ 0 & y_2 & z_2 & 1 \\ 0 & y_3 & z_3 & 1 \\ 0 & y_4 & z_4 & 1 \end{vmatrix}$$

and $$\text{vol. } A'BCD = (1/6) \begin{vmatrix} 0 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

$\therefore$ vol. $AB'C'D'$ + vol. $A'BCD$

$$= (1/6) \left[\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ 0 & y_2 & z_2 & 1 \\ 0 & y_3 & z_3 & 1 \\ 0 & y_4 & z_4 & 1 \end{vmatrix} + \begin{vmatrix} 0 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \right]$$

$$= (1/6) \begin{vmatrix} x_1+0 & y_1 & z_1 & 1 \\ 0+x_2 & y_2 & z_2 & 1 \\ 0+x_3 & y_3 & z_3 & 1 \\ 0+x_4 & y_4 & z_4 & 1 \end{vmatrix}$$ **(See Author's Algebra or Matrices)**

$$= (1/6) \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \quad \text{...(i)}$$

Again if A, B, C, D are coplanar, then

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$ ...See § 3.06 on Plane.

$\therefore$ From (i), we have vol. $AB'C'D'$ + vol. $A'BCD = 0$

or vol. $AB'C'D' = -$ vol. $A'BCD$. Hence proved.

***Ex. 8. If O, A, B, C, D are any five points and p_1, p_2, p_3, p_4 are the projections of OA, OB, OC, OD on any given line, prove that p_1 vol. OBCD $- p_2$ vol. OCDA $+ p_3$ vol. ODAB $- p_4$ vol. OABC $= 0$.**

Sol. Let O be taken as origin and the given line be taken as x–axis or a line parallel to x-axis. Let the co-ordinates of A, B, C and D be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) respectively.

$\therefore$ Projection of OA on the given line $= x_1$ *i.e.* $p_1 = x_1$.

Similarly, $p_2 = x_2$, $p_3 = x_3$ and $p_4 = x_4$.

$\therefore$ p_1 vol. $OBCD - p_2$ vol. $OCDA + p_3$ vol. $ODAB - p_4$ vol. $OABC$

$$= x_1 \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} - x_2 \begin{vmatrix} x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \\ x_1 & y_1 & z_1 \end{vmatrix} + x_3 \begin{vmatrix} x_4 & y_4 & z_4 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} - x_4 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= x_1 \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} - x_2 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} + x_3 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{vmatrix} - x_4 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix},$$

passing a row over the rest of the rows in second and third det. **(Note)**

$$= \begin{vmatrix} x_1 & x_1 & y_1 & z_1 \\ x_2 & x_2 & y_2 & z_2 \\ x_3 & x_3 & y_3 & z_3 \\ x_4 & x_4 & y_4 & z_4 \end{vmatrix}$$

$= 0$, as first two columns are identical. Hence proved.

***Ex. 9. Prove that the volume of a tetrahedron of which a pair of opposite edges is formed by lengths r and r′ on the straight lines whose equations are** $\dfrac{x-a}{l}=\dfrac{y-b}{m}=\dfrac{z-c}{n}$, $\dfrac{x-a'}{l'}=\dfrac{y-b'}{m'}=\dfrac{z-c'}{n'}$ **is**

$$(1/6)\, rr' \begin{vmatrix} a-a' & b-b' & c-c' \\ l & m & n \\ l' & m' & n' \end{vmatrix}$$

Sol. If a vertex of the tetrahedron be (a, b, c) on the line

$$\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n},$$

then the other vertex on this line at a distance r from the vertex (a, b, c) is $(a+lr, b+mr, c+nr)$.

Similarly if (a', b', c') be a vertex of the tetrahedron on the other given line, then the remaining vertex of the tetrahedron at a distance r' from (a', b', c') on this line is $(a'+l'r', b'+m'r', c'+n'r')$.

$\therefore$ Required volume of the tetrahedron

$$=(1/6)\begin{vmatrix} a & b & c & 1 \\ a+lr & b+mr & c+nr & 1 \\ a' & b' & c' & 1 \\ a'+l'r' & b'+m'r' & c'+n'r' & 1 \end{vmatrix}$$

$$=(1/6)\begin{vmatrix} a & b & c & 1 \\ lr & mr & nr & 0 \\ a' & b' & c' & 1 \\ l'r' & m'r' & n'r' & 0 \end{vmatrix}$$, subtracting 1st row from 2nd and 3rd from 4th row

$$=(1/6)\, rr' \begin{vmatrix} a-a' & b-b' & c-c' & 0 \\ l & m & n & 0 \\ a' & b' & c' & 1 \\ l' & m' & n' & 0 \end{vmatrix}$$, subtracting 3rd row from 1st and taking r, r' common from 2nd and 4th rows respectively.

$$=(1/6)\, rr' \begin{vmatrix} a-a' & b-b' & c-c' \\ l & m & n \\ l' & m' & n' \end{vmatrix}$$, expanding with respect to 4th column.

Ex. 10. A point moves so that three mutually perpendicular lines PA, PB, PC may be drawn cutting the axes OX, OY, OZ at A, B, C and the volume of the tetrahedron OABC is constant and equal to (1/6) k^3. Prove that P lies on the surface $(x^2+y^2+z^2)^3=8k^3\,xyz$.

Sol. Let the coordinates of P be (x_1, y_1, z_1) and those of A, B, and C on the coordinate axes be $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

$\therefore$ The direction ratios of the lines PA, PB and PC are

$x_1-a,\ y_1,\ z_1$; $x_1,\ y_1-b,\ z_1$ and $x_1,\ y_1,\ z_1-c$ respectively.

Since these lines PA, PB and PC are mutually perpendicular so from the formula "$a_1a_2 + b_1b_2 + c_1c_2$" $= 0$ we have

$$(x_1 - a) \cdot x_1 + y_1 \cdot (y_1 - b) + z_1 \cdot z_1 = 0$$

$$x_1 \cdot x_1 + (y_1 - b) \cdot y_1 + z_1 \cdot (z_1 - c) = 0$$

and $$x_1 \cdot (x_1 - a) + y_1 \cdot y_1 + (z_1 - c) \cdot z_1 = 0$$

or $$x_1^2 + y_1^2 \quad z_1^2 = ax_1 + by_1; \qquad \text{...(i)}$$

$$x_1^2 + y_1^2 + z_1^2 = by_1 + cz_1 \qquad \text{...(ii)}$$

and $$x_1^2 + y_1^2 + z_1^2 = cz_1 + ax_1. \qquad \text{...(iii)}$$

Also volume of tetrahedron $OABC = (1/6)\, k^3$ (given)

or $$(1/6) \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & z \end{vmatrix} = (1/6)\, k^3 \text{ or } abc = k^3 \qquad \text{...(iv)}$$

Now to get the locus of $P(x_1, y_1, z_1)$ we are to eliminate a, b, c from (i), (ii) and (iii).

Adding (i), (ii) and (iii) we get $3\,(x_1^2 + y_1^2 + z_1^2) = 2\,(ax_1 + by_1 + cz_1)$ ...(v)

Subtracting twice (i) from (v) we get $x_1^2 + y_1^2 + z_1^2 = 2cz_1$. ...(vi)

Similarly from (ii) and (v) we get $x_1^2 + y_1^2 + z_1^2 = 2ax_1$ ...(vii)

and from (iii) and (v) we get $x_1^2 + y_1^2 + z_1^2 = 2by_1$ ...(viii)

Multiplying (vi), (vii) and (viii) we get

$$(x_1^2 + y_1^2 + z_1^2)^3 = 8abc\, x_1 y_1 z_1$$

or $$(x_1^2 + y_1^2 + z_1^2)^3 = 8k^3\, x_1 y_1 z_1, \text{ from (iv).}$$

$\therefore$ The required locus of $P(x_1, y_1, z_1)$ is

$$(x^2 + y^2 + z^2)^3 = 8\, k^3\, xyz$$ Hence proved.

Ex. 11. If the volume of the tetrahedron whose vertices are (a, 1, 2), (3, 0, 1) (4, 3, 6), (2, 3, 2) is 6, find the value of 'a'.

Sol. If V be the volume of the given tetrahedron, then

$$V = (1/6) \begin{vmatrix} a & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ 4 & 3 & 6 & 1 \\ 2 & 3 & 2 & 1 \end{vmatrix}$$, see § 5.01 Pages 1–2 of this chapter

$$= (1/6) \begin{vmatrix} a-2 & -2 & 0 & 0 \\ 1 & -3 & -1 & 0 \\ 2 & 0 & 4 & 0 \\ 2 & 3 & 2 & 1 \end{vmatrix}$$, subtracting R_4 from R_1, R_2, R_3

$$= (1/6) \begin{vmatrix} a-2 & -2 & 0 \\ 1 & -3 & -1 \\ 2 & 0 & 4 \end{vmatrix}$$, expanding w.r. to C_4

$$= (1/6)\begin{vmatrix} a-2 & -2 & 0 \\ 1 & -3 & -1 \\ 2 & -12 & 0 \end{vmatrix}, \text{ replacing } R_4 \text{ by } R_4 + 4R_3$$

$$= -(1/6)\begin{vmatrix} a-2 & -2 \\ 6 & -12 \end{vmatrix}, \text{ expanding w.r. to } C_3$$

$$= (1/6)[12(a-2) - 6.2] = (1/6)[12a - 36] = 2a - 6$$

or $\quad V = 2a - 6.$

But $\quad V = 6$ (given)

$\therefore\ 6 = 2a - 6 \quad$ or $\quad 2a = 12 \quad$ or $\quad a = 6.$ **Ans.**

EXERCISE ON CHAPTER V

Ex. 1. Find the volume of the tetrahedron whose vertices are (0, 0, 0), (1, 2, 3) (0, 0, 1) and (0, 0, 2).

(Hint : Use § 5.01 Page 1 of this chapter) **Ans.** $\frac{1}{2}$

CHAPTER VI

Skew Lines and Change of Axes

SKEW LINES

**§ 6.01 Equations of Two Skew Lines.

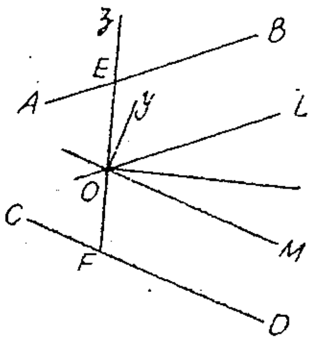
To show that by proper choice of axes equations of two skew lines can be written in a simple form as

$$y = x \tan \alpha,\ z = c \quad \text{and} \quad y = -x \tan \alpha,\ z = -c.$$

Let AB and CD be the skew lines and EF be the shortest distance between them, where $EF = 2c$ (say).

Let O be the mid-point of EF and through O draw OL and OM parallel to AB and CD respectively.

Choose O as origin and the internal and external bisectors of the angle LOM be taken as x and y axes respectively. Let OE be taken as z-axis. If $\angle LOM = 2\alpha$, then the equations of OL and OM are $y = x \tan \alpha$, $z = 0$; $y = -x \tan \alpha$, $z = 0$.



(Fig. 2)

Now AB and CD are lines parallel to OL and OM at distance c above and below xy-plane respectively.

$\therefore$ The equations of AB and CD are given by

$$y = x \tan \alpha,\ z = c \quad \text{and} \quad y = -x \tan \alpha,\ z = -c.$$

*§ 6.02 Locus of a line intersecting three lines.

To find the locus of a line which intersects three lines.

$$u_1 = 0 = v_1 \qquad \text{...(i)}$$

$$u_2 = 0 = v_2 \quad \text{...(ii)} \quad \text{and} \quad u_3 = 0 = v_3 \qquad \text{...(iii)}$$

Equations of any line intersecting the lines (i) and (ii)

[See § 4.12 Case II on Straight Lines]

are
$$u_1 - \lambda_1 v_1 = 0,\ u_2 - \lambda_2 v_2 = 0 \qquad \text{...(iv)}$$

If the lines (iv) intersects the line (iii) also then at the point of intersection, the equations (iii) and (iv) have common values of x, y, z.

$\therefore$ Eliminating x, y, z from (iii) and (iv) we get a relation in λ_1, λ_2 which can be written as
$$f(\lambda_1, \lambda_2) = 0. \qquad \text{...(v)}$$

The required locus is obtained by eliminating λ_1, λ_2 from (iv) and (v) and is
$$f\left(\frac{u_1}{v_1}, \frac{u_2}{v_2}\right) = 0$$

Note : If we are given two lines and a curve (instead of third line), the method of procedure is the same.

Solved Examples on Skew Lines.

Ex. 1. A point moves so that the line joining the feet of the perpendiculars from it to two given straight lines subtends a right angle at mid-point of their S. D. Prove its locus is hyperbolic cylinder.

Sol. Let O the mid-point of the S. D. between given lines AB and CD, be taken as origin and let the equations of AB and CD, (see Fig. 2 P. 16) be

$$\frac{x}{1}=\frac{y}{m}=\frac{z-c}{0}=r \quad ...(i) \quad \text{and} \quad \frac{x}{1}=\frac{y}{-m}=\frac{z+c}{0}=r \quad ...(ii) \quad \textbf{(Note)}$$

respectively.

Let the moving point P be (x_1, y_1, z_1) and feet of perpendicular from P to AB and CD be Q and Q'.

Then from (i) and (ii) Q is (r, mr, c) and Q' is $(r', mr', -c)$.

$\therefore$ The direction ratios of PQ and PQ' are respectively

$$x_1-r, y_1-mr, z_1-c \text{ and } x_1-r', y_1-mr', z_1+c.$$

Now according to the problem PQ is perpendicular to AB and PQ' is perpendicular to CD, so we have

$$1.(x_1-r)+m(y_1-mr)+0.(z_1-c)=0$$

and $$1.(x_1-r')+m(y_1+mr')+0(z_1+c)=0$$

i.e. $r=(x_1+my_1)/(1+m^2)$ and $r'=(x_1-my_1)/(1+m^2)$ ...(iii)

Also QQ' subtends a right angle at O *i.e.* OQ is perpendicular to OQ'.

$\therefore$ $r.r'+mr(-mr')+c(-c)=0$, since the direction ratios of OQ are r, mr, c etc.

or $$rr'(1-m^2)=c^2$$

or $$\left(\frac{x_1+my_1}{1+m^2}\right)\left(\frac{x_1-my_1}{1+m^2}\right)(1-m^2)=c^2, \text{ from (iii)}$$

or $$x_1^2-m^2y_1^2=[c^2(1+m^2)^2/(1-m^2)]$$

$\therefore$ The required locus of $P(x_1, y_1, z_1)$ is

$x^2-m^2y^2=[c^2(1+m^2)^2/(1-m^2)]$, which represents a hyperbolic cylinder.

Ex. 2. In Example 1 above, if QQ′ is equal to the distance of P from the origin O, then prove that P lies on the surface

$$\mathbf{(1-m^2)^2(x^2+y^2)=(1+m^2)^2(4c^2-z^2)}$$

Sol. As in Ex. 1 above we can prove that

Q and Q' are (r, mr, c) and $(r', -mr', -c)$,

where $$r=\frac{(x_1+my_1)}{(1+m^2)},\ r'=\frac{x_1-my_1}{(1+m^2)} \text{ and } P \text{ is } (x_1, y_1, z_1).$$

Now $(QQ')^2=(r-r')^2+(mr+mr')^2+(c+c)^2$

$$=r^2(1+m^2)+r'^2(1+m^2)-2rr'(1-m^2)+4c^2$$

$$= \frac{(x_1 + my_1)^2}{(1+m^2)} + \frac{(x_1 - my_1)^2}{(1+m^2)} - \frac{2(x_1^2 - y_1^2 m^2)(1-m^2)}{(1+m^2)^2} + 4c^2$$

or $\quad (QQ')^2 (1+m^2)^2 = (1+m^2)[(x_1+my_1)^2 + (x_1 - my_1)^2]$
$$- 2(1-m^2)(x_1^2 - m^2 y_1^2) + 4c^2(1+m^2)$$

or $\quad (QQ')^2 (1+m^2)^2 = (1+m^2)[2x_1^2 + 2m^2 y_1^2]$
$$- 2(1-m^2)(x_1^2 - m^2 y_1^2) + 4c^2(1+m^2)^2$$
$$= 4m^2 x_1^2 + 4m^2 y_1^2 + 4c^2(1+m^2)^2 \quad \text{...(iv)}$$

Also $OP^2 = x_1^2 + y_1^2 + z_1^2 = (QQ')^2$ (given)

$\therefore$ From (iv) above we have

$$(x_1^2 + y_1^2 + z_1^2)(1+m^2)^2 = 4m^2 x_1^2 + 4m^2 y_1^2 + 4c^2(1+m^2)^2$$

or $$(x_1^2 + y_1^2)[(1+m^2)^2 - 4m^2] = (1+m^2)^2(4c^2 - z_1^2)$$

or $$(x_1^2 + y_1^2)(1-m^2)^2 = (1+m^2)^2(4c^2 - z_1^2)$$

$\therefore$ The required locus of P is

$$(x^2 + y^2)(1-m^2)^2 = (1+m^2)^2(4c^2 - z^2)$$

Ex. 3. Find the locus of the mid-points of the line whose extremities are on two given lines and which are parallel to a given plane.

Sol. Let the equations of the two lines be taken as

$$\frac{x}{1} = \frac{y}{m} = \frac{z-c}{0} = r \quad \text{...(i)} \qquad \text{and} \qquad \frac{x}{1} = \frac{y}{-m} = \frac{z+c}{0} = r' \quad \text{...(ii)}$$

The coordinates of any two points on these are

$$A(r, mr, c) \quad \text{and} \quad B(r', -mr', -c) \quad \text{...(iii)}$$

$\therefore$ The directon ratios of the variable line AB are

$$r - r',\ mr + mr',\ c + c \quad i.e. \quad r - r',\ m(r+r'),\ 2c \quad \text{...(iv)}$$

Let this line AB be parallel to a given plane

$$Ax + By + Cz + D = 0 \quad \text{...(v)}$$

$\therefore$ From (iv) and (v) we get

$$A(r-r') + Bm(r+r') + C.\,2c = 0 \quad \text{...(vi)}$$

Let $P(x_1, y_1, z_1)$ be the mid-point of AB, then from (iii) we get

$$x_1 = \frac{1}{2}(r+r');\ y_1 = \frac{1}{2}(mr - mr');\ z = \frac{1}{2}(c-c) = 0 \quad \text{...(vii)}$$

Eliminating r and r' from (vi) with the help of (vii) we get

$$A[2y_1/m] + Bm[2x_1] + 2c\,C = 0;\ z_1 = 0 \quad \textbf{(Note)}$$

$\therefore$ The required locus of $P(x_1, y_1, z_1)$ is

$$Bm^2 x + Ay + mcC = 0,\ z = 0$$

which evidently, represents, a line in the plane $z = 0$ (*i.e.* xy-plane).

Ex. 4. A line of constant length has its extremities on two fixed straight lines. Prove that the locus of its mid-point is an ellipse whose axes are equally inclined to the lines.

Sol. If $2l$ be the constant length of the variable line AB then as in Ex. 3. above we can have

$$(2l)^2=(AB)^2=(r-r')^2+(mr+mr')^2+(c+c)^2$$

or $$4l=(r-r')^2+m^2(r+r')^2+4c^2=(2y_1/m)^2+m^2(2x_1)^2+4c^2$$

from result (vii) of last example

or $$l^2=(y_1^2/m^2)+m^2x_1^2+c^2.\ \text{Also } z_1=0\ \text{...See (vii) of last Ex.}$$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ is $m^2x^2+(y^2/m^2)=l^2-c^2,\ z=0$ which represents an ellipse on the xy-plane whose axes lie along the x and y-axes which by definition, are the bisectors of the angles between the given lines.

(see § 6.01 Page 16)

Ex. 5. AB is the S. D. between two given lines and A′, B′ are variable points on them, such that the volume of the tetrahedron ABA′B′ is constant. Prove that the locus of the mid-point of A′B′ is a hyperbola whose asymptotes are parallel to lines.

Sol. Let the given lines be

$$\frac{x}{1}=\frac{y}{m}=\frac{z-c}{0}=r \quad \text{...(i)} \qquad \text{and} \qquad \frac{x}{1}=\frac{y}{-m}=\frac{z+c}{0}=r' \quad \text{...(ii)}$$

Let $AB=2c$, so that A is $(0, 0, c)$ and B $(0, 0, -c)$

Also any point A' on (i) is (r, mr, c) and B' on (ii) is $(r', -mr', -c)$

$\therefore$ Volume of the tetrahedron $ABA'B'$

$$=(1/6)\begin{vmatrix} 0 & 0 & c & 1 \\ 0 & 0 & -c & 1 \\ r & mr & c & 1 \\ r' & -mr' & -c & 1 \end{vmatrix}=(1/6)\begin{vmatrix} 0 & 0 & c & 1 \\ 0 & 0 & -2c & 0 \\ r & mr & 0 & 0 \\ r' & -mr' & -2c & 0 \end{vmatrix},$$

subtracting 1st row from the rest.

$$=-(1/6)\begin{vmatrix} 0 & 0 & -2c \\ r & mr & 0 \\ r' & -mr' & -2c \end{vmatrix}, \text{expanding with respect last column.}$$

$=(1/6).2c.(-2mrr')=-(2/3)\,mcrr'=\lambda$ (say), a constant (given)

$\therefore\ rr'=-(3\lambda)/(2mc).$ ...(iii)

Now let $P(x_1, y_1, z_1)$ be the mid-point of $A'B'$, then

$$x_1=\tfrac{1}{2}(r+r');\ y_1=\tfrac{1}{2}(mr-mr');\ z_1=\tfrac{1}{2}(c-c)$$

or $$r+r'=2x_1,\ r-r'=2y_1/m;\ z_1=0 \quad \text{...(iv)}$$

Now the required locus of P is obtained by eliminating r and r' from (iii) and (iv) and the procedure for the same is as follows :

We know $4rr'=(r+r')^2-(r-r')^2$

$$\therefore\ 4\left[\frac{-3\lambda}{2mc}\right]=(2x_1)^2-\left(\frac{2y_1}{m}\right)^2, \text{ from (iii) and (iv).}$$

Also $$z_1 = 0$$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ is $\left(\dfrac{y}{m}\right)^2 - x^2 = \dfrac{3\lambda}{2mc},\ z = 0$

or $$y^2 - m^2x^2 = (3m\lambda)/(2c),\ z = 0.$$

These equations represent to hyperbola on the xy-plane and whose asymptotes are parallel to the lines $y = \pm mx,\ z = 0$ **(Note)**

i.e. parallel to the given lines (i) and (ii) whose equations can be written as $y = \pm mx,\ z = \pm c$.

Ex. 6. A, B are variable points on two given non-intersecting lines and AB is of constant length 2k. Find the surface generated by AB.

Sol. Let the given lines be

$$\frac{x}{1} = \frac{y}{m} = \frac{z-c}{0} = r;\ \frac{x}{1} = \frac{y}{-m} = \frac{z+c}{0} = r'$$

Then the coordinates of A and B are (r, mr, c) and $(r', -mr', -c)$ respectively. Also $AB = 2k$ (given).

$$\therefore\ (2k)^2 = (AB)^2 = (r - r')^2 + (mr + mr')^2 + (c + c)^2$$

or $$(r - r')^2 + m^2(r + r')^2 + 4c^2 = 4k^2 \quad \text{...(i)}$$

Also the direction ratios of AB are $r - r',\ m(r + r'),\ c + c$.

$\therefore$ The equations of the line AB are

$$\frac{x-r}{r-r'} = \frac{y-mr}{m(r+r')} = \frac{z-c}{2c} \quad \text{...(iii) (Note)}$$

Now the required surface generated by AB is obtained by eliminating r and r' from (i) and (ii).

From (ii) we have

$$\frac{z-c}{2c} = \frac{(y-mr) + m(x-r)}{m(r+r') + m(r-r')} = \frac{(y-mr) - m(x-r)}{m(r+r') - m(r-r')} \quad \textbf{(Note)}$$

or $$\frac{z-c}{2c} = \frac{y+mx-2mr}{2mr} = \frac{y-mx}{2mr'},$$

whence we have $\dfrac{z-c}{2c} = \dfrac{y+mx}{2mr} - 1,\ \dfrac{z-c}{2c} = \dfrac{y-mx}{2mr'}$

or $$\frac{z+c}{c} = \frac{y+mx}{mr},\ \frac{z-c}{c} = \frac{y-mc}{mr'} \quad \textbf{(Note)}$$

or $$r = \frac{c(y+mx)}{m(z+c)};\ r' = \frac{c(y-mx)}{m(z-c)}.$$

Substituting these values of r and r' in (i) and simplifying we get the required surface generated by AB as

$$c^2(mzx - cy)^2 + c^2m^2(yz - mcx)^2 = m^2(z^2 - c^2)^2(k^2 - c^2). \quad \textbf{Ans.}$$

Ex. 7. A and B are variable points on two given non-interseting lines CD and EF. Find the locus of a point P such that PA, PB are perpendicular to one another and perpendicular to CD and EF respectively.

Sol. As in last example if we take the equations of CD and EF as

$$\frac{x}{1}=\frac{y}{m}=\frac{z-c}{0} \quad \text{and} \quad \frac{x}{1}=\frac{y}{-m}=\frac{z+c}{0}$$

Then the coordinates of A and B are $(r, mr\, c)$ and $(r', -mr', -c)$ respectively. Let P be (x_1, y_1, z_1).

Then the direction ratios of PA and PB are

$x_1-r,\ y_1-mr,\ z_1-c$ and $x_1-r', y_1+mr',\ z_1+c$ respectively.

If PA is perpendicular to PB, then

$$(x_1-r)(x_1-r')+(y_1+mr)(y_1+mr')+(z_1-c)(z_1+c)=0 \qquad \text{...(i)}$$

If PA is perpendicular to CD and PB is perpendicular to EF

then $$(x_1-r).1+(y_1-mr).m+(z_1-c).0=0 \qquad \text{...(ii)}$$

and $$(x_1-r').1+(y_1+mr').(-m)+(z_1+c).0=0 \qquad \text{...(iii)}$$

The required locus of P can b obtained by eliminating r and r' from (i), (ii) and (iii).

From (ii) we get $\quad r=(x_1+my_1)/(1+m^2)$

From (iii) we get $\quad r'=(x_1-my_1)/(1+m^2)$

$$\therefore\ x_1-r=x_1-\frac{(x_1+my_1)}{(1+m^2)}=\frac{m\,(mx_1-y_1)}{(1+m^2)};$$

$$x_1-r'=x_1-\frac{(x_1-my_1)}{(1+m^2)}=\frac{m\,(mx_1-y_1)}{(1+m^2)};$$

$$y_1-mr=y_1-\frac{m\,(x_1+my_1)}{1+m^2}=\frac{y_1-mx_1}{1+m^2};$$

and $$y+mr'=y_1+\frac{m\,(x_1+my_1)}{1+m^2}=\frac{y_1+mx_1}{1+m^2}.$$

Substituting these values in (i), we get

$$\frac{m^2\,(m^2x_1^2-y_1^2)}{(1+m^2)^2}+\frac{y_1^2-m^2x_1^2}{(1+m^2)^2}+(z_1^2-c^2)=0$$

or $$(1-m^2)(y_1^2-m^2x_1^2)+(z_1^2-c^2)(1+m^2)^2=0$$

$\therefore$ The locus of $P\,(x_1, y_1, z_1)$ is

$$(1-m^2)(y^2-m^2x^2)+(z^2-c^2)(1+m^2)^2=0. \qquad \textbf{Ans.}$$

Ex. 8 (a). Find the locus of a point which moves so that the ratio of its distance from two given lines is constant.

Sol. Let the variable point be $P\,(x_1, y_1, z_1)$ and the equation of the lines

be $$\frac{x}{1}=\frac{y}{m}=\frac{z-c}{0} \quad \text{and} \quad \frac{x}{1}=\frac{y}{-m}=\frac{z+c}{0}.$$

Let p_1 and p_2 be the lengths of the perpendiculars from P to these lines, then with help of § 4.14 in ch. IV we have

$$p_1^2 = \frac{1}{1+m^2}\left[\begin{vmatrix} x_1 & y_1 \\ 1 & m \end{vmatrix}^2 + \begin{vmatrix} y_1 & z_1 - c \\ m & 0 \end{vmatrix}^2 + \begin{vmatrix} z_1 - c & x_1 \\ 0 & 1 \end{vmatrix}^2\right]$$

or $\quad p_1^2 = [1/(1+m^2)^2]\,[(mx_1 - y_1)^2] + m^2 (z_1 - c)^2 + (z_1 - c)^2]$ ...(i)

Similarly $p_2^2 = \dfrac{1}{1+m^2}\left[(-mx_1 - y_1)^2 + m^2 (z_1 - c)^2 + (z_1 + c)^2\right],$ ...(ii)

substituting $-m$ for m and $-c$ for c in (i).

Also $p_1/p_2 = \text{constant} = \lambda$, say

Then $p_1^2 = \lambda^2 p_2^2$

or $\quad (mx_1 - y_1)^2 + m^2 (z_1 - c)^2 + (z_1 - c)^2 = \lambda\,[(-mx_1 - y_1)^2 + m^2 (z_1 + c)^2 + (z_1 + c)^2]$

or $\quad m^2 x_1^2 (1-\lambda^2) + y_1^2 (1-\lambda^2) - 2m (1+\lambda^2) x_1 y_1 + (1-\lambda^2)(1+m^2) z_1^2 - 2c (m^2+1)(1+\lambda^2) z_1 + (1-\lambda^2) c^2 (1+m^2) = 0$

$\therefore$ **The required locus of $P(x_1, y_1, z_1)$ is**

$$m^2x^2 + y^2 + (1+m^2) z^2 + c^2 (1+m^2) - 2m \frac{(1+\lambda^2)}{(1-\lambda^2)} xy - \frac{2c (m^2+1)(1+\lambda^2) z}{(1-\lambda^2)} = 0$$

***Ex. 9. Show that the locus of lines which meet the lines**

$$\frac{x+a}{0} = \frac{y}{\sin\alpha} = \frac{z}{-\cos\alpha};\quad \frac{x-a}{0} = \frac{y}{\sin\alpha} = \frac{z}{\cos\alpha}$$

at the same angle is xy $\cos\alpha$ – az $\sin\alpha$) (zx $\sin\alpha$ – ay $\cos\alpha$) = 0

(Kanpur 96)

Sol. Let a line meet the given lines in the points

$(-a, r\sin\alpha, -r\cos\alpha)$ and $(a, r'\sin\alpha, r'\cos\alpha)$ **(Note)**

Then the direction ratios of this line are

$$a+a,\ r'\sin\alpha - r\sin\alpha,\ r'\cos\alpha + r\cos\alpha$$

$\therefore$ The equations of this line are

$$\frac{x+a}{2a} = \frac{y - r\sin\alpha}{(r'-r)\sin\alpha} = \frac{z + r\cos\alpha}{(r+r')\cos\alpha} \qquad \text{...(i)}$$

This line meets the given line at the same angle and so on equating the cosine of the angles which this line subtends with the given line, we have

$2a.\,0 + (r'-r)\sin\alpha.\,\sin\alpha + (r'+r)\cos\alpha\,(-\cos\alpha)$

$= \pm\,[2a.0 + (r'-r)\sin\alpha\sin\alpha + (r'+r)\cos\alpha\cos\alpha]$ **(Note)**

or $\quad (r'+r)\cos^2\alpha = 0$, if + sign is taken

and $\quad (r'-r)\sin^2\alpha = 0$, if – sign is taken

i.e. $\quad r' = \pm r$

If r′ = r, then (i) becomes $\dfrac{x+a}{2a} = \dfrac{y - r\sin\alpha}{0} = \dfrac{z + r\cos\alpha}{2r\cos\alpha}$

whence $\quad y = r\sin\alpha$ and $(z + r\cos\alpha)\,2a = (2r\cos\alpha)(x+a)$

or $\quad y = r \sin \alpha \quad$ and $\quad az = xr \cos \alpha$

Eliminating r we get, $az \sin \alpha = xy \cos \alpha$

or $$xy \cos \alpha - az \sin \alpha = 0 \qquad \text{...(ii)}$$

If $\mathbf{r}' = -\mathbf{r}$, then from (i) eliminating r as above we can get

$$zx \sin \alpha - ay \cos \alpha = 0 \qquad \text{...(iii)}$$

Hence from (ii) and (iii) we get the required result.

****Ex. 10 (a). A variable line intersects the x-axis and curve $x = y, y^2 = cz$ and is parallel to the plane $x = 0$. Prove that it generates the paraboloid $xy = cz$.** *(Bundelkhand* 95)

Sol. The plane parallel to the plane $x = 0$ is $x = \lambda$. ...(i)

And the plane containing x-axis *i.e.* $y = 0, z = 0$ is $y = \mu z$ ...(ii)

$\therefore$ The line which intesects the x-axis and is parallel to the plane $x = 0$ is the line of intersection of (i) and (ii).

If this meets the curve $x = y, y^2 = cz$, then with the help of (i) and (ii) we have $\quad x = \lambda = y \quad$ and $\quad \mu = \dfrac{y}{z} = \dfrac{cy}{cz} = \dfrac{cy}{y^2}$, from $y^2 = cz$

or $$\mu = \frac{c}{y} = \frac{c}{\lambda} \quad \text{or} \quad \lambda \mu = c \qquad \text{...(iii)} \quad \textbf{(Note)}$$

This required locus is obtained by eliminating λ and μ from (i), (ii) and (iii) and is $c = \lambda \mu = x.(y/z)$ or $xy = cz$. Hence proved.

***Ex. 10 (b). Prove that the locus of a variable line which inersects the three given lines $y = mx, z = c; y = -mx, z = -c; y = z, mx = -c$ is the surface $y^2 - m^2x^2 = z^2 - c^2$** *(Kanpur* 97)

Sol. The equation of the plane through the line $y = mx, \; z = c$ is

$$(y - mx) + \lambda_1 (z - c) = 0 \qquad \text{...(i)}$$

The equation of the plane through the line $y = -mx, \; z = -c$ is

$$(y + mx) + \lambda_2 (z + c) = 0 \qquad \text{...(ii)}$$

Now any line intersecting the first two given lines is given by plane (i) and (ii). The above two planes intersect in a line and as it meets the third line $y = z, mx = -c$, so putting $mx = -c$ and $z = y$ in (i) and (ii) we get

$$(y + c) + \lambda_1 (y - c) = 0 \text{ and } (y - c) + \lambda_2 (y + c) = 0$$

or $$\left(\frac{y + c}{y - c}\right) = -\lambda_1 \quad \text{and} \quad \left(\frac{y + c}{y - c}\right) = -\frac{1}{\lambda_2} \qquad \textbf{(Note)}$$

$$\therefore -\lambda_1 = -1/\lambda_2 \quad \text{or} \quad \lambda_1 \lambda_2 = 1 \qquad \text{...(iii)}$$

Eliminating λ_1 and λ_2 between (i), (ii) and (iii) we get

$$\left(\frac{y-mx}{z-c}\right)\left(\frac{y+mx}{z+c}\right)=1 \qquad \textbf{(Note)}$$

or
$$y^2-m^2x^2=z^2-c^2$$

Hence proved.

Ex. 10 (c) Find the locus of the variable line which cuts three lines $y=b, z=-c; z=c, x=-a$ and $x=a, y=-b$.

Sol. The equation of the plane through the line $y=b, z=-c$ is

$$(y-b)+\lambda_1(z+c)=0 \qquad \text{...(i)}$$

And the equation of the plane through the line $z=c, x=-a$ is

$$(z-c)+\lambda_2(x+a)=0 \qquad \text{...(ii)}$$

Now any line intersecting the first two given lines is given by planes (i) and (ii). The above two planes intersect in a line and as it meets the third line $x=a, y=-b$ so putting $x=a$ and $y=-b$ in (i) and (ii) we get

$$(-b-b)+\lambda_1(z+c)=0 \quad \text{and} \quad (z-c)+\lambda_2(a+a)=0$$

or
$$z+c=2b/\lambda 1 \quad \text{and} \quad z-c=-2a\lambda_2$$

Subtracting we get $2c=2[(b/\lambda_1)+a\lambda_2]$

or
$$c=(b+a\lambda_1\lambda_2)/\lambda_1 \quad \text{or} \quad c\,\lambda_1=a\lambda_1\lambda_2+b \qquad \text{...(iii)}$$

Eliminating λ_1 and λ_2 between (i), (ii) and (iii) we get

$$c\left[-\left(\frac{y-b}{z+c}\right)\right]=a\left[\left(\frac{y-b}{z+c}\right)\left(\frac{z-c}{x+a}\right)\right]+b$$

or $\quad -c(y-b)(x+a)=a(y-b)(z-c)+b(x+a)(z+c)$

or $\quad ayz+byz+cxy+abc=0$, on simplifying

This is the required locus.

Ans.

***Ex. 10 (d). Prove that the straight lines which intersects the three lines $y-z=1, x=0; z-x=1\ y=0$ and $x-y=1, z=0$ lies on the surface whose equation is $x^2+y^2+z^2-2yz-2zx-2xy=1$.**

Sol. (Refer § 6.02 Page 16 of this chapter.)

The given lines are $y-z-1=0,\ x=0,$ ...(i)

$z-x-1=0, y=0$ ...(ii) and $x-y-1=0,\ z=0$ ...(iii)

Equations of any line intersecting the lines (i) and (ii) are

$$(y-z-1)-\lambda_1 x=0, (z-x-1)-\lambda_2 y=0. \qquad \text{...(iv)}$$

If the line (iv) intersects the line (iii), then we are to eliminate x, y, z from (iii) and (iv).

From (iii) we get $z=0$ and $y=x-1$.

Substracting these values in (iv) we get

$$(x-1-0-1)-\lambda_1 x=0, (0-x-1)-\lambda_2(x-1)=0$$

or $$(1-\lambda_1)x = 2, (1+\lambda_2)x = -(1-\lambda_2)$$

or $$x = 2/(1-\lambda_1), x = -(1-\lambda_2)/(1+\lambda_2).$$

Equating two values of x, we get x, y, z eliminant of (iii) and (iv) as

$$\frac{2}{1-\lambda_1} = -\left(\frac{1-\lambda_2}{1+\lambda_2}\right) \text{ , or } \quad 2(1+\lambda_2) = -(1-\lambda_1)(1-\lambda_2)$$

or $$2 + 2\lambda_2 = -(1-\lambda_2-\lambda_1+\lambda_1\lambda_2)$$

or $$\lambda_1\lambda_2 - \lambda_1 + \lambda_2 + 3 = 0 \quad \text{...(v)}$$

The required locus is obtained by eliminating λ_1, λ_2 from (iv) and (v)

From (iv) w have $\lambda_1 = \dfrac{y-z-1}{x}, \lambda_2 = \dfrac{z-x-1}{y}$.

Substuting these in (v) we get the required locus as

$$\left(\frac{y-z-1}{x}\right)\left(\frac{z-x-1}{y}\right) - \left\{\frac{y-z-1}{x}\right\} + \left\{\frac{z-x-1}{y}\right\} + 3 = 0$$

or $$(y-z-1)(z-x-1) - y(y-z-1) + x(z-x-1) + 3xy = 0$$

or $$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 1.$$ Hence proved.

***Ex. 10 (e). Find the surfarce generated by the lines which intersect the lines $y = mx, z = c; y = -mx, z = -c$ and x-axis.** *(Meerut 97)*

Sol. As in Ex. 10 (b) Page 23 and line intersecting the lines $y = mx, z = c$ and $y = -mx, \ z = -c$ is given by the planes

$$(y-mx) + \lambda_1(z-c) = 0 \quad \text{...(i)}$$

and $$(y+mx) + \lambda_2(z+c) = 0 \quad \text{...(ii)}$$

It meets x–axis *i.e.* $y = 0 = z$, where

$$(0-mx) + \lambda_1(0-c) = 0 \quad \text{and} \quad (0+mx) + \lambda_2(0+c) = 0$$

$$\therefore \lambda_2 = \lambda_1 \quad \text{...(iii)}$$

Eliminating λ_1, λ_2 between (i), (ii) and (iii) we get

$$-\left\{\frac{y-mx}{z-c}\right\} = -\left\{\frac{y+mx}{z+c}\right\}$$

or $$(y-mx)(z+c) = (y+mx)(z-c) \quad \text{or} \quad cy = mzx.$$ **Ans.**

Ex. 11. Find the locus of a straight line that intersects two given lines and makes a right angle with one of them.

Sol. Let the given lines be $\dfrac{x}{1} = \dfrac{y}{m} = \dfrac{z-c}{0}$ and $\dfrac{x}{1} = \dfrac{y}{-m} = \dfrac{z+c}{0}$

or $$y = mx, \ z = c \quad \text{and} \quad y = -mx, \ z = -c.$$

Any line that intersects these lines are given by the planes

$$(y-mx) + k_1(z-c) = 0 \quad \text{and} \quad (y+mx) + k_2(z+c) = 0 \quad \text{...(i)}$$

If λ, μ, ν are the direction ratios of this line, then

$$-m\lambda+\mu+k_1\nu=0 \quad \text{and} \quad m\lambda+\mu+k_2\nu=0$$

Solving these, we get $\dfrac{\lambda}{k_2-k_1}=\dfrac{\mu}{m(k_1+k_2)}=\dfrac{\nu}{-m-m}$

$\therefore$ The direction ratios of the line intersecting the given lines are

$$(k_2-k_1),\ m(k_1+k_2),\ -2m.$$

If this line is perpendicular to the first given line then

$$(k_2-k_1)\cdot 1+m(k_1+k_2)\,m+(-2m)\cdot 0=0$$

or $$(k_2-k_1)+m^2(k_1+k_2)=0. \quad \text{...(ii)}$$

Also from (ii), $k_1=\dfrac{mx-y}{z-c}$ and $k_2=-\dfrac{mx+y}{z+c}$

$\therefore$ From (ii), the required locus is

$$\left[\left\{-\frac{mx+y}{z+c}\right\}-\left\{\frac{mx-y}{z-c}\right\}\right]+m^2\left[\frac{mx-y}{z-c}-\frac{mx+y}{z+c}\right]=0$$

or $$(mx+y)(z-c)+(mx-y)(z+c)=m^2\left[(mx-y)(z+c)-(mx+y)(z-c)\right]$$

or $$(mxz-cy)=m^2(-yz+mcx).$$ **Ans.**

Ex. 12. Find the surface generated by a line which intersects the lines $y=a=z$ and $x+3z=a=y+z$ and is parallel to the plane $x+y=0$.

Sol. Any line that intersects the given line is given by the planes

$$(y-a)+k_1(z-a)=0 \quad \text{and} \quad (x+3z-a)+k_2(y+z-a)=0 \quad \text{...(i)}$$

If λ, μ, ν are direction ratios of this line, then

$$\mu+k_1\nu=0 \quad \text{and} \quad \lambda+k_2\mu+(3+k_2)\nu=0$$

Solving these we get $\dfrac{\lambda}{(3+k_2)-k_1k_2}=\dfrac{\mu}{k_1}=\dfrac{\nu}{-1}$

$\therefore$ The direction ratios of the line intersecting the given lines are

$$3+k_2-k_2k_1,\ k_1,\ -1$$

If this line is parallel to the plane $x+y=0$, then this line is perpendicular to the normal to this plane

We have $$(3+k_2-k_1k_2).1+k_1.1=0 \quad \textbf{(Note)}$$

or $$3+k_1+k_2-k_1k_2=0 \quad \text{...(ii)}$$

Also from (i), we get $k_1=\dfrac{a-y}{z-a},\ k_2=\dfrac{a-x-3z}{y+z-a}$

$\therefore$ From (ii), the required locus is

$$3+\frac{a-y}{z-a}+\frac{a-x-3z}{y+z-a}-\left\{\frac{a-y}{z-a}\right\}\left\{\frac{a-x-3z}{y+z-a}\right\}=0$$

or $$3(z-a)(y+z-a)+(a-y)(y+z-a)+(a-x-3z)(z-a)$$
$$-(a-y)(a-x-3z)=0$$

or $$(y+z-a)[3z-3a+a-y]=(a-x-3z)[a-y-z+a]$$

or $$y^2+yz+xy+xz-2az-2ax=0, \text{ on simplifying}$$

or $$y^2+yz+xy+xz=2a(x+z).$$ **Ans.**

Ex. 13. Find the surface generated by a straight line which intersects the lines $x+y=0=z$, $x-y-z=0=x+y-2a$ and the parabola $y=0=x^2-2az$.

Sol. Given lines are $$x+y=0=z, \quad \text{...(i)}$$
$$x-y-z=0=x+y-2a \quad \text{...(ii)}$$

and the parabola is $$y=0,\ x^2=2az \quad \text{...(iii)}$$

Also line intersecting (i) and (ii) is
$$(x+y)+k_1 z=0,\ (x-y-z)+k_2(x+y-2a)=0 \quad \text{...(iv)}$$

If it meets the parabola (iii), we have to eliminate x, y, z from (iii) and (iv).

Putting $y=0$ from (iii), in (iv), we get
$$x+k_1z=0,\ (x-z)+k_2(x-2a)=0$$

or $$x+k_1z=0 \quad \text{and} \quad (1+k_2)x-z-2ak_2=0$$

Solving these simultaneously, we get
$$\frac{x}{-2ak_1k_2}=\frac{z}{2ak_2}=\frac{1}{-1-k_1(1+k_2)}$$

or $$x=\frac{2ak_1k_2}{1+k_1+k_1k_2},\ z=\frac{-2ak_2}{1+k_1+k_1k_2}$$

Substituting these values of x and z in (iii) we get
$$\left(\frac{2ak_1k_2}{1+k_1+k_1k_2}\right)^2=2a\left(\frac{-2ak_2}{1+k_1+k_1k_2}\right)$$

or $$k_1^2k_2+(1+k_1+k_2k_2)=0, \text{ on simplifying}$$

or $$k_1^2k_2+k_1k_2+k_1+1=0 \quad \text{...(v)}$$

To find the required locus we are to eliminate k_1 and k_2 from (iv) and (v)

Substituting $k_1=-\left(\frac{x+y}{z}\right)$, $k_2=\frac{y+z-x}{x+y-2a}$ in (v) we get the required locus as
$$\left(\frac{x+y}{z}\right)^2\left(\frac{y+z-x}{x+y-2a}\right)-\left(\frac{x+y}{z}\right)\left(\frac{y+z-x}{x+y-2a}\right)-\left(\frac{x+y}{z}\right)+1=0$$

or $$(x+y)^2(y+z-x)-z(x+y)(y+z-x)$$
$$-z(x+y)(x+y-2a)+z^2(x+y-2a)=0$$

or $(x+y)(y+z-x)[(x+y)-z]-z(x+y-2a)[(x+y)-z]=0$

or $(x+y-z)[xy+xz-x^2+y^2+yz-xy-zx-zy+2az]=0$

or $(x+y-z)(2az-x^2+y^2)=0.$ **Ans.**

Ex. 14. Find the surface generated by a straight line which meets the two lines y = mx, z = c and y = – mx, z = – c at the same angle.

Sol. The given lines are $y-mx=0, z-c=0$ ...(i)

$$y+mx=0, z+c=0 \quad \text{...(ii)}$$

Any line intersecting (i) and (ii) is

$$(y-mx)-k_1(z-c)=0, (y+mx)-k_2(z+c)=0 \quad \text{...(iii)}$$

Now we are to find the d-cosines of this line. For this omitting the constant terms we have $mx-y+k_1z=0, mx+y-k_2z=0.$

Solving these simultaneously, we get

$$\frac{x}{k_2-k_1}=\frac{y}{mk_1+mk_2}=\frac{z}{m+m}$$

$\therefore$ d-ratios of the line (iii) are $k_2-k_1, m(k_1+k_2), 2m.$ ...(iv)

Again the equations of the line (i) and (ii) can be written in the symmetric form as

$$\frac{x}{1}=\frac{y}{m}=\frac{z-c}{0} \quad \text{and} \quad \frac{x}{1}=\frac{y}{-m}=\frac{z+c}{0}$$

Since these lines make equal angles α (say) with the line (iii) whose d-ratios are given by (iv), so we have

$$\frac{1.(k_2-k_1)+m\,m(k_1+k_2)+0.(2m)}{\sqrt{(1+m^2+0)}\sqrt{[(k_2-k_1)^2+m^2(k_1+k_2)^2+(2m)^2]}}$$

$$=\frac{1.(k_2-k_1)+(-m)m(k_1+k_2)+0.(2m)}{\sqrt{(1+m^2+0)}.\sqrt{[(k_2-k_1)^2+m^2(k_1+k_2)^2+(2m)^2]}}$$

or $2m^2(k_1+k_2)=0$, on simplifying

or $k_1+k_2=0$

or $\left(\frac{y-mx}{z-c}\right)+\left(\frac{y+mx}{z+c}\right)=0$, substituting values of k_1, k_2, from (iii)

or $(y-mx)(z+c)(y+mx)(z-c)=0$

or $y[(z+c)+(z-c)]-mx[(z+c)-(z-c)]=0$

or $2yz-2cmx=0$ or $yz=cmx$, which is the required surface.

****Ex. 15. Prove that the locus of a line which meets the lines y = ± mx, z = ± c and the circle $x^2+y^2=a^2, z=0$ is**

$$c^2m^2(cy-mxz)^2+c^2(yz-cmx)^2=a^2.m^2(z^2-c^2)^2.$$

(*Garhwal* 94, 91; *Kanpur* 95)

Sol. Given lines are $y-mx=0, z-c=0;$ ...(i)

$$y + mx = 0,\ z + c = 0 \quad \text{...(ii)}$$

and the circle is $\quad x^2 + y^2 = a^2, z = 0 \quad$ ...(iii)

Any line intersecting (i) and (ii) is

$$(y - mx) + k_1 (z - c) = 0, (y + mx) + k_2 (z + c) = 0 \quad \text{...(iv)}$$

If it meets the circle (iii), then we are to eliminate k_1, k_2 from (iii) and (iv).

Putting $z = 0$ in (iv) we get $(y - mx) - k_1 c = 0, (y + mx) + k_2 c = 0$

or $$mx - y + k_1 c = 0,\ mx + y + k_2 c = 0$$

Adding and subtracting these, we get

$$x = -\frac{(k_1 + k_2)\, c}{2m},\ y = \frac{c\,(k_1 - k_2)}{2}$$

Substituting these values of x and y in (iii), we get

$$\frac{(k_1 + k_2)^2\, c^2}{4m^2} + \frac{(k_1 - k_2)^2\, c^2}{4} = a^2$$

or $$[(k_1 + k_2)^2 + m^2 (k_1 - k_2)^2]\, c^2 = 4a^2 m^2. \quad \text{...(v)}$$

or $$\left[\left\{\left(\frac{mx - y}{z - c}\right) + \left(-\frac{mx + y}{z + c}\right)\right\}^2 + m^2 \left\{\left(\frac{mx - y}{z - c}\right) - \left(-\frac{mx + y}{z + c}\right)\right\}^2\right] c^2$$

$$= 4a^2 m^2 \text{ from (iv)}$$

Now simplify and get the result.

***Ex. 16. A straight line is drawn through a variable point on the ellipse $(x^2/a^2) + (y^2/b^2) = 1, z = 0$ to meet two fixed lines $y = mx, z = c$ and $y = -mx, z = -c$. Find the locus of the straight line.** (*Bundelkhand* 95)

Sol. Given fixed lines are $y - mx = 0, z - c = 0$; ...(i)

$$y + mx = 0,\ z + c = 0 \quad \text{...(ii)}$$

and the ellipse is $\quad (x^2/a^2) + (y^2/b^2) = 1,\ z = 0 \quad$ ...(iii)

And line intersecting lines (i) and (ii) is

$$(y - mx) + k_1 (z - c) = 0, (y + mx) + k_2 (z + c) = 0 \quad \text{...(iv)}$$

If it meets the ellipse (iii), then we are to eliminate k_1, k_2 from (iii) and (iv).

Putting $z = 0$ in (iv), we get $y - mx - k_1 c = 0, y + mx + k_2 c = 0$

or $$mx - y + k_1 c = 0,\ mx + y + k_2 c = 0$$

Adding and subtracting these, we get

$$x = -\frac{(k_1 + k_2)\, c}{2m},\ y = \frac{(k_1 - k_2)\, c}{2}$$

Substituting these values of x and y in (iii), we get

$$\frac{(k_1+k_2)^2 c^2}{4a^2m^2}+\frac{(k_1-k_2)^2 c^2}{4b^2}=1$$

or $\quad (k_1+k_2)^2 c^2b^2+(k_1-k_2)^2 c^2 a^2 m^2=4a^2b^2m^2$

or $\quad \left\{\left(\frac{mx-y}{z-c}\right)+\left(-\frac{mx+y}{z+c}\right)\right\}^2 c^2 b^2+\left\{\frac{mx-y}{z-c}+\frac{mx+y}{z+c}\right\}^2 c^2a^2m^2$

$$=4a^2b^2m^2, \text{ from (iv)}$$

or $\quad \{(mx-y)(z+c)-(mx+y)(z-c)\}^2 c^2b^2$

$$+\{(mx-y)(z+c)+(mx+y)(z-c)\}^2 c^2a^2m^2=4a^2b^2m^2(z^2-c^2)^2$$

or $\quad (cmx-yz)^2 c^2b^2+(mxz-cy)^2 c^2a^2m^2=a^2b^2m^2(z^2-c^2)^2,$

which is the required locus. **Ans.**

Exercises on Skew Lines

*Ex. 1. Prove that by proper choice of axes, the equations of two skew lines can be put in the form $y=mx, z=c$ and $y=-mx, z=-c$

or $$\frac{x}{1}=\frac{y}{m}=\frac{z-c}{0} \quad \text{and} \quad \frac{x}{1}=\frac{y}{-m}=\frac{z+c}{0}.$$

[**Hint** : Replace tan α by m in § 6.01 Page 16 of this chapter.]

Ex. 2. Find the locus of the line which moves parallel to the zx-plane and meets the curves $xy=c^2, z=0; y^2=4cz, x=0$

Ex. 3. Find the locus of the line which moves parallel to the yz-plane and meets the curves $x^2+y^2=a^2, z=0; x^2=az, y=0.$

Ex. 4. Find the locus of a point which is equidistant from two given lines

$y=mx, z=c$ and $y=-mx, z=-c.$ **Ans.** $myx+c(1+m^2)z=0$

Ex. 5. Prove that the lines, which meet the lines $y=mx, z=c$; $y=-mx, z=-c$ and the hyperbola $xy=c^2, z=0$ generates the surface

$$(cmx-yz)(mxz-cy)+m(c^2-z^2)^2=0$$

CHANGE OF AXES

§ 6.03. *To change the origin of co-ordinates without changing the direction of the axes.*

Let (x, y, z) be the co-ordinates of a point P referred to original axes of co-ordinates OX, OY and OZ with O as origin.

Let the co-ordinates of a point O' be (α, β, γ) referred to OX, OY and OZ.

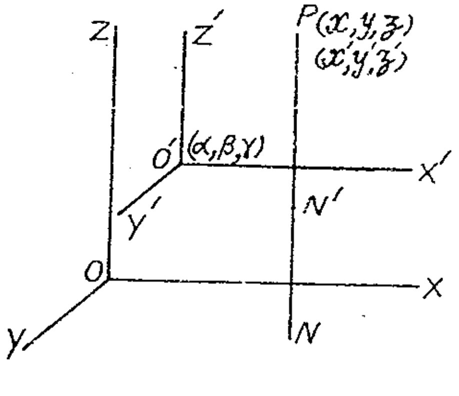
Through O' draw lines $O'X'$, $O'Y'$ and $O'Z'$ parallel to OX, OY and OZ respectively.

Let co-ordinates of P be (x', y', z') referred to these parallel axes through O'.

From P draw PN perpendicular to XOY plane meeting the plane $X'O'Y'$ in N'. Then $PN = z$ and $PN' = z'$.

Also $N'N$ = perpendicular distance between the planes XOY and $X'O'Y'$

= perpendicular distance of O' from XOY plane $= \gamma$.



(Fig. 3)

Now from the figue it is evident that $PN = PN' + N'N$

i.e. $\mathbf{z = z' + \gamma}$.

Similarly we can show that $\mathbf{x = x' + \alpha}$ and $\mathbf{y = y' + \beta}$.

*§ **6.04.** *To change the direction of co-ordinate axes without changing the origin.*

Let OX, OY, OZ and OX', OY', OZ' be two sets of co-ordinate axes through the common origin O.

Let the direction cosines of OX', OY' and OZ' be $l_1, m_1, n_1, l_2, m_2, n_2$ and l_3, m_3, n_3 respectively referred to OX, OY and OZ.

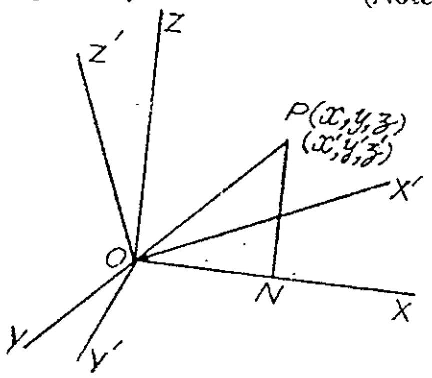
Then the direction cosines of OX, OY, OZ referred to OX', OY', OZ' are evidently l_1, l_2, l_3; $m_1\ m_2, m_3$ and n_1, n_2, n_3 respectively. **(Note)**

Let the co-ordinates of P be (x, y, z) and (x', y', z') referred to the original axes OX, OY, OZ and the new axes OX', OY', OZ' respectively.

From P draw PN perpendicular to OX.

Then $x = ON$ = Projection of OP on OX. ...(i)

Now d.c.'s of OX referred to the new axes are l_1, l_2, l_3 and the co-ordinates of P referred to the new axes are (x', y', z').



(Fig. 4)

∴ From (i), we have

$$x = l_1(x' - 0) + l_2(y' - 0) + l_3(z' - 0). \quad \text{...See § 2.09 Ch. II}$$

or

Similarly

and

$$\left.\begin{aligned} \mathbf{x} &= \mathbf{l_1x' + l_2y' + l_3z'} \\ \mathbf{y} &= \mathbf{m_1x' + m_2y' + m_3z'} \\ \mathbf{z} &= \mathbf{n_1x' + n_2y' + n_3z'} \end{aligned}\right\} \quad \text{...(A)}$$

Multiplying these relations by l_1, m_1, n_1 respectively and adding we get

$$l_1x + m_1y + n_1z = x'(\Sigma l_1^2) + y'(\Sigma l_1l_2) + z'(\Sigma l_1l_3)$$

$$= x'; \quad \Sigma l_1^2 = 1; \quad \Sigma l_1l_2 = 0 = \Sigma l_1l_3$$

or $\qquad x' = l_1x + m_1y + n_1z$

Similarly $\qquad y' = l_2x + m_2y + n_2z$ ...(B)

and $\qquad z' = l_3x + m_3y + n_3z$

The relations (A) express the old co-ordinates x, y, z in tterms of the new co-ordinates x', y', z' and the relations (B) express x', y', z', in terms x, y, z. These relations are also written conveniently with the help of the adjoining table. In this table the horizontal and vertical lines denote the direction cosines of mutually perpendicular axes.

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

(Fig. 5)

How to wirtte the relation (A) or (B) with the help of the above table ?

To get the value of y', we multiply each element of the row of y' with the coresponding elements of first row and add

i.e. $\qquad y' = l_2.x + m_2.y + n_2.z$

Similarly to get the value of x, we multiply each element of the column of x with the corresponding elements of first column and add

i.e. $\qquad x = l_1.x' + l_2.y' + l_3.z'$

***Cor.** The degree of an equation remains unchanged by transformation from one set of axes to another as x, y, z and x', y', z' are connected by inter relations given by (A) or (B).

****§ 6.05. Relation between the direction cosines of three mutually perpendicular lines.**

As in § 6.04 Pages 31-32 of this chapter let l_1, m_1, n_1 ; l_2, m_2, n_2 and l_3, m_3, n_3 be the direction cosines of any three mutually perpendicular lines OX', OY', OZ', referred to rectangular axes OX, OY, OZ.

Hence we have following two sets of relations :—

$$\left.\begin{array}{l} l_1^2 + m_1^2 + n_1^2 = 1, \\ l_2^2 + m_2^2 + n_2^2 = 1, \\ l_3^2 + m_3^2 + n_3^2 = 1, \end{array}\right\} \quad \text{...(i)} \qquad \left.\begin{array}{l} l_1l_2 + m_1m_2 + n_1n_2 = 0, \\ l_2l_3 + m_2m_3 + n_2n_3 = 0, \\ l_3l_1 + m_3m_1 + n_3n_1 = 0, \end{array}\right\} \quad \text{...(ii)}$$

Also in § 6.04 above we have seen that the direction cosines of the lines OX, OY, OZ are l_1, l_2, l_3 ; m_1, m_2, m_3 and n_1, n_2, n_3 referred to OX', OY', OZ'. Hence as above we have

$$\left.\begin{array}{l} l_1^2 + l_2^2 + l_3^2 = 1, \\ m_1^2 + m_2^2 + m_3^2 = 1, \\ n_1^2 + n_2^2 + n_3^2 = 1, \end{array}\right\} \quad \text{...(iii)} \qquad \left.\begin{array}{l} l_1m_1 + l_2m_2 + l_3m_3 = 0, \\ m_1n_1 + m_2n_2 + m_3n_3 = 0, \\ n_1l_1 + n_2l_2 + n_3l_3 = 0 \end{array}\right\} \quad \text{...(iv)}$$

$$\frac{1}{1}=\frac{a+b+c}{a_1+b_1+c_1}=\frac{A+B+C}{A_1+B_1+C_1}=\frac{\Delta}{\Delta_1}$$

$\Rightarrow\ a+b+c=a_1+b_1+c_1, A+B+C=A_1+B_1+C_1$ and $\Delta=\Delta_1$

$\Rightarrow\ a+b+c,\ A+B+C$ and Δ remain unchanged.

Solved Examples on Change of Axes.

***Ex. 1. Two systems of rectangular axes have the same origin. If a plane cuts them at distances a, b, c and a′, b′, c′ from the origin then show that $a^{-2}+b^{-2}+c^{-2}=(a')^{-2}+(b')^{-2}+(c')^{-2}$.**

Sol. Referred to the first system of axes, let the equation of the plane in the intercept form be $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. ...(i)

Replacing x by $l_1x+l_2y+l_3z$, y by $m_1x+m_2y+m_3z$ and z by $n_1x+n_2y+n_3z$ the equation of the plane referred to the second system is

$$\frac{l_1x+l_2y+l_3z}{a}+\frac{m_1x+m_2y+m_3z}{b}+\frac{n_1x+n_2y+n_3z}{c}=1 \quad ...(ii)$$

The intercept on new x-axis by this plane is a' (given), so putting $x=a'$, $y=0=z$ in (ii) we get

$$\left(\frac{l_1}{a}+\frac{m_1}{b}+\frac{n_1}{c}\right)a'=1 \quad \text{or} \quad \frac{1}{a'}=\left(\frac{l_1}{a}+\frac{m_1}{b}+\frac{n_1}{c}\right)$$

Similarly $\frac{1}{b'}=\frac{l_2}{a}+\frac{m_2}{b}+\frac{n_2}{c};\ \frac{1}{c'}=\frac{l_3}{a}+\frac{m_3}{b}+\frac{n_3}{c}$

$$\therefore\ \frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}=\left(\frac{l_1}{a}+\frac{m_1}{b}+\frac{n_1}{c}\right)^2+\left(\frac{l_2}{a}+\frac{m_2}{b}+\frac{n_2}{c}\right)^2+\left(\frac{l_3}{a}+\frac{m_3}{b}+\frac{n_3}{c}\right)^2$$

$$=\frac{1}{a^2}(l_1^2+l_2^2+l_3^2)+\ldots+\ldots+\frac{2}{ab}(l_1m_1+l_2m_2+l_3m_3)+\ldots+\ldots$$

$$=\frac{1}{a^2}(1)+\frac{1}{b^2}(1)+\frac{1}{c^2}(1)+\frac{2}{ab}(0)+\frac{2}{bc}(0)+\frac{2}{ca}(0),\ \Sigma l_1^2=1,\ \Sigma l_1m_1=0.$$

Hence $(a')^{-2}+(b')^{-2}+(c')^{-2}=a^{-2}+b^{-2}+c^{-2}$. Hence proved.

Ex. 2. If $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d$ be transformed by change of coordinates from one set of rectangular axes to another with the same origin, the expression $a+b+c$, $u^2+v^2+w^2$ are invariants.

Sol. Replacing x by $l_1x+l_2y+l_3z$ etc. as in Ex. 1 above the given expression can be transfomed as

$$\Sigma a(l_1x+l_2y+l_3z)^2+\Sigma 2f(m_1x+m_2y+m_3z)(n_1x+n_2y+n_3z)$$
$$+\Sigma 2u(l_1x+l_2y+l_3z)+d$$

$$= a' x^2 + b' y^2 + c' z^2 + 2f' yz + 2g' zx + 2h' xy + 2u' x + 2v' y + 2w' z + d' \quad \text{...(ii)}$$

Comparing coefficients of x^2, y^2, z^2, x, y, z on both sides we get

$a' = al_1^2 + bm_1^2 + cn_1^2 + 2fm_1n_1 + 2gl_1n_1 + 2hl_1m_1$

$b' = al_2^2 + bm_2^2 + cn_2^2 + 2fm_2n_2 + 2gl_2n_2 + 2hl_2m_2$

$c' = al_3^2 + bm_3^2 + cn_3^2 + 2fm_3n_3 + 2gl_3n_3 + 2hl_3m_3$

$u' = ul_1 + vm_1 + wn_1,\ v' = ul_2 + vm_2 + wn_2,\ w' = ul_3 + vm_3 + wn_3$

Now $a' + b' + c' = a\Sigma l_1^2 + b\,\Sigma m_1^2 + c\Sigma n_1^2 + 2f\Sigma m_1n_1$
$+ 2g\,\Sigma n_1l_1 + 2h\Sigma l_1m_1$

$= a(1) + b(1) + c(1) + 2f(0) + 2g(0) + 2h(0)$, from result (iii) and (iv) of § 6.05 Page 32 of this chapter.

i.e. $a' + b' + c' = a + b + c.$ Hence proved.

Also $(u')^2 + (v')^2 + (w')^2 = (ul_1 + vm_1 + wn_1)^2 + (ul_2 + vm_2 + wn_2)^2$
$+ (ul_3 + vm_3 + wn_3)^2$

$= u^2\,\Sigma l_1^2 + v^2\,\Sigma m_1^2 + w^2\,\Sigma n_1^2 + 2uv\,\Sigma l_1m_1 + 2vw\,\Sigma m_1n_1 + 2uw\,\Sigma l_1n_1$

$= u^2(1) + v^2(1) + w^2(1) + 2uv(0) + 2vw(0) + 2uw(0)$, as before

$= u^2 + v^2 + w^2.$ Hence proved.

CHAPTER VII

Sphere

§ 7.01 Definition. *A sphere is the locus of a point which moves such that its distance from a fixed point always remains constant.*

The fixed point is called the *centre* and this constant distance is called the *radius* of the sphere. (*Kanpur* 94; *Purvanchal* 96)

§ 7.02 The equation of a sphere.

(a) When centre and radius are given (Central Form) :

Let $C(a, b, c)$ be the centre and r the radius of the sphere. Then if $P(x, y, z)$ be any point of the surface of the sphere we have

$$CP = \text{radius of the sphere} = r \quad i.e. \quad CP^2 = r^2$$

and therefore $$\boxed{(\mathbf{x}-\mathbf{a})^2 + (\mathbf{y}-\mathbf{b})^2 + (\mathbf{z}-\mathbf{c})^2 = \mathbf{r}^2} \qquad ...(i)$$

For example if the centre of a sphere is (1, 2, 3) and radius in 4. Then its equation is $(x-1)^2 + (y-2)^2 + (z-3)^2 = 4^2$

Particular Case : Let origin $O(0, 0, 0)$ be the centre and r the radius of the sphere. Let $P(x, y, z)$ be any point on this sphere.

Then OP = radius of the sphere = r (given)

or $$OP^2 = r^2 \quad \text{or} \quad (x-0)^2 + (y-0)^2 + (z-0)^2 = r^2$$

or $$\boxed{\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2 = \mathbf{r}^2} \qquad ...(ii)$$

which is called the **standard form** of the equation of a sphere.

(b) General Equation of a sphere. (*Kanpur* 94)

The equation (i) above can be expanded and written as

$$x^2 + y^2 + z^2 - 2ax - 2by - 2cz - (a^2 + b^2 + c^2 - r^2) = 0$$

which is of the form

$$\boxed{\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2 + 2\mathbf{ux} + 2\mathbf{vy} + 2\mathbf{wz} + \mathbf{d} = 0} \qquad ...(iii)$$

and this is known as the **general form** of the equation of a sphere.

Centre and radius for general form.

The equation (iii) can be rewritten as

$$(x+u)^2 + (y+v)^2 + (z+w)^2 = u^2 + v^2 + w^2 - d.$$

or $$[x - (-u)^2] + [y - (-v)]^2 + [z - (-w)]^2 = u^2 + v^2 + w^2 - d.$$

Comparing this with equation (i) above, we find that the centre and radius of the sphere given by (iii) are $(-u, -v, -w)$ and $\sqrt{(u^2 + v^2 + w^2 - d)}$ respectively.

Hence we conclude that for the sphere (iii),

centre is $(-\mathbf{u}, -\mathbf{v}, -\mathbf{w})$ and radius $= \sqrt{(\mathbf{u}^2 + \mathbf{v}^2 + \mathbf{w}^2 - \mathbf{d})}$.

Method of writing the centre and radius of a sphere.

(i) Write down the coefficients of x^2, y^2 and z^2 as 1, if these are not so, by dividing the equation by the coefficient of x^2. For example if the equation of the sphere is given as $2x^2+2y^2+2z^2-10x-12y+16z+23=0$, then we should divide each term by the coefficient 2 of x^2 and write the given equation as

$$x^2+y^2+z^2-5x-6y+8z+(23/2)=0.$$

(ii) Then the coordinates of the centre of the sphere are

$[-\frac{1}{2}$ (coefficient of x), $-\frac{1}{2}$ (coeff. of y), $-(\frac{1}{2}$ coeff. of $z)]$ and the radius of the sphere $=\sqrt{[(\frac{1}{2}\text{ coeff. of }x)^2+(\frac{1}{2}\text{ coeff. of }y)^2+(\frac{1}{2}\text{ coeff. of }z)^2}$

$-\text{(constant term)}]$ **(Remember)**

Conditions for a sphere. From the general form (iii) of the equation of a sphere, we conclude that

(i) The equation of a sphere must be of second degree in x, y and z.

(ii) The coefficient of x^2, y^2 and z^2 must be equal and

(iii) Three should not be terms containing the products xy, yz and zx in this equation. **(Remember)**

Note 1. If $u^2+v^2+w^2-d<0$, then the radius of the sphere (iii) is imaginary whereas the centre is real. Such a sphere is called **pseudo-sphere or a virtual sphere.**

Note 2. The equation (iii) of the sphere contains four unknown constants u, v, w and d and therefore a sphere can be found to satisfy four conditions.

§ 7.03. Equation of a sphere through four given points.

(Four-Points form) (*Agra* 91; *Kanpur* 93)

Let the co-ordinates of the four given points A, B, C and D be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) respectively.

Let the equation of the sphere passing through these four points be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{...(i)}$$

If this sphere passes through the given points A, B, C and D, we have

$$(x_1^2+y_1^2+z_1^2)+2ux_1+2vy_1+2wz_1+d=0\,; \quad \text{...(ii)}$$

$$(x_2^2+y_2^2+z_2^2)+2ux_2+2vy_2+2wz_2+d=0\,, \quad \text{...(iii)}$$

$$(x_3^2+y_3^2+z_3^2)+2ux_3+2vy_3+2wz_3+d=0\,, \quad \text{...(iv)}$$

$$(x_4^2+y_4^2+z_4^2)+2ux_4+2vy_4+2wz_4+d=0 \quad \text{...(v)}$$

Eliminating u, v, w and d from (i), (ii), (iii), (iv) and (v), we get

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 \end{vmatrix}=0$$

Note : In numerical problems, evaluation of determinant takes much time and so the values of u, v, w and d should be found from (ii), (iiii), (iv) and (v) and then these should be substituted in (i), to get the required equation.

****§ 7.04 Equation of a sphere on the line joining two given points as diameter. (Diameter Form).** (*Kumaun* 90)

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two given points.

Let $P(x, y, z)$ be any point on the surface of the sphere, drawn on the line joining A and B as diameter. Then AP is perpendicular to BP.

Now the direction ratios of the lines AP and BP are

$x - x_1, y - y_1, z - z_1$ and $x - x_2, y - y_2, z - z_2$ respectively.

As AP is perpendicular to BP, we have

$$\boxed{(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0,} \qquad ...(i)$$

which is the required equation.

Solved Examples on § 7.01 to § 7.04

Ex. 1. Find the equation of the sphere whose centre is $(2, -3, 4)$ and radius 5.

Sol. Let $P(x_1, y_1, z_1)$ be any point on the sphere, then as $C(2, -3, 4)$ is the given centre of the sphere, so

$$CP = \text{radius of the sphere} = 5 \text{ i.e. } CP^2 = (5)^2$$

or $$(x_1 - 2)^2 + (y_1 + 3)^2 + (z_1 - 4)^2 = 5^2$$

or $$x_1^2 - 4x_1 + 4 + y_1^2 + 6y_1 + 9 + z_1^2 - 8z_1 + 16 = 25$$

or $$x_1^2 + y_1^2 + z_1^2 - 4x_1 + 6y_1 - 8z_1 + 4 = 0.$$

$\therefore$ Locus of P is $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0,$

which is the required equation of the sphere. **Ans.**

Ex. 2. What is the equation of a sphere which passes through $(0, 0, 0)$ and which has its centre at $(\frac{1}{2}, \frac{1}{2}, 0)$.

Sol. Centre of the sphere is $(\frac{1}{2}, \frac{1}{2}, 0)$. As it passes through $(0, 0, 0)$, so

Its radius = distance between $(0, 0, 0)$ and $(\frac{1}{2}, \frac{1}{2}, 0)$ **(Note)**

$$= [(\tfrac{1}{2} - 0)^2 + (\tfrac{1}{2} - 0)^2 + (0 - 0)^2] = \sqrt{(\tfrac{1}{2})}$$

$\therefore$ Required equation of the sphere is

$$(x - 0)^2 + (y - 0)^2 + (z - 0)^2 = [\sqrt{(\tfrac{1}{2})^2}]$$ ...§ 7.02 Page 1 Ch. VII

or $$x^2 + y^2 + z^2 = \frac{1}{2} \quad \text{or} \quad 2x^2 + 2y^2 + 2z^2 = 1.$$ **Ans.**

Ex. 3. Find the centre and the radius of the sphere

$$\mathbf{x^2 + y^2 + z^2 - 2x + 4y - 6z = 11}$$

Sol. Here 'u' $= -1$, 'v' $= 2$, 'w' $= -3$, 'd' $= -11$

...See § 7.02 Page 1 Ch. VII

$\therefore$ Centre is $(-u, -v, -w)$ i.e. $(1, -2, 3)$

and radius $= \sqrt{(u^2+v^2+w^2-d)} = \sqrt{(1+4+9-11)} = \sqrt{3}$. **Ans.**

***Ex. 4. Prove that equation**

$$\mathbf{ax^2 + ay^2 + az^2 + 2ux + 2vy + 2wz + d = 0}$$

represents a sphere and find its centre and radius.

Sol. From the given equation we find that :

(i) this equation is of second degree in x, y and z

(ii) the coefficients of x^2, y^2 and z^2 are equal, each being a and

(iii) the terms containing the product xy, yz and zx are absent.

Hence the given equation represents a sphere (see the conditions for a sphere in § 7.02 on Page 2)

Also as $a \neq 0$, so the given equation of the sphere can be rewritten as $x^2+y^2+z^2+2(u/a)x+2(v/a)y+2(w/a)z+(d/a)=0$, which is of the form "$x^2+y^2+z^2+2ux+2vy+2wz+d=0$"

$\therefore$ Its centre is $\left(-\frac{u}{a}, -\frac{v}{a}, -\frac{w}{a}\right)$ and radius is

$$\sqrt{\left(\frac{u^2}{a^2}+\frac{v^2}{a^2}+\frac{w^2}{a^2}-\frac{d}{a}\right)} \quad i.e. \quad \frac{1}{a}\sqrt{(u^2+v^2+w^2-ad)}$$

Ex. 5. Find the equation of a sphere which passes through the origin and intercepts lengths a, b, c on the x, y and z-axes respectively.

or

Find the equation of the sphere which passes through (a, 0, 0) and (0, b, 0), (0, 0, c) and (0, 0, 0). *(Bundelkhand* 91)

or

The plane ABC whose equation is x/a + y/b + z/c = 1 mets the axes OX, OY, OZ in A, B, C respectively. Find the equation of the sphere O, ABC. *(Meerut* 93)

Find its centre and radius.

Sol. $\because$ The sphere intercepts a length a on x-axis so it passes through the point $(a, 0, 0)$. Similarly it passes through the points $(0, b, 0)$ and $(0, 0, c)$. Also it passes through the origin *i.e.* $(0, 0, 0)$.

Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{...(i)}$$

If it passes through $(0, 0, 0)$ then from (i) we have $d=0$ ...(ii)

If (i) passes through $(a, 0, 0)$, then we get $a^2+2ua+d=0$

or from (ii), $a^2+2ua+0=0$ or $u=-\frac{1}{2}a$, as $a \neq 0$

Similarly as (i) passes through $(0, b, 0)$ and $(0, 0, c)$ we get

$$v=-\frac{1}{2}b \quad \text{and} \quad w=-\frac{1}{2}c$$

Hence from (i), required equation is $x^2+y^2+z^2-ax-by-cz=0$ **Ans.**

Here '$2u$' $=-a$, '$2v$' $=-b$, '$2w$' $=-c$, 'd' $=0$

and radius $=\sqrt{(u^2+v^2+w^2-d)}=\sqrt{[(-\frac{1}{2}a)^2+(-\frac{1}{2}b)^2+(-\frac{1}{2}c)^2-0]}$

$=\frac{1}{2}\sqrt{(a^2+b^2+c^2)}$. **Ans.**

Ex. 6. Find the equation of a sphere which passes through the origin and makes equal intercepts of unit length on the axes.

Hint : Do as Ex. 5 above. Find the equation of the sphere through $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ **Ans.** $x^2+y^2+z^2-x-y-z=0$

Ex. 7 (a). Find the equation to the sphere through the points (0, 0, 0), (0, 1, – 1), (– 1, 2, 0) and (1, 2, 3). (*Rohilkhand* 97)

Sol. Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{...(i)}$$

If it passes through $(0, 0, 0)$, then $d=0$ ...(ii)

If it passes through $(0, 1, -1)$, then

$$0+1+1+0+2v-2w+0=0, \quad \because \quad d=0$$

or $$v-w+1=0 \quad \text{...(iii)}$$

If it passes through $(-1, 2, 0)$, then $1+4+0-2u+4v+0+0=0$.

or $$5-2u+4v=0 \quad \text{...(iv)}$$

If it passes through $(1, 2, 3)$, then

$$1+4+9+2u+4v+6w+0=0, \because d=0$$

or $$u+2v+3w+7=0 \quad \text{...(v)}$$

From (iii), we get $w=v+1$ and from (iv), $u=(5+4v)/2$

Substituting these values in (v) we get

$$[(5+4v)/2]+2v+3(v+1)+7=0$$

or $5+4v+4v+6v+6+7=0$ or $14v=-18$ or $v=-9/7$

$\therefore$ $w=v+1=(-9/7)+1=-2/7$

and $$u=\frac{1}{2}(5+4v)=\frac{1}{2}\left(5-\frac{36}{7}\right)=-\frac{1}{14}$$

$\therefore$ From (i), we get the required equation as

$$x^2+y^2+z^2+2(-1/14)x+2(-9/7)y+2(-2/7)z+0=0$$

or $$14(x^2+y^2+z^2)-2x-36y-8z=0$$ **Ans.**

Ex. 7 (b). Find the equation of the sphere through the points (0, 0, 0); (0, 1, – 1) (– 1, 2, 0) and (1, 2, 1). (*Bundelkhand* 90)

Sol. Do as Ex. 7(a) above. **Ans.** $10(x^2+y^2+z^2)-2x-26y-6z=0$

Ex. 8 (a). Find the equation of the sphere which passes through the points (1, – 3, 4), (1, – 5, 2), (1, – 3, 0) and whose centre lies on the plane x + y + z = 0.

Sol. Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{...(i)}$$

If it passes through $(1,-3,4)$, then

$$1+9+16+2u-6v+8w+d=0 \quad \text{...(ii)}$$

If it passes through $(1,-5,2)$, then

$$1+25+4+2u-10v+4w+d=0 \quad \text{....(iii)}$$

If it passes through $(1,-3,0)$, then $1+9+2u-6v+d=0$ ...(iv)

Again the centre of the sphere is $(-u,-v,-w)$ and this centre lies on the plane $x+y+z=0$. So we have $-u-v-w=0$ or $u+v+w=0$ ...(v)

Solving (ii) and (iv) we get $16+8w=0$ or $w=-2$ ...(vi)

Solving (ii) and (iii) we get $-4+4v+4w=3$

or $$-4+4v-8=0, \text{ from (vi)}$$

or $$4v=12 \quad \text{or} \quad v=3 \quad \text{...(vii)}$$

$\therefore$ From (v), (vi) and (vii) we get $u=-1$ and so from (iv) $d=10$

Hence from (i) we get the required equation as

$$x^2+y^2+z^2-2x+6y-4z+10=0. \quad \textbf{Ans.}$$

Ex. 8 (b) Find the equation of the sphere which passes through the points (3, 0, 2), (– 1, 1, 1), (2, – 5, 4) and whose centre lies on the plane

$$\mathbf{x+y+2z=3.}$$

Hint : Do as Ex. 8 (a) above.

Ans. $8(x^2+y^2+z^2)-x+35y-41z-19=0$

Ex. 8 (c). Show that the equation of the sphere through three points (3, 0, 2), (– 1, 1, 1), (2, – 5, 4) and having its centre on the plane 2x + 3y + 4z = 6 is $\mathbf{x^2+y^2+z^2+4y-6z=1}$ (*Kumaun* 95)

Sol. Do as Ex. 8 (a) above.

***Ex. 9 (a). Find the equation of the sphere circumscribing the tetrahedron whose faces are x = 0, y = 0, z = 0, x/a + y/b + z/c = 1.**

Sol. Taking three planes at a time and solving their equations we get the coordinates of the vertices of this tetrahedron as $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$.

The equation of the sphere circumscribing the tetrahedron *i.e.* through these four points can be obtained as in Ex. 5 Page 4 Ch. VII.

***Ex. 9 (b). Find the equation of the sphere circumscribing the tetrahedron whose faces are y/b + z/c = 0, z/c + x/a = 0, x/a + y/b = 0, x/a + y/b + z/c = 1.**

Sol. Taking three planes at a time and solving their equations, we get the coordinates of the vertices of this tetrahedron as $(0, 0, 0)$, $(a, b, -c)$, $(a, -b, c)$ and $(-a, b, c)$.

Then proceeding as in Ex. 5 on Page 4 Ch. VII we can find that

$$2u=-(a^2+b^2+c^2)/a,$$

$2v = -(a^2+b^2+c^2)/b,\ 2w = -(a^2+b^2+c^2)/c$ and $d = 0$

$\therefore$ The required equation is

$$x^2+y^2+z^2-\frac{(a^2+b^2+c^2)\,x}{a}-\frac{(a^2+b^2+c^2)\,y}{b}-\frac{(a^2+b^2+c^2)\,z}{c}=0$$

or $$\frac{x^2+y^2+z^2}{a^2+b^2+c^2}-\frac{x}{a}-\frac{y}{b}-\frac{z}{c}=0.$$ **Ans.**

***Ex. 10. Obtain the equation of the sphere having its centre on the line $5y+2z=0=2x-3y$ and passing through the points $(0,-2,-4)$ and $(2,-1,-1)$.** (*Garhwal* 95)

Sol. Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad ...(i)$$

Its centre is $(-u,-v,-w)$. If it lies on the lines $5y+2z=0=2x-3y$, then we have $5(-v)+2(-w)=0=2(-u)-3(-v)$

i.e. $5v+2w=0$ ...(ii) and $2u-3v=0$ ...(iii)

If the sphere (i) passes through $(0,-2,-4)$ and $(2,-1,-1)$, then we get

$$0+4+16-4v-8w+d=0 \quad ...(iv)$$

and $$4+1+1+4u-2v-2w+d=0 \quad ...(v)$$

Solving (ii), (iii) and (v) we get

$u=-3;\ v=-2;\ w=5;\ d=12$, which satisfy (iv).

$\therefore$ From (i), the required equation is

$$x^2+y^2+z^2-6x-4y+10z+12=0$$ **Ans.**

***Ex. 11 (a). Find the equation of the sphere with centre at $(2,3,-4)$ and touching the plane $2x+6y-3z+15=0$.** (*Meerut* 96 *P*, 94)

Sol. Centre of the sphere is given as $(2,3,-4)$

Radius of the sphere = length of perpendicular from the centre $(2,3,-4)$ on the given plane

$$=\frac{2(2)+6(3)-3(-4)+15}{\sqrt{[2^2+6^2+(-3)^2]}}=\frac{49}{\sqrt{(49)}}=7$$

$\therefore$ The required equation of the sphere is $(x-2)^2+(y-3)^2+(z+4)^2=7^2$

or $$x^2+y^2+z^2-4x-6y+8z-20=0$$ **Ans.**

Ex. 11 (b). Obtain the equation of the sphere with centre at $(1,-2,3)$ and touching the plane $6x-3y+2z=4$.

Sol. Do as Ex. 11 (a) above **Ans.** $x^2+y^2+z^2-2x+4y-6z+10=0$

***Ex. 12. A sphere of radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2+y^2+z^2)=4k^2$.**

(*Agra* 90; *Kanpur* 96; *Purvanchal* 94; *Rohilkhand* 90)

Sol. Let the coordinates of A, B and C be $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ respectively. The sphere also passes through the origin $O(0,0,0)$. As in Ex. 5

Page 4 Ch. VII we can find the equation of the sphere through O, A, B and C as

$$x^2+y^2+z^2-ax-by-cz=0 \quad ...(i)$$

Its radius $=\sqrt{[(\frac{1}{2}a)^2+(\frac{1}{2}b)^2+(\frac{1}{2}c)^2]}=\frac{1}{2}\sqrt{(a^2+b^2+c^2)}=k$ (given)

or $$a^2+b^2+c^2=4k^2 \quad ...(ii)$$

Also if (x_1, y_1, z_1) be the coordinates of the centroid of ΔABC, then $x_1=\frac{1}{3}a$, $y_1=\frac{1}{3}b$, $z_1=\frac{1}{3}c$ or $a=3x_1, b=3y_1, c=3z_1$.

Substituting these values of a, b and c in (ii) we get

$$(3x_1)^2+(3y_1)^2+(3z_1)^2=4k^2$$

$\therefore$ The required locus of (x_1, y_1, z_1) is $9(x^2+y^2+z^2)=4k^2$. Hence proved.

****Ex. 13. A plane passes through a fixed point (p, q, r) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere OABC is**

$$\frac{\mathbf{p}}{\mathbf{x}}+\frac{\mathbf{q}}{\mathbf{y}}+\frac{\mathbf{r}}{\mathbf{z}}=2.$$

(*Agra* 91; *Avadh* 94, 91; *Bundelkhand* 91; *Garhwal* 94, 92; *Kanpur* 95; *Kumaun* 93; *Meerut* 98, 90 *S*; *Purvanchal* 97, 96, 95, 92)

Sol. Let the coordinates of A, B and C be $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

The equation of the plane ABC is $$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad ...(i)$$

If it passes through (p, q, r), then $$\frac{p}{a}+\frac{q}{b}+\frac{r}{c}=1 \quad ...(ii)$$

Also the equation of the sphere $OABC$ can be found as in Ex. 5 Page 4 Ch. VII as $$x^2+y^2+z^2-ax-by-cz=0.$$

If its centre be (x_1, y_1, z_1) then $x_1=\frac{1}{2}a$, $y_1=\frac{1}{2}b$ and $z_1=\frac{1}{2}c$.

or $$a=2x_1, \quad b=2y_1 \quad \text{and} \quad c=2z_1$$

Substituting these values of a, b and c in (ii), we get

$$(p/2x_1)+(q/2y_1)+(r/2z_1)=1$$

$\therefore$ Locus of the centre (x_1, y_1, z_1) of the sphere $OABC$ is

$$(p/x)+(q/y)+(r/z)=2.$$ Hence proved.

***Ex. 14. Find the equation of the sphere that passes through the points (4, 1, 0), (2, – 3, 4), (1, 0, 0) and touches the plane 2x + 2y – z = 11.**

Sol. Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad ...(i)$$

Its centre is $(-u, -v, -w)$ and radius $=\sqrt{(u^2+v^2+w^2-d)}$

Since the sphere (i) passes through the point (4, 1, 0), so

$$4^2+1^2+0^2+2u(4)+2v(0)+2w(0)+d=0$$

or $$8u+2v+d+17=0 \quad ...(ii)$$

Similarly if the sphere (i) passes through (2, –3, 4) and (1, 0, 0), then we have

$$4u - 6v + 8w + d + 29 = 0 \qquad ...(iii)$$

and

$$2u + d + 1 = 0 \qquad ...(iv)$$

Also as the sphere (i) touches the given plane, so the length of perpendicular from its centre $(-u, -v, -w)$ to the given plane $2x + 2y - z - 11 = 0$ must be equal to the radius $\sqrt{(u^2 + v^2 + w^2 - d)}$ of the sphere (i).

i.e.

$$\frac{2(-u) + 2(-v) - (-w) - 11}{\sqrt{[2^2 + 2^2 + (-1)^2]}} = \sqrt{(u^2 + v^2 + w^2 - d)}$$

or

$$(-2u - 2v + w - 11)^2 = 9(u^2 + v^2 + w^2 - d)$$

or

$$5u^2 + 5v^2 + 8w^2 - 8uv + 4vw + 4uw - 44u - 44v + 22w - 9d - 121 = 0 \qquad ...(v)$$

From (iv), $u = \frac{1}{2}(-d - 1) = -\frac{1}{2}(d + 1)$ ...(vi)

From (ii), $2v = -8u - d - 17 = 4d + 4 - d - 17 = 3d - 13$

or

$$v = \frac{1}{2}(3d - 13) \qquad ...(vii)$$

From (iii), $8w = -4u + 6v - d - 29$

$= (2d + 2) + (9d - 39) - d - 29$, from (vi), (vii)

or

$$8w = 10d - 66 \quad \text{or} \quad w = \frac{1}{4}(5d - 33) \qquad ...(viii)$$

Substituting values of u, v, w from (vi), (vii), (viii) in (v) and simplifying we get $72d^2 - 747d + 1935 = 0$ which gives $d = 5$.

∴ From (vi), (vii) and (viii) we get $u = -3$, $v = 1$, $w = -2$

∴ From (i) the required equation is

$$x^2 + y^2 + z^2 - 6x + 2y - 4z + 5 = 0 \qquad \textbf{Ans.}$$

***Ex. 15 (a). A sphere of constant radius 2k passes through origin and meets the axes in A, B and C. Find the locus of the centroid of the tetrahedron OABC.**

Sol. As in Ex. 13 Page 8 Ch. VII the sphere *OABC* is

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

Its radius $= \sqrt{[(\frac{1}{2}a)^2 + (\frac{1}{2}b)^2 + (\frac{1}{2}c)^2]} = \frac{1}{2}\sqrt{(a^2 + b^2 + c^2)} = 2k$ (given)

or

$$a^2 + b^2 + c^2 = 16k^2 \qquad ...(i)$$

If (x_1, y_1, z_1) be the centroid of the tetrahedron *OABC*, then

$$x_1 = \frac{1}{4}[0 + a + 0 + 0] = \frac{1}{4}a \quad \text{or} \quad a = 4x_1$$

Similarly $\qquad b = 4y_1 \quad$ and $\quad c = 4z_1$

Substituting these values of a, b, c in (i) we get

$$(4x_1)^2 + (4y_1)^2 + (4z_1)^2 = 16k^2 \quad \text{or} \quad x_1^2 + y_1^2 + z_1^2 = k^2$$

∴ The required locus of (x_1, y_1, z_1) is $x^2 + y^2 + z^2 = k^2$ **Ans.**

Ex. 15 (b). A sphere of radius 2 passes the origin O and meets the coordinate axes in A, B and C. Show that the locus of the centroid of the tetrahedron OABC is the sphere $x^2+y^2+z^2=1$ (*Allahabad* 92)

Sol. Do as Ex. 15 (a) above. Here 'k' = 1

****Ex. 16. Find the equation of the sphere which passes through the points (1, 0, 0), (0, 1, 0) and (0, 0, 1) and has its radius as small as possible.** (*Kumaun* 91)

Sol. Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+c=0 \quad ...(i)$$

If it passes through (1, 0, 0), (0, 1, 0) and (0, 0, 1) then

$$1+2u+c=0,\ 1+2v+c=0,\ 1+2w+c=0$$

or
$$u=v=w=-\tfrac{1}{2}(1+c) \quad ...(ii)$$

$\therefore$ If r be the radius of the sphere (i), then

$$r^2=u^2+v^2+w^2-c=R \text{ (say)}$$

or
$$R=\tfrac{3}{4}(1+c)^2-c, \text{ from (ii)}$$

If r is least then R least.

Now $\dfrac{dR}{dc}=\dfrac{3}{2}(1+c)-1$ and $\dfrac{d^2R}{dc^2}=\dfrac{3}{2}=$ positive

Equating $\dfrac{dR}{dc}$ to zero we get $\frac{3}{2}c+\frac{1}{2}=0$ or $c=-\frac{1}{3}$ and $\dfrac{d^2R}{dc^2}$ being positive R is least when $c=-\frac{1}{3}$.

$\therefore$ From (ii) when R *i.e.* r^2 is least we have $u=v=w=-\frac{1}{2}(1-\frac{1}{3})=-\frac{1}{3}$

$\therefore$ From (i), the required equation is $x^2+y^2+z^2-\frac{2}{3}(x+y+z)-\frac{1}{3}=0$

or
$$3(x^2+y^2+z^2)-2(x+y+z)-1=0. \quad \textbf{Ans.}$$

***Ex. 17. A point moves so that the sum of the squares of its distances from the six faces of a cube is constant, show that its locus is a sphere.**

Sol. Take any vertex of the cube as origin and the edges terminating at that vertex as coordinate axes. Then if a be the length of each edge of the cube, then the equations of its six faces are $x=0, y=0, z=0, x=a, y=a$ and $z=a$. **(Note)**

(Students are advised to draw the figure themselves and check the above equations).

Let $P(x_1, y_1, z_1)$ be the moving point, then according to the given problem we have the sum of squares of its distances from the faces of the cube is constant k (say).

i.e. $\left(\dfrac{x_1}{1}\right)^2+\left(\dfrac{y_1}{1}\right)^2+\left(\dfrac{z_1}{1}\right)^2+\left(\dfrac{x_1-a}{1}\right)^2+\left(\dfrac{y_1-a}{1}\right)^2+\left(\dfrac{z_1-a}{1}\right)^2=k.$

or $\quad x_1^2+y_1^2+z_1^2+(x_1^2-2ax_1+a^2)+(y_1^2-2ay_1+a^2)+(z_1^2-2az_1+a^2)=k$

or $\quad 2(x_1^2+y_1^2+z_1^2)-2a(x_1+y_1+z_1)+(3a^2-k)=0$

or $\quad x_1^2+y_1^2+z_1^2-a(x_1+y_1+z_1)+d=0$, where $d=(3a^2-k)/2$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ is $x^2+y^2+z^2-ax-ay-az+d=0$
which evidently represents a sphere as it is second degree equation in x, y, z; coefficients of x^2, y^2, z^2 are equal and terms containing product terms xy, yz, zx are absent. Hence proved.

Ex. 18 (a). A point moves so that the ratio of its two distances from two fixed points is constant. Show that its locus is a sphere.

Sol. Let the moving point be $P(x, y, z)$ and the fixed points A and B be $(a, 0, 0)$ and $(-a, 0, 0)$ respectively.

Given $PA/PB=k$ or $PA^2=k^2 \cdot PB^2$

or $\quad (x-a)^2+(y-0)^2+(z-0)^2=k^2[(x+a)^2+(y-0)^2+(z-0)^2]$

or $\quad x^2(1-k^2)+y^2(1-k^2)+z^2(1-k^2)-2ax(1+k^2)+a^2(1-k^2)=0$

or $\quad x^2+y^2+z^2-\dfrac{2a(1+k^2)}{(1-k^2)}x+a^2=0$, which evidently represents a sphere.

(See Ex. 17 Page 10 for reasons).

Ex 18 (b). A point moves so that sum of squares of its distances from a given number of points is constant. Prove that the locus of this point is a sphere and show that its centre is the centroid of the given points.

Sol. Let there be n given points (x_1, y_1, z_1), (x_2, y_2, z_2) etc.

Let the moving point be (α, β, γ). Then according to the given problem the sum of the squares of its distances from the given points is constant.

i.e. $\quad \Sigma[(\alpha-x_1)^2+(\beta-y_1)^2+(\gamma-z_1)^2]=\text{constant}=k$ (say)

or $\quad n\alpha^2+n\beta^2+n\gamma^2-2\alpha\Sigma x_1-2\beta\Sigma y_1-2\gamma\Sigma z_1+\Sigma x_1^2+\Sigma y_1^2+\Sigma z_1^2=k$

or $\quad \alpha^2+\beta^2+\gamma^2-2\alpha\left(\dfrac{\Sigma x_1}{n}\right)-2\beta\left(\dfrac{\Sigma y_1}{n}\right)-2\gamma\left(\dfrac{\Sigma z_1}{n}\right)+(\ldots)=0$

$\therefore$ Locus of (α, β, γ) is the sphere

$$x^2+y^2+z^2-2\left(\frac{\Sigma x_1}{n}\right)x-2\left(\frac{\Sigma y_1}{n}\right)y-2\left(\frac{\Sigma z_1}{n}\right)z+(\text{constant})=0$$

$\therefore$ Its centre is $\left(\dfrac{\Sigma x_1}{n}, \dfrac{\Sigma y_1}{n}, \dfrac{\Sigma z_1}{n}\right)$ which is evidently the centroid of the points $(x_1, y_1, z_1), (x_2, y_2, z_2) \ldots\ldots, (x_n, y_n, z_n)$ Hence proved.

***Ex. 19. Find the equation of the sphere on the join of $(2, -3, 4)$ and $(-5, 6, -7)$ as diameter.**

Sol. The required eqution of the sphere is

$$(x-2)(x+5)+(y+3)(y-6)+(z-4)(z+7)=0$$

...See § 7.04 Page 3 Ch. VII

or $$x^2+y^2+z^2+3x-3y+3z-49=0.$$ **Ans.**

Ex. 20. OA, OB, OC are three mutually perpendicular lines through the origin having direction cosines l_1, m_1, n_1 ; l_2, m_2, n_2 and l_3, m_3, n_3.

If OA = a, OB = b, OC = c. Find the equation of the sphere OABC.

Sol. Taking OA, OB and OC as coordinate axes the coordinates of A, B and C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively. Consequently the equation of the sphere $OABC$, as in Ex. 5 Page 4 Ch. VII is

$$x^2+y^2+z^2-ax-by-cz=0 \qquad \text{...(i)}$$

Now transferring to original axes by putting $x=l_1x+m_1y+n_1z$, $y=l_2x+m_2y+n_2z$, $z=l_3x+m_3y+n_3z$, we get from (i) the required equation of the sphere as $(l_1x+m_1y+n_1z)^2+(l_2x+m_2y+n_2z)^2+(l_3x+m_3y+n_3z)^2$

$$-a(l_1x+m_1y+n_1z)-b(l_2x+m_2y+n_2z)-c(l_3x+m_3y+n_3z)=0$$

or $$x^2(\Sigma l_1^2)+y^2(\Sigma m_1^2)+z^2(\Sigma n_1^2)+2xy(\Sigma l_1m_1)+2yz(\Sigma m_1n_1)+2zx(\Sigma l_1n_1)$$
$$-(al_1+bl_2+cl_3)x-(am_1+bm_2+cm_3)y-(an_1+bn_2+cn_3)z=0$$

or $$x^2+y^2+z^2=(al_1+bl_2+cl_3)x+(am_1+bm_2+cm_3)y$$
$$+(an_1+bn_2+cn_3)z,$$

since $\Sigma l_1^2=1$ and $\Sigma l_1m_1=0$ etc

Exercises on § 7.01—7.04

Ex. 1. Find the centre and radius of the sphere given by

$2(x^2+y^2+z^2)-2x+4y-6z=1$ **Ans.** $(1/2, -1, 3/2)$; $\sqrt{(11)}$

Ex. 2. Find the centre and radius of the sphere $x^2+y^2+z^2-2y-4z=11$. What is the distance of the centre from the yz-plane ? **Ans.** (0, 1, 2); 4; 0

Ex. 3. Prove that the equation $a(x^2+y^2+z^2)+2(ux+vy+wz)+d=0$, $a \neq 0$ always represents a sphere, provided $u^2+v^2+w^2-ad>0$.

Ex. 4. A sphere has its centre at $(1, -3, 4)$ and passes through $(3, -1, 3)$. Find its equation. Find also the equation of the concentric sphere which touches the plane $2x+5y-5z=21$.

Ans. $x^2+y^2+z^2-2x+6y-8z+17=0$;

$$x^2+y^2+z^2-2x+6y-8z-28=0$$

Ex. 5. Find the centre of the sphere through the four points $(4, -1, 2)$, $(0, -2, 3)$, $(1, -5, 1)$ and $(2, 0, 1)$. **Ans.** $(2, -13/15, 7/5)$

Ex. 6. Find the equations of the spheres through the following points and find their centres and radii :—

(i) $(0, 0, 0)$; $(2, 0, 0)$; $(0, 4, 0)$; $(0, 0, -1)$.

(ii) $(0, 0, 0)$; $(-4, 3, 1)$; $(4, -3, 1)$; $(4, 3, -1)$.

(iii) $(0, 0, 0)$; $(-a, b, c)$; $(a, -b, c)$; $(a, b, -c)$. *(Garhwal 96; Kumaun 92)*

(iv) $(0, 0, 0)$; $(-1, 1, 1)$; $(1, -1, 1)$; $(1, 1, -1)$.

Ans. (i) $x^2+y^2+z^2=2x+4y-z$; $(1, 2, -\frac{1}{2})$; $\frac{1}{2}\sqrt{(21)}$

(ii) $6(x^2+y^2+z^2)=13(3x+4y+12z)$; $\left(\frac{13}{4}, \frac{13}{2}, 13\right)$, $14\frac{1}{13}$

(iii) $\frac{x^2+y^2+z^2}{a^2+b^2+z^2}-\frac{x}{a}-\frac{y}{b}-\frac{z}{c}=0$; $\left(\frac{a^2+b^2+c^2}{2a}, \frac{a^2+b^2+c^2}{2b},\right.$

$$\left.\frac{a^2+b^2+c^2}{2c}\right), \frac{(a^2+b^2+c^2)}{2abc}\sqrt{(a^2b^2+b^2c^2+c^2a^2)}.$$

(iv) $x^2+y^2+z^2=3(x+y+z)$, $(3/2, 3/2, 3/2)$, $3\sqrt{3}/2$.

Ex. 7. Find the equation of the sphere passing through the points $(3, 0, 2)$, $(-1, 1, 1)$, $(2, -5, 4)$ and having its centre on $2x+3y+4z=6$.

Ans. $x^2+y^2+z^2+4y-6z-1=0$

Ex. 8. Find the equation of the sphere through the origin cutting off intercepts 3, 4 and 5 from the coordinate axes.

Ans. $x^2+y^2+z^2-3x-4y-5z=0$

Ex. 9. A plane through a fixed point $(1, 1, 1)$ cuts the axes in A, B, C. Find the locus of the centre of the sphere $OABC$, where O is the origin.

Ans. $x^{-1}+y^{-1}+z^{-1}=2$

Ex. 10. A variable plane cuts the coordinate axes in A, B, C and always passes through the point $(1, 2, 3)$. Find the locus of the centre of the sphere passing through A, B, C and the origin. **Ans.** $x^{-1}+2y^{-1}+3z^{-1}=2$.

Ex. 11. Find the equation of the sphere circumscribing the tetrahedron fromed by the planes $4y+3z=0$, $3x+2y=0$, $z+2x=0$ and $6x+4y+3z=12$.

Ans. $12(x^2+y^2+z^2)=174x+116y+87z$.

Ex. 12. Find the equation of the sphere circumscribing the tetrahedron whose faces are $y+z=0$, $z+x=0$, $x+y=0$ and $x+y+z=1$.

Ans. $x^2+y^2+z^2=3(x+y+z)$

Ex. 13. Find the equation of the sphere the extremities of whose diameter are the ponts $(3, 4, -2)$ and $(-2, -1, 0)$. **Ans.** $x^2+y^2+z^2-x-3y+2z=10$

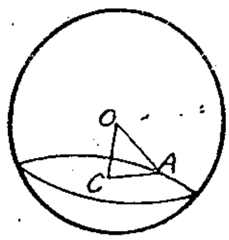
**§ 7.05. Plane section of a sphere.

To prove that the section of a sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad ...(i)$$

by a plane $lx+my+nz=p$ ...(ii)

is a circle and to find its radius and centre.



(Fig. 1)

O is the centre $(-u, -v, -w)$ of the sphere (i) and C is the foot of the perpendicular OC from O to the plane (ii). Let A be any point on the section of the sphere (i) by the plane (ii), then CA is any line through C and as such perpendicular to OC.

Now OA = the radius of the sphere $= \sqrt{(u^2 + v^2 + w^2 - d)}$ and OC = length of the perp. from $O(-u, -v, -w)$ to the plane (iii)

$$= \frac{l(-u) + m(-v) + n(-w) - p}{\sqrt{(l^2 + m^2 + n^2)}} = \frac{lu + mv + nw + p}{\sqrt{(l^2 + m^2 + n^2)}}, \text{ numerically}$$

Now from above fig. it is evident the $CA^2 = OA^2 - OC^2$ ...(iii)

i.e. CA^2 = constant, as the radius of the sphere (i) and the distance of its centre from the plane (ii) are constant.

Hence the distance of any point A on the section of the sphere (i) by the plane (ii) is at a constant distance from the foot C of the perpendicular from centre of (i) to (ii), *i.e.* the locus of A is a circle with centre C and radius CA given by (iii).

Equations of a circle.

From above we have found that the intersection of the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ by the plane $lx + my + nz = p$ is a circle and so the equations of this circle are

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0, \; lx + my + nz = p$$

taken together.

Hence in general, the equations of a circle consist of the equations of a sphere and that of a plane (taken togehter).

Note 1. From § 7.05 above we observe that :—

(i) The foot (The point C in Fig. 1 of § 7.05 above) of perpendicular from the centre (the point O in the above figure) of the sphere on the plane of the circle is the centre of the circle and (ii) and radius of the circle = CA (Fig. 1 Page 13 Ch. VII)

$= \sqrt{(OA^2 - OC^2)} = \sqrt{[(\text{radius of the sphere})^2 - (\text{length of perpendicular from the centre of the sphere on the plane of the circle})^2]}$ **(Remember)**

Note 2. The section of a sphere by a plane passing through the centre of the sphere is called a great circle. Its centre and radius is the same at that of the sphere.

***§ 7.06 Intersection of two sphres.**

Let the equations of the two spheres be

$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$

and

$$S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

The points common to these two circles satisfy both these equations $S_1 = 0$, $S_2 = 0$ and hence satisfy $S_1 - S_2 = 0$.

i.e. $2(u_1 - u_2)x + 2(v_1 - v_2)y + 2(w_1 - w_2)z + 2(d_1 - d_2) = 0$

which being a linear equation in x, y, z represents a plane.

Thus the curve of intersection of the two spheres $S_1=0$, $S_2=0$, is the same as the curve of intersection of any one of these spheres and the plane $S_1-S_2=0$ and so it is a circle.

Note. From § 7.06 above we conclude that the equations of two spheres taken together also represent a circle.

§ 7.07. Sphere through a given circle.

Let the equations of the circle be

$$S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad \text{...(i)}$$

and $$P \equiv lx+my+nz-p=0. \qquad \text{...(ii)}$$

Then the equation $S+\lambda P=0$, ...(iii)

in which λ is a constant represents a sphere as in this equation the coefficient of x^2, y^2, z^2 are equal and the terms containing xy, yz, zx are absent.

Also the equation $S+\lambda P=0$ is satisfied by all points which satisfy both $S=0$ and $P=0$ *i.e.* which lie on the circle $S=0$, $P=0$.

Hence $S+\lambda P=0$ represents a sphere through circle $S=0$, $P=0$.

In a similar manner we can show that the equation $S_1+\lambda S_2=0$ represents a sphere through the circle of intersection of the spheres $S_1=0$ and $S_2=0$.

Solved Examples on § 7.05 to § 7.07.

****Ex. 1 (a). Find the radius and centre of the circle**

$$\mathbf{x^2+y^2+z^2-8x+4y+8z-45=0,\ x-2y+2z=3.}$$

(*Gorakhpur* 92; *Purvanchal* 91; *Rohilkhand* 93, 90)

Sol. Refer Fig. 1. Page 13 of this chapter.

Centre of this sphere $x^2+y^2+z^2-8x+4y+8z-45=0$ is $O(4,-2,-4)$ and radius $OA=\sqrt{[4^2+(-2)^2+(-4)^2+45]}=9$

$\therefore$ Length of perpendicular from $O(4,-2,-4)$ to the given plane $x-2y+2z=3$

$$=OC=\frac{4-2(-2)+2(-4)-3}{\sqrt{[1^2+(-2)^2+2^2]}}=1, \text{ numerically}$$

$\therefore$ **Radius of the circle** $=CA=\sqrt{(OA^2-OC^2)}$

$$=\sqrt{[(9)^2-(1)^2]}=4\sqrt{5}. \qquad \textbf{Ans.}$$

Coordinates of the centre. As OC is perpendicular to the plane $x-2y+2z=3$, so the direction ratios of the line OC are the same as those of the normal to the plane *i.e.* $1, -2, 2$ *i.e.* the coefficients of x, y, z in the equation of this plane.

$\therefore$ The equations of this line OC are

$$\frac{x-4}{1}=\frac{y+2}{-2}=\frac{z+4}{2}=r \text{ (say)}$$

Let $OC = r$, then the coordinates of C are $(4 + r, -2 - 2r, -4 + 2r)$. But C lies on the plane $x - 2y + 2z = 3$

$\therefore (4 + r) - 2(-2 - 2r) + 2(-4 + 2r) = 3$ or $9r = 3$ or $r = \frac{1}{3}$

$\therefore$ Coordinates of centre C of the circle are $(13/3, -8/3, -10/3)$ **Ans.**

****Ex. 1 (b). Find the radius and centre of the circle, determined by the equations $x^2 + y^2 + z^2 = ax + by + cz$, and $(x/a) + (y/b) + (z/c) = 1$.**

(*Avadh* 94, 93)

Sol. Refer Fig. 1 Page 13 of this chapter.

Centre of the sphere $x^2 + y^2 + z^2 - ax - by - cz = 0$ is $O(a/2, b/2, c/2)$ and radius $OA = \sqrt{[(a/2)^2 + (b/2)^2 + (c/2)^2]} = \sqrt{(a^2 + b^2 + c^2)}/2$

$\therefore$ Length of perpendicular from the centre $O(a/2, b/2, c/2)$ to the given plane $(x/a) + (y/b) + (z/c) - 1 = 0$

$$= OC = \frac{(1/a)(a/2) + (1/b)(b/2) + (1/c)(c/2) - 1}{\sqrt{[(1/a)^2 + (1/b)^2 + (1/c)^2]}} \qquad \text{(Note)}$$

$$= abc/[2\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}]$$

$\therefore$ **Radius of the circle** $= CA = \sqrt{(OA^2 - OC^2)}$

$$= \sqrt{\left[\frac{a^2 + b^2 + c^2}{4} - \frac{a^2b^2c^2}{4(b^2c^2 + c^2a^2 + a^2b^2)}\right]}$$

$$= \frac{1}{2}\sqrt{\left[\frac{(a^2 + b^2 + c^2)(b^2c^2 + c^2a^2 + a^2b^2) - a^2b^2c^2}{b^2c^2 + c^2a^2 + a^2b^2}\right]}$$

$$= \frac{1}{2}\sqrt{\left[\frac{(b^2 + c^2)(c^2 + a^2)(a^2 + b^2)}{b^2c^2 + c^2a^2 + a^2b^2}\right]}, \text{ on simplifying}$$

Coordinate of the centre C : As OC is perpendicular to the plane $(x/a) + (y/b) + (z/c) = 1$, so the directon ratios of the line OC are the same as those of the normal to this plane *i.e.* $1/a, 1/b, 1/c$ *i.e.* the coefficients of x, y and z in the equation of this plane.

$\therefore$ Equations of line OC are

$$\frac{x - (a/2)}{1/a} = \frac{y - (b/2)}{1/b} = \frac{z - (c/2)}{1/c} = r \text{ (say)}$$

Let $OC = r$, then C is $\left(\frac{a}{2} + \frac{r}{a}, \frac{b}{a} + \frac{r}{b}, \frac{c}{2} + \frac{r}{c}\right)$

But C lies on the plane $(x/a) + (y/b) + (z/c) = 1$

$$\therefore \frac{1}{a}\left(\frac{a}{2} + \frac{r}{a}\right) + \frac{1}{b}\left(\frac{b}{2} + \frac{r}{b}\right) + \frac{1}{a}\left(\frac{c}{2} + \frac{r}{c}\right) = 1$$

or $r\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=-\frac{1}{2}$ or $r=-\frac{a^2b^2c^2}{2\,(b^2c^2+c^2a^2+a^2b^2)}$ ...(i)

$\therefore C$ is $\left(\frac{a}{2}+\frac{r}{a}, \frac{b}{2}+\frac{r}{b}, \frac{c}{2}+\frac{r}{c}\right)$, where r is given by (i).

Ex. 2. Find the equation of the sphere whose centre is the point $(1, 2, 3)$ and which touches the plane $3x+2y+z+4=0$. Find also the radius of the circle in which the sphere is cut by the plane $x+y+z=0$.

Sol. Since the sphere touches the plane $3x+2y+z+4=0$ ...(i)
so its radius = length of perpendicular from its centre (1, 2, 3) on (i)

$$=\frac{3.1+2.2+1.3+4}{\sqrt{(3^2+2^2+1^2)}}=\frac{14}{\sqrt{(14)}}=\sqrt{(14)}$$

$\therefore$ The required equation of the sphere is

$(x-1)^2+(y-2)^2+(z-3)^2=[\sqrt{(14)^2}]$, ...See § 7.02 (a) P. 1 Ch. VII

or $$x^2+y^2+z^2-2x-4y-6z=0$$

Now for the radius of the circle refer Fig. 1 Page 13 Ch. VII

The coordinates of O, the centre of the sphere are given as (1, 2, 3) and OC being the perpendicular to the plane $x+y+z=0$ its direction ratios are the same as those of the normal to this plane *i.e.* 1, 1, 1 the coefficients of x, y, z in the equation $x+y+z=0$.

$\therefore$ The equation of the line OC are

$$\frac{x-1}{1}=\frac{y-2}{1}=\frac{z-3}{1}=r \text{ (say)}$$

Let $OC=r$, then the coodinates of C are $(r+1, r+2, r+3)$ and C lies on the plane $x+y+z=0$, so we have

$$(r+1)+(r+2)+(r+3)=0 \quad \text{or} \quad 3r+6=0 \quad \text{or} \quad r=-2$$

$\therefore$ The coordinates of C are $(-2+1, -2+2, -2+3)$ or $(-1, 0, 1)$ **Ans.**

****Ex. 3. If r is the radius of the circle**

$$\mathbf{x^2+y^2+z^2+2ux+2vy+2wz+d=0,\ lx+my+nz=0,}$$

prove that $(r^2+d)(l^2+m^2+n^2)=(mw-nv)^2+(nu-lw)^2+(lv-mu)^2$

(*Garhwal* 91)

Sol. The radius of the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is $\sqrt{(u^2+v^2+w^2-d)}=R$ (say) and its centre is $(-u, -v, -w)$.

The length of perpendicular from $(-u, -v, -w)$ to the given plane $lx+my+nz=0$ is $\frac{l(-u)+m(-v)+n(-w)}{\sqrt{(l^2+m^2+n^2)}}=p$ (say)

Then the radius of the circle is given by

$$r^2=R^2-p^2=(u^2+v^2+w^2-d)-\frac{(lu+mv+nw)^2}{(l^2+m^2+n^2)}$$

or $\quad r^2(l^2+m^2+n^2)=(u^2+v^2+w^2-d)(l^2+m^2+n^2)-(lu+mv+nw)^2$

or $\quad (r^2+d)(l^2+m^2+n^2)=(u^2+v^2+w^2)(l^2+m^2+n^2)-(lu+mv+nw)^2$

$$=\Sigma(mw-nv)^2, \text{ by Lagrange's Identity.}$$

Ex. 4 (a). Find the equation of the circle whose centre is (x_1, y_1, z_1) and which lies on the sphere $x^2+y^2+z^2=a^2$.

Sol. The centre of the given plane is $O(0, 0, 0)$ and that of the circle is given to be $C(x_1, y_1, z_1)$.

Now the equation of a plane through $C(x_1, y_1, z_1)$ is

$$\alpha(x-x_1)+\beta(y-y_1)+\gamma(z-z_1)=0 \quad \text{...(i)}$$

If this plane is perpendicular to the line OC whose d.r.'s are x_1-0, y_1-0, z_1-0 *i.e.* x_1, y_1, z_1 then we have $\dfrac{\alpha}{x_1}=\dfrac{\beta}{y_1}=\dfrac{\gamma}{z_1}$ and hence from (i) the equation of plane through $C(x_1, y_1, z_1)$ at right angles to OC is

$$x_1(x-x_1)+y_1(y-y_1)+z_1(z-z_1)=0$$

i.e. $$xx_1+yy_1+zz_1=x_1^2+y_1^2+z_1^2 \quad \text{...(ii)}$$

Also the equation of the given sphere is $x^2+y^2+z^2=a^2$ ...(iii)

The equations (ii) and (iii) together represent the circle whose centre is (x_1, y_1, z_1) and which lies on the sphere (iii). **Ans.**

***Ex. 4 (b). Find the equations of the circle lying on the sphere $x^2+y^2+z^2-2x+4y-6z+3=0$ and having centre at $(2, 3, -4)$.**

(*Allahabad* 92)

Sol. The centre C of the given sphere is $(1, -2, 3)$ and that of circle is given to be $A(2, 3, -4)$.

The equation of any plane through $A(2, 3, -4)$ is

$$\alpha(x-2)+\beta(y-3)+\gamma(z+4)=0 \quad \text{...(i)}$$

If this plane is perpendicular to the line AC, whose d.r.'s are $1-2, -2-3, 3+4$ *i.e.* $-1, -5, 7$, then from (i) we get

$$\frac{\alpha}{-1}=\frac{\beta}{-5}=\frac{\gamma}{7}$$

$\therefore$ From (i), the equation of the plane through $A(2, 3, -4)$ at right angles to AC (*i.e.* the line joining the centres of the circle and sphere) is

$$-(x-2)-5(y-3)+7(z+4)=0$$

i.e. $$x+5y-7z-45=0 \quad \text{...(ii)}$$

Also the given sphere is $x^2+y^2+z^2-2x+4y-6z+3=0$ ...(iii)

$\therefore$ Equations (ii) and (iii) together represent the required circle. **Ans.**

Ex. 4 (c). Find the equations of the circle which has its centre at $(-2, 2, 1)$ and lies on the sphere $x^2+y^2+z^2+5x-7y+2z-8=0$

(*Rohilkhand* 96)

Sol. Do as Ex. 4 (b) above.

Ans. $x^2+y^2+z^2+5x-7y+2z=8,\ x-3y+4z+4=0.$

Ex. 5. Find the equation of the sphere through the circle $x^2+y^2+z^2=9,\ x+y-2z+4=0$ and the origin.

Sol. The equation of any sphere through the given circle is

$$(x^2+y^2+z^2-9)+\lambda(x+y-2z+4)=0 \quad \text{...(i)}$$

If this sphere passes through the origin (0, 0, 0), then we have

$$(0+0+0-9)+\lambda(0+0+0+4)=0 \text{ and } \lambda=9/4$$

$\therefore$ From (i), the required equation of the sphere is

$$(x^2+y^2+z^2-9)+(9/4)(x+y-2z+4)=0$$

or $$4x^2+4y^2+4z^2+9x+9y-18z=0 \quad \textbf{Ans.}$$

***Ex. 6. Prove that the centres of all sections of the sphere $x^2+y^2+z^2=r^2$ by planes through point (x', y', z') lie on the sphere**

$$\Sigma x(x-x')=0.$$ *(Bundelkhand* 95)

Sol. Let $C(x_1, y_1, z_1)$ be the centre of the section and the centre of the sphere $x^2+y^2+z^2=a^2$ is $O(0, 0, 0)$.

$\therefore$ The d.c.'s of the line OC are x_1-0, y_1-0, z_1-0 *i.e.* x_1, y_1, z_1 and so the equation of the plane cutting the given sphere in a circle with centre (x_1, y_1, z_1) is the equation of the plane through (x_1, y_1, z_1) and at right angles to OC and its equation therefore is $x_1(x-x_1)+y_1(y-y_1)+z_1(z-z_1)=0$

If it passes through (x', y', z'), then we get

$$x_1(x'-x_1)+y_1(y'-y_1)+z_1(z'-z_1)=0.$$

$\therefore$ The locus of the centre (x_1, y_1, z_1) is

$$x(x'-x)+y_1(y'-y)+z(z'-z)=0$$

or $$\Sigma x(x-x')=0$$ Hence proved.

***Ex. 7. Prove that the plane $x+2y-z=4$ cuts the sphere $x^2+y^2+z^2-x+z-2=0$ in a circle of radius untiy and find the equations of the sphere which has this circle for one of its great circles.**

(*Garhwal* 94, 92; *Meerut* 97)

Sol. The centre of the given sphere is $(\frac{1}{2}, 0, -\frac{1}{2})$ and its radius

$$=\sqrt{[(\tfrac{1}{2})^2+(0)^2+(-\tfrac{1}{2})^2-(-2)]}=\sqrt{(\tfrac{5}{2})}=R \text{ (say)}$$

Also length of perpendicular from $(\frac{1}{2}, 0, -\frac{1}{2})$ to $x+2y-z-4=0$ is

$$\frac{\frac{1}{2}+2(0)-(-\frac{1}{2})-4}{\sqrt{[2^2+2^2+(-1)^2]}}=\frac{3}{\sqrt{6}}=\frac{1}{2}\sqrt{6}=p \text{ (say)}$$

Then radius of the circle $=\sqrt{(R^2-p^2)}$ ...See § 7.05 Page 13 Ch. VII

$$=\sqrt{(\tfrac{5}{2}-\tfrac{6}{4})}=1.$$ Hence proved.

The equations of the circle are $x^2+y^2+z^2-x+z-2=0,$

$x+2y-z-4=0$ ...See Note § 7.06 Page 15 Ch. VII

$\therefore$ The equation of a sphere through this circle is

$$(x^2+y^2+z^2-x+z-2)+\lambda(x+2y-z-4)=0$$

...See § 7.07 Page 15 Ch. VII

or $$x^2+y^2+z^2+(\lambda-1)x+2\lambda y+(1-\lambda)z-(2+4\lambda)=0 \quad ...(i)$$

Its centre is $[-\frac{1}{2}(\lambda-1),-\lambda,-\frac{1}{2}(1-\lambda)]$. If this circle is a great circle of the sphere (i), then the centre of (i) should lie on the plane of the circle *i.e.* the plane $x+2y-z-4=0$.

$\therefore$ $$-\frac{1}{2}(\lambda-1)+2(-\lambda)+\frac{1}{2}(1-\lambda)-4=0$$

or $$-3\lambda-3=0 \quad \text{or} \quad \lambda=-1.$$

$\therefore$ From (i), the equation of the required sphere is

$$x^2+y^2+z^2-2x-2y+2=0.$$ **Ans.**

****Ex. 8 (a). Find the equation of the sphere for which the circle $x^2+y^2+z^2+7y-2z+2=0$, $2x+3y+4z=8$ is a great circle.**

(*Bundelkhand* 94; *Rohilkhand* 92)

Sol. The equation of any sphere through the given circle is

$$(x^2+y^2+z^2+7y-2z+2)+\lambda(2x+3y+4z-8)=0 \quad ...(i)$$

or $$x^2+y^2+z^2+2\lambda x+(7+3\lambda)y+(4\lambda-2)z+(2-8\lambda)=0$$

Its centre is $[-\lambda,-\frac{1}{2}(7+3\lambda),1-2\lambda]$ ...(ii)

Now if the given circle (which is the section of the sphere $x^2+y^2+z^2-7y-2z+2=0$ by the plane $2x+3y+4z=8$) is a great circle of the sphere (i), then the centre of the sphere (i) must lie on the plane of the circle *i.e.* on the plane $2x+3y+4z=8$.

$\therefore$ From (ii) we get $2(-\lambda)+3[-\frac{1}{2}(7+3\lambda)]+4(1-2\lambda)=8$ **(Note)**

or $$-2\lambda-\frac{21}{2}-\frac{9}{2}\lambda+4-8\lambda=8 \quad \text{or} \quad \lambda=-1$$

Substituting the value of λ in (i), we get the required equation as

$$(x^2+y^2+z^2+7y-2z+2)-(2x+3y+4z-8)=0$$

or $$x^2+y^2+z^2-2x+4y-6z+10=0.$$ **Ans.**

***Ex. 8 (b). Find the equation of the sphere whose great circle is $x^2+y^2+z^2+10y-4z=8$, $x+y+z=3$.** (*Rohilkhand* 95)

Sol. Equation of any sphere through the given circle is

$$(x^2+y^2+z^2+10y-4z-8)+\lambda(x+y+z-3)=0 \quad ...(i)$$

or $$x^2+y^2+z^2+\lambda x+(10+\lambda)y-(4-\lambda)z-(8+3\lambda)=0$$

Its centre is $[-\frac{1}{2}\lambda,-5-\frac{1}{2}\lambda,2-\frac{1}{2}\lambda]$ ...(ii)

Now if the given circle is a great circle of the sphere (i), then the centre of sphere (i) must lie on the plane of the circle *i.e.* on the plane $x + y + z = 3$.

$\therefore$ From (ii), we get $(-\frac{1}{2}\lambda) + (-5 - \frac{1}{2}\lambda) + (2 - \frac{1}{2}\lambda) = 3$

or $$-(3/2)\lambda - 6 = 0 \quad \text{or} \quad \lambda = -4$$

Substituting this value of λ in (i), we get the required equation as

$$(x^2 + y^2 + z^2 + 10y - 4z - 8) - 4(x + y + z - 3) = 0$$

or $$x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0.$$ **Ans.**

****Ex. 9. Prove that the circles $x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0$, $5y + 6z + 1 = 0$ and $x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0$, $x + 2y - 7z = 0$ lies on the same sphere and find its equation. Also find the value of *a* for which $x + y + z = a\sqrt{3}$ touches the sphere.**

(*Avadh* 94; *Bundelkhand* 96, 93; *Garhwal* 93; *Kanpur* 95)

Sol. The equation of any sphere through the first circle is

$$(x^2 + y^2 + z^2 - 2x + 3y + 4z - 5) + \lambda(5y + 6z + 1) = 0 \quad \text{...(i)}$$

Similarly the equation of any sphere through the second circle is

$$(x^2 + y^2 + z^2 - 3x - 4y + 5z - 6) + \mu(x + 2y - 7z) = 0 \quad \text{...(ii)}$$

If the given circles lies on the same sphere then (i) and (ii) should represent the same sphere, so comparing the coefficients of x, y, z and constant terms in (i) and (ii) we get

$-2 = -3 + \mu$ ...(iii); $\quad 3 + 5\lambda = -4 + 2\mu$ ...(iv)

$4 + 6\lambda = 5 - 7\mu$ ...(v) ; and $\quad -5 + \lambda = -6$ ...(vi)

From (iii) and (vi) we get $\mu = -1$, $\lambda = -1$.

These values of λ and μ satisfy (iv) and (v), hence the given circles lie on the same sphere. Putting $\lambda = -1$ in (i) we get the required equation of the sphere as $$x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0 \quad \text{...(vii)}$$

Its centre in (1, 1, 1) and radius $= \sqrt{(1^2 + 1^2 + 1^2 + 6)} = 3$. If the plane $x + y + z = a\sqrt{3}$ ...(viii) touches the sphere (vii), then the length of the perpendicular from the centre (1, 1, 1) of the sphere to the plane (viii) must be equal to the radius of the sphere.

i.e. $$\frac{1 + 1 + 1 - a\sqrt{3}}{\sqrt{(1^2 + 1^2 + 1^2)}} = 3 \quad \text{or} \quad 3 - a\sqrt{3} = \pm 3\sqrt{3}$$

or $$a\sqrt{3} = 3 \pm 3\sqrt{3} \quad \text{or} \quad a = \sqrt{3} \pm 3.$$ **Ans.**

Ex. 10. Find the condition that the circles.

$$\mathbf{x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0,\ lx + my + nz = p;}$$

$$\mathbf{x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0,\ l'x + m'y + n'z = p'}$$

should lie on the same sphere. (*Garhwal* 96)

Sol. The equation of any sphere through the first circle is

$$(x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d) + \lambda(lx + my + nz - p) = 0 \quad \text{...(i)}$$

Similarly the equation of any sphere through second circle is

$$(x^2+y^2+z^2+2u'x+2v'y+2w'z+d')+\mu(l'x+m'y+n'z-p')=0 \quad ...(ii)$$

If the given circles lie on the same sphere, then for some values of λ and μ the equations (i) and (ii) should represent the same sphere. Comparing the coefficients of x, y, z and constant terms in (i) and (ii) we get

$$2u+\lambda l=2u'+\mu l' \quad \text{or} \quad 2(u-u')+\lambda l-\mu l'=0 \quad ...(iii)$$
$$2v+\lambda m=2v'+\mu m' \quad \text{or} \quad 2(v-v')+m-\mu m'=0 \quad ...(iv)$$
$$2w+\lambda n=2w'+\mu n' \quad \text{or} \quad 2(w-w')+\lambda n-\mu n'=0 \quad ...(v)$$

or
$$d-\lambda p=d'-\mu p' \quad \text{or} \quad -(d-d')+\lambda p-\mu p'=0 \quad ...(vi)$$

Eliminating λ and $-\mu$ from (iii), (iv), (v) and (vi) we get required condition as

$$\begin{vmatrix} 2(u-u') & l & l' \\ 2(v-v') & m & m' \\ 2(w-w') & n & n' \\ -(d-d') & p & p' \end{vmatrix}=0$$

Ans.

***Ex. 11. Find the equation of the sphere passing through the circles $y^2+z^2=9, x=4$ and $y^2+z^2=36, x=1$.**

Sol. The equation of any sphere through the circle $y^2+z^2=9, x=4$ is

$$(x^2+y^2+z^2-9-16)+\lambda(x-4)=0 \quad ...(i)$$

(Note the introduction of x^2)

Similarly the equation of any sphere through the circle $y^2+z^2=36, x=1$ is

$$(x^2+y^2+z^2-36-1)+\mu(x-1)=0 \quad ...(ii)$$

If the two given circles lie on the same sphere then (i) and (ii) should represent that sphere on which the given circles lie *i.e.* (i) and (ii) represent the same sphere

$\therefore$ Comparing the coefficients of x and constant terms in (i) and (ii) we get $\lambda=\mu$ and $25+4\lambda=37+\mu$

Solving these we get $\lambda=\mu=4$

$\therefore$ From (i) and (ii) we get the required equation as

$$(x^2+y^2+z^2-25)+4(x-4)=0 \quad \text{or} \quad x^2+y^2+z^2+4x=41. \quad \textbf{Ans.}$$

***Ex. 12. Prove that the sphere**

$$\mathbf{S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0} \textbf{ cuts the sphere}$$

$$\mathbf{S' \equiv x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0} \textbf{ in a great circle if}$$

$$\mathbf{2(u'^2+v'^2+w'^2)-d'=2(uu'+vv'+ww')-d}$$

or if $2(uu'+vv'+ww')=2r'^2+d+d'$, where r' is the radius of the second sphere.

Sol. The equation of the plane through the circle of inttersection of the given sphere is $S-S'=0$

i.e. $$2(u-u')x+2(v-v')y+2(w-w')z+(d-d')=0 \quad ...(i)$$

If the sphere $S=0$ cuts the sphere $S'=0$ in a great circle, then the centre $(-u',-v',-w')$ of the sphere $S'=0$ should lie on the plane (i). **(Note)**

$$\therefore -2(u-u')u'-2(v-v')v'-2(w-w')w'+(d-d')=0$$

or $$2(u'^2+v'^2+w'^2)-d'=2(uu'+vv'+ww')-d \qquad ...(ii)$$

Hence proved.

or $2(r'^2+d')-d'=2(uu'+vv'+ww')-d$, where r' is the radius of the second sphere and so $r'^2=u'^2+v'^2+w'^2-d'$

or $$2r'^2+d'+d=2(uu'+vv'+ww'). \qquad \text{Hence proved.}$$

***Ex. 13. Find the equations of the spheres which pass through circle $x^2+y^2+z^2=5$, $x+2y+3z=3$ and touch the plane $4x+3y=15$.**

(*Kanpur* 97, 91; *Rohilkhand* 94)

Sol. The equation of any sphere through the given circle is

$$(x^2+y^2+z^2-5)+\lambda(x+2y+3z-3)=0$$

or $$x^2+y^2+z^2+\lambda x+2\lambda y+3\lambda z-(5+3\lambda)=0 \qquad ...(i)$$

Its centre is $(-\frac{1}{2}\lambda,-\lambda,-\frac{3}{2}\lambda)$ and its radius

$$=\sqrt{[(-\tfrac{1}{2}\lambda)^2+(-\lambda)^2+(-\tfrac{3}{2}\lambda)^2+(5+3\lambda)]}=\sqrt{(\tfrac{7}{2}\lambda^2+3\lambda+5)}$$

If the sphere (i) touches the given plane $4x+3y-15=0$ ...(ii)

then the length of the perpendicular from the centre of (i) to (ii) must be equal to the radius of (i).

$$\therefore \frac{4(-\frac{1}{2}\lambda)+3(-\lambda)-15}{\sqrt{(4^2+3^2)}}=\sqrt{(\tfrac{7}{2}\lambda^2+3\lambda+5)}$$

or $$(-5\lambda-15)^2=25(\tfrac{7}{2}\lambda^2+3\lambda+5)$$

or $$\lambda^2+6\lambda+9=\tfrac{7}{2}\lambda^2+3\lambda+5 \quad \text{or} \quad 5\lambda^2-6\lambda-8=0$$

or $$(5\lambda+4)(\lambda-2)=0 \quad \text{or} \quad \lambda=2,-\tfrac{4}{5}.$$

Substituting these values of λ in (i), we get required equations as

$$x^2+y^2+z^2+2x+4y+6z-11=0$$

and $$x^2+y^2+z^2-\frac{4}{5}x-\frac{8}{5}y-\frac{12}{5}z-\frac{13}{5}=0. \qquad \text{Ans.}$$

***Ex. 14. Find the equations of the spheres through the circle $x^2+y^2+z^2=1$, $2x+4y+5z=6$ and touching the plane $z=0$.**

(*Purvanchal* 95; *Rohilkhand* 93)

Sol. The equation of any sphere through the given circle is

$$(x^2+y^2+z^2-1)+\lambda(2x+4y+5z-6)=0$$

or $$x^2+y^2+z^2+2\lambda x+4\lambda y+5\lambda z-(1+6\lambda)=0 \qquad ...(i)$$

Its centre is $(-\lambda, -2\lambda, -\frac{5}{2}\lambda)$ and its radius

$= \sqrt{[(-\lambda)^2 + (-2\lambda)^2 + (-\frac{5}{2}\lambda)^2 + (1+6\lambda)]} = \sqrt{(\frac{45}{4}\lambda^2 + 6\lambda + 1)}$

If the sphere (i) touches the palne $z = 0$ ...(ii)

then the length of the perpendicular from the centre of (i) to (ii) must be equal to the radius of (i).

$$\therefore \frac{-\frac{5}{2}\lambda}{\sqrt{(1)^2}} = \sqrt{[(45/4)\lambda^2 + 6\lambda + 1]} \qquad \textbf{(Note)}$$

or $$(25/4)\lambda^2 = (45/4)\lambda^2 + 6\lambda + 1$$

or $5\lambda^2 + 6\lambda + 1 = 0$ or $(5\lambda+1)(\lambda+1) = 0$ or $\lambda = -1, -\frac{1}{5}$.

Substituting these values of λ in (i) we get the required equations as

$$x^2 + y^2 + z^2 - 2x - 4y - 5z + 5 = 0$$

and $$x^2 + y^2 + z^2 - \frac{2}{5}x - \frac{4}{5}y - z + \frac{1}{5} = 0.$$ **Ans.**

***Ex. 15. P is the variable point on the given line and A, B, C are its projections on the axes. Show that the sphere O, ABC passes through a fixed circle.**

Sol. Let the equations of the given line be $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$.

Any variable point on this line may be taken as $P(\alpha + lr, \beta + mr, \gamma + nr)$.

Given that the points A, B, C are the projections of P on the axes, so the coordinates of A, B and C are $(\alpha + lr, 0, 0)$, $(0, \beta + mr, 0)$ and $(0, 0, \gamma + nr)$ respectively. **(Note)**

Now the equation of the sphere thorugh $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ as in Ex. 5 Page 4 of this chapter is $x^2 + y^2 + z^2 - ax - by - cz = 0$

$\therefore$ In this problem the equation of the sphere O, ABC is

$$x^2 + y^2 + z^2 - (\alpha + lr)x - (\beta + mr)y - (\gamma + nr)z = 0$$

or $$(x^2 + y^2 + z^2 - \alpha x - \beta y - \gamma z) - r(lx + my + nz) = 0,$$

which is of the form $S + \lambda P = 0$ where $\lambda = -r$.

$\therefore$ The sphere for all values of r passes through the fixed circle

$$x^2 + y^2 + z^2 - \alpha x - \beta y - \gamma z = 0,\ lx + my + nz = 0.$$ Hence proved.

****Ex. 16 (a). A variable plane is parallel to the given plane $x/a + y/b + z/c = 0$ and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone**

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0.$$

(*Meerut* 91, 90; *Rohilkhand* 91)

Sol. The equation of any plane parallel to the given plane is

$$(x/a)+(y/b)+(z/c)=k \quad ...(i)$$

The plane (i) meets the axes in A, B and C, so the coordinates of A, B and C are $(ak, 0, 0)$; $(0, bk, 0)$ and $(0, 0, ck)$ respectively.

$\therefore$ The equation of the sphere O, ABC as in last example is

$$x^2+y^2+z^2-akx-bky-ckz=0$$

or $$(x^2+y^2+z^2)-k\,(ax+by+cz)=0 \quad ...(ii)$$

The equation of the circle ABC are given by (i) and (ii) hence eliminating k between (i) and (ii), we get the required locus of circle ABC as

$$(x^2+y^2+z^2)-\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)(ax+by+cz)=0$$

or $$\frac{x}{a}(by+cz)+\frac{y}{b}(ax+cz)+\frac{z}{c}(ax+by)=0$$

or $$yz\left(\frac{c}{b}+\frac{b}{c}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{b}{a}+\frac{a}{b}\right)=0.$$ **Ans.**

Ex. 16 (b). A variable plane is parallel to the given plane $(x/a)+(y/b)+(z/c)=1$ and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone $\Sigma\,[yz\,\{(c/b)+(b/c)\}]$. (*Bundelkhand* 90)

Sol. Do as Ex. 16 (a) above.

***Ex. 17. Find the equation of the sphere through the circle $x^2+y^2+z^2=9$, $2x+3y+4z=5$ and the point (1, 2, 3).**

(*Bundelkhand* 92; *Garhwal* 93; *Purvanchal* 97, 94; *Rohilkhand* 94)

Sol. The equation of the sphere through the given circle is

$$(x^2+y^2+z^2-9)+\lambda\,(2x+3y+4z-5)=0, \quad ...(i)$$

where λ is a constant.

If it passes through the given point (1, 2, 3) then

$$(1^2+2^2+3^2-9)+\lambda\,(2+6+12-5)=0$$

or $$5+15\lambda=0 \quad \text{or} \quad \lambda=-\frac{1}{3}$$

$\therefore$ From (i), the required equation of the sphere is

$$(x^2+y^2+z^2-9)-\frac{1}{3}(2x+3y+4z-5)=0$$

or $$3\,(x^2+y^2+z^2)-(2x+3y+4z)-22=0.$$ **Ans.**

****Ex. 18 (a). Find the equation of the sphere which passes through the point (α, β, γ) and the circle $x^2+y^2=a^2$, $z=0$.**

(*Avadh* 92; *Kanpur* 93, 90; *Lucknow* 91; *Meerut* 98, 95)

Sol. The equations of the circle are given as $x^2+y^2=a^2$, $z=0$.

These can be written as $x^2+y^2+z^2-a^2=0$, $z=0$

(Note the introduction of z^2)

$\therefore$ The equation of the sphere through given circle is

$$(x^2+y^2+z^2-a^2)+\lambda z=0, \text{ where } \lambda \text{ is constant.} \quad ...(i) \textbf{ (Note)}$$

If it passes through (α, β, γ) then we have

$$\alpha^2+\beta^2+\gamma^2-a^2+\lambda\gamma=0 \text{ of } \lambda=(a^2-\alpha^2-\beta^2+\gamma^2)/\gamma.$$

Substituting this value of λ in (i) the required equation is

$$(x^2+y^2+z^2-a^2)\gamma+(a^2-\alpha^2-\beta^2-\gamma^2)z=0. \qquad \textbf{Ans.}$$

Ex. 18 (b). Find the equation of the sphere through the circle $x^2+y^2=a^2$, $z=0$ and through the centre of the sphere

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2. \qquad (\textit{Meerut } 92\ P)$$

Sol. The centre of the given sphere is (α, β, γ) and so this example is the same as Ex. 18 (a) above.

Ex. 18 (c). A circle, centre (2, 3, 0) and radius 1, is drawn in the plane $z=0$. Find the equation of the sphere which passes through this circle and the point (1, 1, 1).

Sol. The equations of the given circle are

$$(x-2)^2+(y-3)^2=1^2, z=0. \qquad \textbf{(Note)}$$

which can be rewritten as $(x-2)^2+(y-3)^2+z^2-1=0, z=0.$

(Note the introduction of term z^2)

Equation of any sphere through this circle is

$$[(x-2)^2+(y-3)^2+(z^2-1)]+\lambda z=0. \qquad ...(i)$$

where λ is constant.

If it passes through (1, 1, 1), then we have

$$[(1-2)^2+(1-3)^2+(1^2-1)]+\lambda(1)=0 \quad \text{or} \quad \lambda=-5$$

Substituting this value of λ in (i), the required equation is

$$(x-2)^2+(y-3)^2+z^2-1-5z=0$$

or $$x^2+y^2+z^2-4x-6y-5z+12=0. \qquad \textbf{Ans.}$$

Ex. 19. Find the plane, the centre and the radius of the circle common to the two spheres

$$\mathbf{x^2+y^2+z^2-4z+1=0 \text{ and } x^2+y^2+z^2-4x-2y-1=0.}$$

Sol. The required equation of the plane is

$$(x^2+y^2+z^2-4z+1)-(x^2+y^2+z^2-4x-2y-1)=0 \qquad \textbf{(Note)}$$

or $$4x+2y-4z+2=0 \quad \text{or} \quad 2x+y-2z+1=0 \qquad ...(i)$$

The centre of the sphere $x^2+y^2+z^2-4z+1=0 \qquad ...(ii)$

is $C(0, 0, 2)$ and radius $=\sqrt{[(0)^2+(0)^2+(2)^2-(1)]}=\sqrt{3}=R$ (say).

$\therefore$ Length of perpendicular from $C(0, 0, 2)$ to the plane (i)

$$=\frac{2.0+0-2.2+1}{\sqrt{[2^2+0^2+(-2)^2]}}=\frac{3}{2\sqrt{2}} \text{ numerically } =p \text{ (say)}$$

$\therefore$ Required radius of the circle $= \sqrt{(R^2 - p^2)} = \sqrt{[3 - (9/8)]}$.
$= \sqrt{(15/8)}$. **Ans.**

Let C_1 be the centre of the circle, then CC_1 is perpendicular to the plane (i); so the d.c.'s of the line CC_1 are the same as those of the normal to the plane (i) *i.e.* 2, 1, – 2. ...See Ex. 1 (a). Page 15 of this chapter

$\therefore$ The equation of line CC_1 is $\dfrac{x-0}{2} = \dfrac{y-0}{1} = \dfrac{z-2}{-2} = r$ (say).

Let $CC_1 = r$, then the coordinates of C_1 are $(2r, r, 2-2r)$ and C_1 lies on (i) so we get $2(2r) + (r) - 2(2-2r) + 1 = 0$ or $9r - 3 = 0$ or $r = 1/3$.

$\therefore$ The coordinates of C_1 are $(2/3, 1/3, 4/3)$. **Ans.**

***Ex. 20. POP′ is a variable diameter of the ellipse $z = 0$, $x^2/a^2 + y^2/b^2 = 1$ and a circle is described in the plane PP′ zz′ on PP′ as diameter, prove that as PP′ varies, the circle generates the surface**

$$(x^2 + y^2 + z^2)[(x^2/a^2) + (y^2/b^2)] = x^2 + y^2 \qquad \text{(Garhwal 90)}$$

Sol. From our knowledge of coordinate geometry of two dimensions we know that the parametric coordinates of extremities of a diameter of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ can be taken as $(a \cos\theta, b \sin\theta)$ and $(-a\cos\theta, -b\sin\theta)$.

$\therefore$ Here in this problem (of 3 dimensions), since POP' is a diameter of the ellipse on the plane $z = 0$, so the coordinates of P and P' can be taken as

$(a\cos\theta, b\sin\theta, 0)$ and $(-a\cos\theta, -b\sin\theta, 0)$ **(Note)**

$\therefore$ The equation of the sphere drawn on PP' as diameter is

$$(x - a\cos\theta)(x + a\cos\theta) + (y - b\sin\theta)(y + b\sin\theta) + (z-0)(z-0) = 0$$

or
$$x^2 + y^2 + z^2 = a^2\cos^2\theta + b^2\sin^2\theta \qquad \text{...(i)}$$

The circle referred in the problem is the intersection of this sphere (i) and the plane $PP'zz'$.

Now the equation of any plane through zz' *i.e.* z-axis *i.e.* $x = 0; y = 0$ is

$$x + \lambda y = 0. \qquad \text{...(ii)}$$

If this plane passes through $P(a\cos\theta, b\sin\theta, 0)$ then

$a\cos\theta + \lambda b \sin\theta = 0$ or $\lambda = -(a\cos\theta)/(b\sin\theta)$

$\therefore$ From (ii), the equation of the plane $PP'zz'$ is

$$x - \left(\frac{a\cos\theta}{b\sin\theta}\right)y = 0 \quad \text{or} \quad \frac{x}{a\cos\theta} = \frac{y}{b\sin\theta} \qquad \text{...(iii)}$$

Evidently the coordinates of $P'(-a\cos\theta, -b\sin\theta, 0)$ satisfy it, so (iii) represents the equation of the plane $PP'zz'$.

$\therefore$ The equations of circle described in the plane $PP'zz'$ on PP' as diameter are given by the sphere (i) and the plane (iii). Now in order to find the locus of this circle, we are to eliminate θ from (i) and (ii). From (iii) we get

$$\frac{(x/a)}{\cos\theta}=\frac{(y/b)}{\sin\theta}=\frac{\sqrt{[(x/a)^2+(y/b)^2]}}{\sqrt{(\cos^2\theta+\sin^2\theta)}}=\sqrt{\left[\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2\right]}$$

These give $a\cos\theta=\dfrac{x}{\sqrt{[(x/a)^2+(y/b)^2]}}$, $b\sin\theta=\dfrac{y}{\sqrt{[(x/a)^2+(y/b)^2]}}$

Substituting these values in (i) we get the required locus as

$$(x^2+y^2+z^2)=\frac{x^2+y^2}{(x/a)^2+(y/b)^2} \quad \text{or} \quad (x^2+y^2+z^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=x^2+y^2$$

Hence proved.

R ***Ex. 21. A sphere whose centre lies in the positive octant passes through the origin and cuts the planes $x=0, y=0, z=0$ in circles of radii $a\sqrt{2}, b\sqrt{2}, c\sqrt{2}$ respectively. Find the equation of this sphere.**

Sol. $\because$ The sphere passes through origin, so its equation can be taken as

$$x^2+y^2+z^2+2ux+2vy+2wz=0 \qquad \text{...(i)}$$

As this sphere meets the plane $z=0$; so putting $z=0$ in (i) we get $x^2+y^2+2ux+2vy=0$; which is evidently a circle on xy-plane and its radius $=\sqrt{(u^2+v^2)}$.

But we are given that the sphere meets the plane $z=0$ in a circle of radius $c\sqrt{2}$, so we have $\sqrt{(u^2+v^2)}=c\sqrt{2}$.

or $$u^2+v^2=2c^2 \qquad \text{...(ii)}$$

Similarly as the sphere (i) meets the planes $x=0, y=0$, in circle of radii $a\sqrt{2}, b\sqrt{2}$, so we have $v^2+w^2=2a^2$ ...(iii) and $w^2+u^2=2b^2$ ...(iv)

Adding (ii), (iii) and (iv) we get $2(u^2+v^2+w^2)=2a^2+2b^2+2c^2$

or $$u^2+v^2+w^2=a^2+b^2+c^2. \qquad \text{...(v)}$$

Subtracting (iii), (iv) and (ii) from (v) by turn we get

$$u^2=b^2+c^2-a^2,\ v^2=c^2+a^2-b^2 \text{ and } w^2=a^2+b^2-c^2$$

or $$u=\pm\sqrt{(b^2+c^2-a^2)},\ v=\pm\sqrt{(c^2+a^2-b^2)},\ w=\pm\sqrt{(a^2+b^2-c^2)}$$

Also it is given that the centre of the sphere (i) viz. $(-u,-v,-w)$ lies in the positive octant, so the values of u, v, w must be negative. **(Note)**

$$\therefore\ u=-\sqrt{(b^2+c^2-a^2)},\ v=-\sqrt{(c^2+a^2-b^2)},\ w=-\sqrt{(a^2+b^2-c^2)}$$

Substituting these values of u, v, w in (i), the required equation is

$$x^2+y^2+z^2-2x\sqrt{(b^2+c^2-a^2)}-2y\sqrt{(c^2+a^2-b^2)}$$
$$-2z\sqrt{(a^2+b^2-c^2)}=0. \qquad \textbf{Ans.}$$

P ***Ex. 22. A is point on OX and B on OY, so that the angle OAB is constant and equal to α. On AB as diameter a circle is drawn whose plane**

is parallel to OZ. Prove that as AB varies the circle generates the cone

$$2xy - z^2 \sin 2\alpha = 0. \qquad (Garhwal\ 91;\ Kanpur\ 96)$$

Sol. Let the coordinates of A and B be $(a, 0, 0)$ and $(0, b, 0)$ respectively. Also $\angle AOB = \alpha$ (given), so in right angled triangle AOB with $\angle AOB = \pi/2$ we have

$$\tan \alpha = OB/OA = b/a \qquad \text{...(i)}$$

(Students should draw ΔAOB and verify for themselves).

Now the equation of the plane containing z-axis *i.e.* $x = 0$, $y = 0$ is of the form $x + \lambda y = 0$, whcih does not contain z.

The equation of the plane through A and B and parallel to OZ (*i.e.* z-axis) is

$$(x/a) + (y/b) = 1, \qquad \text{...(ii)}$$

which also does not contain z.

Now the equation of the sphere drawn on AB as diameter is

$$(x - a)(x - 0) + (y - 0)(y - b) + (z - 0)(z - 0) = 0$$

or $$x^2 + y^2 + z^2 = ax + by$$

$\therefore$ The equations of the circle drawn on AB as diameter and plane parallel to z-axis are $x^2 + y^2 + z^2 = ax + by$, $(x/a) + (y/b) = 1$ ...(iii)

The required locus of the circle as AB varies (*i.e.* as a and b vary), is obtained by eliminating a and b between (i) and (iii).

From (iii) we have $x^2 + y^2 + z^2 = (ax + by)\,[(x/a) + (y/b)]$ **(Note)**

or $$x^2 + y^2 + z^2 = x^2 + y^2 + xy\,[(a/b) + (b/a)]$$

or $$z^2 = xy\,[\cot \alpha + \tan \alpha], \text{ from (i)}$$

or $$z^2 = xy\left[\frac{\cos \alpha}{\sin \alpha} + \frac{\sin \alpha}{\cos \alpha}\right] = xy\left[\frac{\cos^2 \alpha + \sin^2 \alpha}{\sin \alpha \cos \alpha}\right]$$

or $$2z^2 \sin \alpha \cos \alpha = 2xy \quad \text{or} \quad z^2 \sin 2\alpha = 2xy. \qquad \text{Hence proved.}$$

Ex. 23. Spheres are described to contain the circle $z = 0$. $x^2 + y^2 = a^2$. Prove that the locus of the extremities of their diameters which are parallel to the x-axis is the rectangular hyperbola $x^2 - z^2 = a^2$, $y = 0$ (*Purvanchal 97*)

Sol. The equation of the sphere through the given circle

$$x^2 + y^2 = a^2, z = 0 \quad \text{is} \quad (x^2 + y^2 + z^2 - a^2) + \lambda z = 0 \qquad \text{...(i)}$$

...See Ex. 18 (a) Page 25 of this chapter

Its centre is $(0, 0, -\lambda/2)$ and radius $= \sqrt{[(-\lambda/2)^2 - (-a^2)]}$

$$= [\sqrt{(\lambda^2 + 4a^2)}]/2$$

Now the equations of the diameter of the sphere (i) and parallel to x-axis *i.e.* the line through the centre $(0, 0, -\lambda/2)$ and parallel to the line with d.c.'s 1, 0, 0 are

$$\frac{x - 0}{1} = \frac{y - 0}{0} = \frac{z + (\lambda/2)}{0}$$

The coordinates of any point on it at a distance r from the centre $(0, 0, -\lambda/2)$ of the sphere (i) are $(r, 0, -\lambda/2)$.

If we take $r = \pm \frac{1}{2}\sqrt{(\lambda^2 + 4a^2)} = \pm$ radius of the sphere then we find that the coordinates of the extremities of the diameter parallel to x-axis are given by

$$x = \pm \frac{1}{2}\sqrt{(\lambda^2 + 4a^2)},\ y = 0,\ z = -\lambda/2 \qquad \text{...(ii)}$$

Required locus is obtained by eliminating λ from (ii).

From (ii) we have $4x^2 = \lambda^2 + 4a^2,\ y = 0,\ 2z = -\lambda$

or $$4x^2 = (-2z)^2 + 4a^2,\ y = 0 \text{ on eliminating } \lambda$$

or $x^2 - z^2 = a^2$, $y = 0$, which is the required locus and is a rectangular hyperbola on the plane $y = 0$.

Hence proved.

***Ex. 24. If a plane cuts the sphere $x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0$ in a circle whose centre is $(13/3, -8/3, -10/3)$, then obtain the equations of the circle and hence its radius.** (*Allahabad* 91)

Sol. Refer Fig. 1 on Page 13.

The centre O of the given sphere is $(4, -2, -4)$ and radius

$$= \sqrt{(4^2 + 2^2 + 4 + 45)} = \sqrt{(81)} = 9 = OA.$$

The centre C of the circle is given as $(13/3, -8/3, -10/3)$

$$\therefore\ OC^2 = \left(\frac{13}{3} - 4\right)^2 + \left(-\frac{8}{3} + 2\right)^2 + \left(-\frac{10}{3} + 4\right)^2$$

or $$OC^2 = (1/9) + (4/9) + (4/9) = 1 \quad \text{or} \quad OC = 1.$$

Also d. ratios of OC are $(13/3) - 4, (-8/3) + 2, (-10/3) + 4$

i.e. $$1/3, -2/3, 2/3 \quad i.e. \quad 1, -2, 2$$

$$\therefore \text{ d. cosines of } OC \text{ are } \frac{1, -2, 2}{\sqrt{[1^2 + (-2)^2 + 2^2]}} \quad i.e. \quad \frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$$

$\therefore$ Equations of the plane through $C(13/3, -8/3, -10/3)$ and perpendicular to OC is

$$(1/3)[x - (13/3)] - (2/3)[y + (8/3)] + (2/3)[z + (10/3)] = 0$$

or $$(3x - 13) - 2(3y + 8) + 2(3z + 10) = 0$$

or $$3x - 6y + 6z - 9 = 0 \quad \text{or} \quad x - 2y + 2z = 3$$

$\therefore$ Required equations of the circle is

$$x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0,\ x - 2y + 2z = 3$$

Also its raadius $= CA = \sqrt{(OA^2 - OC^2)} = \sqrt{[9^2 - 1^2]} = \sqrt{(80)} = 4\sqrt{5}$. **Ans.**

Exercises on § 7.05—§ 7.07.

Ex. 1. Obtain the radius and centre of the circle $x^2 + y^2 + z^2 + x + y + z - 4 = 0, x + y + z = 0$. (*Bundelkhand* 96, 94)

Ans. 2 ; (0, 0, 0)

Ex. 2. Find the centre and radius of the circle in which the sphere $x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0$ is cut by the plane $x + 2y + 2z + 7 = 0$.

Ans. $\left(-\frac{7}{3}, -\frac{5}{3}, -\frac{2}{3}\right)$; $(5/3)\sqrt{5}$.

Ex. 3. Find the radius of the circle in which the sphere $x^2+y^2+z^2=4$ is cut by the plane $x+y+z=1$. **Ans.** $\sqrt{(11/3)}$

Ex. 4. Find the centre and radius of the circle

$$x^2+y^2+z^2-8x+4y+8z-45=0,\ x-2y+3z=3.$$

Ans. $(\frac{9}{2}, -3, -\frac{5}{2})$; $\sqrt{(155/2)}$

Ex. 5. Find the radius and centre of the circle of intersection of the sphere $x^2+y^2+z^2-2y-4z-11=0$ and the plane $x+2y+2z=15$.

Ans. $\sqrt{7}$ and $(1, 3, 4)$

Ex. 6. Find the radius of the circle given by the equation

$$3x^2+3y^2+3z^2+x-5y-2=0,\quad x+y=2.$$ **Ans.** $1/\sqrt{2}$.

Ex. 7. Find out the equation of the sphere having the circle given by $x^2+y^2+z^2=9$, $x-2y-2z=5$ for a great circle. Also find out its centre and radius. **Ans.** $9(x^2+y^2+z^2)-10x+20y-20z=31$;

$(5/9,\ -10/9,\ 10/9)$; $\frac{2}{3}\sqrt{(14)}$.

Ex. 8. Find the equation of the sphere having the circle $x^2+y^2+z^2-2x+4y-6z+7=0$, $2x-y+2z=5$ for a great circle.

Ans. $9(x^2+y^2+z^2)+2x+26y-34z+13=0$.

Ex. 9. Find the equation of the sphere passing through the circle $x^2+z^2=25$, $y=2$ and $x^2+z^2=16$, $y=3$.

Ex. 10. Find the equations of the spheres which passes through the circle $x^2+y^2+z^2-2x+2y+4z=3$; $2x+y+z+4=0$ and touch the plane $3x+4y=14$.

Ans. $x^2+y^2+z^2-2x+2y+4z=3$; $x^2+y^2+z^2+34x+20y-22z+69=0$.

Ex. 11. Find the equation of the spheres which pass through the circle $x^2+y^2+z^2-2x+2z=2$, $y=0$ and touch the plane $y+z=7$.

Ans. $x^2+y^2+z^2-2x-4y+2z=2$; $x^2+y^2+z^2-2x+28y+2z=2$.

Ex. 12. Find the equations of the spheres which pass through the circle $x^2+y^2+z^2=2x+4y$; $x+2y+3z=8$ and touching the plane $4x+3y=25$.

Ans. $x^2+y^2+z^2+6z-16=0$; $5x^2+5y^2+5z^2-14x-28y-12z+32=0$

Ex. 13. Find the equation of the sphere through the circle $x^2+y^2+z^2=9$; $2x+3y+4z=5$ and the point $(1, 2, 3)$.

[Hint : Use $S+\lambda P=0$]. *(Bundelkhand* 92)

Ans. $3(x^2+y^2+z^2)-2x-3y-4z=22$.

Ex. 14. Show that the two circles $x^2+y^2+z^2+3x-4y-3z=0$, $x-y+2z=4$ and $2(x^2+y^2+z^2)+8x-18y+17z=17$, $2x+y-3z+1=0$ lie on the same sphere. Find its equation. **Ans.** $x^2+y^2+z^2+5x-6y+7x=8$

**Ex. 15. Show that the circles, $x^2+y^2+z^2-y+2z=0$, $x-y+z=2$ and $x^2+y^2+z^2+x-3y+z=5$, $2x-y+4z=1$ lie on the same sphere. Find the equation of the sphere. *(Avadh 91; Gorakhpur 90; Kanpur 94)*

Ans. $x^2+y^2+z^2+3x-4y+5z=6$.

§ 7.08 Intersection of a straight line and a sphere.

Let the equations of the sphere and the straight line be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad \text{...(i)}$$

and

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \text{ (say)} \qquad \text{...(ii)}$$

Any point on the line (ii) is $(\alpha+lr, \beta+mr, \gamma+nr)$.

If this point lies on the sphere (i), then we have

$$(\alpha+lr)^2+(\beta+mr)^2+(\gamma+nr)^2+2u(\alpha+lr)+2v(\beta+mr)$$
$$+2w(\gamma+nr)+d=0$$

or

$$r^2(l^2+m^2+n^2)+2r[l(u+\alpha)+m(v+\beta)+n(w+\gamma)]$$
$$+(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d)=0 \qquad \text{...(iii)}$$

This is a quadratic equation in r and so gives two values of r and therefore the line (ii) meets the sphere (i) in two points which may be real, coincident or imaginary according as roots of (iii) are so.

Note :—If l, m, n are the actual direction cosines of the line (iii), then $l^2+m^2+n^2=1$ and then the equation (iii) can be simplified.

****§ 7.09. Equation of the tangent plane.**

(Agra 92; Kanpur 94; Kumaun 94, 92)

(A) Let us find the equation of the tangent plane to the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad \text{...(i)}$$

at the point (α, β, γ).

As (α, β, γ) is the point on the sphere (i), so we have

$$\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d=0. \qquad \text{...(ii)}$$

The equations of any line through the point (α, β, γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{...(iii)}$$

The points of intersection of the line (iii) and the sphere (i) as in § 7.08 above are given by

$$r^2(l^2+m^2+n^2)+2r\{l(\alpha+u)+m(\beta+v)+n(\gamma+w)\}$$
$$+(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d)=0$$

or $\quad r^2(l^2+m^2+n^2)+2r\{l(\alpha+u)+m(\beta+v)+n(\gamma+w)\}=0$, from (ii)

As one of the roots of (iv) is zero, so one of the points of intersection of the sphere (i) and the line (iii) coincides with the point (α, β, γ). If the line (iii) is a tangent line to the sphere (i) at (α, β, γ), then the other point of intersection should also coincide with (α, β, γ), *i.e.* the second root of (iv) should also vanish and for this form (iv), we have

$$l(\alpha+u)+m(\beta+v)+n(\gamma+w)=0 \quad \text{...(v)}$$

$\therefore$ The line (iii) is a tangent line to the sphere (i) if the d.c.'s of the line (iii) viz. l, m, n should satisfy the condition (v).

The tangent plane at (α, β, γ) to the sphere (i) is the locus of all such tangents and its equation is obtained by eliminating l, m, n between (v) and the equation (iii) of the line.

$\therefore$ The required equation of the tangent plane is

$$(x-\alpha)(\alpha+u)+(y-\beta)(\beta+v)+(z-\gamma)(\gamma+w)=0$$

or $\quad x(\alpha+u)+y(\beta+v)+z(\gamma+w)-(\alpha^2+\beta^2+\gamma^2+u\alpha+v\beta+w\gamma)=0$

or $\quad x(\alpha+u)+y(\beta+v)+z(\gamma+w)+(u\alpha+v\beta+w\gamma+d)=0,$

with the help of (ii)

or $\quad x\alpha+y\beta+z\gamma+u(x+\alpha)+v(y+\beta)+w(z+\gamma)+d=0. \quad \text{...(vi)}$

The equation (vi) is the required equation of the tangent plane to the sphere (i) at the point (α, β, γ).

Note :—The method of writing the equation of the tangent plane to the sphere at (α, β, γ) is the same as used in writing the equations of the tangent to a conic in two dimensional co-ordinate geometry *i.e.* write x^2 in the equation of the sphere as $x \cdot x$ and change one x into α [*i.e.* the x-coordinate of the point (α, β, γ)] and similarly for y^2 and z^2 write $y\beta$ and $z\gamma$; write $2ux$ as $u(x+x)$ and change one x into α *i.e.* write $u(x+\alpha)$ for $2ux$. Similarly write $v(y+\beta)$, $w(z+\gamma)$ for $2vy$, $2wz$ respectively.

Cor. 1. *Tangent line at any point is perpendicular to the radius through that point.*

From § 7.08, we know that if $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ is a tangent line to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ at (α, β, γ), then

$$l(u+\alpha)+m(v+\beta)+n(w+\gamma)=0. \quad \text{...(i)}$$

Also the direction ratios of the radius through (α, β, γ) are proportional to $\alpha+u, \beta+v, \gamma+w$ as $(-u, -v, -w)$ are the coordinates of the centre of the sphere.

Hence the result (i) shows that the line with d.r.'s $\alpha+u, \beta+v, \gamma+w$ is at right angles to the tangent line $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ at (α, β, γ).

Cor 2. *The tangent palne at any point is perpendicular to any radius through that point.*

The d.c.'s of the normal to the tangent plane, given by (vi) of § 7.09 are proportional to the coefficients of x, y, z in the equaton (vi) *i.e.* $\alpha+u$, $\beta+v$ and

$\gamma + w$. But these are also the d.r.'s of the radius through α, β, γ of the sphere as shown in cor. 1. above. Hence the normal to the tangent plane at (α, β, γ) is parallel to radius of the sphere at the point *i.e.* the tangent at (α, β, γ) is perpendicular to the radius through that point.

(B) Let us find the equation of the tangent plane to the sphere

$$x^2 + y^2 + z^2 = a^2 \qquad \text{...(i)}$$

at the point (α, β, γ).

As (α, β, γ) is a point on the sphere (i), so we get $\alpha^2 + \beta^2 + \gamma^2 = a^2$ ...(ii)

The equations of any line through (α, β, γ) are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)} \qquad \text{...(iii)}$$

The point of intersection of the line (iii) and the sphere (i) are given by

$$(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr)^2 = a^2$$

or $$r^2 (l^2 + m^2 + n^2) + 2r (l\alpha + m\beta + n\gamma) + (\alpha^2 + \beta^2 + \gamma^2 - a^2) = 0$$

or $$r^2 (l^2 + m^2 + n^2) + 2r (l\alpha + m\beta + n\gamma) = 0 \text{, from (ii)} \qquad \text{...(iv)}$$

As one of the roots of (iv) is zero, so one of the points of intersection of (i) and (ii) coincides with the point (α, β, γ). If the line (iii) is a tangent line to (i) at (α, β, γ) then the other point of intersection should also coincide with $(\alpha; \beta, \gamma)$, *i.e.* the second root of (iv) should also vanish and so from (v) we have

$$l\alpha + m\beta + n\gamma = 0 \qquad \text{...(v)}$$

$\therefore$ The line (iii) is tangent line to (i) if d.c.'s of the line (iii) viz. l, m, n should satisfy the condition (v).

The tangent plane at (α, β, γ) to the sphere (i) is the locus of all such tangents and its equation is obtained by eliminating l, m, n between (iii) and (v).

$\therefore$ The required equation of the tangent plane is

$$(x-\alpha)\,\alpha + (y-\beta)\,\beta + (z-\gamma)\,\gamma = 0$$

or $$\alpha x + \beta y + \gamma z = \alpha^2 + \beta^2 + \gamma^2 = a^2 \text{, from (iii)}$$

or $\alpha x + \beta y + \gamma z = a^2$, which is the required equation of the tangent plane to the sphere (i) at (α, β, γ).

****§ 7.10 Condition for the plane $lx + my + nz = p$ to touch the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.**

(*Avadh* 92; *Gorakhpur* 92; *Kumaun* 93)

The centre of the given sphere is $(-u, -v, -w)$ and its radius $= \sqrt{(u^2 + v^2 + w^2 - d)}$. If the given plane touches the given sphere, then the **length** of the perpendicular from the centre $(-u, -v, -w)$ of the sphere to the given plane must be equal to the radius of the sphere.

i.e. $$\frac{l(-u) + m(-v) + n(-w) - p}{\sqrt{(l^2 + m^2 + n^2)}} = \sqrt{(u^2 + v^2 + w^2 - d)}$$

or $$(lu + mv + nw + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d),$$
which is the required condition.

Solved Examples on § 7.08 to § 7.10.

***Ex. 1. Find the equation of the tangent plane at the point (1, 2, 3) to the sphere $3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$** *(Kumaun 96)*

Sol. Equation of the sphere is
$$x^2 + y^2 + z^2 - (2/3)x - y - (4/3)z - (22/3) = 0$$
The equation of the tangent plane to this sphere at (1, 2, 3) is
$$x.1 + y.2 + z.3 - (1/3)(x + 1) - (1/2)(y + 2) - (2/3)(z + 3) - (22/3) = 0$$
...See § 7.09 (Note) Page 33 Ch. VII

or $$x + 2y + 3z - (1/3)x - (1/3) - (1/2)y - 1 - (2/3)z - 2 - (22/3) = 0$$
or $$(2/3)x + (3/2)y + (7/3)z - (32/3) = 0$$
or $$4x + 9y + 14z - 64 = 0$$ **Ans.**

***Ex. 2 (a). Find the condition for the plane $lx + my + nz = p$ to touch the sphere $x^2 + y^2 + z^2 = a^2$.**

Sol. The centre of the given sphere is (0, 0, 0) and its radius is a.

Now if the given plane $lx + my + nz - p = 0$ touches the given sphere, then the length of the perpendicular from the centre (0, 0, 0) of the sphere to this plane must be equal to the radius a of the sphere.

i.e. $$\frac{l.0 + m.0 + n.0 - p}{\sqrt{(l^2 + m^2 + n^2)}} = a \quad \text{or} \quad -p = a\sqrt{(l^2 + m^2 + n^2)}$$

or $p^2 = a^2(l^2 + m^2 + n^2)$, which is the required condition. **Ans.**

Ex. 2 (b). Find the condition for the plane $ax + by + cz + k = 0$ to be a tangent plane to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

(Kumaun 93)

Sol. The centre of the given sphere is $(-u, -v, -w)$ and its radius
$$= \sqrt{(u^2 + v^2 + w^2 - d)}$$
If the given plane touches the given sphere, then the length of perpendicular from the centre $(-u, -v, -w)$ of the sphere to the given plane must be equal to the radius of the sphere.

i.e. $$\frac{a(-u) + b(-v) + c(-w) + k}{\sqrt{(a^2 + b^2 + c^2)}} = \sqrt{(u^2 + v^2 + w^2 - d)}$$

or $$(au + bv + cw - k)^2 = (a^2 + b^2 + c^2)(u^2 + v^2 + w^2 - d),$$
which is the required condition. **Ans.**

Ex. 3 (a). Show that the plane $2x + y - z = 12$ touches the sphere $x^2 + y^2 + z^2 = 24$ and find its point of contact.

(Bundelkhand 95, 92; Purvanchal 93)

(b) Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$. Also find the point of contact.

(Kumaun 90; Lucknow 91)

Sol. (a). Let the required point of contact be (x_1, y_1, z_1). Then the equation of the tangent plane to the given sphere at (x_1, y_1, z_1) is

$$xx_1 + yy_1 + zz_1 = 24 \qquad \text{...(i)}$$

This should be the same as the given plane $2x + y - z = 12$ ...(ii)

Comparing (i) and (ii) we get $\dfrac{x_1}{2} = \dfrac{y_1}{1} = \dfrac{z_1}{-1} = \dfrac{24}{12}$.

These give $x_1 = 4, y_1 = 2, z_1 = -2$ ...(iii)

Also (x_1, y_1, z_1) is a point on the given sphere so $x_1^2 + y_1^2 + z_1^2 = 24$ and the same is satisfied by the values of x_1, y_1, z_1 given by (iii).

Hence the plane (ii) touches the given sphere at the point

$$(x_1, y_1, z_1) \quad i.e. \quad (4, 2, -2)$$ **Ans.**

(b). Do as part (a) above **Ans.** $(-1, 4, -2)$

***Ex. 3 (c). Show that the plane $2x - 2y + z + 16 = 0$ touches the sphere $x^2 + y^2 + z^2 + 2x - 4y + 2z - 3 = 0$ and find the coordinates of the point of contact.** *(Rohilkhand 97)*

Sol. Do as Ex. 3 (a) above. **Ans.** $(-3, 4, -2)$

***Ex. 4. Obtain the equation of the sphere inscribed on the line joining the points A (3, 4, 1) and B (– 1, 0, 5) as diameter. Find also the equation of the tangent plane at B.**

Sol. The equation of the sphere described on the line joining the given points A (3, 4, 1) and B (– 1, 0, 5) is

$$(x-3)[x-(-1)] + (y-4)(y-0) + (z-1)(z-5) = 0$$

or $$x^2 + y^2 + z^2 - 2x - 4y - 6z + 2 = 0, \text{ on simplifying.}$$ **Ans.**

The tangent plane to this sphere at any point (α, β, γ) is

$$x\alpha + y\beta + z\gamma - (x+\alpha) - 2(y+\beta) - 3(z+\gamma) + 2 = 0 \qquad \textbf{(Note)}$$

...See § 7.09 (A) Page 32 of this chapter.

$\therefore$ The required tangent plane to this sphere at $B(-1, 0, 5)$ is given by

$$x(-1) + y.0 + z(5) - [x + (-1)] - 2[y + 0] - 3[z + 5] + 2 = 0$$

or $$-x + 5z - x + 1 - 2y - 3z - 15 + 2 = 0 \quad \text{or} \quad x + y - z + 6 = 0.$$ **Ans.**

***Ex. 5. Find the equation of the tangent planes to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$, which are parallel to the plane $2x + y - z = 0$.** *(Purvanchal 97)*

Sol. Let the tangent plane to the given sphere parallel to the given plane be

$$2x + y - z + k = 0. \qquad \text{...(i)}$$

The centre of the sphere is $(2, -1, 3)$

and radius $= \sqrt{(2^2 + 1^2 + 3^2 - 5)}$ or $\sqrt{(4 + 1 + 9 - 5)}$ or 3.

If the plane (i) touches the given sphere, then the length of perpendicular from the centre $(2, -1, 3)$ to (i) must be equal to the radius 3.

i.e. $$\frac{2(2) + (-1) - (3) + k}{\sqrt{(2^2 + 1^2 + 1^2)}} = 3 \quad \text{or} \quad k = \pm 3\sqrt{6} \qquad \textbf{(Note)}$$

$\therefore$ From (i) the required planes are $2x + y - z \pm 3\sqrt{6} = 0$ **Ans.**

Ex. 6 (a). Find the equation of the sphere which touches the sphere $4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0$ at $(1, 2, -2)$ and passes through the point $(-1, 0, 0)$ *(Bundelkhand 93)*

Sol. The tangent plane at $(1, 2, -2)$ to the given sphere

$$x^2 + y^2 + z^2 + \frac{5}{2}x - \frac{25}{4}y - \frac{1}{2}z = 0 \qquad \textbf{(Note)}$$

is $$x.1 + y.2 + z.(-2) + \frac{5}{4}(x+1) - \frac{25}{8}(y+2) - \frac{1}{4}(z-2) = 0$$

or $2x - y - 2z - 4 = 0$, on simplifying.

$\therefore$ The equation of the sphere touching the given sphere at $(1, 2, -2)$ is

$$(x^2 + y^2 + z^2 + \frac{5}{2}x - \frac{25}{4}y - \frac{1}{2}z) + \lambda(2x - y - 2z - 4) = 0 \qquad ...(i)$$

If it passes through $(-1, 0, 0)$, then we have

$$(1 - \frac{5}{2}) + \lambda(-2 - 4) = 0 \quad \text{or} \quad \lambda = -\frac{1}{4}.$$

$\therefore$ From (i), the required equation is

$$(x^2 + y^2 + z^2 + \frac{5}{2}x - \frac{25}{4}y - \frac{1}{2}z) - \frac{1}{4}(2x - y - 2z - 4) = 0$$

or $$(x^2 + y^2 + z^2) + 2x - 6y + 1 = 0. \qquad \textbf{Ans.}$$

Ex. 6 (b). Find the equation of the sphere which touches the sphere $x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$ at $1, 2, -2$ and passes through the point $(1, -1, 0)$.

Sol. Do as Ex. 6 (a) above. **Ans.** $x^2 + y^2 + z^2 + 24x - 17y - 22z - 43 = 0$

***Ex. 6 (c). Find the equation of the sphere which touches the sphere $x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0$ at the point $(1, 1, -1)$ and passes through the origin.** *(Allahabad 91; Gorakhpur 91; Kanpur 90)*

Hint. Do as Ex. 6 (a) above. **Ans.** $2x^2 + 2y^2 + 2z^2 - 3x + y + 4z = 0.$

***Ex. 7 (a). If any tangent plane to the sphere $x^2 + y^2 + z^2 = r^2$ makes intercepts a, b and c on the coordinates axes prove that**

$$a^{-2} + b^{-2} + c^{-2} = r^{-2}.$$

Sol. The equation of the tangent plane at (α, β, γ) to the given sphere is

$$x\alpha + y\beta + z\gamma = r^2. \qquad ...(i)$$

Given that a is the intercept made by the plane (i) on x-axis.

So we have $a\alpha + 0 + 0 = r^2$ or $\alpha = r^2/a.$

Similarly $\beta = r^2/b$ and $\gamma = r^2/c$, as b and c are the intercepts made by the plane (i) on y and z axes respectively.

Also as (α, β, γ) is a point on the sphere, so we have

$$\alpha^2 + \beta^2 + \gamma^2 = r^2 \quad \text{or} \quad (r^2/a)^2 + (r^2/b)^2 + (r^2/c)^2 = r^2$$

or $$a^{-2} + b^{-2} + c^{-2} = r^{-2}$$ Hence proved.

Ex. 7 (b). Find the equation of the tangent planes to the sphere $x^2+y^2+z^2+2x-4y+6z-7=0$ which intersect in the line

$$6x-3y-23=0=3z+2.$$

Sol. The equation of any plane through the line is

$$(6x-3y-23)+\lambda(3z+2)=0$$

or $$6x-3y+(3\lambda)z+(2\lambda-23)=0. \quad \text{...(i)}$$

If this plane touches the given sphere, then the length of the perpendicular from the centre $(-1, 2, -3)$ of the sphere to this plane must be equal to the radius $\sqrt{[(-1)^2+(2)^2+(-3)^2-(-7)]}$ *i.e.* $\sqrt{(21)}$ of the sphere

$$\therefore \frac{6(-1)-3(2)+(3\lambda)(-3)+(2\lambda-23)}{\sqrt{[6^2+(-3)^2+(3\lambda)^2]}}=\sqrt{(21)}$$

or $$-6-6-9\lambda+2\lambda-23=\sqrt{(21)}\sqrt{(45+9\lambda^2)}$$

or $$(-7\lambda-35)^2=21(45+9\lambda^2)$$

or $$140\lambda^2-490\lambda-280=0 \text{ or } 2\lambda^2-7\lambda-4=0$$

or $$\lambda=\frac{1}{4}[7\pm\sqrt{(49+32)}]=\frac{1}{4}(7\pm 9)=4, -\frac{1}{2}.$$

$\therefore$ The required equations of the tangent planes from (i) are

$$6x-3y+12z-15=0 \text{ and } 6x-3y-\frac{3}{2}z=24$$

or $$2x-y+4z=5 \text{ and } 4x-2y-z=16. \qquad \textbf{Ans.}$$

***Ex. 8 (a). If three mutually prpendicular chords of lengths d_1, d_2, d_3 be drawn through the point (α, β, γ) to the sphere $x^2+y^2+z^2=a^2$, prove that $d_1^2+d_2^2+d_3^2=12a^2-8(\alpha^2+\beta^2+\gamma^2)$.** *(Rohilkhand 91)*

Sol. Let the equation of a chord through $A(\alpha, \beta, \gamma)$ be

$$\frac{x-\alpha}{l_1}=\frac{y-\beta}{m_1}=\frac{z-\gamma}{n_1}=r \text{ (say)}$$

The point of intersection of this line and the sphere $x^2+y^2+z^2=a^2$ are given by $(\alpha+l_1r)^2+(\beta+m_1r)^2+(\gamma+n_1r)^2=a^2$

or $$r^2(l_1^2+m_1^2+n_1^2)+2r(l_1\alpha+m_1\beta+n_1\gamma)+(\alpha^2+\beta^2+\gamma^2-a^2)=0$$

or $$r^2+2r(l_1\alpha+m_1\beta+n_1\gamma)+(\alpha^2+\beta^2+\gamma^2-a^2)=0, \quad \text{...(i)}$$

if l_1, m_1, n_1 be the actual d.c.'s of the chord.

If this chord through $A(\alpha, \beta, \gamma)$ meets the given sphere in B and C, then $AB=r_1$ and $AC=r_2$, where r_1 and r_2 are the roots of (i).

Also if $PQ=d_1$, then $d_1=AQ-AP=r_2-r_1$.

$$\therefore d_1^2=(r_2-r_1)^2=(r_2+r_1)^2-4r_1r_2$$

$$=[2(l_1\alpha+m_1\beta+n_1\gamma)]^2-4[\alpha^2+\beta^2+\gamma^2-a^2], \text{ from (i)}$$

or $$d_1^2=4(l_1\alpha+m_1\beta+n_1\gamma)^2-4(\alpha^2+\beta^2+\gamma^2-a^2),$$

Similarly $d_2^2=4(l_2\alpha+m_2\beta+n_2\gamma)^2-4(\alpha^2+\beta^2+\gamma^2-a^2)$

and $\quad d_3^2 = 4\,(l_3\alpha + m_3\beta + n_3\gamma)^2 - 4\,(\alpha^2 + \beta^2 + \gamma^2 - a^2)$

$$\therefore\ d_1^2 + d_2^2 + d_3^2 = 4\,[\alpha^2\,(l_1^2 + l_2^2 + l_3^2) + \beta^2\,(m_1^2 + m_2^2 + m_3^2) + \gamma^2\,(n_1^2 + n_2^2 + n_3^2) + 2\alpha\beta\Sigma\, l_1 m_1 + 2\beta\gamma\,\Sigma\, m_1 n_1 + 2\gamma\alpha\Sigma\, l_1 n_1 - 3\,(\alpha^2 + \beta^2 + \gamma^2 - a^2)],$$

$$= 4\,[\alpha^2 + \beta^2 + \gamma^2 - 3\,(\alpha^2 + \beta^2 + \gamma^2 - a^2)],$$

$\because \Sigma\, l_1^2 = 1$ etc. and $\Sigma\, l_1 m_1 = 0$

$$= 12a^2 - 8\,(\alpha^2 + \beta^2 + \gamma^2).$$ Hence proved.

Ex. 8 (b). Find the length of the chord intercepted by the line $(x - \alpha)/l = (y - \beta)/m = (z - \gamma)/n$ on the sphere $x^2 + y^2 + z^2 = a^2$.

Sol. As in Ex. 8 (a) above we can prove that the required length of the chord is given by $d^2 = 4\,[(l\alpha + m\beta + n\gamma)^2 - (\alpha^2 + \beta^2 + \gamma^2 - a^2)]$

or $$d = 2\,[(l\alpha + m\beta + n\gamma)^2 - (\alpha^2 + \beta^2 + \gamma^2 - a^2)]^{1/2}.$$ **Ans.**

Ex. 9. Find the points of intersection of the line $\frac{1}{2}(x - 1) = \frac{1}{3}(y - 2) = \frac{1}{4}(z - 3)$ with the sphere $x^2 + y^2 + z^2 - 4y - 7 = 0$.

Sol. Any point on the given line is $(1 + 2r, 2 + 3r, 3 + 4r)$ ...(i)

If this point lies on the given sphere, then we have

$$(1 + 2r)^2 + (2 + 3r)^2 + (3 + 4r)^2 - 4\,(2 + 3r) - 7 = 0$$

or $\quad 29r^2 + 28r - 1 = 0 \quad$ or $\quad (29r - 1)(r + 1) = 0 \quad$ or $\quad r = -1, 1/29.$

$\therefore$ From (i) the required points are

$$(1 - 2, 2 - 3, 3 - 4) \text{ and } (1 + \tfrac{2}{29},\ 2 + \tfrac{3}{29},\ 3 + \tfrac{4}{29})$$

or $$(-1, -1, -1) \text{ and } (\tfrac{31}{29},\ \tfrac{61}{29},\ \tfrac{91}{29}).$$ **Ans.**

Ex. 10. Find the equations of the tangent line to the circle $3x^2 + 3y^2 + 3z^2 - 2x - 3y - 4z - 22 = 0$, $3x + 4y + 5z - 26 = 0$ at the point (1, 2, 3).

Sol. The equation of the tangent plane to the sphere

$$x^2 + y^2 + z^2 - \tfrac{2}{3}x - y - \tfrac{4}{3}z - \tfrac{22}{3} = 0 \text{ at } (1, 2, 3) \text{ is}$$

$$x\,(1) + y\,(2) + z\,(3) - \tfrac{1}{3}(x + 1) - \tfrac{1}{2}(y + 2) - \tfrac{2}{3}(z + 3) - \tfrac{22}{3} = 0$$

or $$4x + 9y + 14z - 64 = 0.$$

The required tangent line is the line of intersection of the planes

$$4x + 9y + 14z - 64 = 0 \text{ and } 3x + 4y + 5z - 26 = 0$$

$\therefore$ If l, m, n be the direction ratios of this line [which evidently passes through (1, 2, 3)], then we have $4l + 9m + 14n = 0,\ 3l + 4m + 5n = 0$

$$\therefore\ \frac{l}{45 - 56} = \frac{m}{42 - 20} = \frac{n}{16 - 27} \quad \text{or} \quad \frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$$

$\therefore$ The required tangent line at $(1, 2, 3)$ is

$$x-1=-\frac{1}{2}(y-2)=(z-3).$$ **Ans.**

***Ex. 11. Find the equation of a sphere touching the three co-ordinate planes. How many such spheres can be drawn?**

Sol. Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0. \quad ...(i)$$

If the sphere touches the yz-plane *i.e.*, $x=0$, then the length of the perpendicular from its centre $(-u,-v,-w)$ to the plane $x=0$ must be equal to its radius $=(u^2+v^2+w^2-d)$

i.e. $$\frac{-u}{1}=\sqrt{(u^2+v^2+w^2-d)} \quad \text{or} \quad u^2=u^2+v^2+w^2-d$$

or $$v^2+w^2=d. \quad ...(ii)$$

Similarly if the sphere (i) touches zx and xy-planes then we shall have

$w^2+u^2=d$ ...(iii) and $u^2+v^2=d$ ...(iv)

Adding (ii), (iii) and (iv) we get $2(u^2+v^2+w^2)=3d$

or $$u^2+v^2+w^2=\frac{3}{2}d \quad ...(v)$$

or $$u^2=\frac{1}{2}d, \text{ from (ii)}$$

Similarly from (iii), (iv) and (v), we get $v^2=\frac{1}{2}d=w^2$

$\therefore u^2=\frac{1}{2}d=v^2=w^2=\lambda^2$ (say) or $u=\pm\lambda=v=w$.

Hence from (i) the required equation is

$$x^2+y^2+z^2\pm 2\lambda(x+y+z)+2\lambda^2=0.$$ **Ans.**

Since λ can take an infinite number of values, so an infinite number of such spheres can be drawn but if the radius of the sphere is given then λ can be expressed in terms of the given radius and then only eight such spheres can be possible as the sets of values of u, v and w can be taken in eight different ways.

Ex. 12. A sphere touches the three coordinate planes and passes through the point (2, 1, 5). Find its equation. (*Kanpur* 92)

Sol. Let the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0. \quad ...(i)$$

Its centre is $(-u,-v,-w)$ and radius $\sqrt{(u^2+v^2+w^2-d)}$.

If the sphere touches the yz-plane *i.e.* $x=0$, then the length of the perpendicular drawn from the centre $(-u,-v,-w)$ of the sphere to the plane $x=0$ must be equal to the radius $\sqrt{(u^2+v^2+w^2-d)}$ of the sphere. **(Note)**

$$\therefore \frac{-u}{\sqrt{(1^2+0^2+0^2)}}=\sqrt{(u^2+v^2+w^2-d)} \quad \text{or} \quad u^2=u^2+v^2+w^2-d$$

or $$v^2+w^2=d. \quad ...(ii)$$

Similarly if the sphere (i) touches planes $y=0$ and $z=0$ (the other ordinate planes) then we shall get

$u^2+w^2=d$ ...(iii) and $u^2+v^2=d$ ...(iv)

Adding (ii), (iii) and (iv) we get $u^2+v^2+w^2=\frac{3}{2}d$ ...(v)

$\therefore$ From (v) we get $u^2=\frac{1}{2}d$.

Similarly from (iii), (iv) and (v) we get $v^2=\frac{1}{2}d=w^2$.

$\therefore$ From (i) the equation of the sphere reduces to

$$x^2+y^2+z^2+2\sqrt{(\tfrac{1}{2}d)}\,(x+y+z)+d=0, \qquad \text{...(vi)}$$

on substituting values of u, v and w.

If this sphere passes through (2, 1, 5) then we get

$$4+1+25+\sqrt{(2d)}\,(2+1+5)+d=0$$

$d+8\sqrt{2}\sqrt{d}+30=0$ or $\sqrt{d}=\frac{1}{2}[-8/\sqrt{2}\pm\sqrt{(128-120)}]$

$\sqrt{d}=\frac{1}{2}[-8\sqrt{2}\pm 2\sqrt{2}]=-3\sqrt{2},-5\sqrt{2}$ or $d=18, 50$

$\therefore$ From (vi), the required equations of the sphere are

$$x^2+y^2+z^2-6(x+y+z)+18=0$$

and $$x^2+y^2+z^2-10(x+y+z)+30=0. \qquad \textbf{Ans.}$$

***Ex. 13 (a). Find the equations of the spheres which pass through the circle $x^2+y^2+z^2=5$, $2x+3y+z=3$ and touch the plane $3x+4y=15$.**

Sol. The equation of any sphere through the given circle is

$$(x^2+y^2+z^2-5)+\lambda(2x+3y+z-3)=0$$

or $$x^2+y^2+z^2+2\lambda x+3\lambda y+\lambda z-(3\lambda+5)=0 \qquad \text{...(i)}$$

If this sphere touches the plane $3x+4y-15=0$ then the length of perpendicular from its centre $(-\lambda,-3\lambda/2,-\lambda/2)$ to this plane must be equal to its radius $\sqrt{[(-\lambda)^2+(-3\lambda/2)^2+(-\lambda/2)^2+(3\lambda+5)]}$

i.e. $$\sqrt{[(7\lambda^2/2)+(3\lambda+5)]} \qquad \textbf{(Note)}$$

i.e. $$\frac{3(-\lambda)+4(-3\lambda/2)-15}{\sqrt{(3^2+4^2)}}=\sqrt{\left[\left(\frac{7\lambda}{2}\right)^2+(3\lambda+5)\right]}$$

or $$\frac{-(9\lambda+15)}{5}=\sqrt{\left(\frac{7\lambda^2+6\lambda+10}{2}\right)}$$

or $18(3\lambda+5)^2=25(7\lambda^2+6\lambda+10)$, squaring both sides

or $13\lambda^2-390\lambda-200=0$, on simplifying.

Substituting the two values of λ obtained from here in (i) by turn, we get the required equations of spheres. **Ans.**

Ex. 13 (b). Find the equation of the spheres through the circle $x^2 + y^2 + z^2 = 1$, $2x + 4y + 5z = 6$ and touching the plane $z = 0$.

Sol. Do as Ex. 13 (a) above. **Ans.** $5(x^2 + y^2 + z^2) - 2x - 4y - 5z + 1 = 0$

Ex. 14 (a). Find the equation of a sphere inscribed in the tetrahedron whose faces are $x = 0$, $y = 0$, $z = 0$, $2x - 6y + 3z + 6 = 0$.

Sol. The equation of the sphere touching the coordinate planes is

$$x^2 + y^2 + z^2 - 2\lambda x + 2\lambda y - 2\lambda z + 2\lambda^2 = 0$$

...See Ex. 11 Page 40 Ch. VII **(Note)**

noting that the fourth plane meets the x, y and z axes in the $-$ve, $+$ ve and $-$ve directions.

Its centre is $(\lambda, -\lambda, \lambda)$ and radius $= \sqrt{(\lambda^2 + \lambda^2 + \lambda^2 - 2\lambda^2)} = \lambda$

If this sphere touches the plane $2x - 6y + 3z + 6 = 0$, then the length of the perpendicular from the centre $(\lambda, -\lambda, \lambda)$ of the sphere to this plane must b equal to its radius λ.

i.e. $\dfrac{2\lambda + 6\lambda + 3\lambda + 6}{\sqrt{(2^2 + 6^2 + 3^2)}} = \lambda$ or $\dfrac{6 + 11\lambda}{7} = -\lambda$ or $\lambda = \dfrac{-1}{3}$ **(Note)**

The required equation is $x^2 + y^2 + z^2 + (2/3)(x - y + z) + 2(1/9) = 0$

or $$9(x^2 + y^2 + z^2) + 6(x - y + z) + 2 = 0$$ **Ans.**

Ex. 14 (b). Find the equation of the sphere in the positive octant touching the coordinates planes and the plane $2x + 3y + 6z - 24 = 0$

Sol. Do as Ex. 14 (a) above.

***Ex. 15. Prove that the centres of the spheres which touch the lines $y = mx$, $z = c$; $y = -mx$, $z = -c$ lie upon the conicoid $mxy + cz(1 + m^2) = 0$.**

(*Garhwal* 90)

Sol. Let the equation of the sphere which touches the given lines be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad ...(i)$$

If this sphere touches the line $y = mx$, $z = c$, then we have

$$x^2 + m^2x^2 + c^2 + 2ux + 2vmx + 2wc + d = 0,$$

putting $y = mx$, $z = c$ in (i)

or $$x^2(1 + m^2) + 2(u + vm)x + (c^2 + 2wc + d) = 0. \quad ...(ii)$$

If the line $y = mx$, $z = c$ touches the sphere (i) then the roots of (ii) must be coincident and the condition for the same is

$$[2(u + vm)]^2 = 4(1 + m^2)(c^2 + 2wc + d) \quad ... \text{“}b^2 = 4ac\text{”}$$

or $$(u + vm)^2 = (1 + m^2)(c^2 + 2wc + d). \quad ...(iii)$$

Similarly if the line $y = -mx$, $z = -c$ touches the sphere (i), then we shall have $$(u - vm)^2 = (1 + m^2)(c^2 - 2wc + d), \quad ...(iv)$$

putting $-m$ for m and $-c$ for c in (iii)

Subtracting (iv) from (iii), we get $4uvm = (1 + m^2)(4wc)$

or $(-u)(-v)\,m + (1 + m^2)(-w)\,c = 0.$ **(Note)**

$\therefore$ The locus of the centre $(-u, -v, -w)$ of the sphere is

$$xym + (1 + m^2)\,zc = 0$$ Hence proved.

Ex. 16. In one end of a diameter of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z - 11 = 0$ be (1, 2, 4) find the coordinates of the other end.

Sol. If C be the centre of the given sphere then C is $(1, -2, 3)$

Also let A be the point $(-1, 2, 4)$. Then the direction ratios of the line AC are $(1 + 1, -2 - 2, 3 - 4)$ or $(2, -4, -1)$.

$\therefore$ The equation of the line AC is $\dfrac{x+1}{2} = \dfrac{y-2}{-4} = \dfrac{z-4}{-1}$

Any point on this line is $B\,(-1 + 2r, 2 - 4r, 4 - r)$. ...(i)

If the point B is the other end of the diameter through A, then B must lie on the sphere, and so we have

$$(-1 + 2r)^2 + (2 - 4r)^2 + (4 - r)^2 - 2(-1 + 2r) + 4(2 - 4r) - 6(4 - r) - 11 = 0$$

or $21r^2 - 42r - 4 = 0$ or $r = 1 + [5/\sqrt{(21)}]$, taking + ve value.

$\therefore$ The required point B is

$$\left[1 + \frac{10}{\sqrt{(21)}}, -2 - \frac{20}{\sqrt{(21)}}, 3 - \frac{25}{\sqrt{(21)}}\right]$$ Ans.

****Ex. 17. Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ and find the point of contact.** (*Agra* 92)

Sol. If the plane $2x - 2y - z + 12 = 0$ ...(i)

touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ ...(ii)

then the length of the perpendicular from the centre $(1, 2, -1)$ of the sphere (ii) to the plane (i) must be equal to the radius

$\sqrt{[(-1)^2 + 2^2 + (-1)^2 - (-3)]} = \sqrt{(9)} = 3$ of the sphere (ii)

i.e. $$\frac{2(1) - 2(2) + 1(-1) + 12}{\sqrt{[2^2 + (-2)^2 + (1)^2]}} = 3 \quad \text{or} \quad 9/3 = 3.$$

which being true the plane (i) touches the sphere (ii).

Also if C be the centre of the sphere and P the required point of contact, then the d.r.'s of the line CP are same as those of the normal to the plane (i) *i.e.* $2, -2, 1$. Also C is $(1, 2, -1)$.

Hence the equation of the line CP is $\dfrac{x-1}{2} = \dfrac{y-2}{-2} = \dfrac{z+1}{1} = r$ (say)

If $CP = r$, the coordinates of P are $(2r + 1, -2r + 2, r - 1)$ and P lies on (i), so we have

$$2(2r + 1) - 2(-2r + 2) + (r - 1) + 12 = 0 \quad \text{or} \quad 9r + 9 = 0 \quad \text{or} \quad r = -1.$$

$\therefore$ The coordinates of P are

$[2(-1) + 1, -2(-1) + 2, -1 - 1]$ or $(-1, 4, -2)$ Ans.

****Ex. 18. Find the locus of the centres of spheres of constant radius which pass through a given point and touch a given line.**

Sol. Take x-axis as the given line and $(0, 0, a)$ as the given point. **(Note)**

Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad ...(i)$$

As it passes through the given point $(0, 0, a)$ so $a^2+2wa+d=0$ ...(ii)

Also the radius of the sphere (i) is given as constant k (say).

Then $$u^2+v^2+w^2-d=k^2 \quad ...(iii)$$

The sphere (i) touches the given line which we have chosen as x-axis *i.e.* $y=0=z$ at the points given by $x^2+2ux+d=0$ ...(iv)

Since the sphere (i) touches the line $y=0=z$, so the roots of (iv) must be equal and therefore using '$b^2=4ac$' we have

$$(2u)^2=4.\,1.\,d \quad \text{or} \quad u^2=d \quad ...(v)$$

Eliminating d from (ii), (iii) and (v) we get

$$a^2+2wa+u^2=0,\ v^2+w^2=k^2$$

$\therefore$ The required locus of the centre $(-u, -v, -w)$ of the sphere (i) is given by the equations $a^2+2(-z)a+(-x)^2=0,\ (-y)^2+(-z)^2=k^2$

or $$x^2-2az+a^2=0,\ y^2+z^2=k^2,$$

which is the curve of intersection of two quadratic surfaces

$$x^2-2az+a^2=0 \quad \text{and} \quad y^2+z^2=k^2 \qquad \textbf{Ans.}$$

***Ex. 19. Find the locus of the centres of spheres which pass through a given point and touch a given plane.**

Sol. Take $z=0$ as the given plane and $(0, 0, a)$ as the given point. **(Note)**

Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad ...(i)$$

As it passes through $(0, 0, a)$, so we have $a^2+2wa+d=0$ ...(ii)

Also if the sphere (i) touches the plane $x=0$ then the length of perpendicular from its centre $(-u, -v, -w)$ to the plane $z=0$ must be equal to its radius $\sqrt{(u^2+v^2+w^2-d)}$

i.e. $$\frac{(-w)}{\sqrt{(1^2)}}=\sqrt{(u^2+v^2+w^2-d)} \text{ i.e. } u^2+v^2=d \quad ...(iii)$$

Eliminating d from (ii) and (iii) we get $a^2+2wa+u^2+v^2=0$ ...(iv)

$\therefore$ Locus of the centre $(-u, -v, -w)$ of the sphere (i) from (iv) is $a^2+2(-z)a+(-x)^2+(-y)^2=0$ or $x^2+y^2-2az+a^2=0$.

Exercises on § 7.10

Ex. 1. Show the equation of the tangent plane to the sphere $x^2+y^2+z^2=9$ at $(1, -2, 2)$ is $x-2y+2z=9$.

Ex. 2. Find the equation of the sphere which touches the sphere $x^2+y^2+z^2+2x-6y+1=0$ at $(1, 2, -2)$ and passes through $(0, 0, 0)$.

Ans. $x^2+y^2+z^2+(5/2)\,x-(25/4)\,y-(1/2)\,z=0$.

Ex. 3. Find the equation of the sphere which touches the sphere $x^2+y^2+z^2-x+3y+2z-3=0$ at $(1, 1, -1)$ and passes through $(2, 0, 1)$.

Ans. $2\,(x^2+y^2+z^2)-x+11y+4z-12=0$.

Ex. 4. If a tangent plane to the sphere $k^2\,(x^2+y^2+z^2)=r^2$ makes intercepts a, b, c on the axes of coordinates, prove that

$$k^{-2}\,(a^{-2}+b^{-2}+c^{-2})=r^{-2}$$

Ex. 5. Obtain the equations of the tangent planes to the sphere $x^2+y^2+z^2+6x-2z+1=0$ which passes through the line

$$3\,(16-x)=3z=2y+30$$

Ans. $x+2y-2z+14=0,\ 2x+2y-z-2=0$.

Ex. 6. Find the equations of the sphere passing through the circle $x^2+y^2+z^2-4x-y+3z+12=0,\ 2x+3y+7z=10$ and touching the plane $x-2y+2z=1$.

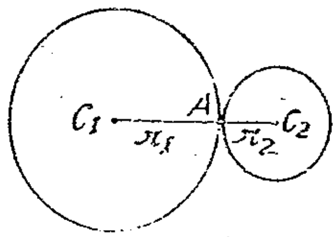
Ans. $x^2+y^2+z^2-2x+2y-4z+2=0,\ x^2+y^2+z^2-6x-4y+10z+22=0$.

Ex. 7. Find the equations of the spheres passing through $x^2+y^2+z^2=5$, $x+2y+3z=3$ and touch the plane $4x+3y=15$.

Ans. $x^2+y^2+z^2+2x+4y+6z=11,\ 5\,(x^2+y^2+z^2)-4x-8y-12z=13$.

Ex. 8. A sphere is inscribed in the tetrahedron whose faces are $x=0$, $y=0$, $z=0$ and $2x+6y+3z=14$. Find its centre, radius and equation.

Ans. $(7/9, 7/9, 7/9)$; $7/9$; $81\,(x^2+y^2+z^2)=128\,(x+y+z)-98$.



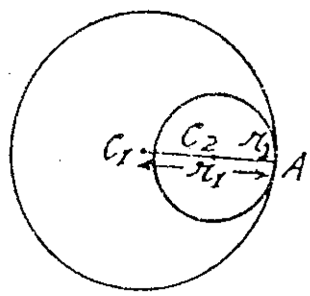
(Fig. 34)

§ 7.11 Touching spheres.

(i) Two spheres are said to touch **externally**, if the distance between their centres is equal to the sum of their radii.

(Remember)

In this case the point of contact of the sphere divides the line joining the centres internally in the ratio of their radii. [In adjoining Fig. 34 the point of contact A divides the line joining centres C_1 and C_2 internally in the ratio $r_1 : r_2$.]



(Fig. 35)

(ii) Two spheres are said to touch **internally**, if the distance between their centres is equal to the difference of their radii. **(Remember)**

In this case the point of contact of the spheres divides the line joining the centres externally in the ratio of their radii. [In Fig. 35 P. 45 Ch. VII the point of contact A divides line joining centres C_1 and C_2 externally in ratio $r_1 : r_2$].

Solved Examples on § 7.11.

Ex. 1 (a). Show that the spheres $x^2+y^2+z^2=25$ and $x^2+y^2+z^2-24x-40y-18z+225=0$ touch externally and find the point of contact.

Sol. Let the given spheres touch each other at (α, β, γ). The equations of the tangent planes of the spheres at (α, β, γ) are $x\alpha+y\beta+z\gamma=25$

and $\quad x\alpha+y\beta+z\gamma-12(x+\alpha)-20(y+\beta)-9(z+\gamma)+225=0$

or $$x\alpha+y\beta+z\gamma=25 \qquad \text{...(i)}$$

and $$(\alpha-12)x+(\beta-20)y+(\gamma-9)z=12\alpha+20\beta+9\gamma-225 \qquad \text{...(ii)}$$

If the two spheres touch each other, then (i) and (ii) represent the same plane, hence comparing the coefficients of x, y, z and constant terms in these equations, we get

$$\frac{\alpha-12}{\alpha}=\frac{\beta-20}{\beta}=\frac{\gamma-9}{\gamma}=\frac{12\alpha+20\beta+9\gamma-225}{25}=k \text{ (say)}$$

Then $\alpha-12=k\alpha$ or $\alpha=12/(1-k)$

Similarly $\beta=20/(1-k)$ and $\gamma=9/(1-k)$

Also $12\alpha+20\beta+9\gamma-225=25k$

or $$12\left(\frac{12}{1-k}\right)+20\left(\frac{20}{1-k}\right)+9\left(\frac{9}{1-k}\right)-225=25k$$

or $$144+400+81-225(1-k)=25k(1-k)$$

or $$k^2+8k+16=0 \quad \text{or} \quad (k+4)^2=0 \quad \text{or} \quad k=-4$$

$$\therefore \alpha=\frac{12}{1+4}=\frac{12}{5};\ \beta=\frac{20}{1+4}=\frac{20}{5}=4,\ \gamma=\frac{9}{1+4}=\frac{9}{5}$$

$\therefore$ The required point of contact is $(12/5, 4, 9/5)$. **Ans.**

Also the radii of the spheres are $5, \sqrt{[(12)^2+(20)^2+(9)^2-225]}$ *i.e.* 20.

$\therefore$ Sum of radii $=5+20=25$ ...(iii)

Distance between the centres $(0, 0, 0)$ and $(12, 20, 9)$ of these spheres

$$=\sqrt{[(12)^2+(20)^2+(9)^2]}=\sqrt{(625)}=25$$

$=$ sum of radii of the spheres, from (iii)

Hence spheres touch externally.

***Ex. 1 (b). Show that the spheres $x^2+y^2+z^2=100$ and $x^2+y^2+z^2-24x-30y-32z+400=0$ touch externally and find their point of contact.** (*Kanpur* 94)

Sol. Do as Ex. 1 (a) above. **Ans.** Point of contact is (24/5, 6, 32/5).

Ex. 2. Show that the spheres $x^2+y^2+z^2-2x-3=0$ and $x^2+y^2+z^2+6x+6y+9=0$ touch externally.

Sol. The centre and radius of sphere $x^2+y^2+z^2-2x-3=0$ are $(1, 1, 0)$ and $\sqrt{(1+3)}$ *i.e.* 2.

The centre and radius of the sphere $x^2+y^2+z^2+6x+6y+9=0$ are $(-3,-3,0)$ and $\sqrt{(9+9-9)}$ *i.e.* 3.

Now the distance between the centres

$$=\sqrt{[(1+3)^2+(0+3)^2+(0-0)^2]}=\sqrt{(16+9)}=5$$

And the sum of radii of the spheres $=2+3=5$

$\because$ The sum of the radii = distance between the centres so two spheres touch externally.

Ex. 3. Show that the spheres $x^2+y^2+z^2=64$ and

$$x^2+y^2+z^2-12x+4y-6z+48=0$$

touch internally and find their point of contact.

Sol. For the point of contact proceed as in Ex. 1 (a) Page 46.

For the other part *i.e.* (touching internally) show that the distance between the centres = difference of their radii. **Ans.** $(48/7, -1/7, 24/7)$

Exercise on § 7.11.

Ex. Show that the spheres $x^2+y^2+z^2=25$, $x^2+y^2+z^2-18x-24y-40z+225=0$ touch and show that their point of contact is $(9/5, 12/5, 4)$.

§ 7.12. Plane of contact.

Let the tangent plane at (α, β, γ) to the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad ...(i)$$

pass through a point (x_1, y_1, z_1) external to this sphere.

The equation of the tangent plane at (α, β, γ) to the sphere (i) is

$$x\alpha+y\beta+z\gamma+u(x+\alpha)+v(y+\beta)+w(z+\gamma)+d=0$$

or $$x(u+\alpha)+y(v+\beta)+z(w+\gamma)+(u\alpha+v\beta+w\gamma+d)=0$$

If it passes through (x_1, y_1, z_1), then

$$x_1(u+\alpha)+y_1(v+\beta)+z_1(w+\gamma)+(u\alpha+v\beta+w\gamma+d)=0$$

or $$\alpha(x_1+u)+\beta(y_1+v)+\gamma(z_1+w)+(ux_1+vy_1+wz_1+d)=0$$

This shows that (α, β, γ) lies on the plane

$$x(x_1+u)+y(y_1+v)+z(z_1+w)+(ux_1+vy_1+wz_1+d)=0$$

or $$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$$

which is known as the *plane of contact* of the point (α, β, γ).

And the locus of (α, β, γ) is the circle of intersection of this plane and the sphere.

§ 7.13 The Polar Plane.

Definition. *If the line* $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ *(where l, m, n are its actual direction cosines) drawn through the point $A(\alpha, \beta, \gamma)$ meets the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ in points P and Q and a point R lies on* *this such that* $\dfrac{1}{AP}+\dfrac{1}{AQ}=\dfrac{2}{AR}$

(*i.e. AR is the harmonic mean of AP and AQ*) *then the locus of R is defined as the polar plane of* (α, β, γ) *with respect of the sphere.* (*Kumaun* 93)

The coordinates of any point on the given line are $(\alpha+lr, \beta+mr, \gamma+nr)$.

The distances of the points of intersection of the line and the sphere are given by the equation

$$(\alpha+lr)^2+(\beta+mr)^2+(\gamma+nr)^2+2u(\alpha+lr)+2v(\beta+mr)$$
$$+2w(\gamma+nr)+d=0$$

or $$r^2+2r[l(\alpha+u)+m(\beta+v)+n(\gamma+w)]$$
$$+(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d)=0, \quad \text{...(i)}$$

remembering that $\qquad l^2+m^2+n^2=1.$

As this line meets the sphere in P and Q so the roots of the equation (i) are AP and AQ

Also $$\frac{2}{AR}=\frac{1}{AP}+\frac{1}{AQ}=\frac{AP+AQ}{AP\,.\,AQ}=\frac{\text{sum of the roots of (i)}}{\text{product of the roots of (i)}}$$
$$=\frac{-2[l(\alpha+u)+m(\beta+v)+n(\gamma+w)]}{\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d}$$

or $$\alpha^2+\beta^2+2u\alpha+2v\beta+2w\gamma+d$$
$$=-[l\,.\,AR(\alpha+u)+m\,.\,AR(\beta+v)+n\,.\,AR(\gamma+w)] \quad \text{...(ii)}$$

Now let R be (x, y, z) and its distance from $A(\alpha, \beta, \gamma)$ is AR.

Then from the equation of the line, we have

$$x-\alpha=l\,.\,AR,\ y-\beta=m\,.\,AR \text{ and } z-\gamma=n\,.\,AR \qquad \textbf{(Note)}$$

$\therefore$ From (ii), we get $\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d$
$$=-[(x-\alpha)(\alpha+u)+(y-\beta)(\beta+v)+(z-\gamma)(\gamma+w)]$$

or $$x\alpha+y\beta+z\gamma+u(x+\alpha)+v(y+\beta)+w(z+\gamma)+d=0, \quad \text{...(iii)}$$

which is the equation of the locus of R *i.e.* from definition the equation of the polar of $A(\alpha, \beta, \gamma)$ with respect to the sphere.

The point $A(\alpha, \beta, \gamma)$ is called the **pole of the plane (iii)** with respect to the sphere. (*Kumaun* 93)

To find the pole of a given plane with respect to a given sphere.

Let the pole be (x_1, y_1, z_1). Then compare the given plane with the polar plane of (x_1, y_1, z_1) with respect to the given sphere and evaluate x_1, y_1 and z_1.

§ 7.14 Properties of pole and polar.

Property I. *The distance of the points from the centre of a sphere are proportional to the distance of each from the polar plane of the other.*

(*Salmon's Theorem*)

Let the points be $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ whose polar planes with respect to the sphere $x^2+y^2+z^2=a^2$ are

$xx_1 + yy_1 + zz_1 = a^2$ ...(i) and $xx_2 + yy_2 + zz_2 = a^2$. ...(ii)

$$\therefore \frac{\text{Distance of } P \text{ from the polar plane of } Q}{\text{Ditance of } Q \text{ from the polar plane of } P}$$

$$= \frac{(x_1x_2 + y_1y_2 + z_1z_2 - a^2)/\sqrt{(x_2^2 + y_2^2 + z_2^2)}}{(x_2x_1 + y_2y_1 + z_2z_1 - a^2)/\sqrt{(x_1^2 + y_1^2 + z_1^2)}} \quad \textbf{(Note)}$$

$$= \frac{\sqrt{(x_1^2 + y_1^2 + z_1^2)}}{\sqrt{(x_2^2 + y_2^2 + z_2^2)}} = \frac{OP}{OQ},$$

where O (0, 0, 0) is the centre of the sphere.

Property II. *If the polar plane of a point $P(x_1, y_1, z_1)$ passes through another point $Q(x_2, y_2, z_2)$, then the polar plane of Q will pass through P and such points are known as* **conjugate points.**

The proof of this and property III below are left as exercises for the reader.

Property III. If the pole of a given plane α lies on another plane β, then the pole of the plane β will lie on the plane α and such planes are known as **conjugate planes.**

Property IV. *The polar plane of a point with respect to a sphere is perpendicular to the line joining the point to the centre of the sphere.*

Let the point be $P(x_1, y_1, z_1)$ and then the equation of its polar with respect to the sphere $x^2 + y^2 + z^2 = r^2$ is $xx_1 + yy_1 + zz_1 = r^2$.

$\therefore$ The direction ratios of the normal to this polar plane are x_1, y_1, z_1 which are also the direction ratios of the line joining $P(x_1, y_1, z_1)$ and the centre O (0, 0, 0) of the sphre. Hence OP is perpendicular to the polar plane of P with respect to the sphere.

Property V. *If the line joining the centre of the sphere and point P meets the polar plane of P in Q, then $OP.OQ = (radius)^2$.*

Let P be (x_1, y_1, z_1) and the equation of the sphere be

$$x^2 + y^2 + z^2 = r^2 \quad \text{...(i)}$$

Then the radius of the sphere $= r$ and its centre is (0, 0, 0).

$\therefore OP = \sqrt{(x_1^2 + y_1^2 + z_1^2)}$ and the equations of polar plane of P with respect to the sphere (i) is $xx_1 + yy_1 + zz_1 = r^2$ ...(ii)

$\therefore OQ =$ perp. distance of O (0, 0, 0) from the plane (ii)

$$= \frac{r^2}{\sqrt{(x_1^2 + y_1^2 + z_1^2)}} = \frac{r^2}{OP}.$$ Hence $OP.OQ = r^2$.

Property VI. Polar lines. Definition. *Two lines which are such that the polar of any point on any one passes through the other are known as polar lines.*

Let AB and CD be two polar lines, with respect to the sphere

$$x^2+y^2+z^2=a^2 \quad \text{...(i)}$$

Let the line AB be $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$

Then any point on the line AB is $(\alpha+lr, \beta+mr, \gamma+nr)$. Its polar with respect to the sphere (i) is $x(\alpha+lr)+y(\beta+mr)+z(\gamma+nr)=a^2$

or $$(\alpha x+\beta y+\gamma z-a^2)+r(lx+my+nz)=0$$

or $P+rQ=0$, which for all values of r passes through the line $P=0$, $Q=0$ *i.e.*, the line $\alpha x+\beta y+\gamma z-a^2=0$, $lx+my+nz=0$, which are therefore the equations of the polar line CD of the line AB.

Solved Examples on § 7.12—§7.14.

****Ex. 1. Find the pole of the plane $lx+my+nz=p$ with respect to the sphere** $\mathbf{x^2+y^2+z^2=a^2}$ (*Kumaun* 95)

Sol. Let the pole of the plane $lx+my+nz=p$ ...(i)

with respect to the sphere $x^2+y^2+z^2=a^2$ ...(ii)

be (x_1, y_1, z_1).

Then the polar of (x_1, y_1, z_1) with respect to the sphere (ii) is

$$x_1x+y_1y+z_1z=a^2 \quad \text{...(iii)}$$

Now (i) and (iii) represent the same plane, so comparing them we have

$$\frac{x_1}{l}=\frac{y_1}{m}=\frac{z_1}{n}=\frac{a^2}{p}$$

whence $x_1=a^2l/p$; $y_1=a^2m/p$; $z_1=a^2n/p$.

$\therefore$ The required pole is $\left(\frac{a^2l}{p}, \frac{a^2m}{p}, \frac{a^2n}{p}\right)$. **Ans.**

Ex. 2. Prove that the polar plane of any point on the line $\frac{1}{2}x=\frac{1}{3}(y-1)=\frac{1}{4}(z+3)$ with respect to the sphere $x^2+y^2+z^2=1$ passes through the line $(1/13)(2x+3)=-\frac{1}{3}(y-1)=-z$.

Sol. Any point on the line $\frac{x}{2}=\frac{y-1}{3}=\frac{z+3}{4}$ is $P(2r, 1+3r, -3+4r)$

$\therefore$ The polar plane of the point P with respect to the sphere $x^2+y^2+z^2=1$ is $x.(2r)+y.(1+3r)+z.(-3+4r)=1$ ...(i)

If this plane (i) passes through the given line

$$\frac{2x+3}{13}=\frac{y-1}{-3}=\frac{z}{-1} \quad \text{or} \quad \frac{x+\frac{3}{2}}{13}=\frac{y-1}{-6}=\frac{z-0}{-2} \quad \text{...(ii)}$$

then the point $(-3/2, 1, 0)$ on this line must lie on the plane (i) and so we have $-\frac{3}{2}(2r)+1(1+3r)+0(-3+4r)=1$, which being true for all values of r the point $(-\frac{3}{2}, 1, 0)$ lies on the plane (i).

Also the normal to the plane (i) must be perpendicular to this line with direction ratios 13, – 6, – 2.

i.e. $$(2r)(13) + (1 + 3r)(-6) + (-3 + 4r)(-2) = 0$$

which is also satisfied for all values of r.

Hence the polar plane (i) passes through the given line (ii). Hence proved.

Exercises on § 7.12–§ 7.14

Ex. 1. Show that the equation of the polar plane of (x_1, y_1, z_1) with respect to the sphere $x^2 + y^2 + z^2 = a^2$ is $xx_1 + yy_1 + zz_1 = a^2$.

Ex. 2. Show that the equation of the polar plane of

$\frac{1}{2}(x+1) = \frac{1}{3}(y-2) = z + 3$ with respect to the sphere $x^2 + y^2 + z^2 = 1$ is

$$x - 2y + 3z + 1 = 0 = 2x + 3y + z.$$

Ex. 3. Show that every straight line through the point A meets a given sphere in two points P and Q. (*Kanpur* 92)

§ 7.15. Angle of intersectin of two spheres.

The angle of intersectin of two spheres is the angle between the tangent planes to them at their common point of intersection. As the radii of the spheres at this common point are normal to the tangent planes so this angle is also equal to the angle between the radii of the spheres at their common point of intersection.

If the angle of intersection of two spheres is a right angle, the spheres are said to be **orthogonal.**

****§ 7.16. Condition for orthogonality of the spheres.** (*Agra* 91, 90)

Sol. Let the equations of the two spheres be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{...(i)}$$

and $$x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0 \quad \text{...(ii)}$$

Let P be a common point of intersection of these spheres and C and C' be their centres.

Then $CP = \sqrt{(u^2 + v^2 + w^2 - d)}$ and $C'P = \sqrt{(u'^2 + v'^2 + w'^2 - d')}$,

Also CC' = distance between $C(-u, -v, -w)$ and $C'(-u', -v', -w')$

$$= \sqrt{[(u-u')^2 + (v-v')^2 + (w-w')^2]}$$

If the two spheres (i) and (ii) cut orthgonally, then

$$\angle CPC' = \frac{1}{2}\pi \quad \text{or} \quad C'C^2 = CP^2 + C'P^2$$

or $$(u-u')^2 + (v-v')^2 + (w-w')^2 = (u^2 + v^2 + w^2 - d) + (u'^2 + v'^2 + w'^2 - d')$$

or $\boxed{2uu' + 2vv' + 2ww' = d + d', \text{ which is the required condition.}}$

Solved Examples on § 7.15–7.16.

****Ex. 1 (a). Show that the two spheres $x^2 + y^2 + z^2 + 6y + 2z + 8 = 0$ and $x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0$ are orthogonal.**

Find their plane of intersection.

Sol. Here we have $u = 0, v = 3, w = 1, d = 8$ and $u' = 3, v' = -4, w' = 2, d' = 20.$

$$\therefore\ 2uu' + 2vv' + 2ww' = 2(0)(3) + 2(3)(4) + 2(1)(2) = 24 + 4$$
$$= 28 = 8 + 20 = d + d'$$

Hence the spheres cut orthogonally.

Also the plane of intersection of two spheres $S_1 = 0$ and $S_2 = 0$ is given by $S_1 - S_2 = 0$, the coefficients of x^2, y^2, z^2 in each of S_1 and S_2 must be unity.

$\therefore$ Required equation of plane of intersection is

$$(x^2 + y^2 + z^2 + 6x + 8y + 4z + 20) - (x^2 + y^2 + z^2 + 6y + 2z + 8) = 0$$

or $\quad 6x + 2y + 2z + 12 = 0$ or $3x + y + z + 6 = 0.$ **Ans.**

Ex. 1 (b). Find the condition that the spheres $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ may cut orthogonally.

Sol. Here we have $u = 0, v = 0, w = 0, d = -a^2$ and $u' = u,\ v' = v,\ w' = w, d' = d$.

Now the condition for orthogonality is $2uu' + 2vv' + 2ww' = d + d'$

or $\quad 2.0\, u + 2.0 v + 2.0\, w = -a^2 + d$ or $d - a^2 = 0$ or $d = a^2.$ **Ans.**

***Ex. 2. Two points P and Q are conjugate with respect to a sphere S; prove that the sphere on PQ as diameter cuts S orthogonally.**

Sol. Let the points P and Q be (x_1, y_1, z_1) and (x_2, y_2, z_2) and the equation of the sphere be $\qquad x^2 + y^2 + z^2 = r^2.$ ...(i)

The polar plane of $P(x_1, y_1, z_1)$ with respect to the sphere (i) is $xx_1 + yy_1 + zz_1 = r^2$ and if $Q(x_2, y_2, z_2)$ lies on this plane then

$$x_1x_2 + y_1y_2 + z_1z_2 = r^2 \qquad \text{...(ii)}$$

Now the equation of the sphere on PQ as diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

or
$$x^2 + y^2 + z^2 - (x_1 + x_2)x - (y_1 + y_2)y - (z_1 + z_2)z$$
$$+ (x_1x_2 + y_1y_2 + z_1z_2) = 0 \qquad \text{...(iii)}$$

If this sphere (iii) cuts the sphere (i) orthogonally, then

"$2uu' + 2vv' + 2ww' = d + d'$"

i.e. $\quad 0 = (-r^2) + (x_1x_2 + y_1y_2 + z_1z_2), \quad \because\ u' = 0 = v' = w'$, from (i)

or $\quad x_1x_2 + y_1y_2 + z_1z_2 = r^2$. which is true by virtue of (ii).

Hence the spheres (i) and (iii) cut orthogonally.

Ex. 3. Find the equation of the sphere that passes through the circle $x^2 + y^2 + z^2 - 2x + 3y - 4z + 9 = 0, 3x - 4y + 5z - 15 = 0$ and cuts the sphere $x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$ orthogonally. (*Purvanchal* 90)

Sol. The equation of the sphere through the given circle is

$$(x^2+y^2+z^2-2x+3y-4z+9)+\lambda(3x-4y+5z-15)=0$$

or $$x^2+y^2+z^2+(3\lambda-2)x+(3-4\lambda)y+(5\lambda-4)z+(9-15\lambda)=0 \quad ...(i)$$

The other given sphere is $x^2+y^2+z^2+2x+4y-6z+11=0$ ...(ii)

The condition of orthogonal intersection of the spheres (i) and (ii) is

"$2uu'+2vv'+2ww'=d+d'$"

or $$2\frac{1}{2}(3\lambda-12).1+2.\frac{1}{2}(3-4\lambda)2+2.\frac{1}{2}(5\lambda-4)(-3)=(6-15\lambda)+11$$

or $(3\lambda-2)+2(3-4\lambda)-3(5\lambda-4)=17-15\lambda$ or $\lambda=-1/5$

$\therefore$ From (i) the required sphere is

$$5(x^2+y^2+z^2)-13x+19y-25z+45=0. \qquad \textbf{Ans.}$$

***Ex. 4. Find the equation of the sphere which touches the plane $3x+2y-z+2=0$ at the point $(1,-2,1)$ and cuts orthogonally the sphere $x^2+y^2+z^2-4x+6y+4=0$.** (*Kanpur* 95; *Lucknow* 90; *Rohilkhand* 95)

Sol. As the plane $3x+2y-z+2=0$ is a tangent plane to the required sphere at $A(1,-2,1)$, so the line joining the centre C of the sphere and the point $A(1,-2,1)$ must be at right angles to this plane.

Hence the equation of the line AC is $\frac{1}{3}(x-1)=\frac{1}{2}(y+2)=-(z-1)$, ...(i)

the d.c.'s of the line AC are the coefficients of x, y, z in the given plane.

Any point on this line (i) is $(1+3r,-2+2r,1-r)$ and can be taken as the centre C of the sphere. Also the radius of the sphere is CA

i.e. $\sqrt{[(1+3r-1)^2+(-2+2r+2)^2+(1-r-1)^2]}$ *i.e.* $r\sqrt{14}$.

Now the centre and radius of the given sphere are $C'(2,-3,0)$ and

$$\sqrt{(2^2+3^2-4)}=3$$

If the two spheres cut orthogonally, then we have

$$(CP)^2+(C'P)^2=(CC')^2$$ ...See § 7.16 Page 51 Ch. VII

i.e., $$(r\sqrt{14})^2+(3)^2=(1+3r-2)^2+(-2+2r+3)^2+(1-r-0)^2$$

or $$14r^2+9=(3r-1)^2+(2r-1)^2+(1-r)^2 \text{ or } r=-3/2.$$

$\therefore$ The centre C of the sphere is $(1+3r,-2+2r,1-r)$,

where $r=-3/2$

i.e. $[1-(9/2),-2-3,1+(3/2)]$ *i.e.* $(-7/2,-5,5/2)$,

and radius is $r\sqrt{(14)}$ *i.e.* $(3/2)\sqrt{(14)}$, numerically.

$\therefore$ The equation of the sphere is

$$\{x+(7/2)\}^2+(y+5)^2+\{z-(5/2)\}^2=[-(3/2)\sqrt{(14)}]^2$$

or $$x^2+y^2+z^2+7x+10y-5z+12=0 \qquad \textbf{Ans.}$$

****Ex. 5. Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of the common circle is $r_1r_2/\sqrt{(r_1^2+r_2^2)}$.**

(*Agra* 92; *Avadh* 95, 93; *Garhwal* 90; *Kanpur* 97, 91; *Kumaun* 91; *Meerut* 96; *Parvanchal* 96)

Sol. Let the equation of the common circle be $x^2+y^2=a^2, z=0.$ ...(i)
Its radius is evidently a and we are to evaluate it.

Now let the equations of the two given spheres through this circle be

$$(x^2+y^2-a^2)+2\lambda z+z^2=0 \quad \text{...(ii)}$$

and $$(x^2+y^2-a^2)+2\mu z+z^2=0 \quad \text{...(iii) (Note)}$$

[Here an extra term z^2 has been introduced in each equation, so that it may represent a sphere].

From (ii) the radius of the sphere

$$=\sqrt{[(-\lambda)^2-(-a^2)]}=\sqrt{(\lambda^2+a^2)}=r_1 \text{ (given)}$$

and similarly from (iii) the radius of the sphere $=\sqrt{(\mu^2+a^2)}=r_2$ (given).

Also as the sphere (ii) and (iii) cut each other orthogonally, so we have

$$2\lambda\mu=(-a^2)+(-a^2) \quad \text{or} \quad \lambda^2\mu^2=a^4, \text{ squaring both sides}$$

or $$(r_1^2-a^2)(r_2^2-a^2)=a^4, \quad \because \lambda^2+a^2=r_1^2, \mu^2+a^2=r_2^2$$

or $$r_1^2r_2^2=a^2(r_1^2+r_2^2) \quad \text{or} \quad a=r_1r_2/\sqrt{(r_1^2+r_2^2)}$$

***Ex. 6. Show that every sphere through the circle $x^2+y^2-2ax+r^2=0,\ z=0$ cuts orthogonally every sphere through the circle $x^2+z^2=r^2, y=0$.**

Sol. The spheres through the given circles are

$$(x^2+y^2+z^2-2ax+r^2)+2\lambda z=0 \quad \text{...(i)}$$

and $$(x^2+y^2+z^2-r^2)+2\mu y=0 \quad \text{...(ii) (Note)}$$

If (i) and (ii) cut orthogonally, then we have

$$2(-a)0+2(0)(\mu)+2(\lambda)0=r^2-r^2,$$ ...see § 7.16 Page 51

which is true for all values of λ and μ. Hence proved.

P **Ex. 7 (a). Find the equation of a sphere which cuts four given spheres orthogonally.**

Sol. Let the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0,$ ...(i)
cut the four given spheres $x^2+y^2+z^2+2u_ix+2v_iy+2w_iz+d_i=0,$ where $i=1,2,3,4,$ orthogonally.

Then we have the conditions $2uu_i+2vv_i+2ww_i=d+d_i, i=1,2,3,4$ which can be rewritten as

$$-d_1+2uu_1+2vv_1+2ww_1-d=0, \quad \text{...(ii)}$$
$$-d_2+2uu_2+2vv_2+2ww_2-d=0, \quad \text{...(iii)}$$
$$-d_3+2uu_3+2vv_3+2ww_3-d=0, \quad \text{...(iv)}$$
and $$-d_4+2uu_4+2vv_4+2ww_4-d=0. \quad \text{...(v)}$$

Eliminating u, v, w and d from (i), (ii), (iii), (iv), and (v), we get the required equation as

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ -d_1 & u_1 & v_1 & w_1 & -1 \\ -d_2 & u_2 & v_2 & w_2 & -1 \\ -d_3 & u_3 & v_3 & w_3 & -1 \\ -d_4 & u_4 & v_4 & w_4 & -1 \end{vmatrix} = 0$$

Ans.

Ex. 7 (b). Find the equation of the sphere which cuts orthogonally each of the four spheres $x^2+y^2+z^2+2ax=a^2$; $x^2+y^2+z^2+2by=b^2$; $x^2+y^2+z^2+2cz=c^2$ and $x^2+y^2+z^2=a^2+b^2+c^2$. (*Garhwal* 95)

Sol. Let the required equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad ...(A)$$

If this sphere cuts the first of the given spheres orthogonally, then

$2ua=d-a^2$ ...(i), using "$2uu'+2vv'+2ww'=d+d'$".

Similarly if its cuts the second and third of the given spheres orthogonally, then we have

$2bv=d-b^2$ ...(ii) and $2wc=d-c^2$ ...(iii)

Also if this sphere cuts the last of the given spheres orthogonally, then we have

$2u\,.\,0+2v\,.\,0+2w\,.\,0=d-a^2-b^2-c^2$ or $d=a^2+b^2+c^2$ ...(iv)

From (i), (ii), (iii), with the help of (iv), we have

$$2u=(b^2+c^2)/a,\ 2v=(c^2+a^2)/b,\ 2w=(a^2+b^2)/c.$$

Hence the required sphere from (A) is

$$x^2+y^2+z^2+\left(\frac{b^2+c^2}{a}\right)x+\left(\frac{c^2+a^2}{b}\right)y+\left(\frac{a^2+b^2}{c}\right)z+(a^2+b^2+c^2)=0$$

Ex. 8. If the variable sphere $x^2+y^2+z^2+2ux+2vy+2wz+2=0$ always cuts the sphere $3(x^2+y^2+z^2)-6x+10y+z=8$ at right angles, then show that the point (u, v, w) moves on a fixed plane.

Sol. The given spheres are $x^2+y^2+z^2+2ux+2vy+2wz+2=0$ ...(i)

and $x^2+y^2+z^2-2x+(10/3)\,y+(1/3)\,z-(8/3)=0$ ...(ii)

If the spheres (i) and (ii) cut each other at right angles, then

$$2u(-1)+2v(5/3)+2w(1/6)=2+(-8/3),$$

using $2uu'+2vv'+2ww'=d+d'$

or $u-10v-w=2$, on simplifying.

$\therefore$ The locus of the point (u, v, w) is $6x-10y-z=2$, which represents a fixed plane. Hence proved.

Ex. 9. Find the general equation of all spheres through the given points A (a, 0, 0), B (0, b, 0) and C (0, 0, c).

Also find the condition that this sphere may cut orthgonally the sphere $x^2+y^2+z^2-2ax-2by-2cz=0$

Sol. Let the fourth point on the sphere be taken as origin. Then the equation of the sphere through O, A, B and C is

$$S \equiv x^2+y^2+z^2-ax+by-cz=0 \qquad \text{...(i)} \quad \text{...See Ex. 5 Page 4}$$

Also the equation of the plane ABC is

$$P \equiv (x/a)+(y/b)+(z/c)-1=0 \qquad \text{...(iii)}$$

Now the equation of any sphere through the intersection of (i) and (ii) is given by $S+\lambda P=0$

i.e. $$(x^2+y^2+z^2-ax-by-cz)+\lambda(x/a+y/b+z/c-1)=0. \qquad \text{...(iii)}$$

which is the required general equation.

Also (iii) can be rewritten as

$$x^2+y^2+z^2-(a-\lambda/a)\,x-(b-\lambda/b)\,y-(c-\lambda/c)\,z-\lambda=0. \qquad \text{...(iv)}$$

If the given sphere and sphere (iv) cut orthogonally, then we have

$$-(a-\lambda/a)(-a)-(b-\lambda/b)(-b)-(c-\lambda/c)(-c)=-\lambda+0$$

or $\lambda=\frac{1}{4}(a^2+b^2+c^2)$, which is the required condition.

Ex. 10. Prove that the spheres, that can be drawn through the origin and each set of points where the planes parallel to the plane $x/a+y/b+z/c=0$ cut the co-ordinate axes, form a system of spheres which are cut orthogonally by the sphere $x^2+y^2+z^2+2ux+2vy+2wz=0$ if $au+bv+cw=0$.

Sol. The equation of any plane parallel to the plane

$$(x/a)+(y/b)+(z/c)=0 \text{ is } (x/a)+(y/b)+(z/c)=k. \qquad \text{...(i)}$$

This plane cuts the axes at $A\,(ak, 0, 0)$, $B\,(0, bk, 0)$ and $C\,(0, 0, ck)$.

The equation of sphere through $O\,(0, 0, 0)$, A, B and C is

$$x^2+y^2+z^2-akx-bky-ckz=0 \qquad \text{...See Ex. 5 Page 4 Ch. VII}$$

If this sphere cuts the given sphere orthogonally, then we have

$$2u\,(-\tfrac{1}{2}ak)+2v\,(-\tfrac{1}{2}bk)+2w\,(-\tfrac{1}{2}ck)=0+0$$

or $au+bv+cw=0$, for all values of k. Hence proved.

Ex. 11. Prove that a sphere $S=0$ which cuts the two spheres $S_1=0$ and $S_2=0$ at right angles will also cut the sphere $\lambda_1 S_1+\lambda_2 S_2=0$ at right angles.

Sol. Let $S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$$S_1 \equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$$

and $$S_2 \equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$$

Then if $S=0$ cuts $S_1=0$ and $S_2=0$ orthogonally (*i.e.* at right angles), we get

$$2uu_1+2vv_1+2ww_1=d+d_1 \qquad \text{...(i)}$$

and $$2uu_2+2vv_2+2ww_2=d+d_2. \qquad \text{...(ii)}$$

Now the equation of the sphere $\lambda_1 S_1 + \lambda_2 S_2 = 0$ reduces to

$$x^2 + y^2 + z^2 + 2\left(\frac{\lambda_1 u_1 + \lambda_2 u_2}{\lambda_1 + \lambda_2}\right)x + 2\left(\frac{\lambda_1 v_1 + \lambda_2 v_2}{\lambda_1 + \lambda_2}\right)y + 2\left(\frac{\lambda_1 w_1 + \lambda_2 w_2}{\lambda_1 + \lambda_2}\right)z$$
$$+ 2\left(\frac{\lambda_1 d_1 + \lambda_2 d_2}{\lambda_1 + \lambda_2}\right) = 0$$

If $S = 0$ cuts this sphere $\lambda_1 S_1 + \lambda_2 S_2 = 0$ orthogonally, then we have

$$2u\left(\frac{\lambda_1 u_1 + \lambda_2 u_2}{\lambda_1 + \lambda_2}\right) + 2v\left(\frac{\lambda_1 v_1 + \lambda_2 v_2}{\lambda_1 + \lambda_2}\right) + 2w\left(\frac{\lambda_1 w_1 + \lambda_2 w_2}{\lambda_1 + \lambda_2}\right)$$
$$= d + \left(\frac{\lambda_1 d_1 + \lambda_2 d_2}{\lambda_1 + \lambda_2}\right)$$

or $\quad \lambda_1 (2uu_1 + 2vv_1 + 2ww_1) + \lambda_2 (2uu_2 + 2vv_2 + 2ww_2)$
$$= \lambda_1 (d + d_1) + \lambda_2 (d + d_2)$$

or $\lambda_1 (d + d_1) + \lambda_2 (d + d_2) = \lambda_1 (d + d_1) + \lambda_2 (d + d_2)$, from (i) and (ii) and is evidently true for all values of λ_1 and λ_2. Hence proved.

Exercises on § 7.15—7.16.

Ex. 1. Obtain the condition that the spheres $a(x^2 + y^2 + z^2) + 2lx + 2my + 2nz + p = 0$ and $b(x^2 + y^2 + z^2) = k^2$ may cut orthogonally. **Ans.** $ak^2 = bp$.

Ex. 2. Obtain the condition that the spheres $a(x^2 + y^2 + z^2) + 2ux + 2vy + 2wz + d = 0$ and $a'(x^2 + y^2 + z^2) + 2u'x + 2v'y + 2w'z + d' = 0$ may cut orthogonally. **Ans.** $2(uu' + vv' + ww') = da' + d'a$

§ 7.17. Length of the tangent.

Let $P(x_1, y_1, z_1)$ be a point outside the sphere

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0. \qquad \text{...(i)}$$

Its centre is $C(-u, -v, -w)$ and radius $= \sqrt{(u^2 + v^2 + w^2 - d)}$.

Now let the tangent from $P(x_1, y_1, z_1)$ to the sphere meet at T, then the radius CT at T must be at right angles to the tangent PT. Therefore ΔPCT is a right angled triangle with $\angle T = 90°$.

$$\therefore PT^2 = (PC)^2 - (CT)^2$$
$$= [(x_1 + u)^2 + (y_1 + v)^2 + (z_1 + w)^2] - (u^2 + v^2 + w^2 - d)$$
$$= x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d \equiv S_1, \text{ from (i)}$$

This gives the square of the length of the tangent PT. This is also called power of the point P with respect to the sphere $S = 0$.

Rule for finding the length of the tangent. In the left hand expression of given equation of the sphere, (right hand side being zero) put the coordinates of the point, from which length of the tangent is to be calculated, in place of x, y, z and the result so obtained is the square of the length of the tangent.

Solved Example on § 7.17.

Ex. 1. Find the length of the tangent drawn from the point (1, 2, 3) to the sphere $5(x^2+y^2+z^2)-x+10y+20z+8=0$.

Sol. Let $P(1, 2, 3)$ be the given point and let the tangent from P to the given sphere $x^2+y^2+z^2-(1/5)x+2y+4z+(8/5)=0$ meet at T, then square of the required length of the tangent $= PT^2$

$= 1^2+2^2+3^2-(1/5)(1)+2(2)+4(3)+(8/5)$

$= 1+4+9-(1/5)+4+12+(8/5)=30+(7/5)=(157/5)$

or Required length of the tangent $= PT = \sqrt{(157/5)}$. **Ans.**

§ 7.18. Radical plane.

Definition. *The radical plane of two spheres is defined as the locus of a point whose powers with respect to the spheres are equal i.e. the locus of a point from where the square of the lengths of the tangents to the two spheres are equal.* (*Kumaun* 93)

Let the equations of the two spheres be

$$S_1 \equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0 \quad \text{...(i)}$$

and
$$S_2 \equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0 \quad \text{...(ii)}$$

Let $P(x_1, y_1, z_1)$ be a point which is such that its powers with respect to the two spheres are equal *i.e.* the squares of the lengths of tangents from $P(x_1, y_1, z_1)$ to the spheres $S_1=0$ and $S_2=0$ are equal.

Then $x_1^2+y_1^2+z_1^2+2u_1x_1+2v_1y_1+2w_1z_1+d_1$

$$= x_1^2+y_1^2+z_1^2+2u_2x_1+2v_2y_1+2w_2z_1+d_2$$

or
$$2(u_1-u_2)x_1+2(v_1-v_2)y_1+2(w_1-w_2)z_1+(d_1-d_2)=0$$

$\therefore$ The equation of the radical plane of the given spheres or the locus of $P(x_1, y_1, z_1)$ is

$$\mathbf{2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+(d_1-d_2)=0} \quad \text{...(iii)}$$

or
$$\boxed{\mathbf{S_1-S_2=0.}}$$

****Properties of Radical plane.**

Prop I. *The radical plane of two spheres passes through their point of intersecttion.*

The radical plane of the spheres $S_1=0$ and $S_2=0$ is $S_1-S_2=0$. The points which satisfy both $S_1=0$ and $S_2=0$, satisfy $S_1-S_2=0$ hence the points of intersection of $S_1=0$ and $S_2=0$ lie on the radical plane $S_1-S_2=0$.

Also two spheres intersect in a circle which from above lies on the radical plane of the spheres.

Prop. II. *The radical planes of two spheres is at right angles to the line joining their centres.*

The direction ratios of the line joining the centres

$$(-u_1, -v_1, -w_1) \text{ and } (-u_2, -v_2, -w_2)$$

of the spheres $S_1 = 0$, $S_2 = 0$ respectively are $u_1 - u_2$, $v_1 - v_2$, $w_1 - w_2$ which are also the direction ratios of the normal to the radical plane $S_1 - S_2 = 0$ given by (iii) above. Hence the property.

§ 7.19. Radical axis (or radical line) and radical centre.

Radical axis : *The radical planes of three spheres taken two by two pass through a line which is called the radical axis or the radical line of the three spheres.*

Let the three spheres be given by the equations $S_1 = 0$, $S_2 = 0$ and $S_3 = 0$. The radical planes of these three spheres taken two by two are $S_1 - S_2 = 0$; $S_2 - S_3 = 0$ and $S_3 - S_1 = 0$, and these three radical planes evidently pass through the line $S_1 = S_2 = S_3$, which is defined as the radical axis of the spheres $S_1 = 0$, $S_2 = 0$ and $S_3 = 0$.

Radical centre. The four radical axes of the four spheres taken three at a time meet in a point which is called radical centre.

If $S_1 = 0$, $S_2 = 0$, $S_3 = 0$, $S_4 = 0$ be the four spheres, then the four radical axes taking three spheres at a time are given by

$S_1 = S_2 = S_3$; $S_1 = S_2 = S_4$; $S_1 = S_3 = S_4$ and $S_2 = S_3 = S_4$.

These radical axes evidently pass through the point given by $S_1 = S_2 = S_3 = S_4$ which is defined as the radical centre.

Note. This point also lies on the radical planes of the four spheres taken two by two.

****§ 7.20 Coaxial system of spheres.**

Definition. *A system of spheres is said to be coaxial system of spheres if any two spheres of the system have the same radical plane.*

General Equation of coaxial system of spheres :

If $S_1 = 0$ and $S_2 = 0$ be the equations of two spheres, then the equation $S_1 + \lambda S_2 = 0$ always represent a system of spheres for all values of λ except -1 and then it degenerates into a plane and all the spheres of this system passes, through the circle of intersection of the spheres $S_1 = 0$ and $S_2 = 0$ and hence any two spheres of this system have the same radical planes $S_1 - S_2 = 0$.

Again if $S_1 = 0$ be the equation of sphere and $P = 0$ be the equation of a plane, then the equation $S_1 + \lambda P = 0$ represents a sphere for all values of λ and pases through the intersection of $S_1 = 0$ and $P = 0$.

Let $S_1 + \lambda_1 P = 0$ and $S_1 + \lambda_2 P = 0$ be two members of this system of spheres $S_1 + \lambda P = 0$. Their radical plane is given by

$(S_1 + \lambda_1 P) - (S_1 + \lambda_2 P) = 0$ or $P = 0$, $\lambda_1 \neq \lambda_2$

Hence the system of spheres $S_1 + \lambda P = 0$ form a system of coaxial spheres as any two spheres of the system have the same radical plane $P = 0$.

***§ 7.21. The equation of coaxial spheres in the simplest form.**

(*Kanpur* 97)

Let us suppose the centre of all the spheres of the system lie on x-axis, then y and z-coordinates of the centres of all the spheres are zero whereas the x-coordinates differ.

$\therefore$ The equation of such a system of sphere can be written as

$$x^2+y^2+z^2+2ux+d=0 \quad \text{...(i)}$$

The radical plane of two spheres

$$x^2+y^2+z^2+2u_1x+d_1=0 \text{ and } x^2+y^2+z^2+2u_2x+d_2=0$$

of the system (i) is $\quad 2(u_1-u_2)\,x+(d_1-d_2)=0 \quad$ ...(ii)

Now if yz-plane is chosen as the radical plane then the equation (ii) should reduce to $x=0$, the condition for which is

$d_1-d_2=0$ or $d_1=d_2$ *i.e.* d is absolute constant.

Hence the equation of coaxial system of spheres can be written as

$$x^2+y^2+z^2+2\lambda x+d=0, \quad \text{...(iii)}$$

the common radical plane being $x=0$ *i.e.* yz-plane.

***§ 7.22 Limiting Points**

Definition : *The centre of spheres of zero radii of a coaxial system of spheres are called the Limiting points of the system.*

Let the equation of the coaxial system of spheres be

$$x^2+y^2+z^2+2\lambda x+d=0, \quad \text{...(i)}$$

where λ is a parameter and d an absolute constant.

This can be rewritten as $(x+\lambda)^2+(y-0)^2+(z-0)^2=(\lambda^2-d)$

$\therefore$ The radius of a sphere of this system is $\sqrt{(\lambda^2-d)}$ and centre is $(-\lambda, 0, 0)$. If this radius is zero, then we geat $\lambda^2=d$ or $\lambda=\pm\sqrt{d}$. Hence the limiting points of the system as defined above are $(\pm\sqrt{d}, 0, 0)$.

Also the section of the system of the coaxial spheres [given by (i)] by the radical plane $x=0$ is the circle

$$x^2+y^2+z^2+2\lambda x+d=0,\ x=0$$

or in the plane geometry by the circle $y^2+z^2=-d$ *i.e.* a circle in yz-plane of radius $\sqrt{(-d)}$.

If d is negative, then the radius of the circle is real *i.e.* the circle is real *i.e.* the spheres intersect.

If $d=0$, then the radius of the circle is zero *i.e.* the spheres intersect in a point circle *i.e.* the spheres touch each other.

If d is positive, the radius of the circle is imaginary *i.e.* the circle is imaginary *i.e.* the spheres do not intersect. (*Kanpur 97*)

Hence the two spheres of the coaxial system of spheres intersect, touch or do not intersect each other according as d is negative, zero or positive.

Rule for finding the limiting points : Equate to zero the radius of the coaxial system of spheres.

Solved Examples on § 7.17 to § 7.22.

Ex. 1. Prove that every sphere that passes through the limiting points of a coaxial system cuts every sphere of that system orthogonally.

(*Lucknow* 92)

Sol. Let the equation of the coaxial system of spheres be

$$x^2+y^2+z^2+2\lambda x+d=0. \quad \text{...(i)}$$

Its limiting points are $(\sqrt{d}, 0, 0)$ and $(-\sqrt{d}, 0, 0)$.

Let any sphere be $x^2+y^2+z^2+2ux+2vy+2wz+k=0$ ...(ii)

It is passes through $(\sqrt{d}, 0, 0)$ and $(-\sqrt{d}, 0, 0)$, then

$$d+2u\sqrt{d}+k=0 \quad \text{and} \quad d-2u\sqrt{d}+k=0.$$

Solving we get $u=0$ and $k=-d$.

Hence (ii) becomes $x^2+y^2+z^2+2vy+2wz-d=0.$ ...(iii)

If (i) and (iii) cut orthogonally, then $2\lambda(0)+2.0\,v+2.0\,w=d-d$ which is true for all values of v and w. Hence proved.

Ex. 2. Find the equation of the radical axis in the symmetric form of the spheres $S_1 \equiv x^2+y^2+z^2+2x+2y+2z+2=0,$

$$S_2 \equiv x^2+y^2+z^2+4x+4z+4=0$$

and $S_3 \equiv x^2+y^2+z^2+x+6y-4z-2=0$

Sol. The radical plane of $S_1=0$ and $S_2=0$ is

$$S_1-S_2=0 \quad i.e. \quad 2x-2y+2z+2=0 \quad i.e. \quad x-y+z+1=0 \quad \text{...(i)}$$

And the radical plane of $S_2=0$ and $S_3=0$ is

$$S_2-S_3=0 \quad i.e. \quad 3x-6y+8z+6=0 \quad \text{...(ii)}$$

Now putting $z=0$ in (i) and (ii), we get

$$x-y+1=0,\ 3x-6y+6=0.$$

Solving these we get $x=0, y=1$. Therefore the planes (i) and (ii) pass through the point (0, 1, 0) *i.e.* (0, 1, 0) is a point on the radical axis given by (i) and (ii) of the given spheres. Also from (i) and (ii) we find that

$l-m+n=0$ and $3l-5m+8n=0$, where l, m, n are the d.c.'s of the radical axis.

Solving these we get $l/2=m/5=n/3$.

$\therefore$ The required equation is $\frac{1}{2}(x-0)=\frac{1}{5}(y-1)=\frac{1}{3}(z-0)$ **Ans.**

***Ex. 3. A sphere of radius r passes through the origin. Show that the ends of the diameter which is parallel to x-axis lie on each of the spheres**

$$x^2+y^2+z^2 \pm 2rx=0$$

(*Lucknow* 92)

Sol. The equation of any sphere of radius r through the origin is

$$x^2+y^2+z^2+2ux+2vy+2wz=0, \quad \text{...(i)}$$

where
$$r^2 = u^2 + v^2 + w^2 \quad ...(ii)$$

Its centre is $(-u, -v, -w)$. The equations of its diameter parallel to x-axis, whose d.c.'s are $(1, 0, 0)$ are

$$\frac{x-(-u)}{1} = \frac{y-(-v)}{0} = \frac{z-(-w)}{0} = \pm r,$$

since its extremities are at distance $+r$ or $-r$ from the centre.

Co-ordinates of its extremities are $(r-u, -v, -w)$ and $(-r-u, -v, -w)$.

Let one extremity be (x_1, y_1, z_1), then $x_1 = r-u$, $y_1 = -v$ and $z_1 = -w$

or
$$u = r - x_1, \quad v = -y_1 \text{ and } w = -z_1.$$

Substituting these values of u, v and w in (ii), we get

$$r^2 = (r-x_1)^2 + (-y_1)^2 + (-z_1)^2 \quad \text{or} \quad x_1^2 + y_1^2 + z_1^2 - 2rx_1 = 0$$

$\therefore$ The locus of this extremity is $x^2 + y^2 + z^2 = 2rx$. Similarly the locus of the other extremity is $\quad x^2 + y^2 + z^2 = -2rx.$ Hence proved.

***Ex. 4. Prove that the spheres cutting two given spheres along a great circle pass throguh two fixed points.** (*Garhwal* 91)

Sol. Let x-axis be the line of centres and yz-plane be the radical plane, then the equation of the two given spheres can be taken as

$$x^2 + y^2 + z^2 + 2u_1x + d = 0 \quad ...(i)$$

and
$$x^2 + y^2 + z^2 + 2u_2x + d = 0 \quad ...(ii)$$

Let the equation of another sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + k = 0 \quad ...(iii)$$

Now (iii) cuts (i) along a great circle *i.e.* the radical plane of (i) and (iii) viz. $2(u-u_1)x + 2vy + 2wz + (k-d) = 0$ passes through the centre $(-u_1, 0, 0)$ of the sphere (i)

i.e.
$$-2(u-u_1)u_1 + (k-d) = 0. \quad ...(iv)$$

Similarly if (iii) cuts (ii) along a great circle, we have

$$-2(u-u_2)u_2 + (k-d) = 0. \quad ...(v)$$

Subtracting (v) from (iv), we get $u = u_1 + u_2$ and so from (iv) we get

$$k = d + 2u_1u_2$$

Since (i) and (ii) spheres are given so u_1, u_2 and d are constants and therefore u and k are also constants.

Now the x-axis meets (iii) in points whose absciassae are given by $x^2 + 2ux + k = 0$, the roots of which depend upon the values of u and k and hence constants. Thus every sphere [of the form (iii)] cuts the x-axis in the same points.

****Ex. 5 (a). Find the limiting points of coaxial systems defined by the spheres $x^2 + y^2 + z^2 + 2x + 2y + 4z + 2 = 0$ and $x^2 + y^2 + z^2 + x + y + 2z + 2 = 0$** (*Avadh* 95)

Sol. The equation of the coaxial system of spheres is

$$(x^2+y^2+z^2+2x+2y+4z+2)+\lambda(x^2+y^2+z^2+x+y+3z+2)=0$$

or $$x^2+y^2+z^2+\left(\frac{2+\lambda}{1+\lambda}\right)x+\left(\frac{2+\lambda}{1+\lambda}\right)y+\left(\frac{4+2\lambda}{1+\lambda}\right)z+2=0 \quad \text{...(i)}$$

Its centre is $\left(-\frac{2+\lambda}{1+\lambda}, -\frac{2+\lambda}{1+\lambda}, -\frac{4+2\lambda}{1+\lambda}\right)$

and equating its radius to zero, we get

$$\frac{(2+\lambda)^2}{(1+\lambda)^2}+\frac{(2+\lambda)^2}{(1+\lambda)^2}+\frac{(4+2\lambda)^2}{(1+\lambda)^2}-2=0$$

or $$6(2+\lambda)^2=2(1+\lambda)^2 \quad \text{or} \quad 3(\lambda^2+4\lambda+4)=\lambda^2+2\lambda+1$$

or $$2\lambda^2+10\lambda+11=0 \quad \text{or} \quad \lambda=\frac{-10\pm\sqrt{(100-88)}}{4}=\frac{-5\pm\sqrt{3}}{2}$$

or $$\lambda=\frac{-5+\sqrt{3}}{2},\ \frac{-5-\sqrt{3}}{2}$$

Substituting these values of λ in the coordinates of the above centre, the required limiting points are

$(1/\sqrt{3}, 1/\sqrt{3}, 2/\sqrt{3})$ and $(-1/\sqrt{3}, -1/\sqrt{3}, -2/\sqrt{3})$ **Ans.**

****Ex. 5 (b). Find the limiting points of the co-axial system of spheres determined by $x^2+y^2+z^2-20x+30y-40z+29=0$ and $x^2+y^2+z^2-18x+27y-36z+29=0$.**

Ans. $(4, -6, 8), (2, -3, 4)$

Sol. Do as Ex. 5 (a) above.

Ex. 6. Obtain the radical axis for the spheres.

$$(x-2)^2+y^2+z^2=1;\ x^2+(y-3)^2+z^2=6;$$
$$(x+2)^2+(y+1)^2+(z-2)^2=6. \qquad (Kanpur\ 92)$$

Sol. The given spheres are

$(x-2)^2+y^2+z^2=1$ ...(i); $x^2+(y-3)^2+z^2=6$ ...(ii)

and $(x+2)^2+(y+1)^2+(z-2)^2=6$ ...(iii)

The radical plane of spheres (i) and (ii) is

$(x-2)^2+y^2+z^2-1=x^2+(y-3)^2+z^2-6$ ...See § 7.18 P. 58. Ch. VII

or $4x-6y=0$ or $2x-3y+0.z=0$ ...(iv)

The radical plane of spheres (i) and (iii) is

$(x-2)^2+y^2+z^2-1=(x+2)^2+(y+1)^2+(z-2)^2-6$

or $8x+2y-4z=0$ or $4x+y-2z=0$ ...(v)

From (iv) and (v) it is evident (as none contains a constant term) that both the radical planes pass through (0, 0, 0) *i.e.* the required radical axis (see § 7.19 Page 59) passes through (0, 0, 0).

If l, m, n be the d.c.'s of this radical axis, then from (iv) and (v) we get

$$2l-3m+0.n=0,\ 4l+m-2n=0$$

Solving these we have $\frac{l}{6}=\frac{m}{4}=\frac{n}{14}$ or $\frac{l}{3}=\frac{m}{2}=\frac{n}{7}$

Hence the required axis is given by

$$\frac{1}{3}(x-0)=\frac{1}{2}(y-0)=\frac{1}{7}(z-0) \quad \text{or} \quad \frac{x}{3}=\frac{y}{2}=\frac{z}{7}$$ **Ans.**

Exercises on § 7.17—§ 7.22.

Ex. 1. Prove that the radical planes of three spheres taken two by two pass through one line.

Ex. 2. Show that the limiting points of the co-axial system of spheres determined by $x^2+y^2+z^2+3x-3y+6=0$ and $x^2+y^2+z^2-6y-6z+6=0$ are $(-1, 2, 1)$ and $(-2, 1, -1)$.

MISCELLANEOUS SOLVED EXAMPLES

****Ex. 1. Obtain the equations of the spheres which pass through the circle $y^2+z^2=4$, $x=0$ and are cut by the plane $2x+2y+z=0$ in a circle of radius 3.**

Sol. The equation of any sphere through the given circle is

$$(x^2+y^2+z^2-4)+\lambda x=0$$

or $$x^2+y^2+z^2+\lambda x-4=0$$ **(Note x^2)**

Its centre C is $(-\frac{1}{2}\lambda, 0, 0)$ and radius $\sqrt{(\frac{1}{4}\lambda^2+4)}$. ...(i)

Let A be any point on the circumference of the circle in which the sphere (i) is cut by the plane $2x+2y+z=0$ and B be the centre of this circle.

Then CA = radius of sphere (i) $=\sqrt{(\frac{1}{4}\lambda^2+4)}$. ...(ii)

and CB = length of perp. from from C to the plane $2x+2y+z=0$

i.e. $$CB=\frac{2(-\frac{1}{2}\lambda)+2.0+1.0}{\sqrt{(2^2+2^2+1)}}=\frac{-\lambda}{3}.$$ ...(iii)

Also given that the radius of the circle is 3 *i.e.* $BA=3$.

Since ABC is a right angled triangle, with $\angle B=\frac{1}{2}\pi$,

so $CA^2=CB^2+BA^2$ or $(\frac{1}{2}\lambda^2+4)=\frac{1}{9}\lambda^2+9$ from (ii), (iii)

or $\left[\frac{1}{4}-\frac{1}{9}\right]\lambda^2=9-4$ or $\lambda^2=36$ or $\lambda=\pm 6$

$\therefore$ From (i), the required equations of the spheres are

$$x^2+y^2+z^2\pm 6x-4=0$$ **Ans.**

****Ex. 2. A sphere of constant radius r passes through the origin O and cuts the axes in A, B, C. Find the locus of the foot of the perpendicular from O to the plane ABC.**

(*Bundelkhand* 96; *Garhwal* 90; *Kumaun* 96, 94; *Meerut* 93, 91, *S*, 91, 90)

Sol. Let A, B and C be $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$.

Then the equation of the sphere through O, A, B, C is

$$x^2+y^2+z^2-ax-by-cz=0,$$ ...(i).

where $$4r^2 = a^2 + b^2 + c^2 \quad \text{...(ii)}$$

Also the equation of the plane ABC is $x/a + y/b + z/c = 1$ ...(iii)

The equation of the line through O perpendicular to the plane (iii) is

$$\frac{x}{(1/a)} = \frac{y}{(1/b)} = \frac{z}{(1/c)} \quad \text{...(iv)}$$

Any point on it $(k/a, k/b, k/c)$ and if it is the foot (x_1, y_1, z_1) of the perpendicular then $x_1 = k/a,\ y_1 = k/b,\ z_1 = k/c$

or $$a = k/x_1,\ b = k/y_1,\ c = k/z_1$$

Therefore from (ii) we get $4r^2 = k^2 (x_1^{-2} + y_1^{-2} + z_1^{-2})$ ...(v)

Also from (iv) we get $\dfrac{x}{(1/a)} = \dfrac{y}{(1/b)} = \dfrac{z}{(1/c)} = k$

or $$k = \frac{x^2}{x/a} = \frac{y^2}{y/b} = \frac{z^2}{z/c} \quad \textbf{(Note)}$$

$$= \frac{x^2 + y^2 + z^2}{(x/a + y/b + z/c)} = \frac{x^2 + y^2 + z^2}{1}, \text{ from (iii)}$$

or $$k = x^2 + y^2 + z^2 \quad \text{...(vi)}$$

Now from (v) the locus of (x_1, y_1, z_1) is $4r^2 = k^2 (x^{-2} + y^{-2} + z^{-2})$

or $$4r^2 = (x^2 + y^2 + z^2)^2 (x^{-2} + y^{-2} + z^{-2}), \text{ from (vi)} \quad \textbf{Ans.}$$

***Ex. 3. Find the plane and centre and radius of the circle common to the two spheres $x^2 + y^2 + z^2 - 4z + 1 = 0$ and $x^2 + y^2 + z^2 - 4x - 2y - 1 = 0$**

Sol. The equation of the plane of the circle in which the given spheres intersect is $(x^2 + y^2 + z^2 - 4z + 1) - (x^2 + y^2 + z^2 - 4x - 2y - 1) = 0$

or $$4x + 2y - 4z + 2 = 0 \quad \text{or} \quad 2x + y - 2z + 1 = 0 \quad \text{...(i)}$$

Also the centre of the sphere $x^2 + y^2 + z^2 - 4z + 1 = 0$ ...(ii)

is $C(0, 0, 2)$ and its radius $= \sqrt{(2^2 - 1)} = \sqrt{3}$.

Let A and B be respectively any point on the circumference and the centre of the circle in which the plane (i) interect the sphere (ii), then $CA = \sqrt{3}$ and CB = length of perpendicular from C to plane (i)

$$= \frac{2(0) + 0 - 2(2) + 1}{\sqrt{[2^2 + 1^2 + (-2)^2]}} = -\frac{3}{3} = 1, \text{ numerically}$$

$\therefore$ The required radius of the circle $= BA = \sqrt{(CA^2 - CB^2)}$

$$= \sqrt{(3 - 1)} = \sqrt{2}. \quad \textbf{Ans.}$$

Also CB is a line through $C(0, 0, 2)$ perpendicular to the plane given by (i) and so its equation is $$\frac{x - 0}{2} = \frac{y - 0}{1} = \frac{z - 2}{-2} \quad \textbf{(Note)}$$

Any point of this line is $(2r, r, 2 - 2r)$

If this point is B, then it must lie on (i) and so we have

$$2(2r) + (r) - 2(2 - 2r) + 1 = 0$$

or $$4r + r - 4 + 4r + 1 = 0 \quad \text{or} \quad 9r = 3 \quad \text{or} \quad r = \frac{1}{3}$$

The centre B is $\left(\frac{2}{3}, \frac{1}{3}, 2 - \frac{2}{3}\right)$ or $\left(\frac{2}{3}, \frac{1}{3}, \frac{4}{3}\right)$. **Ans.**

****Ex. 4 (a). Find the equation of the sphere which touches the sphere $x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0$ at the point $(1, 1, -1)$ and passes through the origin.** *(Rohilkhand 96)*

Sol. Equation of the tangent plane to the given sphere

$$x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0 \qquad \text{...(i)}$$

at $(1, 1, -1)$ is $x.1 + y.1 + z(-1) - \frac{1}{2}(x+1) + \frac{3}{2}(y+1) + (z-1) - 3 = 0$

or $$x + 5y - 6 = 0 \qquad \text{...(ii)}$$

The required sphere is the sphere through the **point circle** (or a circle of zero radius) of intersection of the sphere (i) and the plane (ii) and also passing through the point (0, 0, 0). **(Note)**

The equation of any sphere through the circle of intersection (point circle) is $(x^2 + y^2 + z^2 - x + 3y + 2z - 3) + \lambda(x + 5y - 6) = 0$. ...(iii)

If it passes through (0, 0, 0), then $-3 - 6\lambda = 0$ or $\lambda = -1/2$

Substituting this value of λ in (iii), the required equation is

$$(x^2 + y^2 + z^2 - x + 3y + 2z - 3) - (1/2)(x + 5y - 6) = 0$$

or $$2(x^2 + y^2 + z^2 - x + 3y + 2z - 3) - (x + 5y - 6) = 0$$

or $$2(x^2 + y^2 + z^2) - 3x + y + 4z = 0$$ **Ans.**

***Ex. 4 (b). Find the equations of the sphere which touches the sphere $x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$ at the point $(1, 2, -2)$ and passes through the origin.**

Sol. Do as Ex. 4 (a) above **Ans.** $4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0$

***Ex. 5. Prove that the plane $x + 2y - z = 0$ cuts the sphere $x^2 + y^2 + z^2 - x + z - 2 = 0$ in a circle of radius unity and find the equation of the sphere which has this circle for one of its great circles.**

Sol. Refer fig. 1. Page 13 Chapter VII.

If O be the centre of the sphere, then O is $(\frac{1}{2}, 0, -\frac{1}{2})$ and radius OA of the sphere $= \sqrt{[(\frac{1}{2})^2 + (0)^2 + (-\frac{1}{2})^2 - (-2)]} = \sqrt{\left(\frac{5}{2}\right)}$.

Also OC = length of perpendicular from O on the plane

$$x + 2y - z = 4 \quad i.e. \quad -x - 2y + z + 4 = 0 \qquad \textbf{(Note)}$$

i.e. $$OC = \frac{-\frac{1}{2} - 2(0) + (-\frac{1}{2}) + 4}{\sqrt{[(-1)^2 + (-2)^2 + (1)^2]}} = \frac{3}{\sqrt{6}} = \frac{\sqrt{3}}{\sqrt{2}}$$

$\therefore$ The radius of the circle $= CA = \sqrt{(OA^2 - OC^2)}$

$= \sqrt{[(5/2) - (3/2)]} = 1.$ Hence proved.

Now the equation of any sphere through the given circle is

$$(x^2 + y^2 + z^2 - x + z - 2) + \lambda (x + 2y - z - 4) = 0 \quad ...(i)$$

or $$x^2 + y^2 + z^2 + (\lambda - 1) x + 2\lambda y + (1 - \lambda) z - (2 + 4\lambda) = 0 \quad ...(ii)$$

Its centre is $[-\frac{1}{2}(\lambda - 1), -\lambda, -\frac{1}{2}(1 - \lambda)]$

Now if the given circle is a great circle of the sphere (i), then the centre of (i) must lie on the plane $x + 2y - z = 4$ of the circle.

$$\therefore -\frac{1}{2}(\lambda - 1) + 2(-\lambda) + \frac{1}{2}(1 - \lambda) = 4 \Rightarrow -3\lambda = 3 \Rightarrow \lambda = -1$$

$\therefore$ From (i) the required sphere is

$$(x^2 + y^2 + z^2 - x + z - 2) - (x + 2y - z - 4) = 0$$

or $$x^2 + y^2 + z^2 - 2x - 2y + 2z + 2 = 0$$ **Ans.**

****Ex. 6. Show that the locus of the centre of the circle of radius a which always intersects the co-ordinates axes (rectangular), is**

$$\mathbf{x \sqrt{(a^2 - y^2 - z^2)} + y \sqrt{(a^2 - z^2 - x^2)} + z \sqrt{(a^2 - x^2 - y^2)} = a^2.}$$

Sol. Let $C(x_1, y_1, z_1)$ be the centre of the circle which cuts the co-ordinates axes at $A(p, 0, 0)$, $B(0, q, 0)$ and $C(0, 0, r)$.

Then the centre (x_1, y_1, z_1) lies on the plane ABC, whose equation is

$$x/p + y/q + z/r = 1$$

$$\therefore (x_1/p) + (y_1/q) + (z_1/r) = 1. \quad ...(i)$$

Also $a = \text{radius } CA = \sqrt{[(x_1 - p)^2 + y_1^2 + z_1^2]}$

or $(x_1 - p)^2 = a^2 - y_1^2 - z_1^2$ or $p = x_1 - \sqrt{(a^2 - y_1^2 - z_1^2)}$

Similarly $q = y_1 - \sqrt{(a^2 - z_1^2 - x_1^2)}$, $r = z_1 - \sqrt{(a^2 - x_1^2 - y_1^2)}$

$$\therefore \quad \frac{1}{p} = \frac{1}{x_1 - \sqrt{(a^2 - y_1^2 - z_1^2)}} = \frac{x_1 + \sqrt{(a^2 - y_1^2 - z_1^2)}}{x_1^2 - a^2 + y_1^2 + z_1^2}$$

Similarly $\frac{1}{q} = \frac{y_1 + \sqrt{(a^2 - z_1^2 - x_1^2)}}{x_1^2 + y_1^2 + z_1^2 - a^2}$; $\frac{1}{r} = \frac{z_1 + \sqrt{(a^2 - x_1^2 - y_1^2)}}{x_1^2 + y_1^2 + z_1^2 - a^2}$

Substituting these values of $1/p$, $1/q$, $1/r$ in (i) and simplifying we can get the required locus of (x_1, y_1, z_1).

Ex. 7. If O be the centre of a sphere of radius unity and A and B be two points in a line with O such that OA . OB = 1 and if P be any variable point on the sphere, show that PA : PB = constant.

Sol. The equation of the sphere is $x^2 + y^2 + z^2 = 1$ ...(i)

Let A be the point (x_1, y_1, z_1) then the direction ratios of OA are x_1, y_1, z_1 so that the equation of line OA is $x/x_1 = y/y_1 = z/z_1$.

If B be the point on this line at a distance r from O, then the coordinates of B are (rx_1, ry_1, rz_1).

Given that the $OA \cdot OB = 1$, which reduces to

$$\sqrt{(x_1^2 + y_1^2 + z_1^2)} \cdot \sqrt{(r^2x_1^2 + r^2y_1^2 + r^2z_1^2)} = 1$$

or $$r(x_1^2+y_1^2+z_1^2)=1. \qquad ...(ii)$$

If the coordinates of any variable point P on the sphere be (x_2, y_2, z_2), then from (i) we get $$x_2^2+y_2^2+z_2^2=1. \qquad ...(iii)$$

Now $$\frac{PA^2}{PB^2}=\frac{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}{(rx_1-x_2)^2+(ry_1+y_2)^2+(rz_1-z_2)^2}$$
$$=\frac{(x_1^2+y_1^2+z_1^2)-2(x_1x_2+y_1y_2+z_1z_2)+(x_2^2+y_2^2+z_2^2)}{r^2(x_1^2+y_1^2+z_1^2)-2r(x_1x_2+y_2y_2+z_1z_2)+(x_2^2+y_2^2+z_2^2)}$$
$$=\frac{(1/r)-2(x_1x_2+y_1y_2+z_1z_2)+1}{r^2(1/r)-2r(x_1x_2+y_1y_2+z_1z_2)+1}, \quad \text{from (ii) and (iii)}$$
$$=\frac{[1-2r(x_1x_2+y_1y_2+z_1z_2)+r]/r}{r-2r(x_1x_2+y_1y_2+z_1z_2)+1}=\frac{1}{r}=\text{constant}.$$

Hence $PA : PB$ is constant.

***Ex. 8. Show that the circle in which the spheres $x^2+y^2+z^2-2x-4y-11=0$ and $x^2+y^2+z^2+2x-y+12z+5=0$ intersect each other lie in the plane $4x+3y+12z+16=0$.**

Show also that this plane is perpendicular to the line joining the centres of these spheres.

Sol. The equation of the plane common to given spheres is
$$(x^2+y^2+z^2-2x-4y+11)-(x^2+y^2+z^2+2x-y+12z+5)=0$$
or $$-4x-3y-12z-16=0 \quad \text{or} \quad 4x+3y+12z+16=0 \qquad ...(i)$$

The d. ratios of the normal to this plane are 4, 3, 12. ...(ii)

Also the centres of the given spheres are $(1, 2, 0)$, $(-1, \frac{1}{2}, -6)$.

$\therefore$ The direction ratios of the line joining these centres are
$$1+1, 2-\tfrac{1}{2}, 0+6 \quad \text{or} \quad 2, 3/2, 6 \quad \text{or} \quad 4, 3, 12$$
which are the same as given by (ii).

Hence the normal to the plane (i) is parallel to the line joining the centres of the given spheres *i.e.* the plane (i) is perpendicular to the line joining the centres of the given spheres. Hence proved.

Ex. 9. A plane passes through a fixed point (a, b, c), show that the locus of the foot of the perpendicular to it from the origin is the sphere OABC i.e. $x^2+y^2+z^2-ax-by-cz=0$.

Sol. The equation of any plane through (a, b, c) is
$$l(x-a)+m(y-b)+n(z-c)=0 \qquad ...(i)$$

The direction ratios of the normal to this plane are l, m, n.

$\therefore$ The equations of the line through (0, 0, 0) and perpendicular to the plane (i) are $$x/l=y/m=z/n \qquad ...(ii)$$

The point of intersection of (i) and (ii) is the foot of the perpendicular whose locus is required and to get the same we should eliminate l, m, n between (i) and (ii).

Hence the required equation is $x(x-a)+y(y-b)+z(z-c)=0$

or $$x^2+y^2+z^2-ax-by-cz=0$$ Hence proved.

Ex. 10. Find the equation of the circumcircle of the triangle ABC, whose vertices are A (a, 0, 0), B (0, b, 0) and C (0, 0, c). Find also the coordinates of its centre. **Or**

The equation of the plane ABC is x/a + y/b + z/c = 1. Obtain the equations of the circle passing through A, B and C, where A, B, C are the points of intersection of the plane with coordinates axes.

Sol. Equation of the plane ABC is $(x/a)+(y/b)+(z/c)=1$. ...(i)

Let the fourth point be $O(0,0,0)$. Then the equation of the sphere through O, A, B, C (as in Ex. 5 Page 4 Chapter VII) is

$$x^2+y^2+z^2-ax-by-cz=0. \quad ...(ii)$$

The equations (i) and (ii) together give the required circumcircle of ΔABC.

The equations of a line through the centre $(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c)$ of the sphere and perpendicular to (i) is

$$\frac{x-\frac{1}{2}a}{1/a}=\frac{y-\frac{1}{2}b}{1/b}=\frac{z-\frac{1}{2}c}{1/c}=r\text{ (say)} \quad \text{(Note)}$$

Any point on this line is $(\frac{1}{2}a+r/a, \frac{1}{2}b+r/b, \frac{1}{2}c+r/c.)$, If this is the centre of the circle, then it must lie on (i) and so we get

$$\left(\frac{1}{2}+\frac{r}{a^2}\right)+\left(\frac{1}{2}+\frac{r}{b^2}\right)+\left(\frac{1}{2}+\frac{r}{c^2}\right)=1 \text{ or } r=\frac{-1}{2(a^{-2}+b^{-2}+c^{-2})}$$

$$\therefore \frac{1}{2}a+\frac{r}{a}=\frac{1}{2}a-\frac{1}{2a(a^{-2}+b^{-2}+c^{-}2)}$$

$$=\frac{a^2(a^{-2}+b^{-2}+c^{-2})-1}{2a(a^{-2}+b^{-2}+c^{-2})}=\frac{a(b^{-2}+c^{-2})}{2(a^{-2}+b^{-2}+c^{-2})}$$

Similarly $\frac{1}{2}b+\frac{r}{b}=\frac{b(c^{-2}+a^{-2})}{2(a^{-2}+b^{-2}+c^{-2})}$; $\frac{1}{2}c+\frac{r}{c}=\frac{c(a^{-2}+b^{-2})}{2(a^{-2}+b^{-2}+c^{-2})}$.

$\therefore$ The required centre of the circle is

$$\left[\frac{a(b^{-2}+c^{-2})}{2(a^{-2}+b^{-2}+c^{-2})}, \frac{b(c^{-2}+a^{-2})}{2(a^{-2}+b^{-2}+c^{-2})}, \frac{c(a^{-2}+b^{-2})}{2(a^{-2}+b^{-2}+c^{-2})}\right] \quad \text{Ans.}$$

Ex. 11. Find the diameter of the circumcircle of ΔABC in Ex. 10 above.

Sol. As in Ex. 10 above we can prove the equation of the plane O, A, B, C is $\frac{1}{2}\sqrt{(a^2+b^2+c^2)}$and the centre of this sphere is $(\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c)$.

$\therefore$ The length of perpendicular from the centre of this sphere to the plane

$$ABC = \frac{[(\frac{1}{2}a/a)+(\frac{1}{2}b/b)+(\frac{1}{2}c/c)-1}{\sqrt{[(1/a)^2+(1/b)^2+(1/c)^2]}} = \frac{1}{2\sqrt{(a^{-2}+b^{-2}+c^{-2})}} \qquad \textbf{(Note)}$$

Now if R be the radius of the circumcircle of ΔABC we have

$R^2 =$ (radius of the sphere $)^2$ – (length of the perpendicular from centre of the sphere on the plane of the circle$)^2$

...See § 7.05 Note 1 Page 14 Ch. VII

i.e. $$R^2 = [\tfrac{1}{2}\sqrt{(a^2+b^2+c^2)}] - [1/2\sqrt{(a^{-2}+b^{-2}+c^{-2})}]^2$$

$$= \frac{(a^2+b^2+c^2)}{4} - \frac{1}{4(a^{-2}+b^{-2}+c^{-2})}$$

or $$R^2 = \frac{a^2+b^2+c^2}{4} - \frac{a^2b^2c^2}{4(b^2c^2+c^2a^2+a^2b^2)}$$

or $$4R^2 = \frac{(a^2+b^2+c^2)(b^2c^2+c^2a^2+a^2b^2)-a^2b^2c^2}{(b^2c^2+c^2a^2+a^2b^2)}$$

or $$(2R)^2 = \frac{(b^2+c^2)(c^2+a^2)(a^2+b^2)}{(b^2c^2+c^2a^2+a^2b^2)}, \text{ on simplifying}$$

or the required diameter $= 2R = \sqrt{\left[\frac{(b^2+c^2)(c^2+a^2)(a^2+b^2)}{(b^2c^2+c^2a^2+a^2b^2)}\right]}$

Ex. 12. Find the centre and radius of the circle $x^2+y^2+z^2 = ax+by+cz$, $(x/a)+(y/b)+(z/c)=1$ (*Purvanchal* 93)

Sol. Same as Ex. 10 and 11 above.

Ex. 13. If d be the distance between the centres of two spheres of radii r_1 and r_2, prove that the angle between them is

$$\cos^{-1}[(r_1^2+r_2^2-d^2)/2r_1r_2].$$

Hence find the angle of intersection of the sphere $x^2+y^2+z^2-2x-4y-6z+10=0$ with the sphere, the extremities of whose diameter are $(1, 2, -3)$ and $(5, 0, 1)$.

Sol. Let C_1, C_2 be the centres of the spheres and P be their point of intersection. Then the angle between the spheres is the angle between their radii C_1P and C_2P.

$\therefore$ In ΔC_1PC_2, $C_1P = r_1$, $C_2P = r_2$ and $C_1C_2 = d$.

$\therefore$ If θ be the required angle, then

$$\cos\theta = \cos\angle C_1PC_2 = \frac{C_1P^2+C_2P^2-C_1C_2^2}{2C_1P \cdot C_2P} = \frac{r_1^2+r_2^2-d^2}{2r_1r_2}$$

Now for the second part, the given spheres are

$$x^2+y^2+z^2-2x-4y-6z+10=0 \quad \text{...(i)}$$

and $$(x-1)(x-5)+(y-2)(y-0)+(z+3)(z-1)=0$$

or $$x^2+y^2+z^2-6x-2y+2z+2=0. \quad \text{...(ii)}$$

Centre and radius of (i) are (1, 2, 3) and 2.

Centre and radius of (ii) are (3, 1, – 1) and 3.

$\therefore$ Here $r_1=2, r_2=3, d^2=[(3-1)^2+(1-2)^2+(-1-3)^2]=21.$

The required angle $=\cos^{-1}\left[\dfrac{r_1^2+r_2^2-d^2}{2r_1r_2}\right]=\cos^{-1}\left[\dfrac{4+9-21}{2.2.3}\right]$

$=\cos^{-1}(-2/3)$ **Ans.**

Ex. 13 (b). Find the angle of intersection of the spheres

$$\mathbf{x^2+y^2+z^2-2x-4y-6z+10=0}$$

and $$\mathbf{x^2+y^2+z^2-6x-2y-2z+2=0}$$

Sol. Same as second part of Ex. 13 (a) above. **Ans.** $\cos^{-1}(-2/3)$

***Ex. 14. Show that the two spheres each of which pass through (0, 0, 0), (2a, 0, 0) and (0, 2b, 0) touch the line $\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$.**

Also find the distance between the centres of these spheres.

Sol. Let the given points be O (0, 0, 0), A ($2a$, 0, 0) and B (0, $2b$, 0).

Let us suppose that the spheres cut z-axis at $C(0, 0, 2\lambda)$.

Then as in Ex. 5 Page 4 Ch. VII the equtition of the sphere through O, A, B, and C is $x^2+y^2+z^2-2ax-2by-2\lambda z=0.$ ...(i)

Any point on the given line is $(a+lr, b+mr, c+nr)$. If this lies on (i), then we have

$$(a+lr)^2+(b+mr)^2+(c+nr)^2-2a(a+r)-2b(b+mr)-2\lambda(c+nr)=0$$

or $$r^2+2rn(c-\lambda)-(a^2+b^2+2\lambda c-\ )=0, \quad \because \ \Sigma l^2=1.$$

If the given line touches the sphere), then the roots of this equation must be coincident and the condition for th same is

$$\text{'}b^2=4ac\text{'} \ i.e.\ [2n(c-\lambda)]^2-4(a^2+b^2+2\lambda c-c^2)$$

or $$n^2\lambda^2-2c(n^2-1)\lambda+[a^2b^2+c^2(n^2-1)]=0, \quad \text{...(ii)}$$

which being a quadratic equation in gives the values λ_1 and λ_2 of λ. Corresponding to these two values of we have two spheres from (i) whose centres are (a, b, λ_1) and (a, b, λ_2).

$\therefore$ The required istance betw^n these centres

$=(\lambda_1-\lambda_2)=\sqrt{[(\lambda_1+\lambda_2)^2-4\lambda_1\lambda_2]}$

$=\sqrt{\left[\left\{\dfrac{2c(n^2-1)}{n^2}\right\}^2-4\left\{\dfrac{a^2+b^2+c^2(n^2-1)}{n^2}\right\}\right]}$, from (ii)

$= (2/n^2)\sqrt{[c^2(n^2-1)^2 - n^2\{a^2+b^2+c^2(n^2-1)\}]}$

$= (2/n^2)\sqrt{[c^2 - n^2(a^2+b^2+c^2)]}$, on simplifying **Ans.**

Ex. 14. Prove that the plane $(x-\alpha)(u+\alpha)+(y-\beta)(v+\beta)+(z-\gamma)(w+\gamma)=0$ cuts the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ in circle whose centre is (α, β, γ) and the equation of the sphere which has this circle as a great circle is

$$x^2+y^2+z^2-2\alpha(x-u)-2\beta(y-v)-2\gamma(z-w)+2\alpha^2+2\beta^2+2\gamma^2-d=0.$$

Sol. Refer Fig. 1 Page 13 Chapter VII.

The co-ordinates of the centre O of the given sphere are $(-u, -v, -w)$.

The line OC being perpendicular to the given plane, its d.r.'s will be the same as those of the normal to the given plane

i.e. d.r.'s of OC are $(u+\alpha, v+\beta, w+\gamma)$. **(Note)**

$\therefore$ The equations of OC (a line through O and perpendicular to the given plane) are

$$\frac{x+u}{(u+\alpha)/\sqrt{[\Sigma(u+\alpha)^2]}} = \frac{y+v}{(v+\beta)/\sqrt{[\Sigma(u+\alpha)^2]}} = \frac{z+w}{(w+\gamma)/\sqrt{[\Sigma(u+\alpha)^2]}}$$

$\therefore$ Any point on this line is

$$\left[-u+\frac{(u+\alpha)r}{\sqrt{[\Sigma(u+\alpha)^2]}}, -v+\frac{(v+\beta)r}{\sqrt{[\Sigma(u+\alpha)^2]}}, -w+\frac{(w+\gamma)r}{\sqrt{[\Sigma(u+\alpha)^2]}}\right]$$

If it lies on the given plane, then

$$\Sigma\left[(u+\alpha)\left\{-u+\frac{(u+\alpha)r}{\sqrt{[\Sigma(u+\alpha)^2]}}-\alpha\right\}\right]=0$$

or $$\Sigma\left[(u+\alpha)\left\{\frac{r}{\sqrt{[\Sigma(u+\alpha)^2]}}-1\right\}\right]=0$$

or $$r\left[\frac{(u+\alpha)^2+(v+\beta)^2+(w+\gamma)^2}{\sqrt{[\Sigma(u+\alpha)^2]}}\right]-[(u+\alpha)^2+(v+\beta)^2+(w+\gamma)^2]=0$$

or $$r=\sqrt{[\Sigma(u+\alpha)^2]}.$$

$\therefore$ From (i) the co-ordinates of C, the centre of the circle are $\{-u+(u+\alpha), -v+(v+\beta), -w+(w+\gamma)\}$ *i.e.* (α, β, γ)

The equation of the sphere through the given circle is

$$[x^2+y^2+z^2+2ux+2vy+2wz+d]$$
$$+\lambda[(x-\alpha)(u+\alpha)+(y-\beta)(v+\beta)+(z-\gamma)(w+\gamma)]=0 \quad ...(ii)$$

or $x^2+y^2+z^2+x\{2u+\lambda(u+\alpha)\}+y\{...\}+z\{...\}+\{...\}=0$;

$\therefore$ Its centre is

$$[-\tfrac{1}{2}\{2u+\lambda(u+\alpha)\}, -\tfrac{1}{2}\{2v+\lambda(v+\beta)\}, -\tfrac{1}{2}\{2w+\lambda(w+\gamma)\}]$$

or $$[-\tfrac{1}{2}(2u+\lambda u+\alpha\lambda), -\tfrac{1}{2}(2v+\lambda v+\beta\lambda), -\tfrac{1}{2}(2w+\lambda w+\gamma\lambda)]$$

If the given circle is a great circle of (ii), then the centre of (ii) must lie on the given plane of the circle.

$$\therefore\ \Sigma\,[\{-\tfrac{1}{2}(2u+\lambda u+\alpha\lambda)-\alpha\}\,(u+\alpha)]=0$$

or $$\Sigma\,[\{-u-\tfrac{1}{2}\lambda u-\tfrac{1}{2}\alpha\lambda-\alpha\}\,(u+\alpha)]=0$$

or $$\Sigma\,[\{(u+\alpha)+\tfrac{1}{2}\lambda\,(u+\alpha)\}\,(u+\alpha)]=0 \quad \text{or} \quad \Sigma\,[(u+\alpha)^2\,(1+\tfrac{1}{2}\lambda)]=0$$

or $$1+(1/2)\,\lambda=0 \quad \text{or} \quad \lambda=-2.$$

$\therefore$ From (ii), required sphere is

$$[x^2+y^2+z^2+2ux+2vy+2wz+d]-2\,[\Sigma\,(x-\alpha)\,(u+\alpha)]=0$$

or $$x^2+y^2+z^2+2\,\Sigma\,[x\,\{u-(u+\alpha)\}]+2\,\Sigma\,[\alpha\,(u+\alpha)]+d=0$$

or $$x^2+y^2+z^2+2\Sigma\,(-x\alpha)+\Sigma\,(\alpha u+\alpha^2)+d=0$$

or $$x^2+y^2+z^2-2\,(\alpha x+\beta y+\gamma z)+2\,(\alpha u+\beta v+\gamma w+\alpha^2+\beta^2+\gamma^2)+d=0$$

or $$x^2+y^2+z^2-2\alpha\,(x-u)-2\beta\,(y-v)-2\gamma\,(z-w)+2\alpha^2+2\beta^2+2\gamma^2+d=0.$$

Hence proved.

Ex. 15. The tangents from a point P to a sphere are equal to the distance of P from a fixed tangent plane to the sphere. Show that the locus of P is the surface $x^2+y^2+z^2+2a\,(x+a)=0$.

Sol. Do yourself.

Ex. 16. Prove that the tangent planes to the spheres

$$\mathbf{x^2+y^2+z^2+2ux+2vy+2wz+d=0}$$

and $$\mathbf{x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0}$$

at any common point are at right angles if

$$\mathbf{2uu_1+2vv_1+2ww_1=d+d_1}$$

Sol. The given spheres are $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ ...(i)

and $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ ...(ii)

The centre and radius of (i) are $(-u,-v,-w)$ and $\sqrt{(u^2+v^2+w^2-d)}$ and those of (ii) are $(-u_1,-v_1,-w_1)$ and $\sqrt{(u_1^2+v_1^2+w_1^2-d_1)}$.

Now if the tangent planes to the spheres (i) and (ii) at any common point are at right angles then the tangent plane of each sphere at such a point must pass through the centre of the other sphere (students may draw the figure themselves) and as such the sum of the squares of the radii of the two spheres must be equal to the square of the distance between their centres. **(Note)**

i.e. $$(u^2+v^2+w^2-d)+(u_1^2+v_1^2+w_1^2-d_1)$$

$$=(-u+u_1)^2+(-v+v_1)^2+(-w+w_1)^2$$

or $$2uu_1+2vv_1+2ww_1=d+d_1.$$ Hence proved

Ex. 17. Prove that the sum of the squares of the intercepts by a given sphere on any three mutually perpendicular lines through a fixed point is constant.

Sol. Let the given sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ ...(i)

Let the three mutually perpendicular lines be the co-ordinate axes which pass through a fixed point (0, 0, 0). **(Note)**

Now the equatons of the x-axis are $y=0, z=0$. ...(ii)

$\therefore$ From (i) and (ii), we find that x-axis meets the sphere (i) at points whose abscissae are given by $x^2+2ux+d=0$. ...(iii)

If these points be $(x_1, 0, 0)$ and $(x_2, 0, 0)$, then from (iii), we have

$$x_1+x_2=-2u \text{ and } x_1x_2=d.$$

$\therefore (x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2=(-2u)^2-4d=4(u^2-d)$

i.e. the square of the intercept by the sphere (i) on the x-axis

$$=4(u^2-d).$$

Similarly the squares of the intercepts by the sphere (i) on y and z axes are $4(v^2-d)$ and $4(w^2-d)$ respectively.

$\therefore$ Required sum of squares of the intercepts

$$=4(u^2-d)+4(v^2-d)+4(w^2-d)$$

$$=4(u^2+v^2+w^2-3d)=\text{constant.}$$ Hence proved.

***Ex. 18. Find the radius and the coordinates of the centre of the sphere $x^2+y^2+z^2-4x+6y+9=0$. Determine whether the point P (1, – 2, 1) lies inside or outside the sphere. Find the values of k for which the plane x + y + kz = 3 is tangent to this sphere.**

Sol. Given sphere is $x^2+y^2+z^2-4x+6y+9=0$ ...(i)

Its centre is $(-u, -v, -w)$ *i.e.* $(2, -3, 0)$.

and radius $=\sqrt{(u^2+v^2+w^2-d)}=\sqrt{(2^2+3^2+0^2-9)}=2.$

Also substituting the coordinates of $P(1, -2, 1)$ on the left hand side expression of the equation (i), we get

$1^2+(-2)^2+1^2-4(1)+6(-2)+9$

$$=1+4+1-4-12+9=-1=\text{negative}$$

$\therefore$ The point $P(1, -2, 1)$ is inside the sphere given by (i).

Again if the plane $x+y+kz=3$ ...(ii)

is a tangent plane to the sphere (i), then the length of perpendicular from the centre $(2, -3, 0)$ of the sphere (i) to the plane (ii) must be equal to the radius of the sphere (i)

i.e. $$\frac{(2)+(-3)+k(0)-3}{\pm\sqrt{(1^2+k^2+1^2)}}=2. \text{ or } k=\pm\sqrt{2}$$ **Ans.**

Ex. 19. The plane x + 2y + 3z = 12 cuts the axes of coordinates in A, B, C. Find the equation of the circle circumscribing the triangle ABC.

Sol. The given plane $x+2y+3z=12$ ...(i)

meets the x-axis *i.e.* $y=0, z=0$ in the point A whose coordinates are $(12, 0, 0)$

Similarly the coordinates of B and C where the given plane meets y and z-axes are (0, 6, 0) and (0, 0, 4) respectively.

Thus the point A, B, C are (12, 0, 0), (0, 6, 0) and (0, 0, 4) respectively.

Let the equation of the circle circumscribing the triangle ABC be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad \text{...(ii)}$$

If it passes through A, B, C, then we have

$$(12)^2+2u(12)+d=0 \quad \text{or} \quad 24u+d+144=0 \qquad \text{...(iii)}$$

$$(6)^2+2v(6)+d=0 \quad \text{or} \quad 12v+d+36=0 \qquad \text{...(iv)}$$

and $$(4)^2+2w(4)+d=0 \quad \text{or} \quad 8w+d+16=0 \qquad \text{...(v)}$$

From (iii), (iv) and (v) we get

$2u=-[12-(d/12)]$, $2v=-[6-(d/6)]$ and $2w=-[4-(d/4)]$.

Substituting these values of $2u, 2v, 2w$ in (ii), we get the required equation as

$$x^2+y^2+z^2-[12-(d/12)]\,x-[6-(d/6)]\,y-[4-(d/4)]\,z+d=0.$$

where d can take any value. **Ans.**

EXERCISES ON CHAPTER VII

Ex. 1. Find the co-ordinates of the centre and the radius of the circle $x+2y+2z=15, x^2+y^2+z^2-2y-4z=11$. **Ans.** (1, 2, 3) and $\sqrt{7}$.

***Ex. 2.** Find the equation to the sphere which passes through the circle $x^2+y^2+z^2=4$, $z=0$ and is cut by the plane $x+2y+2z=0$ in a circle of radius 3.

Ans. $x^2+y^2+z^2\pm 6z=4$.

Ex. 3. Find the equations of the tangent planes to the sphere $x^2+y^2+z^2-2(x+y+z)=0$ perpendicular to the line with direction numbers 1, 1, 1.

Ex. 4. Prove that in general two spheres can be drawn through a given point to touch the coordinate planes and find for what positions of the point the spheres are (i) real ; (ii) coincident.

Ex. 5. Find the equation of the circle with centre (α, β, γ), radius k and the direction cosines of the normal to the plane are l, m, n.

Ex. 6. Find the locus of the point from which equal tangents may be drawn to the spheres $x^2+y^2+z^2=1$,

$$x^2+y^2+z^2+2x-2y+2z-1=0, \quad x^2+y^2+z^2-x+4y-6z-2=0$$

Ans. $\frac{1}{2}(x-1)=\frac{1}{5}(y-2)=\frac{1}{3}(z-1)$.

Ex. 7. Find the equation of the sphere which touches the planes $x-2z=8$ and $2x-z+5=0$ and has centre on the line $x+z=0=y$.

Ex. 8. If two spheres S_1 and S_2 are orthogonal, the polar plane of any point P on S_1 with respect to S_2 passes through the other end of the diameter of S_1 through P.

Ex. 9. Show that the equations of any two spheres can be put in the form $x^2+y^2+z^2+2\lambda_1 x+d=0$, $x^2+y^2+z^2+2\lambda_2 x+d=0$.

Ex. 10. Prove that only one tangentt plane to $x^2+y^2+z^2-2x+6y+2z+8=0$ can pass through the straight line $x=4t+4,\ y=3t+1,\ z=t+1$.

Ex. 11. Rectify the mistakes if any, otherwise assert correctness of the following statement giving adequate reasons :– The plane $3x+4y+5z=49$ intersects the sphere $x^2+y^2+z^2=50$ in a point circle.

****Ex. 12.** State which of the following statements are true and which are false. Justify your answer :—

(i) One and only one sphere of given radius R can be made to pass through a circle of radius $r<R$.

(ii) A circle on sphere is uniqueluy determined if we know its radius and the coordinates of its centre.

Ex. 13. A circle with centre (2, 3, 0) and radius 1, is drawn in the plane $z=0$. Find the equation of the sphere which passes through this circle and through the point (1, 1, 1).

Ex. 14. A variable sphere passes through the point $(0, 0, \pm c)$ and cuts the line $y=x\tan\alpha,\ z=c\ ;\ y=-x\tan\alpha,\ z=-c$ in the points P, P'. If PP' has constant length $2a$, show that the centre of the sphere lies on the circle $z=0;\ x^2+y^2=(a^2-c^2)\operatorname{cosec}^2 2\alpha$.

Ex. 15. Find the equation of the plane which cuts the sphere $x^2+y^2+z^2-2x+4y-6z+3=0$ in a circle whose centre is $(2, 3, -4)$. Also find the radius of this circle.

Ex. 16. Show that the centres of the spheres which cut both the spheres $x^2+y^2+z^2+2\lambda x+d=0$ and $x^2+y^2+z^2+2\mu x+d=0$ in great circles, lies on the plane $x+\lambda+\mu=0$.

Ex. 17. The equation of a sphere with the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) as diameter is

(i) $\Sigma (x-x_1)(x-x_2)=0$; (ii) $\Sigma (x-x_1)(y-y_1)=0$

(iii) $\Sigma (x+x_1)^2 (y-y_1)=0$; (iv) $\Sigma (x-x_2)(y-y_1)^2=0$ **Ans.** (i)

Ex. 18. Find the smallest sphere (*i.e.* the sphere of smallet radius) which touches the lines

$$\frac{1}{2}(x-5)=-(y-2)=-(z-5),\ -\frac{1}{3}(x+4)=-\frac{1}{6}(y+5)=\frac{1}{4}(z-4)$$

(*Rohilkhand* 92)

CHAPTER VIII

The Cone and Cylinder

CONE

§ 8.01. Definition. (*Kumaun* 95; *Rohilkhand* 92)

A **cone** *is a surface generated by a variable striaght line passing through a fixed point and satsifying one more condition i.e. intersecting a given curve or touching a given surface.*

The fixed point is called the *vertex* and the given curve (or surface) is called **the** *guiding curve* (or *guiding surface*) of the cone. The variable straight line is known as the *generator* of the cone.

A cone whose equation is of second degree is known as **quartic cone** or **quadratic cone.**

§ 8.02 Cone with vertex at the origin. *To prove that the equtaion of the cone with vertex at the origin is a homogeneous second degree equation in x, y and z.* (*Agra* 91; *Kanpur* 93)

Let the general equation of the second degree in x, y, z viz.

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0 \quad \text{...(i)}$$

represent a cone with vertex at the origin O.

Let $P(x_1, y_1, z_1)$ be any point on the cone. The equation of the generator OP is

$$\frac{x-0}{x_1}=\frac{y-0}{y_1}=\frac{z-0}{z_1} \quad \text{...(ii)}$$

Any point Q on this generator is (rx_1, ry_1, rz_1). As OP is a generator of the cone (i), so every point on it like Q must lie on cone (i) for all values of r, which means that $r^2(ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1)$

$$+2r(ux_1+vy_1+wz_1)+d=0,$$

must be an identity and the conditions for the same are

$$ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1=0 \quad \text{...(iii)}$$

$$ux_1+vy_1+wz_1=0 \quad \text{...(iv)} \qquad \text{and} \qquad d=0 \quad \text{...(v)}$$

From (iv) we conclude that $P(x_1, y_1, z_1)$ is a point on a plane $ux+vy+wz=0$ if u, v and w are not all zero and this is against hypothesis. So u, v and w must be all zero. Also (v) is obvious as the cone passes through the origin.

Hence, from (i) the equation of the cone with vertex at the origin is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0, \quad \text{...(vi)}$$

which is a homogeneous equation of second degree in x, y and z.

Conversely, *every homegenous equation of second degree in x, y and z reperesents a cone with its vertex at the origin.*

It is obvious that if $P(x_1, y_1, z_1)$ satisfies the equation (vi) then for all values of r, the point (rx_1, ry_1, rz_1) also satisfies (vi) *i.e.* all points on the generator OP lies on the surface of the cone *i.e.* the line OP lies entirely on the cone.

Thus the surface is generated by straight lines through the origin and hence it is a cone with its vertex at the origin.

COROLLARY. *If the line $x/l = y/m = z/n$ is a generator of the cone given by* $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, ...(i)
then its direction cosines viz. l, m, n satisfy the equation of the cone.

Any point on the given generator is (lr, mr, nr). If this lies on the cone (i), then we have $al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0$ ...(ii)

Conversely if the relation (ii) holds good *i.e.* if the direction ratios of a staright line which always passes through a fixed point satisfy a homogeneous equation, then this line is a generator of a cone whose vertex is at the fixed point.

****§ 8.03. General equation of a cone of second degree which passes through the coordinate axes.** (*Purvanchal* 94; *Rohilkhand* 92)

If the cone passes through the coordinates axes, then its vertex must be at the origin and as such its equation must be of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad \text{...(i)}$$

Also the d.c.'s of the coordinate axes are 1, 0, 0 : 0, 1, 0 and 0, 0, 1 which should satisfy (i). (see cor. of § 8.02 above)

So we get $a = 0, b = 0$ and $c = 0$. Hence the required equation of the cone through the coordinate axes is $\boxed{\mathbf{fyz + gzx + hxy = 0.}}$

Solved Examples on § 8.01 to § 8.03.

***Ex. 1. Show that $2x^2 + 3y^2 - 5z^2 = 0$ represents a cone with its vertex at the origin. What relation must the direction cosines of the generators of this cone satisfy ?**

Sol. As $2x^2 + 3y - 5z^2 = 0$ ...(i)
is a homogeneous equation of second degree in x, y and z, so it represents a cone with its vertex at origin. (See § 8.02 converse above)

Let l, m, n be the *d.* cosines of any generator of the cone given by (i), then l, m, n must satisfy (i) [See § 8.02 cor. above]. So the required relation is

$$2l^2 + 3m^2 - 5n^2 = 0. \qquad \text{Ans.}$$

***Ex. 2 (a). Find the equation of the cone with vertex at the origin and which passes through the curve $ax^2 + by^2 + cz^2 = 1$; $\alpha x^2 + \beta y^2 = 2z$.**
(*Gorakhpur* 92; *Meerut* 92)

Sol. Let the equations of the curve be written in the homogeneous form as

$ax^2 + by^2 + cz^2 = t^2$ ...(i), $\alpha x^2 + \beta y^2 = 2zt$ ...(ii)

Here a new variable t has been introduced to make them homogeneous. **(Note)**

From (ii), $t = (\alpha x^2 + \beta y^2)/2z$. Substituting this value of t in (i) we get the required cone as $(ax^2 + by^2 + cz^2)\, 4z^2 = (\alpha x^2 + \beta y^2)^2$. **Ans.**

***Ex. 2 (b). Find the equation of the cone with vertex at the origin and which passes through the curve given by**

$$(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1,\ (x^2/\alpha^2) + (y^2/\beta^2) = 2z.$$

(*Avadh* 90; *Lucknow* 92; *Meerut* 98)

Sol. Let the equations of the curve be written in the homogeneous form by introducing a new variable t as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = t^2 \quad \text{...(i)} \qquad \text{and} \qquad \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 2zt \quad \text{...(ii)}$$

From (ii), $t = \dfrac{\beta^2 x^2 + \alpha^2 y^2}{2\alpha^2 \beta^2 z}$. Substituting this value of t in (i), we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \left(\frac{x^2\beta^2 + \alpha^2 y^2}{2\alpha^2\beta^2 z}\right)^2$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{1}{4z^2}\left(\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2}\right)^2, \qquad \textbf{Ans.}$$

which is the required equation of the cone.

Ex. 2 (c). Find the equation to the cone whose vertex is the point (0, 0, 0) and whose base is the curve $x^2 + y^2 = 4$, $z = 2$.

Sol. Let the equations of the given curve be written in the homogenous form by introducing a new variable t as

$$x^2 + y^2 = 4t^2 \quad \text{...(i)} \qquad \text{and} \qquad z = 2t \quad \text{...(ii)}$$

Substituting the value of t from (ii) in (i) we get the required equation as

$$x^2 + y^2 = z^2 \qquad \textbf{Ans.}$$

Ex. 2 (d). Find the equation of the cone with vertex at (0, 0, 0) and passing through the circle given by $x^2 + y^2 + z^2 + x - 2y + 3z - 4 = 0$, $x - y + z = 2$.

Sol. The equations of the circle are given to be

$$x^2 + y^2 + z^2 + x - 2y + 3z - 4 = 0 \quad \text{...(i)}$$

and

$$x - y + z = 2. \quad \text{...(ii)}$$

Making (i) homogeneous with the help of (ii) we get the required equation as

$$(x^2 + y^2 + z^2) + (x - 2y + 3z)\left[\tfrac{1}{2}(x - y + z)\right] - 4\left[\tfrac{1}{2}(x - y + z)\right]^2 = 0 \qquad \textbf{(Note)}$$

or

$$2(x^2 + y^2 + z^2) + (x - 2y + 3z)(x - y + z) - 2(x - y + z)^2 = 0$$

or $$x^2+2y^2+z^2-xy-3yz+2zx=0.$$ **Ans.**

Ex. 2 (e). Find the equation to the cone whose vertex is at (0, 0, 0) and which passes through the curve

$$\mathbf{x^2+y^2+z^2+x-2y+3z=4,\ x^2+y^2+z^2+2x-3y+4z=5.}$$

Sol. The curve is given by $x^2+y^2+z^2+x-2y+3z=4$. ...(i)

and $$x^2+y^2+z^2+2x-3y+4z=5$$ ...(ii)

Substracting (i) from (ii) we get $x-y+z=1$. ...(iii)

Now making (i) homogeneous with the help of (iii) we get the required equation as $x^2+y^2+z^2+(x-2y+3z)(x-y+z)=4(x-y+z)^2$

or $$2x^2+y^2-5xy-3yz+4zx=0$$ **Ans.**

***Ex. 3 (a). Find the equation of the cone whose vertex is (0, 0, 0) and which passes through the curve of intersection of the plane lx + my + nz = p and the surface $ax^2+by^2+cz^2=1$.** *(Allahabad 92; Meerut 97, 94)*

Sol. Making the equation of the given surface homogeneous with the help of the equation of the given plane, we have

$$ax^2+by^2+cz^2=[(lx+my+nz)/p]^2$$ **(Note)**

or $$p^2(ax^2+by^2+cz^2)=(lx+mx+nz)^2,$$

which is the required equation. **Ans.**

Ex. 3 (b). Find the equation to the cone which has vertex at the origin and passes through the curve given by

$$\mathbf{ax^2+by^2=2z,\ lx+my+nz=p.}$$ *(Avadh 90)*

Sol. Making the equation of the given surface homogeneous with the help of the equation of the given plane, we have

$$ax^2+by^2=2z\,[(lx+my+nz)/p]$$

or $$p(ax^2+by^2)=2z(lx+my+nz),$$

which is the required equation. **Ans.**

***Ex. 4. Find the equation of the cone whose vertex is the origin and base the circle x = a, $y^2+z^2=b^2$ and show that the section of the cone by a plane parallel to the plane XOY is a hyperbola.**

Sol. The equation of the cone is $y^2+z^2=b^2(x/a)^2$, making the equations of the circle homogeneous

i.e. $$a^2(y^2+z^2)=b^2x^2 \quad \text{or} \quad b^2x^2+a^2y^2-a^2z^2=0.$$ **Ans.**

Its section by a plane $z=c$ parallel to the plane XOY is the conic $b^2x^2-a^2y^2-a^2c^2=0,\ z=c$, which is a hyperbola. **Ans.**

****Ex. 5. Prove that the equation of the cone whose vertex is (0, 0, 0) and base the curve z = k, f (x, y) = 0 is f [(xk/z), (yk/z)] = 0, where f (x, y) ≡ $ax^2+2hxy+by^2+2gx+2fy+c=0$.**

Sol. Making $f(x, y) = 0$, homogeneous with the help of $z = k$, we get the required equation of the cone as

$$ax^2 + 2hxy + by^2 + 2gx\,(z/k) + 2fy\,(z/k) + c\,(z^2/k^2) = 0$$

or $$ax^2k^2 + 2hxyk^2 + by^2k^2 + 2gxzk + 2fyzk + cz^2 = 0$$

or $$a\,(xk/z)^2 + 2h\,(xk/z)\cdot(yk/z) + b\,(yk/z)^2 + 2g\,(xk/z) + 2f\,(yk/z) + c = 0$$

or $$f[(xk/z),\ (yk/z)] = 0$$ Hence proved.

****Ex. 6. The plane $x/a + y/b + z/c = 1$ meets the coordinate axes in A, B, C. Prove that the equation of the cone generated by lines drawn from O to meet the circle ABC is**

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0.$$ *(Kanpur* 97; *Meerut* 91, 90)

or, Show that the equation of the cone whose vertex is the origin and whose base is the circle through the points (a, 0, 0), (0, b, 0), (0, 0, c) is

$$\Sigma a\,(b^2 + c^2)\,yz = 0.$$

or The plane $x/a + y/b + z/c = 1$ cuts the coordinate axes in A, B, C. Prove that the lines passing through the origin are intersecting the circle ABC generate the following cone.

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0.$$ *(Garhwal* 95)

Sol. The plane ABC is $x/a + y/b + z/c = 1$ ...(i)

It meets the axes at $A\,(a, 0, 0)$, $B\,(0, b, 0)$, and $C\,(0, 0, c)$. The equation of the sphere $OABC$ (To be proved in Exam.) is

$$x^2 + y^2 + z^2 - ax - by - cz = 0, \quad \text{...(ii)}$$

as proved in Chapter VII on sphere.

The required cone is generated by the lines drawn from O to meet the circle ABC [given by (i) and (ii) together] and will be homogeneous. So making (ii) homogeneous with the help of (i), we get the required equation as

$$x^2 + y^2 + z^2 - (ax + by + cz)\,(x/a + y/b + z/c) = 0$$

or $$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0.$$ Proved.

***Ex. 7. Planes through OX, OY include as angle α. Show that their line of intersection lies on the cone**

$$z^2\,(x^2 + y^2 + z^2) = x^2y^2\tan^2\alpha.$$ *(Meerut* 92 *P)*

Sol. The equation of any plane through OX *i.e.* $y = 0$, $z = 0$ is

$$y + \lambda z = 0 \quad \text{or} \quad 0\,.\,x + 1\,.\,y + \lambda\,.\,z = 0 \quad \text{...(i)}$$

Similarly the equation of the plane through OY is

$$z + \mu x = 0 \quad \text{or} \quad \mu\,.\,x + 0\,.\,y + 1\,.\,z = 0. \quad \text{...(ii)}$$

Since α is the angle between these planes, so we have

$$\cos\alpha = \frac{0.\mu + 1.0 + \lambda.1}{\sqrt{(0^2+1^2+\lambda^2)}\,.\,\sqrt{(\mu^2+0^2+1^2)}} = \frac{\lambda}{\sqrt{(1+\lambda^2)}\sqrt{(1+\mu^2)}}$$

or $$\sec^2\alpha = \frac{(1+\lambda^2)(1+\mu^2)}{\lambda^2} = \frac{1+\lambda^2+\mu^2+\lambda^2\mu^2}{\lambda^2}$$

or $\lambda^2(1+\tan^2\alpha) = 1+\lambda^2+\mu^2+\lambda^2\mu^2$ or $\lambda^2\tan^2\alpha = 1+\mu^2+\lambda^2\mu^2$ ...(iii)

Now from (i) and (ii), we get $\lambda = -y/z,\ \mu = -z/x$,

Substituting these values of λ and μ in (iii), we get

$$(y^2/z^2)\tan^2\alpha = 1 + (z^2/x^2) + (y^2/z^2)(z^2/x^2)$$

or $$x^2y^2\tan^2\alpha = z^2(x^2+y^2+z^2),$$

which being a homogeneous equation represents a cone with its vertex at (0, 0, 0). Hence proved.

****Ex. 8. Show that a cone of the second degree can be found to pass through any two sets of rectangular axes through the same origin.**

or, Show that a cone can be found to contain any two sets of three mutually perpendicular concurrent lines as generators.

Sol. As in § 8.03, we can prove that the equation of a cone passing through one set of rectangular axes is $fyz + gzx + hxy = 0$. ...(i)

Let the direction ratios of another set of axes through the same origin be

$$l_1, m_1, n_1;\ l_2, m_2, n_2 : l_3, m_3, n_3.$$

The cone (i) passes through first two axes of this second set if

$$fm_1n_1 + gn_1l_1 + hl_1m_1 = 0 \qquad ...(ii)$$

and $$fm_2n_2 + gn_2l_2 + hl_2m_2 = 0 \qquad ...(iii)$$

Adding (ii) and (iii), we get $\Sigma f(m_1n_1 + m_2n_2) = 0$ ...(iv)

Also as these axes of second set are mutually perpendicular, so we have

$$m_1n_1 + m_2n_2 + m_3n_3 = 0 \text{ etc.}$$

Using these relations (iv) reduces to $f(m_3n_3) + g(n_3l_3) + h(l_3m_3) = 0$ which shows that the cone (i) passes through the third axis of the second set.

Hence proved.

Ex. 9 (a). Show that the equation of the cone which contains the coordinate axes and the lines through the origin having direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 is $\Sigma l_1l_2(m_1n_2 - m_2n_1)yz = 0$.

Sol. Proceed exactly as in Ex. 8 above. Here there are two lines instead of three in the second set. Solving (ii) and (iii) of last example, we get

$$\frac{f}{n_1l_1l_2m_2 - l_1m_1n_2l_2} = \frac{g}{(l_1m_1m_2n_2 - m_1n_1l_2m_2)} = \frac{h}{(\ldots)}$$

or $$\frac{f}{-l_1l_2(m_1n_2 - n_1m_2)} = \frac{g}{-m_1m_2(n_1l_2 - l_1n_2)} = \frac{h}{(\ldots)}$$

Substituting these proportionate values of f, g, h in (i) of last example, we get the required equation.

****Ex. 9 (b). Find the equation of the cone which passes through three coordinates axes and the lines**

$$x/1 = y/(-2) = z/3 \; ; \; x/3 = y/2 = z/(-1) \qquad \textit{(Avadh 94)}$$

Sol. If the cone passes through the coordinate axes, then its vertex must be the origin and so its equation must be of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad \text{...(i)}$$

Also the d.c.'s of the coordinate axes are (1, 0, 0), (0, 1, 0) and (0, 0, 1) which should satisfy (i). ...See cor. § 8.02 Page 2 Ch VIII

So we get $a = 0, b = 0, c = 0$ and so from (i), the equation of the cone reduces to

$$fyz + gzx + hxy = 0 \qquad \text{...(ii)}$$

Now if the given lines $x/1 + y/(-2) = z/3$ and $x/3 = y/2 = z/(-1)$ whose d.c.'s are 1, – 2, 3 and 3, 2, – 1 also be on (ii), then

$$f(-2)(3) + g(3)(1) + h(1)(-2) = 0$$

and

$$f(2)(-1) + g(-1)(3) + h(3)(2) = 0$$

or

$$6f - 3g + 2h = 0 \quad \text{and} \quad 2f + 3g - 6h = 0$$

Solving these, we get $\dfrac{f}{18-6} = \dfrac{g}{4+36} = \dfrac{h}{18+6}$ or $\dfrac{f}{3} = \dfrac{g}{10} = \dfrac{h}{6}$

$\therefore$ From (ii), required equation is $3yz + 10zx + 6xy = 0$ **Ans.**

****Ex. 9 (c). Find the equation of the cone which passes through the coordinate axes as well as the lines**

$$x/1 = y/(-2) = z/3 \; ; \; x/3 = y/(-1) = z/1$$

Sol. Do as Ex. 9 (b) above. **Ans.** $3yz + 16zx + 15xy = 0$

***Ex. 10. Show that a cone of second degree can be found to pass through any five concurrent lines.**

Sol. Let the point of concurrence of the five lines be the origin. Then their equations can be taken as

$$\frac{x}{l_r} = \frac{y}{m_r} = \frac{z}{n_r} \; ; \; r = 1, 2, 3, 4, 5. \qquad \text{...(i)}$$

Also we know that the general second degree equation of a cone is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

or

$$x^2 + \frac{b}{a}y^2 + \frac{c}{a}z^2 + 2\left(\frac{f}{a}\right)yz + 2\left(\frac{g}{a}\right)zx + 2\left(\frac{h}{a}\right)xy = 0,$$

dividing each term by a.

or $\quad x^2 + b_1y^2 + c_1z^2 + 2f_1yz + 2g_1zx + 2h_1xy = 0$, where $b_1 = b/a$ etc.

This equation of the cone contains five arbitrary constants $b_1, c_1, f_1\, g_1$ and h_1 and so it can satisfy five independent conditions. **(Note)**

Now we also know (see cor. § 8.02 Page 2) that the d.c.'s of generators of a cone satisfy the equation of the cone, so any five lines passing through the origin given by (i) are sufficient to evaluate the five arbitrary constants.

Hence a cone of second degree can be found to pass through any five concurrent lines.

Hence proved.

***Ex. 11. Show that the lines drawn through the point (α, β, γ) whose direction cosines satisfy $al^2 + bm^2 + cn^2 = 0$ generate the cone**

$$a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 = 0$$

Sol. Any line through (α, β, γ) is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$. ...(i)

Its direction cosines satisfy $al^2 + bm^2 + cn^2 = 0$ ...(ii)

Eliminating l, m, n between (i) and (ii), we get the required equation as

$$a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 = 0.$$ Hence proved.

Ex. 12. Find the equation of the cone with vetex at the origin and direction cosines of its generators satisfying the relation

$$3l^2 - 4m^2 + 5n^2 = 0.$$ *(Kumaun 90)*

Sol. Any line through the origin is $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$. ...(i)

Its direction cosines satisfy $3l^2 - 4m^2 + 5n^2 = 0$ ...(ii)

Eliminating l, m, n between (i) and (ii) we get the required equation as

$$3(x)^2 - 4(y)^2 + 5(z)^2 = 0 \quad \text{or} \quad 3x^2 - 4y^2 + 5z^2 = 0.$$ Ans.

****Ex. 13. OP and OQ are two lines which remain perpendicular and move so that the plane OPQ passes through OZ. If OP describes the cone $f(y/x, z/x) = 0$, prove that OQ describes the cone**

$$f\left\{\frac{y}{x}, \left(-\frac{x}{z} - \frac{y^2}{zx}\right)\right\} = 0.$$ *(Rohilkhand 91)*

Sol. Let the direction cosines of the lines OP and OQ passing through the origin be l_1, m_1, n_1 and l_2, m_2, n_2.

Since OP is perpendicular to OQ (given) so we have

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$ ...(i)

Also the equation of any plane through z-axis (*i.e.* $x = 0, y = 0$) is

$$x + \lambda y = 0$$ ...(ii)

If (ii) represents the plane OPQ, then the line OP and OQ lie on the plane given by (ii) and so we have

$l_1 + \lambda m_1 = 0$ ...(iii) and $l_2 + \lambda m_2 = 0$ ...(iv)

Also it is given that OP describes the cone $f(y/x, z/x) = 0$ ...(v)

$\therefore$ The direction cosines l_1, m_1, n_1 of the generator OP must satisfy (v) and so we have $f(m_1/l_1, n_1/l_1) = 0$ ...(vi)

Now if we eliminate l_1, m_1, n_1 from (iii), (iv) and (vi) and obtain a relation between l_2, m_2, n_2, then we can find the locus of OQ whose d.c.'s are l_2, m_2, n_2.

From (iii) and (iv) we get $\frac{m_1}{l_1} = -\frac{1}{\lambda} = \frac{m_2}{l_2}$ ...(vii)

Dividing each term of (i) by $l_1 n_2$ we get

$$\frac{l_2}{n_2}+\frac{m_1}{l_1}\cdot\frac{m_2}{n_2}+\frac{n_1}{l_1}=0 \quad \text{or} \quad \frac{l_2}{n_2}+\frac{m_2}{l_2}\cdot\frac{m_2}{n_2}+\frac{n_1}{l_1}=0, \text{ from (vii)}$$

or
$$\frac{n_1}{l_1}=-\left(\frac{l_2}{n_2}+\frac{m_2^2}{l_2 n_2}\right) \quad \text{...(viii)}$$

Substituting the values of m_1/l_1 and n_1/l_1 in terms of l_2, m_2, n_2 in (vi) we have

$$f\left[\frac{m_2}{l_2}, -\left(\frac{l_2}{n_2}+\frac{m_2^2}{l_2 n_2}\right)\right]=0 \quad \text{...(ix)}$$

Also the equations of the line OQ whose d.c.'s ar l_2, m_2, n_2 and which passes through O (0, 0, 0) are $x/l_2 = y/m_2 = z/n_2$ ...(x)

$\therefore$ With the help of (x) the locus of OQ from (ix) is

$$f\left[\frac{y}{x}, -\left(\frac{x}{z}+\frac{y^2}{zx}\right)\right]=0.$$ Hence proved.

Exercises on § 8.01 — § 8.03

Ex. 1. Find the equation to the cone whose vertex is (0, 0, 0) and which contains the curve given by $x^2 - y^2 + 4ax = 0, x + y + z = b.$

Ans. $(b + 4a)\, x^2 - by^2 + 4ay + 4az = 0.$

Ex. 2. Find the equation of the cone whose vertex is the origin and whose generators pass through the section of the sphere $x^2 + y^2 + z^2 + 2x + 2y + 2z + 5 = 0$ by the plane $x + y + z = 1.$ **Ans.** $4\,(x^2 + y^2 + z^2) + 7\,(xy + yz + zx) = 0$

Ex. 3. Show that the line $x/l = y/m = z/n$, whose $l^2 + 2m^2 - 3n^2 = 0$, is a generator of the cone $x^2 + 2y^2 - 3z^2 = 0.$

Ex. 4. Find the equation of the cone generated by straight lines drawn through the point (1, 2, 3) whose direction ratios satisfy the relation

$2l^2 + 3m^2 - 4n^2 = 0$ **Ans.** $2x^2 + 3y^2 - 4z^2 - 4x - 12y + 24z = 22.$

*§ 8.04. **Equation of the cone with a given vertex and a given conic for its base.** *To find the equation to the cone whose vertex is* (α, β, γ) *and the base conic* $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,\ z = 0.$

(*Avadh* 92; *Kanpur* 96; *Kumaun* 95; *Purvanchal* 91; *Rohilkhand* 93)

Sol. Any line through (α, β, γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{...(i)}$$

It meets the plane $z = 0$ at the point given by

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \text{ i.e. } \left[\alpha-\frac{l\gamma}{n}, \beta-\frac{m\gamma}{n}, 0\right]$$

If this point lies on the given conic, we have

$$a\left(\alpha-\frac{l\gamma}{n}\right)^2+2h\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)+b\left(\beta-\frac{m\gamma}{n}\right)^2$$
$$+2g\left(\alpha-\frac{l\gamma}{n}\right)+2f\left(\beta-\frac{m\gamma}{n}\right)+c=0. \quad \text{...(ii)}$$

The equation (ii) is the equation for the line (i) to intersect the given conic and hence the locus of the line (i) is obtained by eliminating l, m, n between (i) and (ii) by writing $\frac{x-\alpha}{z-\gamma}$ and $\frac{y-\beta}{z-\gamma}$ for l/n and m/n respectively.

Hence the required equation of the cone is

$$a\left[\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]^2+2h\left[\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]$$
$$+b\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2+2g\left[\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]+2f\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]+c=0$$

Simplifying [multiplying each term by $(z-\gamma)^2$], we get

$$a(\alpha z-\gamma x)^2-2h(\alpha z-\gamma x)(\beta z-\gamma y)+b(\beta z-\gamma y)^2+2g(\alpha z-\gamma x)(z-\gamma)$$
$$+2f(\beta z-\gamma y)(z-\gamma)+c(z-\gamma)^2=0.$$

****§ 8.05. Condition for the general equation of the second degree to represent a cone and to find its vertex.**

(*Gorakhpur* 91, 90; *Kanpur* 97; *Lucknow* 91, 90; *Rohilkhand* 94)

Let the general equation of second degree viz.

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0 \quad \text{...(i)}$$

represent a cone with its vertex at (α, β, γ).

Shifting the origin to (α, β, γ), the equation (i) transforms into

$$a(x+\alpha)^2+b(y+\beta)^2+c(z+\gamma)^2+2f(y+\beta)(z+\gamma)+2g(z+\gamma)(x+\alpha)$$
$$+2h(x+\alpha)(y+\beta)+2u(x+\alpha)+2v(y+\beta)+2w(z+\gamma)+d=0$$

or
$$(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)+2x(a\alpha+h\beta+g\gamma+u)$$
$$+2y(h\alpha+b\beta+f\gamma+v)+2z(g\alpha+f\beta+c\gamma+w)+(a\alpha^2+b\beta^2+c\gamma^2$$
$$+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta+2u\alpha+2v\beta+2w\gamma+d)=0 \quad \text{...(ii)}$$

Since (ii) represents a cone with vertex at the origin, so it must be homogeneous and therefore the coefficients of x, y, z and absolute term in (ii) must vanish separately

i.e.
$$a\alpha+h\beta+g\gamma+u=0, \quad \text{...(iii)}$$
$$h\alpha+b\beta+f\gamma+v=0 \quad \text{...(iv)}; \qquad g\alpha+f\beta+c\gamma+w=0 \quad \text{...(v)}$$

and
$$a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta+2u\alpha+2v\beta+2w\gamma+d=0 \quad \text{...(vi)}$$

Now (vi) can be written as

$$\alpha(a\alpha+h\beta+g\gamma+u)+\beta(h\alpha+b\beta+f\gamma+v)+\gamma(g\alpha+f\beta+c\gamma+w)$$
$$+(u\alpha+v\beta+w\gamma+d)=0$$

or with the help of (iii), (iv) and (v) this reduces to

$$u\alpha + v\beta + w\gamma + d = 0. \qquad ...(vii)$$

The required condition obtained by eliminating α, β, γ between (iii), (iv), (v) and (vii) is

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0. \qquad ...(viii)$$

If this condition holds good, the coordinates (α, β, γ) of the vertex are found by solving any three of equations (iii), (iv), (v) and (vii).

Working rule. We denote the given equations of the second degree by $F(x, y, z) = 0$, and introduce a variable t so as to make $F(x, y, z)$ homogeneous in x, y, z, t. Then the equations (iii), (iv), (v) and (vii) of the above article are given by

$$\frac{\partial F}{\partial x} = 0, \ \frac{\partial F}{\partial y} = 0, \ \frac{\partial F}{\partial z} = 0, \ \frac{\partial F}{\partial t} = 0,$$

where t is replaced by unity after differentiation.

Solve any three of these equations and if the fourth is satisfied by the values of x, y and z so obtained, then the given equation represents a cone with its vertex at (x, y, z).

Solved Examples on § 8.04 and § 8.05.

Ex. 1 (a). Find the equation of the cone whose vertex is the point (1, 1, 0) and whose guiding curve is $x^2 + z^2 = 4, y = 0$. (*Kumaun* 96, 94)

Sol. Any line through (1, 1, 0) is $(x-1)/l = (y-1)/m = z/n$ ...(i)

It meets the plane $y = 0$ at $\left(-\frac{l}{m} + 1, 0, -\frac{n}{m}\right)$

and if this point lies on $x^2 + z^2 = 4, \ y = 0$, then we have

$$\left(-\frac{l}{m} + 1\right)^2 + \left(-\frac{n}{m}\right)^2 = 4 \qquad ...(ii)$$

Also from (i), we have $\frac{l}{m} = \frac{x-1}{y-1}$ and $\frac{n}{m} = \frac{z}{y-1}$ ...(iii)

Eliminating l, m, n between (i) and (ii) with the help of (iii), we get

$$\left[-\frac{x-1}{y-1} + 1\right]^2 + \left[-\frac{z}{y-1}\right]^2 = 4 \quad \text{or} \quad \left[\frac{-x+1+y-1}{y-1}\right]^2 + \frac{z^2}{(y-1)^2} = 4$$

or $(y-x)^2 + z^2 = 4(y-1)^2$ or $x^2 - 3y^2 + z^2 - 2xy + 8y - 4 = 0$ **Ans.**

Ex. 1 (b). Find the equation to the cone whose vertex is the point (0, 0, 3) and guiding curve is the circle $x^2 + y^2 = 4, \ z = 0$.

Sol. Do as Ex. 1 (a) above **Ans.** $9(x^2+y^2)-16z^2+24z=36$

***Ex. 1 (c). Prove that a line which passes through (α, β, γ) and intersects the parabola $z^2 = 4ax, y = 0$ lies on the cone**

$$(\beta z - \gamma y)^2 = 4b\,(\beta - y)\,(\beta x - \alpha y).$$

Sol. Any line through (α, β, γ) is $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ ...(i)

It meets the plane $y=0$ at $\left(\alpha-\dfrac{l\beta}{m}, 0, \gamma-\dfrac{n\beta}{m}\right)$ and if it lies on $z^2=4ax, y=0$, then we have $\left(\gamma-\dfrac{n\beta}{m}\right)^2=4a\left(\alpha-\dfrac{l\beta}{m}\right)$ ...(ii)

Eliminating l, m, n between (i) and (ii) by putting $\dfrac{x-\alpha}{y-\beta}$ and $\dfrac{z-\gamma}{y-\beta}$ for $\dfrac{l}{m}$ and $\dfrac{n}{m}$ respectively, we get the equation of the required cone as $\left[\gamma-\left(\dfrac{z-\gamma}{y-\beta}\right)\beta\right]^2=4a\left[\alpha-\left(\dfrac{x-\alpha}{y-\beta}\right)\beta\right]$

or $$(\gamma y-\beta z)^2=4a\,(\alpha y-\beta x)\,(y-\beta).$$ Hence proved.

****Ex. 1 (d). Find the equaton of a cone whose vertex is (α, β, γ) and base $y^2 = 4ax, z = 0$.** (*Avadh* 95; *Bundelkhand* 95, 91; *Lucknow* 91, 90; *Purvanchal* 95, 92; *Rohilkhand* 90)

Sol. Any line through (α, β, γ) is $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ ...(i)

It meets the plane $z=0$ at $\left(\alpha-\dfrac{l\gamma}{n}, \beta-\dfrac{m\gamma}{n}, 0\right)$ and if it lies on $y^2=4ax, z=0$, then $\left(\beta-\dfrac{m\gamma}{n}\right)^2=4a\left(\alpha-\dfrac{l\gamma}{n}\right)$. ...(ii)

Eliminating l, m, n between (i) and (ii) by putting $\dfrac{x-\alpha}{z-\gamma}$ and $\dfrac{y-\beta}{z-\gamma}$ for $\dfrac{l}{n}$ and $\dfrac{m}{n}$ respectively, we get the required cone as

$$\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2=4a\left[\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]$$

or $$(\beta z-y\gamma)^2=4a\,(\alpha z-x\gamma)\,(z-\gamma)$$ **Ans.**

Ex. 1 (e). Find the equation to the cone whose vertex is the origin and base the circle $x = a, y^2 + z^2 = b^2$ and show that the section of the cone by a plane parallel to the plane XOY is a hyperbola. (*Avadh* 93)

Sol. Any line through (0, 0, 0) is $x/l=y/m=z/n$. ...(i)

It meeet the plane $x = a$ at $\left(a, \frac{am}{l}, \frac{an}{l}\right)$ and if it lies on $y^2 + z^2 = b^2, x = a$

then we have $$\frac{b^2 m^2}{l^2} + \frac{a^2 n^2}{l^2} = b^2 \quad \text{...(ii)}$$

Eliminating l, m, n between (i) and (ii) we have

$$a^2 \left(\frac{y^2}{x^2} + \frac{z^2}{x^2}\right) = b^2 \quad \text{or} \quad a^2 (y^2 + z^2) = b^2 x^2, \quad \text{...(iii)}$$

which is the required equation of the cone.

Again consider a plane $z = c$ which is parallel to the plane XOY. The section of the cone (iii) by this plane is $a^2 (y^2 + c^2) = b^2 x^2$

or $b^2 x^2 - a^2 y^2 = a^2 c^2$, which is evidently a hyperbola.

Ex. 2 (a). Find the equation of a cone whose vertex is the point (α, β, γ) and whose generating lines pass through the conic i.e. whose base curve is $x^2/a^2 + y^2/b^2 = 1, z = 0$. *(Meerut 92, 92 P)*

Sol. Any line through (α, β, γ) is $$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \quad \text{...(i)}$$

It meets the plane $z = 0$ at $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$ and if it lies on the given

conic we have $$\frac{1}{a^2}\left(\alpha - \frac{l\gamma}{n}\right)^2 + \frac{1}{b^2}\left(\beta - \frac{m\gamma}{n}\right)^2 = 1. \quad \text{...(ii)}$$

Eliminating l, m, n between (i) and (ii) as in Ex. 1 (d) P. 12 we get the equation of the required cone as

$$\frac{1}{a^2}\left[\alpha - \left(\frac{x - \alpha}{z - \gamma}\right)\gamma\right]^2 + \frac{1}{b^2}\left[\beta - \left(\frac{y - \beta}{z - \gamma}\right)\gamma\right]^2 = 1$$

or $$b^2 (\alpha z - \gamma x)^2 + a^2 (\beta z - \gamma y)^2 = a^2 b^2 (z - \gamma)^2 \quad \textbf{Ans.}$$

***Ex. 2 (b). Find the equation of the cone whose vertex is (α, β, γ) and base $ax^2 + by^2 = 1, z = 0$.** *(Meerut 93; Rohilkhand 97)*

Sol. Any line through (α, β, γ) is $$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \quad \text{...(i)}$$

It meets the plane $z = 0$ at $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$ and if it lies on the given

conic we have $$a\left(\alpha - \frac{l\gamma}{n}\right)^2 + b\left(\beta - \frac{m\gamma}{n}\right)^2 = 1 \quad \text{...(ii)}$$

Eliminating l, m, n between (i) and (ii) as in Ex. 1 (d) Page 12, we get the equation of the required cone as

$$a\left[\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]^2+b\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2=1$$

or $$a\,(\alpha z-\gamma x)^2+b\,(\beta z-\gamma y)^2=(z-\gamma)^2$$ **Ans.**

Ex. 3. If P be the vertex of the cone found in Ex. 2 (a) P. 13 and the section of this cone by the plane x = 0 is a rectangular hyperbola, then find the locus of P.

Sol. The equation of the cone is given by (iii) of the last example.

Its section by the plane $x=0$ is obtained by putting $x=0$ in (iii) and is

$$\frac{1}{a^2}(\alpha z-0)^2+\frac{1}{b^2}(\beta z-\gamma y)^2=(z-\gamma)^2.$$

If it represents a rectangular hypebola in the yz-plane, then

the coefficient of y^2 + coefficient of $z^2=0$.

i.e. $$\left(\frac{\gamma^2}{b^2}\right)+\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-1\right]=0.$$

$\therefore$ Required locus of $P\,(\alpha,\beta,\gamma)$ is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b}+\dfrac{z^2}{c^2}=1.$ **Ans.**

Ex. 4. Two cones with a common vertex pass through the curves $z^2=4ax$, $y=0$ and $z^2=4by$, $x=0$. The plane $z=0$ meets them in two conics which intersect in four concylic points. Show that the vertex lies on the surface $z\,(x/a+y/b)=4\,(x^2+y^2)$.

Sol. Let (α,β,γ) be the common vertex of the cones, then their equations as in Ex. 1 (b) Page 11 can be found as $(y\gamma-z\beta)^2=4a\,(\alpha y-\beta x)\,(y-\beta)$

and $$(x\gamma-\alpha z)^2=4a\,(x\beta-\alpha y)\,(x-\alpha).$$

The plane $z=0$ *i.e.* xy-plane meets these cones in conics given by

$$S\equiv y^2\gamma^2-4a\,(\alpha y-\beta x)\,(y-\beta)=0,\ z=0$$

and $$S'\equiv x^2\gamma^2-4b\,(x\beta-\alpha y)\,(x-\alpha)=0,\ z=0.$$

The equations of any curve through the points of intersection of these conics $S=0$ and $S'=0$ in the xy-plane is $S+\lambda S'=0$, $z=0$,

or $$[y^2\gamma^2-4a\,(\alpha y-\beta x)\,(y-\beta)]+\lambda\,[x^2y^2-4b\,(x\beta-\alpha y)\,(x-\alpha)]=0,\ z=0.$$

If this curve is a circle, then the coefficients of x^2 and y^2 must be equal and that of xy should vanish which gives $\lambda\,(\gamma^2-4b\beta)=(\gamma^2-4a\alpha)$ ...(i)

and $$(4a\beta)+\lambda\,(4b\alpha)=0.$$ ...(ii)

Eliminating λ between (i) and (ii), we get

$$[-(4a\beta)/(4b\alpha)]\,(\gamma^2-4b\beta)=(\gamma^2-4a\alpha)$$

or $a\beta\,(\gamma^2-4b\beta)+b\alpha\,(\gamma^2-4a\alpha)=0$ or $\gamma^2\,(\alpha/a+\beta/b)=4\,(\alpha^2+\beta^2)$.

$\therefore$ The locus of the common vertex (α,β,γ) is

$$z^2 (x/a + y/b) = 4 (x^2 + y^2).$$ Hence proved.

***Ex. 5. Prove that the line $x = pz + q$, $y = rz + s$ intersects the conic $z = 0$, $ax^2 + by^2 = 1$ if $aq^2 + bs^2 = 1$.**

Hence show that the coordinates of any point on a line which intersects the conic and passes through the point (α, β, γ) satisfy the equation $$a (\gamma x - \alpha z)^2 + b (\gamma y - \beta z)^2 = (z - \gamma)^2$$

Sol. The given line $x = pz + q,\ y = rz + s$ ...(i)

meets the plane $z = 0$ at $x = q$, $y = s$ *i.e.* in the point $(q, s, 0)$.

If this point $(q, s, 0)$ lies on the given conic

$$ax^2 + by^2 = 1,\ z = 0 \quad \text{...(ii)}$$

then we have $$aq^2 + bs^2 = 1. \quad \text{...(iii)}$$

which is the required condition. Hence proved.

Again if the line (i) passes through (α, β, γ),

then $$\alpha = p\gamma + q,\ \beta = r\gamma + s \quad \text{...(iv)}$$

From (i) and (iv) we get

$$x\gamma - \alpha z = \gamma (pz + q) - z (p\gamma + q) = q (\gamma - z)$$

and $$y\gamma - \beta z = \gamma (rz + s) - z (r\gamma + s) = s (\gamma - z)$$

These give $$q = \frac{\gamma x - \alpha z}{\gamma - z},\quad s = \frac{\gamma y - \beta z}{\gamma - z}$$

Substituting these values in (iii), we obtain the required locus as

$$a\left(\frac{\gamma x - \alpha z}{\gamma - z}\right)^2 + b\left(\frac{\gamma y - \beta z}{\gamma - z}\right)^2 = 1$$

or $$a (\gamma x - \alpha z)^2 + b (\gamma y - \beta z)^2 = (z - \gamma)^2$$ Hence proved.

****Ex. 6 (a). A cone has as base the circle $x^2 + y^2 + 2ax + 2by = 0$, $z = 0$ and passes through the fixed point $(0, 0, c)$. If the section of the cone by zx-plane is a rectangular hyperbola, prove that the vertex lies on a fixed circle.**

Sol. Let (α, β, γ) be the vertex of the cone. Any line through (α, β, γ) is

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \quad \text{...(i)}$$

It meets the plane $z = 0$ in $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$ and if this point lies on the given conic, we have

$$\left(\alpha - \frac{l\gamma}{n}\right)^2 + \left(\beta - \frac{m\gamma}{n}\right)^2 + 2a\left(\alpha - \frac{l\gamma}{n}\right) + 2b\left(\beta - \frac{m\gamma}{n}\right) = 0. \quad \text{...(ii)}$$

Eliminating l, m, n between (i) and (ii), the equation of the cone is

$$\left[\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]^2+\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2+2a\left[\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]$$
$$+2b\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]=0$$

or $$(\alpha z-x\gamma)^2+(\beta z-\gamma y)^2+2a\,(\alpha z-xy)\,(z-\gamma)+2b\,(\beta z-y\gamma)\,(z-\gamma)=0.$$

If this cone passes through $(0, 0, c)$, then

$$(\alpha c)^2+(\beta c)^2+2a\,(\alpha c)\,(c-\gamma)+2b\,(\beta c)\,(c-\gamma)=0 \qquad \text{...(iii)}$$

Again the section of the cone by zx-plane *i.e.* $y=0$ is

$$(\alpha z-\gamma x)^2+(\beta z)^2+2a\,(\alpha z-x\gamma)\,(z-\gamma)+2b\,(\beta z)\,(z-\gamma)=0$$

and if this section is a rectangular hyperbola in the zx-plane, then the sum of the coefficients of x^2 and z^2 should be zero

i.e. $$\gamma^2+(\alpha^2+\beta^2+2a\alpha+2b\beta)=0 \qquad \text{...(iv)}$$

$\therefore$ The locus of (α, β, γ) from (iii) and (iv) is

$$c\,(x^2+y^2)+2ax\,(c-z)+2by\,(c-z)=0 \qquad \text{...(v)}$$

and $$x^2+y^2+z^2+2ax+2by=0 \qquad \text{...(vi)}$$

Multiplying (vi) by c and subtracting (v) from the result so obtained, we get

$$cz^2+2azx+2byz=0 \quad \text{or} \quad 2ax+2by+cz=0, \qquad \text{...(vii)}$$

which is the equation of a plane. Therefore the required locus of the vertex is given by (vi) [or (v)] and (vii) which taken together represent a circle.

****Ex. 6 (b). The section of a cone with vertex at P and guiding curve $(x^2/a^2)+(y^2/b^2)=1$, $z=0$ by the plane $x=0$ is a rectangular hyperbola. Show that the locus of P is $(x^2/a^2)+\{(y^2+z^2)/b^2\}=1$.**

Sol. Let the vertex P of the cone by (α, β, γ).

Any line through $P(\alpha, \beta, \gamma)$ is $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ ...(i)

This line meets the plane $z=0$ is $\left(\alpha-\dfrac{l\gamma}{n}, \beta-\dfrac{m\gamma}{n}, 0\right)$ and if this point lies on the given curve $(x^2/a^2)+(y^2/b^2)=1$, we have

$$\frac{1}{a^2}\left(\alpha-\frac{l\gamma}{n}\right)^2+\frac{1}{b^2}\left(\beta-\frac{m\gamma}{n}\right)^2=1. \qquad \text{...(ii)}$$

Eliminating l, m, n between (i) and (ii), the equation of the cone is

$$\frac{1}{a^2}\left[\alpha-\left(\frac{x-a}{z-\gamma}\right)\gamma\right]^2+\frac{1}{b^2}\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2=1$$

or $$b^2\,(\alpha z-x\gamma)^2+a^2\,(\beta z-y\gamma)^2=a^2b^2\,(z-\gamma)^2 \qquad \text{...(iii)}$$

The section of this cone by the plane $x = 0$ gives the conic on yz-plane as

$$b^2a^2z^2 + a^2(\beta z - y\gamma)^2 = a^2b^2(z - \gamma)^2$$

or $$a^2\gamma^2y^2 + (b^2\alpha^2 + a^2\beta^2 - a^2b^2)z^2 - 2a^2\beta\gamma yz + 2a^2b^2\gamma z - a^2b^2\gamma^2 = 0.$$

If it represents a rectangular hyperbola on the yz-plane, then the sum of the coefficients of y^2 and z^2 must be zero

i.e. $$a^2\gamma^2 + (b^2\alpha^2 + a^2\beta^2 - a^2b^2) = 0 \qquad \textbf{(Note)}$$

or $$(\alpha^2/a^2) + [(\beta^2 + \gamma^2)/b^2] = 1$$

$\therefore$ The locus of $P(\alpha, \beta, \gamma)$ is $(x^2/a^2) + [(y^2 + z^2)/b^2] = 1$. Hence proved.

***Ex. 7. The vertex of cone is (a, b, c) and the yz-plane cuts it in the curve F (y, z) = 0, x = 0, show that the xz-plane cuts it in the curve.**

$$\mathbf{y = 0, F\left[\frac{bx}{x-a}, \frac{cx-az}{x-a}\right] = 0.}$$

Sol. Any line through (a, b, c) is $\dfrac{x-a}{l} = \dfrac{y-b}{m} = \dfrac{z-c}{n}$

It meets $x = 0$ in the point $\left[0, b - \dfrac{am}{l}, c - \dfrac{an}{l}\right]$ and if it lies on the given curve $F(y, z) = 0$, then we get $F\left[b - \dfrac{am}{l}, c - \dfrac{an}{l}\right] = 0$ ...(ii)

Eliminating l, m, n between (i) and (ii), we get

$$F\left[b - a\left(\frac{y-b}{x-a}\right), c - a\left(\frac{z-c}{x-a}\right)\right] = 0 \quad \text{or} \quad F\left[\frac{bx-ay}{x-a}, \frac{cx-az}{x-a}\right] = 0.$$

It meets zx-plane *i.e.* $y = 0$ in the curve

$$F\left[\frac{bx}{x-a}, \frac{cx-az}{x-a}\right] = 0, y = 0$$ Hence proved.

****Ex. 8. Find the equation of the cone with vertex at (2a, b, c) and passing through the curve $x^2 + y^2 = 4a^2$ and $z = 0$. Find b and c if the cone also passes through the curve $y^2 = 4a(z + a)$, $x = 0$. Also show that the cone is cut by the plane $y = 0$ in two straight lines and the angle θ between them is given by $\tan\theta = 2$.**

Sol. Any line through $(2a, b, c)$ is $\dfrac{x-2a}{l} = \dfrac{y-b}{m} = \dfrac{z-c}{n}$ ...(i)

It meets $z = 0$ in the point $\left[2a - \dfrac{lc}{n}, b - \dfrac{mc}{n}, 0\right]$ and if it lies on the curve $x^2 + y^2 = 4a^2$, $z = 0$, then we get $\left(2a - \dfrac{lc}{n}\right)^2 + \left(b - \dfrac{mc}{n}\right)^2 = 4a^2.$ ...(ii)

Eliminating l, m, n between (i) and (ii) we get the required cone as

$$\left[2a-\left(\frac{x-2a}{z-c}\right)c\right]^2+\left[b-\left(\frac{y-b}{z-c}\right)c\right]^2=4a^2$$

or
$$(2az-cx)^2+(bz-yc)^2=4a^2(z-c)^2. \quad \text{...(iii)}$$

If this cone passes through $y^2=4a(z+a), x=0$, then putting $x=0$ in (iii), we get
$$(2az)^2+(bz-yc)^2=4a^2(z-c)^2$$
or
$$b^2z^2+c^2y^2-2bcyz-4a^2c^2+8a^2zc=0 \quad \text{...(iv)}$$

If it is the same as $y^2=4a(z+a)$, then comparing this with (iv) we have $b^2=0$ *i.e.* $b=0$ which reduces (iv) to
$$c^2y^2=4a^2c^2-8a^2zc \quad \text{or} \quad y^2=-(8a^2/c)(z-\tfrac{1}{2}c).$$

Comparing this with $y^2=4a[z+a]$, we get $-(8a^2/c)=4a$ and $-\frac{1}{2}c=a$ which gives $c=-2a$. Hence $b=0, c=-2a$.

Substituting these values of b and c in (iii), the equation of cone intersecting the given conics reduces to $(2ax+2az)^2+4a^2y^2=4a^2(z+2a)^2$

or
$$x^2+y^2+2zx-4az-4a^2=0. \quad \text{...(v)}$$

The plane $y=0$ cuts cone (v) in $x^2+2zx-4az-4a^2=0, y=0$

or
$$(x^2-4a^2)+2z(x-2a)=0, y=0$$
or
$$(x-2a)[x+2a+2z]=0, y=0$$
or $x-2a=0, y=0$ and $x+2a+2z=0$, $y=0$. which give the required lines. These lines lie in the plane $y=0$ and their combined equation is given by
$$y=0, x^2+2zx-4az-4a^2=0.$$

$\therefore$ If θ be the angle between these lines, then
$$\tan\theta=\frac{\text{“}2\sqrt{(h^2-ab)}\text{”}}{a+b}=\frac{2\sqrt{(1^2-0)}}{1+0}=2. \qquad \text{Hence proved.}$$

Ex. 9. Find the equation of the cone whose vertex is (1, 2, 3) and guiding curve is the circle $x^2+y^2+z^2=4, x+y+z=1$. (*Bundelkhand* 93)

Sol. Any generator through (1, 2, 3) is
$$\frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}=\frac{x+y+z-6}{l+m+n} \quad \text{...(i)}$$

If it meets the plane $x+y+z=1$, then from (i) we have
$$\frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}=\frac{1-6}{l+m+n}=\frac{-5}{l+m+n}$$
which gives $x=1-[5l/(l+m+n)]$, $y=2-[5m/(l+m+n)]$ and

$z=3-[5n/(l+m+n)]$, *i.e.* the generator (i) meets the plane $x+y+z=1$ in the point $\left[\frac{m+n-4l}{l+m+n}, \frac{2l-3m+2n}{l+m+n}, \frac{3l+3m-2n}{l+m+n}\right]$

If this point lies on $x^2+y^2+z^2=4$ we get

$$(m+n-4l)^2+(2l-3m+2n)^2+(3l+3m-2n)^2=4\,(l+m+n)^2 \quad ...(ii)$$

Eliminating l, m, n between (i) and (ii) we get the required equation as

$$[(y-2)+(z-3)-4\,(x-1)]^2+[2\,(x-1)-3\,(y-2)+2\,(z-3)]^2$$
$$+[3\,(x-1)+3\,(y-2)-2\,(z-3)]^2=4\,[(x-1)+(y-2)+(z-3)]^2$$

or $$(y+z-4x-1)^2+(2x-3y+2z-2)^2+(3x+3y-2z-3)^2=4\,(x+y+z-6)^2$$

or $$5x^2+3y^2+z^2-6yz-4zx-2xy+6x+8y+10z-26=0$$ **Ans.**

****Ex. 10 (a). Prove that the equation**

$$\mathbf{4x^2-y^2+2z^2+2xy-3yz+12x-11y+6z+4=0}$$

represents a cone. Hence find its vetex.

(*Kanpur* 91; *Meerut* 95; *Purvanchal* 93)

Sol. Making the given equation homogeneous, we get

$$F\,(x, y, z, t)\equiv 4x^2-y^2+2z^2+2xy-3yz+12xt-11yt+6zt+4t^2=0$$

$\therefore$ $\dfrac{\partial F}{\partial x}=0$ gives $8x+2y+12t=0$ or $4x+y+6t=0$,

$\dfrac{\partial F}{\partial y}=0$ gives $-2y+2x-3z-11t=0$ or $2x-2y-3z-11t=0$

$\dfrac{\partial F}{\partial z}=0$ gives $4z-3y+6t=0$ or $3y-4z-6t=0$

and $\dfrac{\partial F}{\partial t}=0$ gives $12-11y+6z+8t=0$

Putting $t=1$, these equations become

$4x+y+6=0$ ...(i) ; $2x-2y-3z-11=0$ ...(ii)

$3y-4z-6=0$ ...(iii) ; $12x-11y+6z+8=0$ ...(iv)

From (ii) we get $4x-4y-6z-22=0$

Subtracting (i) from it we get $5y+6z+28=0$

or $10y+12z+56=0$ ...(v)

Multiplying (iii) by 3 we get $9y-12z-18=0$ ...(vi)

Adding (v) and (vi) we get $19y+38=0$ or $y=-2$

$\therefore$ From (iii) we get $3\,(-2)-4z-6=0$ or $z=-3$

From (i) we get $4x+(-2)+6=0$ or $x=-1$

These values viz. $x=-1, y=-2, z=-3$ satisfy (iv) and so the given equation represents a cone and its vertex is $(-1,-2,-3)$. **Ans.**

***Ex. 10 (b). Show that the equation $2y^2-8yz-4zx-8xy+6x-4y-2z+5=0$ represents a cone whose vertex is $(-7/6, 1/3, 5/6)$.**

(*Bundelkhand* 92)

Sol. Making the given equation homogeneous, we get

$$F\,(x, y, z, t)\equiv 2y^2-8yz-4zx-8xy+6xt-4yt-2zt+5t^2=0$$

$\therefore \dfrac{\partial F}{\partial x} = 0$ gives $-4z - 8y + 6t = 0$ or $2z + 4y - 3t = 0$

$\dfrac{\partial F}{\partial y} = 0$ gives $4y - 8z - 8x - 4t = 0$ or $y - 2z - 2x - t = 0$

$\dfrac{\partial F}{\partial z} = 0$ gives $-8y - 4x - 2t = 0$ or $4y + 2x + t = 0$

$\dfrac{\partial F}{\partial t} = 0$ gives $6x - 4y - 2z + 10t = 0$ or $3x - 2y - z + 5t = 0$

Putting $t = 1$, we get $2z + 4y - 3 = 0$ ...(i)

$y - 2z - 2x - 1 = 0$ ...(ii); $4y + 2x + 1 = 0$ ...(iii)

and $3x - 2y - z + 5 = 0$ ...(iv)

Adding (ii) and (iii), we get $5y - 2z = 0$ ...(v)

Adding (i) and (v), we get $9y - 3 = 0$ or $y = 1/3$

$\therefore$ From (v) $2z = 5y = 5\,(1/3)$ or $z = 5/6$

And from (iii), $2x = -4y - 1 = -4\,(1/3) - 1 = -7/3$ or $x = -7/6$

These values viz. $x = -7/6$, $y = 1/3$, $z = 5/6$ satisfy (iv) and so the given equation represents a cone and its vertex is $(-7/6, 1/3, 5/6)$. Hence proved.

Ex. 10 (c). Prove that the equation $2x^2 + 2y^2 + 7z^2 - 10yz - 10zx + 2x + 2y + 26z - 17 = 0$ represents a cone whose vertex is at (2, 2, 1).

(*Garhwal* 94, 92)

Sol. Do as Ex. 10 (b) above.

****Ex. 11 (a). Prove that the equation**

$$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$$

represents a cone if $u^2/a + v^2/b + w^2/c = d$.

(*Agra* 92; *Avadh* 95; *Kanpur* 90; *Meerut* 94, 91, 90)

Sol. Let $F(x, y, z, t) = ax^2 + by^2 + cz^2 + 2uxt + 2vyt + 2wzt + dt^2 = 0$

$\therefore \dfrac{\partial F}{\partial x} = 0$ for $t = 1$ gives $2ax + 2u = 0$ or $x = -u/a$ ...(i)

Similarly $\dfrac{\partial F}{\partial y} = 0$ for $t = 1$ gives $y = -v/b$; ...(ii)

$\dfrac{\partial F}{\partial z} = 0$ for $t = 1$ gives $z = -w/c$ ...(iii)

and $\dfrac{\partial F}{\partial t} = 0$ for $t = 1$ gives $ux + vy + wz + d = 0$ ...(iv)

Substituting the values x, y, z from (i), (ii), (iii) in (iv) we get the required condition as $u(-u/a) + v(-v/b) + w(-w/c) + d = 0$

or $(u^2/a) + (v^2/b) + (w^2/c) = d$. Hence proved.

Ex. 11 (b). Prove that the equation

$$(x^2/a^2) + (y^2/b^2) + (z^2/c^2) + 2ux + 2vy + 2wz + d = 0$$

represents a cone if $a^2u^2 + b^2v^2 + c^2w^2 = d$. (*Kanpur* 92)

Sol. Do e actly as Ex. 11 (a) above.

Exercises on § 8.04 — § 8.05.

Ex. 1 (a). Find the equation of the cone with vertex (0, 0, 1) and passing through the points of the circle $x^2 + y^2 = 1, z = 0$ **Ans.** $x^2 + y^2 + z^2 + 2z - 1 = 0$

Ex. 1 (b). Find the equation of the cone with vertex at (1, 1, 1) which passes through the curve $x^2 + y^2 = 4, z = 6$ (*Kumaun* 93)

Ans. $25x^2 + 25y^2 - 2z^2 + 10yz + 10zx - 60x - 60y - 16z + 68 = 0$

Ex. 2. Find the equation to the cone whose vertex is the point $P(a, b, c)$ and whose generating line intersects the conic $px^2 + qy^2 = 1, \ z = 0$.

Ans. $c^2 (px^2 + qy^2) + (a^2 p + b^2 q - 1)^2 - 2c(apzx + bqyz - z) = c^2$

Ex. 3. Find the equation of the cone with vertex (5, 4, 3) and with $3x^2 + 2y^2 = 6, y + z = 0$ as base.

Ans. $147x^2 + 87y^2 + 101z^2 + 90yz - 210(zx + xy) + 84(y + z) = 294$.

Ex. 4. Find the cone whose vertex is (1, 1, 0) and whose guiding curve is $y = 0, x^2 + z^2 = 4$.

Ex. 5. Find the equation of the cone whose vertex is (1, – 2, 3) and whose guiding curve is $x^2 + y^2 = 9, z = 0$.

Ans. $9x^2 + 9y^2 - 4z^2 + 12yz - 6zx + 54z - 81 = 0$

Ex. 6. Prove that the equation

$$x^2 - 2y^2 + 3z^2 - 4xy + 5yz - 6zx + 8x - 19y - 2z = 20$$

represents a cone. Show that the coordinates of its vertex are (1, – 2, 3). (*Meerut* 91 *S*)

Ex. 7. Prove that the equation $2x^2 - 8xy - 4yz - 4x - 2y + 6z + 35 = 0$ represents a cone with its vertex at (4, 3/2, – 17/2).

***§ 8.06. Angle between the lines in which a plane cuts a cone.**

Let the plane be $$ux + vy + wz = 0 \quad \text{...(i)}$$

and the cone be $$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \text{...(ii)}$$

The vertex of this cone is (0, 0, 0) which evidently lies on (i). Let l, m, n be the d.c.'s of the line through the vertex in which the plane (i) cuts the cone (ii), then its equations are $$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \text{...(iii)}$$

As (iii) is generator of the cone (ii), so its d.c.'s will satisfy (ii) and so we get $$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0 \quad \text{...(iv)}$$

Also the line (iii) lies on plane (i), hence we have

$$ul + vm + wn = 0 \quad \text{...(v)}$$

Eliminating n between (iv) and (v), we get

$$al^2 + bm^2 + c\left(-\frac{ul+vm}{w}\right)^2 + 2hlm + 2(gl+fm)\left(-\frac{ul+vm}{w}\right) = 0$$

or $$l^2(aw^2 + cu^2 - 2guw) + 2lm(cuv - fuw - gvw + hw^2) + m^2(bw^2 + cv^2 - 2fvw) = 0$$

or $$(l^2/m^2)(aw^2 + cu^2 - 2guw) + 2(l/m)(cuv - fuw - gvw + hw^2) + (bw^2 + cv^2 - 2fvw) = 0 \quad ...(vi)$$

This equation being quadratic in (l/m) indicates that the plane (i) cuts the cone (ii) in two lines and if their d.c.'s are l_1, m_1, n_1 and l_2, m_2, n_2, then

$$\frac{l_1 l_2}{m_1 m_2} = \text{product of roots of (vi)} = \frac{bw^2 + cv^2 - 2fvw}{aw^2 + cu^2 - 2guw}$$

$$\therefore \frac{l_1 l_2}{bw^2 + cv^2 - 2fvw} = \frac{m_1 m_2}{cu^2 + aw^2 - 2guw} = \frac{n_1 n_2}{av^2 + bu^2 - 2hvw} \quad ...(vii)$$

$= \lambda$ (say), by symmetry third fraction is written

Also from (vi), $\frac{l_1}{m_1} + \frac{l_2}{m_2}$ = sum of roots of (vi)

or $$\frac{l_1 m_2 + l_2 m_1}{m_1 m_2} = \frac{-2(cuv - fuw - gvw + hw^2)}{aw^2 + cu^2 - 2guw}$$

or $$\frac{l_1 m_2 + l_2 m_1}{-2(cuv - fuw - gvw + hw^2)} = \frac{m_1 m_2}{cu^2 + aw^2 - 2guw} = \lambda, \text{ from (vi)} \quad ...(viii)$$

$$\therefore (l_1 m_2 - l_2 m_1)^2 = (l_1 m_2 + m_1 l_2)^2 - 4 l_1 l_2 m_1 m_2$$
$$= 4\lambda^2 (cuv - fuw - gvw + hw^2)^2 - 4\lambda^2 (bw^2 + cv^2 - 2fvw) \times (cu^2 + aw^2 - 2guw), \text{ from (vii) and (viii)}$$
$$= 4w^2\lambda^2 [-(Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv)]$$
$$= 4w^2\lambda^2 P^2,$$

where the capital letters A, B, C, etc. are the cofactors of the correspoinding small letters a, b, c, etc. in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \text{ and } P^2 = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}$$

$$\therefore l_1 m_2 - l_2 m_1 = 2w\lambda P \quad ...(ix)$$

Again if θ be the angle between the lines of intersection then we have $\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$.

$$= \Sigma\lambda(bw^2 + cv^2 - 2fvw), \text{ from (vii)}$$
$$= \lambda[(b+c)u^2 + (c+a)v^2 + (a+b)w^2 - 2fvw - 2gwu - 2huv]$$
$$= \lambda[(a+b+c)(u^2 + v^2 + w^2)$$

$$-(au^2 + bv^2 + cw^2 + 2fvw + 2gwu + 2huv)]$$

$$= \lambda\,[(a+b+c)(u^2+v^2+w^2) - f(u, v, w)] \qquad \text{...(x)}$$

Also $\sin^2\theta = \Sigma\,(l_1m_2 - l_2m_1)^2 = (2\lambda wP)^2 + (2\lambda\, vP)^2 + (2\lambda\, uP)^2$

or $$\sin\theta = 2\lambda\, P\,\sqrt{(u^2+v^2+w^2)} \qquad \text{...(xi)}$$

$$\therefore\ \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2P\sqrt{(u^2+v^2+w^2)}}{[(a+b+c)(u^2+v^2+w^2) - f(u, v, w)]}.$$

Cor 1. If the plane (i) cuts the cone (ii) in two perpendicular generators, then $\theta = \pi/2$ or $\cancel{\cos\theta = 0}$

or $$\boxed{(a+b+c)(u^2+v^2+w^2) - f(u, v, w) = 0,} \text{ from (x).}$$

Particular Case. If the d.c.'s of two perpendicular lines be given by

$$ul + vm + wn = 0 \quad \text{and} \quad al^2 + bm^2 + cn^2 = 0$$

i.e. the plane $ux + vy + wz = 0$ cuts the cone $ax^2 + by^2 + cz^2 = 0$ in perpendicular generators, then

$$(a+b+c)(u^2+v^2+w^2) - (au^2+bv^2+cw^2) = 0 \qquad \textbf{(Note)}$$

or $$(b+c)\,u^2 + (c+a)\,v^2 + (a+b)\,w^2 = 0.$$

Cor. 2. If the plane (i) cuts the cone (ii) in two coincident generators then

$\sin\theta = 0$ *i.e.* $P = 0$, from (xi)

i.e. $$\boxed{Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv = 0}$$

Solved Examples on § 8.06. leave

***Ex. 1 (a). Find the equations to the planes in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$.**

Find also the angle between the lines of section.

Sol. Let the plane $$2x + y - z = 0 \qquad \text{...(i)}$$

cut the cone $$4x^2 - y^2 + 3z^2 = 0 \qquad \text{...(ii)}$$

in a line $$x/l = y/m = z/n \qquad \text{...(iii)}$$

Then $2l + m - n = 0$ and $4l^2 - m^2 + 3n^2 = 0$ ...(iv)

Eliminating n between these relations, we get

$$4l^2 - m^2 + 3\,(2l+m)^2 = 0 \qquad \textbf{(Note)}$$

or $$16l^2 + 12l\,m + 2m^2 = 0 \quad \text{or} \quad 8\,(l/m)^2 + 6\,(l/m) + 1 = 0$$

or $$l/m = (1/16)\,[-6 \pm \sqrt{\{36 - 32\}}] = -\tfrac{1}{4}, -\tfrac{1}{2}$$

or $$l = -\tfrac{1}{4}m \quad \text{and} \quad l = -\tfrac{1}{2}m \qquad \text{...(v)}$$

From (iv), $n = 2l + m = 2\,(-\frac{1}{4}m) + m$, when $l = -\frac{1}{4}m$

or $n = \frac{1}{2}m$. $\quad\therefore\quad l : m : n = -\frac{1}{4}m : m : \frac{1}{2}m$

i.e. $$l : m : n = -1 : 4 : 2 \qquad \text{(Note)}$$

Again when $l = -(1/2)\,m$, then from (iv), we get

$$n = 2l + m = 2\,(-\tfrac{1}{2}m) + m = 0.\ m$$

$\therefore l : m : n = -(1/2)\, m : m : 0 . m = -1 : 2 : 0$

Hence the required equations of the lines are

$$\frac{x}{-1} = \frac{y}{4} = \frac{z}{2} \quad \text{and} \quad \frac{x}{-1} = \frac{y}{2} = \frac{z}{0}. \qquad \textbf{Ans.}$$

Also if θ be the required angle between these lines, then

$$\cos\theta = \frac{(-1)(-1) + (4)(2) + (2)(0)}{\sqrt{[(-1)^2 + 4^2 + 2^2]}\sqrt{[(-1)^2 + 2^2 + 0^2]}} = \frac{9}{\sqrt{(21)}\sqrt{(5)}} \qquad \textbf{Ans.}$$

Ex. 1 (b). Find the angle between the lines whose d.c.'s are given by the equations $ul + vm + wn = 0$ and $al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0$.

Sol. Here we are to find the angle between the lines in which the plane $ux + vy + wz = 0$ cuts the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ and for this proceed as in § 8.06 Page 21 of this chapter.

****Ex. 2. Prove that the plane $ax + by + cz = 0$ cuts the cone $yz + zx + xy = 0$ in perpendicular lines if $1/a + 1/b + 1/c = 0$.**

(Agra 90; Kanpur 92)

Sol. Let the plane $ax + by + cz = 0$ cut the cone

$yz + zx + xy = 0$ in a line $x/l = y/m = z/n$.

Then $mn + nl + lm = 0$ and $al + bm + cn = 0$

Eliminating n between these relations, we get

$$(m + l)\,[-(al + bm)/c] + lm = 0$$

or $\quad al^2 + (a + b - c)\, lm + bm^2 = 0 \quad$ or $\quad a\,(l/m)^2 + (a + b - c)\,(l/m) + b = 0$

If the roots of this equation are l_1/m_1 and l_2/m_2, then

$$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \text{product of the roots} = \frac{b}{a}$$

or $\quad \dfrac{l_1 l_2}{1/a} = \dfrac{m_1 m_2}{1/b} = \dfrac{n_1 n_2}{1/c}$, by symmetry

If these lines are at right angles, then $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

or $\quad (1/a) + (1/b) + (1/c) = 0. \qquad$ Hence proved.

Ex. 3 (a). Prove that the condition that the plane $ux + vy + wz = 0$ may cut the cone $ax^2 + by^2 + cz^2 = 0$ in perpendicular generators if

$$\mathbf{(b + c)\, u^2\, (c + a)\, v^2 + (a + b)\, w^2 = 0}$$

Sol. Let $x/l = y/m = z/n$ be one of the lines in which the plane $ux + vy + wz = 0$ meets the cone $ax^2 + by^2 + cz^2 = 0$, then we have

$ul + vm + wn = 0 \quad$ ...(i) $\quad$ and $\quad al^2 + bm^2 + cn^2 = 0 \quad$ ...(ii)

Eliminating n between (i) and (ii), we get

$$al^2 + bm^2 + c\,[-(ul + vm)/w]^2 = 0$$

or $\quad (aw^2 + cu^2)\, l^2 + 2cuvlm + (bw^2 + cv^2)\, m^2 = 0$

or $\quad (aw^2 + cu^2)\,(l^2/m^2) + 2cuv\,(l/m) + (bw^2 + cv^2) = 0$

If its roots are l_1/m_1 and l_2/m_2, then we have

$$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2}=\text{product of the roots}=\frac{bw^2+cv^2}{aw^2+cu^2}$$

or $$\frac{l_1l_2}{bw^2+cv^2}=\frac{m_1m_2}{cu^2+aw^2}=\frac{n_1n_2}{av^2+bu^2}, \text{ by symmetry.}$$

If the lines are perpendicular then $l_1l_2+m_1m_2+n_1n_2=0$

i.e. $$(bw^2+cv^2)+(cu^2+aw^2)+(av^2+bu^2)=0$$

or $$(b+c)\,u^2+(c+a)\,v^2+(a+b)\,w^2=0.$$ Hence proved.

Ex. 3 (b). Find the condition that the plane $ux+vy+wz=0$ may meet the cone $f(x,y,z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ in two mutually perpendicular lines.

Sol. Let $x/l=y/m=z/n$ be one of the lines in which the plane $ux+vy+wz=0$ meets the cone $f(x,y,z)=0$, then we have

$$ul+vm+wn=0 \qquad ...(i)$$

and $$f(l,m,n)\equiv al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0 \qquad ...(ii)$$

Eliminating n between (i) and (ii) we get

$$al^2+bm^2+c\left(-\frac{ul+vm}{w}\right)^2+2fm\left(-\frac{ul+vm}{w}\right)+2g\left(-\frac{ul+vm}{w}\right)l+2hlm=0$$

or $$aw^2l^2+bw^2m^2+c\,(ul+vm)^2-2fwm\,(ul+vm)-2gwl\,(ul+vm)+2hw^2lm=0$$

or $$(aw^2+cu^2-2guw)\,l^2+2\,(cuv-fuw-gvw+hw^2)\,lm+(bw^2+cv^2-2fvw)\,m^2=0$$

or $$(aw^2+cu^2-2guw)\,(l/m)^2+2\,(cuv-fuw-gvw+hw^2)\,(l/m)+(bw^2+cv^2-2fvw)=0,$$

which is a quadratic equation in l/m and if its roots are l_1/m_1 and l_2/m_2, then

we get $$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2}=\text{product of the roots}=\frac{bw^2+cv^2-2fvw}{aw^2+cu^2-2guw}$$

or $$\frac{l_1l_2}{bw^2+cv^2-2fvw}=\frac{m_1m_2}{cu^2+aw^2-2gwu}=\frac{n_1n_2}{av^2+bu^2-2luv}, \text{ by symmetry,}$$

(Note)

If the lines of intersection of the given plane and cone are mutually perpendicular, then $l_1l_2+m_1m_2+n_1n_2=0$

i.e. $$(bw^2+cv^2-2fvw)+(cu^2+aw^2-2gwu)+(av^2+bu^2-2huv)=0$$

or $$a\,(v^2+w^2)+b\,(w^2+u^2)+c\,(u^2+v^2)=2\,(fvw+gwu+huv),$$

which is the required condition.

$\boxed{(a+b+c)(u^2+v^2+w^2)=f(u,v,w)}$

Ex. 4 (a). Prove that the angle between the lines given by $x+y+z=0$, $ayz+bzx+cxy=0$ is $\pi/2$ if $a+b+c=0$ and $\pi/3$ if $1/a+1/b+1/c=0$. *(Meerut 95)*

Sol. Let the plane $x+y+z=0$ cut the cone $ayz+bzx+cxy=0$ in a line

$$x/l=y/m=z/n.$$

Then $l+m+n=0$ and $amn+bnl+clm=0$.

Eliminating n between these relations, we get

$$(am+bl)(-l-m)+clm=0 \quad \text{or} \quad bl^2+(a+b-c)\,lm+am^2=0$$

or $$b\,(l/m)^2+(a+b-c)\,(l/m)+a=0 \qquad \text{...(i)}$$

If the roots of this equation are l_1/m_1 and l_2/m_2, then

$$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2}=\text{product of the roots}=\frac{a}{b}$$

or $$\frac{l_1l_2}{a}=\frac{m_1m_2}{b}=\frac{n_1n_2}{c} \text{ (by symmetry)}=k \text{ (say)}, \qquad \text{...(ii)}$$

If the angle between the lines is $\pi/2$, then we get

$$l_1l_2+m_1m_2+n_1n_2=0 \text{ or } a+b+c=0, \text{ from (ii)}.$$

Again from (i) we get

$$\frac{l_1}{m_1}+\frac{l_2}{m_2}=\text{sum of the roots}=\frac{c-b-a}{b}$$

or $$\frac{l_1m_2+l_2m_1}{m_1m_2}=\frac{c-b-a}{b} \text{ or } \frac{l_1m_2+l_2m_1}{c-b-a}=\frac{m_1m_2}{b}=k, \text{ from (ii)}.$$

Now $$(l_1m_2-l_2m_1)^2=(l_1m_2+l_2m_1)^2-4l_1l_2.m_1m_2$$
$$=k^2(c-b-a)^2-4ak\,.\,bk=k^2[(c-b-a)^2-4ab]$$
$$=k^2[a^2+b^2+c^2-2ab-2bc-2ca],$$

Now $$\tan\theta=\frac{\sqrt{[\Sigma(l_1m_2-l_2m_1)^2]}}{l_1l_2+m_1m_2+n_1n_2}$$
$$=\frac{\sqrt{[k^2\{3(a^2+b^2+c^2-2bc-2ca-2ab)\}]}}{k(a+b+c)} \qquad \textbf{(Note)}$$

If $\theta=\pi/3$, then

$$\tan^2(\pi/3)=3(a^2+b^2+c^2-2bc-2ca-2ab)/(a+b+c)^2$$

or $3(a+b+c)^2=3(a^2+b^2+c^2-2bc-2ca-2ab)$; $\because \tan(\pi/3)=\sqrt{3}$

or $4(bc+ca+ab)=0$ or $1/a+1/b+1/c=0$. Hence proved.

Ex. 4 (b). If the plane $2x-y+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines, find the value of c. *(Meerut 96)*

Sol. Let the plane $2x-y+cz=0$ cut the cone $yz+zx+xy=0$ in a line

$$x/l=y/m=z/n.$$

Then $2l-m+cn=0$ and $mn+nl+lm=0$ ...(i)

Eliminating m between these relations, we get

$$(2l + cn)\, n + nl + l\,(2l + cn) = 0$$

or $\quad 2l^2 + (c+3)\, ln + cn^2 = 0$ or $2\,(l/n)^2 + (c+3)\,(l/n) + c = 0$ ...(ii)

If the roots of this equation are (l_1/n_1) and (l_2/n_2), then

$$\frac{l_1}{n_1} \cdot \frac{l_2}{n_2} = \text{product of the roots} = \frac{c}{2} \quad \text{or} \quad \frac{l_1 l_2}{c} = \frac{n_1 n_2}{2} \qquad \text{...(iii)}$$

Eliminating l between the relations (i) we get

$$2nm + n\,(m - cn) + m\,(m - cn) = 0 \qquad \textbf{(Note)}$$

or $\quad m^2 + (3 - c)\, mn - cn^2 = 0$ or $c\,(n/m)^2 + (c-3)\,(n/m) - 1 = 0$

$\therefore$ If the roots of this equation are (n_1/m_1) and (n_2/m_2) then

$$\frac{n_1}{m_1} \cdot \frac{n_2}{m_2} = \text{product of the roots} = -\frac{1}{c}.$$

or
$$\frac{n_1 n_2}{1} = \frac{m_1 m_2}{-c} \quad \text{or} \quad \frac{n_1 n_2}{2} = \frac{m_1 m_2}{-2c} \qquad \text{...(iv)}$$

From (iii) and (iv) we get $\dfrac{l_1 l_2}{1} = \dfrac{m_1 m_2}{-2c} = \dfrac{n_1 n_2}{2} = k$ (say).

If the angle between the lines is a right angle, then we have

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ or $c + (-2c) + 2 = 0$ or $c = 2$. **Ans.**

Ex. 5. Find the angle between the lines given by $x + y + z = 0$ and

&8

$$\frac{yz}{q - r} + \frac{zx}{r - p} + \frac{xy}{p - q} = 0.$$

Sol. Find as in Ex. 4 (a) Page 26 the angle between the lines given by

$$x + y + z = 0 \quad \text{and} \quad ayz + bzx + cxy = 0,$$

where $\quad a = 1/(q - r),\ b = 1/(r - p),\ c = 1/(p - q)$

Also here $1/a + 1/b + 1/c = 0$, which show that the angle between those lines is $\pi/3$. **Ans.**

Ex. 6 (a). Find the angle between the lines of section of the plane $6x - y - 2z = 0$ and the cone $108x^2 - 7y^2 - 20z^2 = 0$.

Sol. Let $x/l = y/m = z/n$ be one of the lines in which the given plane cuts the given cone, then we have $\quad 6l - m - 2n = 0$ ...(i)

and $\quad 108l^2 - 7m^2 - 20n^2 = 0$ ...(ii)

Eliminating n between (i) and (ii) we get

$$108l^2 - 7m^2 - 20\,[\tfrac{1}{2}\,(6l - m)]^2 = 0$$

or $\quad 6l^2 - 5lm + m^2 = 0$ or $(3l - m)\,(2l - m) = 0$

When $3l - m = 0$, from (i) we have $3l - 2n = 0$

$\therefore\ 3l = m = 2n$ or $\dfrac{l}{(1/3)} = \dfrac{m}{1} = \dfrac{n}{(1/2)}$

When $2l - m = 0$, from (i) we have $2l - n = 0$

$\therefore\ 2l = m = n$ or $\dfrac{l}{(1/2)} = \dfrac{m}{1} = \dfrac{n}{1}$

$\therefore$ If θ be the required angle, then

$$\cos\theta = \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{(l_1^2+m_1^2+n_1^2)}\sqrt{(l_2^2+m_2^2+n_2^2)}}$$

$$= \frac{\frac{1}{3}\cdot\frac{1}{2}+1\,.\,1+\frac{1}{2}\,.\,1}{\sqrt{(\frac{1}{9}+1+\frac{1}{4})}\sqrt{(\frac{1}{4}+1+1)}} = \frac{\frac{1}{6}+1+\frac{1}{2}}{\sqrt{(\frac{49}{36})}\,.\,\sqrt{(\frac{9}{4})}} = \frac{(10/6)}{\frac{7}{6}\cdot\frac{3}{2}} = \frac{20}{21}$$

or $\theta = \cos^{-1}(20/21)$ **Ans.**

***Ex. 6 (b). Find the condition that the lines of section of the plane $lx + my + nz = 0$ and cones $fyz + gzx + hxy = 0$, $ax^2 + by^2 + cz^2 = 0$ should be coincident.**

Sol. Let $x/\lambda = y/\mu = z/\nu$ be one of the lines in which the given plane $lx + my + nz = 0$ cuts the cone $fyz + gzx + hxy = 0$,

then we have $l\lambda + m\mu + n\nu = 0$ ...(i) and $f\mu\nu + g\nu\lambda + h\lambda\mu = 0$ ...(ii)

Eliminating ν between (i) and (ii), we get

$$f\mu[-(l\lambda + m\mu)/n] + g\lambda[-(l\lambda + m\mu)/n] + h\lambda\mu = 0$$

or $$-lf\lambda\mu + mf\mu^2 - gl\lambda^2 - gm\lambda\mu + nh\,\lambda\mu = 0$$

or $$gl\lambda^2 + (lf + gm - nh)\,\lambda\mu + mf\mu^2 = 0$$

or $$gl\,(\lambda/\mu)^2 + (lf + gm - nh)\,(\lambda/\mu) + mf = 0. \quad \text{...(iii)}$$

If $x/\lambda_1 = y/\mu_1 = z/\nu_1$ and $x/\lambda_2 = y/\mu_2 = z/\nu_2$ be two lines of section then from (iii) we get $\frac{\lambda_1}{\mu_1}\cdot\frac{\lambda_2}{\mu_2}$ = product of roots $= \frac{mf}{gl}$

or $$\frac{\lambda_1\lambda_2}{f/l} = \frac{\mu_1\mu_2}{g/m} = \frac{\nu_1\nu_2}{h/n}, \text{ by symmetry} \quad \text{...(iv)}$$

Again if $x/p = y/q = z/r$ be one of the lines in which the plane $lx + my + nz = 0$ cuts the cone $ax^2 + by^2 + cz^2 = 0$

then we have $$lp + mq + nr = 0 \quad \text{...(v)}$$

and $$ap^2 + bq^2 + cr^2 = 0 \quad \text{...(vi)}$$

Eliminating r between (v) and (vi), we get

$$ap^2 + bq^2 + c\,[-(lp - mq)/n]^2 = 0$$

or $$(an^2 + cl^2)\,p^2 - 2clm\,pq + (bn^2 + cm^2)\,q^2 = 0$$

or $$(an^2 + cl^2)\,(p/q)^2 - 2clm\,(p/q) + (bn^2 + cm^2) = 0 \quad \text{...(vii)}$$

If $x/p_1 = y/q_1 = z/r_1$ and $x/p_2 = y/q_2 = z/r_2$ be the two lines of section then from (vii) we get $\frac{p_1}{q_1}\cdot\frac{p_2}{q_2}$ = product of the roots $= \frac{bn^2 + cm^2}{an^2 + cl^2}$

or $$\frac{p_1p_2}{bn^2 + cm^2} = \frac{q_1q_2}{cl^2 + an^2} = \frac{r_1r_2}{am^2 + bl^2}, \text{ by symmetry} \quad \text{...(viii)}$$

Now if the lines of section of the plane $lx + my + nz = 0$ and cones $fyz + gzx + hxy = 0$ and $ax^2 + by^2 + cz^2 = 0$ are coincident then we must have $$\frac{\lambda_1\lambda_2}{p_1p_2} = \frac{\mu_1\mu_2}{q_1q_2} = \frac{\nu_1\nu_2}{r_1r_2}$$

or $$\frac{f/l}{bn^2 + cm^2} = \frac{g/m}{cl^2 + an^2} = \frac{h/n}{am^2 + bl^2}$$

or $$\frac{fmn}{bn^2 + cm^2} = \frac{gln}{cl^2 + an^2} = \frac{hlm}{am^2 + bl^2}$$ are the required conditions. **Ans.**

***Ex. 7. Show that the plane $ax + by + cz = 0$ cuts the cone $yz + zx + xy = 0$ in two lines inclined at an angle**

$$\tan^{-1}\left[\frac{\sqrt{\{(a^2 + b^2 + c^2)(a^2 + b^2 + c^2 - 2bc - 2ca - 2ab)\}}}{bc + ca + ab}\right]$$

and by considering the value of this experssion when $a + b + c = 0$, show that the cone is of revolution and that its axis is $x = y = z$ and vertical angle $\tan^{-1}(-2/\sqrt{2})$.

Sol. Let $x/l = y/m = z/n$ be one of the lines in which the plane $ax + by + cz = 0$ cuts the cone $yz + zx + xy = 0$, then we have

$al + bm + cn = 0$ ...(i) and $mn + nl + lm = 0$ ...(ii)

Eliminating n between (i) and (ii), we get

$$(m + l)[-(al + bm)/c] + lm = 0$$

or $$al^2 + (a + b - c)\, lm + bm^2 = 0$$

or $$a(l/m)^2 + (a + b - c)(l/m) + b = 0$$

If its roots are l_1/m_1 and l_2/m_2, then we have

$$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \text{product of the roots} = \frac{b}{a}$$

or $$\frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c} \text{ (by symmetry)} = k \text{ (say).} \quad \text{...(iii)}$$

Also $$\frac{l_1}{m_1} + \frac{l_2}{m_2} = \text{sum of the roots} = \frac{c - b - a}{a}$$

or $$\frac{l_1 m_2 + l_2 m_1}{m_1 m_2} = \frac{c - b - a}{a} \quad \text{or} \quad \frac{l_1 m_2 + l_2 m_1}{c - b - a} = \frac{m_1 m_2}{a}$$

or $$\frac{l_1 m_2 + l_2 m_1}{(c - b - a)/(ab)} = \frac{m_1 m_2}{1/b} = k, \text{ from (iii)} \quad \textbf{(Note)}$$

$$\therefore (l_1 m_2 - l_2 m_1) = (l_1 m_2 + l_2 m_1)^2 - 4 l_1 l_2 \cdot m_1 m_2$$
$$= k^2 [(c - b - a)/(ab)]^2 - 4(k/a)(k/b)$$
$$= (k^2/a^2 b^2)[(c - b - a)^2 - 4ab]$$
$$= (k^2/a^2 b^2)[\lambda^2] = k^2 \lambda^2/(a^2 b^2),$$

where $$\lambda^2 = a^2 + b^2 + c^2 - 2bc - 2ca - 2ab \quad \text{...(iv)}$$

$\therefore$ If θ be the angle between these two lines, then we have

$$\tan\theta = \frac{\sqrt{[\Sigma (l_1 m_2 - l_2 m_1)^2]}}{l_1 l_2 + m_1 m_2 + n_1 n_2} = \frac{k\lambda\,[(1/a^2 b^2) + (1/b^2 c^2) + (1/c^2 a^2)]}{k\,[1/a + 1/b + 1/c]}$$

or $\tan\theta = \dfrac{\lambda\sqrt{(c^2+a^2+b^2)}}{abc\,(1/a+1/b+1/c)}$ or $\theta = \tan^{-1}\left[\dfrac{\lambda\sqrt{(c^2+a^2+b^2)}}{bc+ca+ab}\right]$, ...(v)

where λ is given by (iv). Hence proved.

Again if $a+b+c=0$, then $(a+b+c)^2=0$

or $$a^2+b^2+c^2=-2\,(bc+ca+ab)$$ **(Note)**

$\therefore$ From (iv), we get $\lambda^2=-4\,(bc+ca+ab)$.

Substituting these values of λ and $(a^2+b^2+c^2)$ in (v), we get

$$\theta=\tan^{-1}\left[\frac{\sqrt{\{-4\,(bc+ca+ab)\}}\,\sqrt{\{-2\,(bc+ca+ab)\}}}{(bc+ca+ab)}\right]$$

$$=\tan^{-1}[\sqrt{(8)}]=\tan^{-1}(-2\sqrt{2}). \quad ...(vi)$$ Hence proved.

Again if $a+b+c=0$, it means that the line $\dfrac{x}{1}=\dfrac{y}{1}=\dfrac{z}{1}$ lies on the plane $ax+by+cz=0$ and we are to prove that this line is the axis of the cone *i.e.* the angle between this line and any generator of the cone is half of angle θ given by (vi).

Now if $x/l=y/m=z/n$ is a generator of the cone, then $mn+nl+lm=0$ and if α be the angle between this generator and the axis, then

$$\cos\alpha=\frac{1.l+1.m+1.n}{\sqrt{(1^2+1^2+1^2)}\sqrt{(l^2+m^2+n^2)}}=\frac{\Sigma l}{\sqrt{3}\sqrt{[(\Sigma l)^2-2\Sigma mn]}}$$

$$=1/\sqrt{3}, \quad \because \Sigma mn=0.$$

$\therefore \tan\alpha=\sqrt{2}$ or $\tan 2\alpha=\dfrac{2\tan\alpha}{1-\tan^2\alpha}=\dfrac{2\sqrt{2}}{1-2}=-2\sqrt{2}$.

$\therefore$ From (vi) $\tan 2\alpha=\tan\theta$ or $\alpha=(1/2)\,\theta$. Hence proved

Ex. 8. Find the equation of the cone generated by the straight lines drawn from the origin to cut the circle through the points A (1, 0, 0), B (0, 2, 0), C (2, 1, 1) and prove that the acute angle between two straight lines in which the plane x = 2y cuts the cone is $\cos^{-1}\sqrt{(5/14)}$.

Sol. Let the equation of sphere through A, B, C and origin be

$$x^2+y^2+z^2+2ux+2vy+2wz=0. \quad ...(i)$$

If it passes through A (1, 0, 0), then $1+2u=0$ or $2u=-1$

If it passes through B (0, 2, 0), then $4+4v=0$ or $v=-1$

If it passes through C (2, 1, 1), then $6+4u+2v+8w=0$

or $$6+4\left(-\tfrac{1}{2}\right)+2\,(-1)+8w=0 \quad \text{or} \quad w=-1$$

$\therefore$ From (i) the equation of the sphere ABC is

$$x^2+y^2+z^2-x-2y-2z=0 \quad (ii)$$

Also the equation of the plane through $A(1, 0, 0)$ and $B(0, 2, 0)$ can be taken as

$$\frac{x}{1}+\frac{y}{2}+\frac{z}{c}=1. \quad \textbf{(Note)}$$

If it passes through $C(2, 1, 1)$ then $2+\frac{1}{2}+(1/c)=1$ or $c=-\frac{2}{3}$.

$\therefore$ The plane ABC is given by $\frac{x}{1}+\frac{y}{2}-\frac{3z}{2}=1$

or
$$2x+y-3z=2 \quad \text{...(iii)}$$

$\therefore$ Making (ii) homogeneous with the help of (iii) we get the equation of the required cone as $x^2+y^2+z^2-(x+2y+2z)\left[\frac{1}{2}(2x+y-3z)\right]=0$

or
$$8z^2+4yz-zx-5xy=0. \quad \text{...(iv)}$$

If the plane $x=2y$ cuts this cone along a line $x/l=y/m=z/n$, then

$$8n^2+4mn-nl-5lm=0 \quad \text{and} \quad l=2m \quad \text{...(v)} \quad \textbf{(Note)}$$

Eliminating l, we get $8n^2+4mn-n(2m)-5(2m)m=0$

or $8n^2+2mn-10m^2=0$ or $2(n-m)(4n+5m)=0$

When $n=m$, then from (v) we get $l/2=m/1=n/1$.

And when $4n+5m=0$, then from (v) we get $l=2m=-(8/5)n$

or
$$\frac{l}{2}=\frac{m}{1}=\frac{n}{-(5/4)}$$

$\therefore$ If θ be the required angle between these two lines, then

$$\cos\theta=\frac{\Sigma l_1 l_2}{\sqrt{(\Sigma l_1^2)}\sqrt{(\Sigma l_2^2)}}=\frac{2.2+1.1+1.(-5/4)}{\sqrt{(2^2+1^2+1^2)}\sqrt{[2^2+1^2+(-5/4)^2]}}$$

or $\cos\theta=\frac{(15/4)}{\sqrt{6}\sqrt{(105/16)}}=\sqrt{\left(\frac{5}{14}\right)}$ or $\theta=\cos^{-1}\sqrt{\left(\frac{5}{14}\right)}$. **Ans.**

Ex. 9. Prove that the equation of the planes through the origin perpendicularr to the lines of section of the plane $lx+my+nz=0$ and the cone $ax^2+by^2+cz^2=0$ is $x^2(bn^2+cm^2)+y^2(cl^2+an^2)+z^2(am^2+bl^2)-2amnyz-2bnlzx-2clmxy=0$.

Sol. Let $x/\lambda=y/\mu=z/\nu$ be one of the lines in which the given plane and the given cone intersect. Then we have

$$l\lambda+m\mu+n\nu=0 \quad \text{and} \quad a\lambda^2+b\mu^2+c\nu^2=0$$

Eliminating ν between these, we get

$$a\lambda^2+b\mu^2+c[(-\lambda l+\mu)/n]^2=0$$

or
$$(\lambda^2/\mu^2)(an^2+cl^2)+2lmc(\lambda/\mu)+(bn^2+cm^2)=0$$

$$\therefore \quad \frac{\lambda_1}{\mu_1}\cdot\frac{\lambda_2}{\mu_2}=\text{product of roots}=\frac{bn^2+cm^2}{an^2+cl^2}.$$

or
$$\frac{\lambda_1\lambda_2}{bn^2+cm^2}=\frac{\mu_1\mu_2}{cl^2+an^2}=\frac{\nu_1\nu_2}{am^2+bl^2} \text{ (by symmetry)} = K \text{ (say)} \quad \text{...(i)}$$

Also $\dfrac{\lambda_1}{\mu_1}+\dfrac{\lambda_2}{\mu_2}=$ sum of the roots $=\dfrac{-2lmc}{an^2+cl^2}$

or $\dfrac{\lambda_1\mu_2+\lambda_2\mu_1}{\mu_1\mu_2}=\dfrac{-2lmc}{an^2+cl^2}$ or $\dfrac{\lambda_1\mu_2+\lambda_2\mu_1}{-2lmc}=\dfrac{\mu_1\mu_2}{an^2+cl^2}$

or $\dfrac{\lambda_1\mu_2+\lambda_2\mu_1}{-2lmc}=\dfrac{\mu_1\mu_2}{an^2+cl^2}=k$, from (i)

$$=\frac{\mu_1\nu_2+\mu_2\nu_1}{-2mna}=\frac{\nu_1\lambda_2+\nu_2\lambda_1}{-2nlb} \quad \text{...(ii)}$$

Now these lines are $x/\lambda_1=y/\mu_1=z/\nu_1$; $x/\lambda_2=y/\mu_2=z/\nu_2$

The planes through origin at right angles to these lines are

$$(\lambda_1x+\mu_1y+\nu_1z)(\lambda_2x+\mu_2y+\nu_2z)=0$$

or $\lambda_1\lambda_2x^2+\mu_1\mu_2y^2+\nu_1\nu_2z^2+(\lambda_1\mu_2+\lambda_2\mu_1)xy+(\mu_1\nu_2+\mu_2\nu_1)yz$
$+(\lambda_1\nu_2+\lambda_2\nu_1)xz=0$

or $(bn^2+cm^2)x^2+\ldots+\ldots+(-2mlc)xy+\ldots+\ldots=0$, from (i) and (ii).

Hence proved.

pr **Ex. 10. Prove that the locus of the line of intersection of tangent planes to the cone $ax^2+by^2+cz^2=0$ which touch along perpendicular generators is the cone**

$$a^2(b+c)x^2+b^2(c+a)y^2+c^2(a+b)z^2=0 \quad (\textit{Kanpur } 95)$$

Sol. We know that the tangent plane to a cone at any point touches it along the generator through that point (see § 8.08 cor. P. 39 Ch. VIII). Let $x/l=y/m=z/n$ be the line of intersection of two tangent planes which touch the cone along perpendicular generators.

$\therefore$ The equations of the plane containing these two perpendicular generators is $\quad ax.l+by.m+cz.n=0 \quad$ ...(i) **(Note)**

Also the equation of the cone is $\quad ax^2+by^2+cz^2=0. \quad$ ...(ii)

Now as in Ex. 3 (a) Page 24 Ch. VIII we can prove that the plane

$$ux+vy+wz=0$$

cuts the cone $ax^2+by^2+cz^2=0$ in pepndicular generators if

$$(b+c)u^2+(c+a)v^2+(a+b)w^2=0.$$

$\therefore$ The plane (i) will cut the cone (ii) in perpendicular generators if

$$(b+c)(al)^2+(c+a)(bm)^2+(a+b)(cn)^2=0 \quad \textbf{(Note)}$$

$\therefore$ The locus of the line $x/l=y/m=z/n$ is

$$(b+c)a^2x^2+(c+a)b^2y^2+(a+b)c^2z^2=0. \quad \text{Hence proved.}$$

pp **Ex. 11. Prove that the equation of the cone through the coordinate axes and the lines through which the plane $ux+vy+wz=0$ cuts the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ is**

$$\mathbf{u\,(bw^2 + cv^2 - 2fwv)\,yz + v\,(cu^2 + aw^2 - 2guw)\,zx}$$
$$\mathbf{+ w\,(av^2 + bu^2 - 2hvu)\,xy = 0} \quad (\textit{Kanpur } 97)$$

Sol. We know that the equation of the cone which passes through the coordinate axes is $Fyz + Gzx + Hxy = 0$ ...(i)

...See § 8.03 Page 2 Ch VIII, where f, g, h have been replaced by other three quantities F, G, H respectively.

Again if the plane $ux + vy + wz = 0$ cuts the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ in two lines whose d.c.'s are l_1, m_1, n_1 and l_2, m_2, n_2 then as in § 8.06 Page 21 Ch. VIII, we can prove that

$$\frac{l_1l_2}{bw^2 + cv^2 - 2fvw} = \frac{m_1m_2}{cu^2 + aw^2 - 2guw} = \frac{n_1n_2}{av^2 + bu^2 - 2huv} \quad ...(ii)$$

and $l_1m_2 - l_2m_1 = 2w\lambda P$ etc. which give

$$\frac{l_1m_2 - l_2m_1}{w} = \frac{m_1n_2 - m_2n_1}{u} = \frac{n_1l_2 - n_2l_1}{v} \quad ...(iii)$$

Also if these two lines with d.c.'s l_1, m_1, n_1, and l_2, m_2, n_2 lie on the cone (i), then their d.c.'s must satisfy the equation (i)

...See § 8.02 Cor. Page 2 Ch VIII

$\therefore$ $$F\,m_1n_1 + G\,n_1l_1 + H\,l_1m_1 = 0 \quad ...(iv)$$

and $$F\,m_2n_2 + G\,n_2l_2 + H\,l_2m_2 = 0 \quad ...(v)$$

Solving (iv) and (v) we get

$$\frac{F}{l_1l_2\,(m_1n_2 - m_2n_2)} = \frac{G}{m_1m_2\,(n_1l_2 - n_2l_1)} = \frac{H}{n_1n_2\,(l_1m_2 - l_2m_1)}$$

or $$\frac{F}{(bw^2 + cv^2 - 2fvw)\,u} = \frac{G}{(cu^2 + aw^2 - 2gwu)\,v} = \frac{H}{(av^2 + bu^2 - 2huv)\,w}$$

(Note)

Substituting these proportionate values of F, G and H in (i), the required equation is $u\,(bw^2 + cv^2 - 2fvw)\,yz + v\,(cu^2 + aw^2 - 2gwu)\,zx + w\,(av^2 + bu^2 - 2huv)\,xy = 0$ Hence proved.

Exercises on § 8.06

Ex. 1. Find the angle between the generators in which the plane $x - 3y + z = 0$ cuts the cone $x^2 - 5y^2 + z^2 = 0$. **Ans.** $\cos^{-1}(5/6)$

Ex. 2. Find the equations of the line along which the plane $x + y + z = 0$ cuts the cone $x^2 + 2y^2 + z^2 = 0$. Also find the acute angle betwen them.

Ex. 3. Find the angle between the two lines in which the plane $6x - 10y - 7z = 0$ cuts the cone $108x^2 - 20y^2 + 7z^2 = 0$. Ans. $\cos^{-1}(16/21)$

§ 8.07. Condition for the cone to have three mutually perpendicular generators. (*Allahabad* 92; *Kumaun* 95)

Let the cone be

$$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0. \quad \text{...(i)}$$

Let $x/l = y/m = z/n$ be one of its generators, then its direction cosines will satisfy (i) and therefore we have

$$f(l, m, n) \equiv al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0. \quad \text{...(ii)}$$

Now the plane through the vertex (0, 0, 0) and perpendicular to this generator $x/l = y/m = z/n$ is $lx + my + nz = 0$. ...(iii)

The plane (iii) cuts the cone (i) in two lines (or generators) which are at right angle to each other if

$$(a + b + c)(l^2 + m^2 + n^2) - f(l, m, n) = 0$$

...See Cor. 1 § 8.06 Page 23 Ch. VIII

or $$(a + b + c)(l^2 + m^2 + n^2) = 0, \text{ from (ii)}$$

or $$a + b + c = 0, \quad \because \; l^2 + m^2 + n^2 \neq 0.$$

This condition $a + b + c = 0$ is independent of l, m, n and therefore we conclude that the plane through vertex and perpendicular to any generator cuts the cone in two generators which are mutually perpendicular and each of them perpendicular to the first generator. Hence the cone has a set of three mutually perpendicular generators and as the condition $a + b + c = 0$ is independent of l, m, n so there can be **an infinite number of set of three mutually perpendicular generators** of the cone. (*Kanpur* 97)

Solved Examples on § 8.07.

Ex. 1. Show that the cone whose vertex is at the origin and which passes through the curve of intersection of the sphere $x^2 + y^2 + z^2 = 3a^2$ and any plane distance 'a' from the origin has three mutually perpendicular generators.

Sol. Any plane at a distance a from the origin is given by

$$lx + my + nz = a, \quad \text{...(i)}$$

where l, m, n are the actual d.c.' of its normal.

Also the given sphere is $x^2 + y^2 + z^2 = 3a^2$ ...(ii)

$\therefore$ The equation of the cone with vertex at (0, 0, 0) and the curve of intersection of (i) and (ii) as the guiding curve is

$$x^2 + y^2 + z^2 = 3(lx + my + nz)^2, \quad \text{...(iii)}$$

making (ii) homogeneous with the help of (i).

If the cone given by (iii) has three mutually perpendicular generators, then $a + b + c = 0$" ...See § 8.07 above.

i.e. $(3l^2 - 1) + (3m^2 - 1) + (3n^2 - 1) = 0$ or $3(l^2 + m^2 + n^2) - 3 = 0$,

which is true as $l^2 + m^2 + n^2 = 1$. Hence proved.

***Ex. 2. Prove that the angle between the lines in which the plane $x+y+z=0$ cuts the cone $ayz+bzx+cxy=0$ will be $\pi/2$ if $a+b+c=0$.**

Sol. From the equation of the cone we observe that the sum of the coefficients of x^2, y^2 and z^2 is zero. *i.e.*, the given curve has three mutually perpendicular generators. If the two lines in which the plane $x+y+z=0$ cuts the given cone are at right angles, then the third generator for the cone will be the normal to the given plane through the vertex *i.e.* $x/1=y/1=z/1$ and as such its direction cosines will satisfy the equation of the cone and so we have

$$a.1.1+b.1.1+c.1.1=0 \quad \text{or} \quad a+b+c=0. \qquad \text{Hence proved.}$$

****Ex. 3 (a). If $x/1=y/2=z/3$ represent one of a set of three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$, find the equations of the other two.** (*Agra* 91; *Bundelkhand* 92)

Sol. The equation of the given cone as in last example suggests that it has three mutually perpendicular generators, one of them is normal to a plane which cuts the cone in two mutually perpendicular generators.

Thus if $x/1=y/2=z/3$ is one the three mutually perpendicular generators, then it is normal to the plane through the vertex cutting the cone in two perpendicular generators and therefore the equation of the plane is

$$x+2y+3z=0 \qquad \text{...(i)}$$

Now we are to find the lines of intersection of this plane and the given cone and let one of the lines of intersection be

$$x/l=y/m=z/n.$$

Then we have $\quad l+2m+3n=0 \quad$ and $\quad 5mn-8nl-3lm=0 \qquad$ ...(ii)

Eliminating l between these we get

$$5mn-(8n+3m)[-(2m+3n)]=0 \quad \text{or} \quad 24n^2+30mn+6m^2=0$$

or $$m^2+5mn+4n^2=0 \quad \text{or} \quad (m+n)(m+4n)=0$$

When $m=-n$, from (ii) we get $l+n=0$ or $l=-n$

$$\therefore \qquad l/1=m/1=n/(-1) \qquad \text{...(iii)}$$

When $m=-4n$, from (ii) we get $l-5n=0$ or $l=5n$

$$\therefore \qquad l/5=m/(-4)=n/1 \qquad \text{...(iv)}$$

Hence from (iii) and (iv) the other two generators are

$$x/1=y/1=z/(-1) \quad \text{and} \quad x/5=y/(-4)=z/1$$

and evidently these are perpendicular as $1.5+1.(-4)+(-1)(1)=0$ and also each one of them is perpendicular to the given generator.

Ex. 3 (b). If $x/1=y/1=z/2$ be one of a set of three mutually perpendicular generators of the cone $3yz-2zx-2xy=0$. Find the equations of other two generators.

Sol. If $x/1=y/1=z/2$ be one of the three mutually perpendicular generators, then it is normal to the plane through the vertex cutting the cone in two perpendicular generators and therefore the equation of the plane is

$$x + y + 2z = 0.$$

Now we are to find the lines of intersection of this plane and the given cone and let one of the lines of intersection be $x/l = y/m = z/n$.

Then we have $\quad l + m + 2n = 0 \quad$ and $\quad 3mn - 2nl - 2lm = 0$. ...(ii)

Eliminating l between these we get

$$3mn - 2(n+m)[-(m+2n)] = 0 \quad \text{or} \quad 4n^2 + 9mn + 2m^2 = 0$$

or $$(2m+n)(m+4n) = 0 \quad \text{or} \quad m = -4n, \ -\frac{1}{2}n$$

When $m = -4n$, from (ii) we get $l - 2n = 0$ or $l = 2n$

$\therefore 2l = -m = 4n$ or $l/2 = m/(-4) = n/1$ ...(iii)

When $m = -\frac{1}{2}n$, from (ii) we get $l + \frac{3}{2}n = 0$ or $2l = -3n$

$\therefore 2l = 6m = -3n$ or $l/3 = m/1 = n/-(2)$ ...(iv)

Hence from (iii) and (iv) the other two generators are

$$x/2 = y/(-4) = z/1 \quad \text{and} \quad x/3 = y/1 = z/(-2). \qquad \textbf{Ans.}$$

And evidently these are perpendicular as $2.3 + (-4)\,1 + 1\,(-2) = 0$ and also each one of them is perpendicular to the given generator.

Ex. 4. Find the condition that the paine $lx + my + nz = 0$ cuts the cone

$$(\mathbf{b - c})\,\mathbf{x^2} + (\mathbf{c - a})\,\mathbf{y^2} + (\mathbf{a - b})\,\mathbf{z^2} + \mathbf{2fyz + 2gzx + 2hxy = 0} \qquad ...(i)$$

in perpendicular lines.

Sol. The sum of the coefficients of x^2, y^2 and z^2 in the equation of the cone $= (b-c) + (c-a) + (a-b) = 0$, therefore it has an infinite set of three mutually perpendicular generators. (see § 8.07 Page 34)

Now if the plane $lx + my + nz = 0$ cuts the cone (i) in two perpendicular generators then the third generator is the normal to this plane through the vertex (0, 0, 0) of the cone

i.e. the line $\quad x/l = y/m = z/n.$

And as it is a generator of the cone (i), so its d.c.'s will satisfy the equation of the cone $\therefore$ The required condition is

$$(b-c)\,l^2 + (c-a)\,m^2 + (a-b)\,n^2 + 2fmn + 2gnl + 2hlm = 0 \qquad \textbf{Ans.}$$

***Ex. 5. Show that the locus of points from which three mutually perpendicular lines can be drawn to intersect a given circle $x^2 + y^2 = a^2$, $z = 0$ is a surface of revolution.**

Sol. Let $P(\alpha, \beta, \gamma)$ be the point whose locus is to be found.

Any line through $P(\alpha, \beta, \gamma)$ is $\quad \dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ ...(i)

and this meets the plane $z = 0$ in $\left(\alpha - \dfrac{l\gamma}{n}, \beta - \dfrac{m\gamma}{n}, 0\right)$ which lies on the circle if

$$\left(\alpha - \frac{l\gamma}{n}\right)^2 + \left(\beta - \frac{m\gamma}{n}\right)^2 = a^2 \qquad ...(ii)$$

Generator of Contact

Again the locus of all such tangent lines at P *i.e.* the tangent plane to the cone at P is obtained by eliminating l, m, n between (iv) and (ii). Thus we have

$$(x-\alpha)(a\alpha+h\beta+g\gamma)+(y-\beta)(h\alpha+b\beta+f\gamma)+(z-\gamma)(g\alpha+f\beta+c\gamma)=0$$

or $\quad x(a\alpha+h\beta+g\gamma)+y(h\alpha+b\beta+f\gamma)+z(g\alpha+f\beta+c\gamma)=0,$

since $f(\alpha, \beta, \gamma)=0$

or $\quad a\alpha x+b\beta y+c\gamma z+f(\gamma y+\beta z)+g(\alpha z+\gamma x)+h(\beta x+\alpha y)=0,$

which is the equation of the tangent plane to the cone at $P(\alpha, \beta, \gamma)$.

(*Bundelkhand* 90)

Cor. This plane passes through the vertex $O(0, 0, 0)$ of the cone and thus the generator OP lies on this cone. Also we can show that the equation of the tangent plane at $(k\alpha, k\beta, k\gamma)$ is the same as that of this plane and hence this plane is the tangent plane to the surface at every point of this generator OP and is called the 'generator of contact.'

All tangentt planes of the cone pass through the vertex and it is called a singular point of the surface.

§ 8.09. Condition of Tangency. (*Agra* 9)

To find the condition that the plane $\quad ux+vy+wz=0$ ...(i)

be a tangent plane to the cone

$$f(x, y, z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad \text{...(ii)}$$

Let the given plane be a tangent plane to the cone at the point $P(\alpha, \beta, \gamma)$, say. The equation of the tangent plane to (ii) at $P(\alpha, \beta, \gamma)$ is

$$a\alpha x+b\beta y+c\gamma z+f(y\gamma+z\beta)+g(z\alpha+x\gamma)+h(x\beta+y\alpha)=0$$

or $\quad (a\alpha+h\beta+g\gamma)x+(h\alpha+b\beta+f\gamma)y+(g\alpha+f\beta+c\gamma)z=0$ ...(iii)

If (i) touches (ii) at $P(\alpha, \beta, \gamma)$, then (i) and (iii) represent the same plane and hence comparing the coefficients in (i) and (iii) we get

$$\frac{a\alpha+h\beta+g\gamma}{u}=\frac{h\alpha+b\beta+f\gamma}{v}=\frac{g\alpha+f\beta+c\gamma}{w}=k \text{ (say)}$$

$$\therefore a\alpha+h\beta+g\gamma-uk=0 \qquad \text{...(iv)}$$

$$h\alpha+b\beta+f\gamma-vk=0 \qquad \text{...(v)}$$

$$g\alpha+f\beta+c\gamma-wk=0 \qquad \text{...(vi)}$$

Also $P(\alpha, \beta, \gamma)$ lies on (i), so we have $u\alpha+v\beta+w\gamma=0$ ...(vii)

Eliminating $\alpha, \beta, \gamma, -k$ between (iv), (v), (vi) and (vii) we get

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}=0 \qquad \text{...(viii)}$$

or $\quad Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv=0,$ ...(ix)

on expanding the determinant and where capital letters denote the cofactors of the corresponding small letters in the determinant

$$D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

i.e. $A = bc - f^2, B = ca - g^2, C = ab - h^2, F = gh - af, G = hf - bg$

and $H = fg - ch$. **(See Author's Matrices or Algebra).**

Hence (viii) or (ix) is the condition that the plane (i) touches the cone (ii).

****§ 8.10. Reciprocal Cone.**

Definition. *The locus of the lines through the vertex, at right angles to the tangent planes of the given cone is called* ***the reciprocal cone of the given*** *cone.* (*Kumaun* 94, 92)

Let
$$ux + vy + wz = 0 \quad \text{...(i)}$$
be a tangent plane to the cone
$$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \text{...(ii)}$$
Then (see § 8.09 above) we have
$$Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv = 0 \quad \text{...(iii)}$$
Now the equation of the normal to the plane (i) passing through the vertex (0, 0, 0) of the cone (ii) are
$$\frac{x}{u} = \frac{y}{v} = \frac{z}{w} \quad \text{...(iv)}$$
$\therefore$ The equation of the locus of (iv) is obtained by eliminating u, v and w between (iii) and (iv) and is
$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0, \quad \text{...(v)}$$
which is again a cone with its vertex at the origin.

The cone whose equation is given by (v) is known as the reciprocal cone of the given by (ii).

Working Rule. To write down the equation of the reciprocal cone of the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, we should simply replace the small letters a, b, c etc. by the capital letters A, B, C etc.; where capital letters denote the cofactors of the corresponding small letters in the determinant
$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

To find the reciprocal cone of $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$, *where* A, B, C *etc. are the cofactors of the corresponding small letters in the determinant* $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Let A', B', C' etc. be the cofactors of A, B, C etc. in the determinant
$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

Then $A' = Bc - F^2 = (ca - g^2)(ab - h^2) - (gh - af)^2$ ...See § 8.09 P. 39

$= a^2bc - abg^2 - cah^2 + g^2h^2 - g^2h^2 - a^2f^2 + 2fgah$

$= a(abc + 2fgh - af^2 - bg^2 - ch^2) = aD$

Similarly we can prove (see § 8.09 also) that $B' = bD$, $C' = cD$, $F' = fD$, $G' = gD$ and $H' = hD$.

Now by the working rule given above the equation of the reciprocal cone of $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$ is

$$A'x^2 + B'y^2 + C'z^2 + 2F'yz + 2G'zx + 2H'xy = 0, \quad \text{...(i)}$$

where A', B', C' etc. are the cofactors of A, B, C etc. in the determinant

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

and from above we know $A' = aD$, $B' = bD$ etc.

$\therefore$ From (i), the required reciprocal cone is

$$(aD)x^2 + (bD)y^2 + (cD)z^2 + 2(fD)yz + 2(gD)xz + 2(hD)xy = 0$$

or $$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0, \; D \neq 0 \quad \text{...(ii)}$$

Hence we observe that the cones

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

and $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$ are such that each is the locus of the normals drawn through the vertex (the origin) to the tangent planes to the other and due to this property they are known as reciprocal cones.

****Cor.** The condition for the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ to have three mutually perpendicular tangent planes is the same as that of the reciprocal cone $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$ to have three mutually perpendicular generators and that is given by

$$A + B + C = 0 \quad \text{...See § 8.07 Page 34.}$$

i.e. $$(bc - f^2) + (ca - g^2) + (ab - h^2) = 0 \quad \text{...See § 8.09 Page 39.}$$

i.e. $$bc + ca + ab = f^2 + g^2 + h^2 \quad \text{(Kumaun 90)}$$

Solved Examples on § 8.08 to § 8.10

****Ex. 1 (a). Prove that the cones $ax^2 + by^2 + cz^2 = 0$ and $x^2/a + y^2/b + z^2/c = 0$ are reciprocal to each other.**

(*Agra* 92; *Bundelkhand* 95, 91, 90; *Kanpur* 96, 90)

Sol. Let the cone reciprocal to $ax^2 + by^2 + cz^2 = 0$ ...(i)

be $$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0, \quad \text{...(ii)}$$

where $A =$ "$bc - f^2$" $= bc$, here $f = 0$. Similarly $B = ca$ and $C = ab$.

Also $F =$ "$gh - af$" $= 0$, $G = 0$, $H = 0$.

$\therefore$ From (ii) the equation of the cone reciprocal to (i) is

$bcx^2 + cay + abz^2 = 0$ or $x^2/a + y^2/b + z^2/c = 0$. Hence proved.

Ex. 1 (b). Find the reciprocal cone of the cone

$$(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=0 \qquad \textit{(Bundelkhand 96)}$$

Sol. Let the cone reciprocal to $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=0$ ...(i)

be $Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$ ...(ii)

where $A=\text{"}bc-f^2\text{"}=(1/b^2)(1/c^2)-(0)^2=1/(b^2c^2)$

Similarly $B=1/(c^2a^2)$ and $C=1/(a^2b^2)$

Also $F=\text{'}gh-af\text{'}=0$. Similarly $G=0$, $H=0$

$\therefore$ From (ii), the equation of cone reciprocal to (i) is

$$(1/b^2c^2)x^2+(1/c^2a^2)y^2+(1/a^2b^2)z^2=0$$

or $a^2x^2+b^2y^2+c^2z^2=0$ **Ans.**

Ex. 2. Prove that the perpendicular drawn from the origin to the tangent planes to the cone $ax^2+by^2+cz^2=0$ lies on the cone

$$x^2/a+y^2/b+z^2/c=0.$$

Sol. We know that the locus of the perpendicular (*i.e.* normal) drawn from the origin, (*i.e.* vertex) to the tangent planes to a given cone is called reciprocal cone. Hence we are to find the reciprocal cone to the cone $ax^2+by^2+cz^2=0$, which can be done as in Ex. 1 (a) above.

Ex. 3. Prove that the perpendicular drawn from the origin to tangent planes to the cone $3x^2+4y^2+5z^2+2yz+4zx+6xy=0$ lie on the cone

$$19x^2+11y^2+3z^2+6yz-10zx-26xy=0.$$

Sol. As in Ex. 2 above we are to find the reciprocal cone of the cone

$$3x^2+4y^2+5z^2+2yz+4zx+6xy=0. \qquad ...(i)$$

Here $a=3, b=4, c=5, f=1, g=2, h=3$

$\therefore$ $A=bc-f^2=4.5-1=19, B=ca-g^2=15-4=11;$

$C=ab-h^2=12-9=3; F=gh-af=6-3=3;$

$G=hf-bg=3-8=-5; H=fg-ch=2-15=-13.$

$\therefore$ The equation of the reciprocal cone to (i) is

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$$

or $19x^2+11y^2+3z^2+6yz-10zx-26xy=0$ Hence proved.

Ex. 4 (a). Find the condition that the plane $ux+vy+wz=0$ may touch the cone $ax^2+by^2+cz^2=0$ *(Meerut 91 S)*

Sol. The d.c.'s of the normal to the plane $ux+vy+wz=0$ are u, v, w and if this plane touches the given cone then,

$$Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv=0. \qquad ...(1)$$

...See § 8.09 Page 39 Ch. VIII

Here $A=bc-f^2=bc$, since $f=0$.

Similarly $B=ca, C=ab, F=\text{"}gh-af\text{"}=0; G=0, H=0.$

$\therefore$ From (i) the required condition is

$$bcu^2 + cav^2 + abw^2 = 0 \quad \text{or} \quad u^2/a + v^2/b + w^2/c = 0. \qquad \textbf{Ans.}$$

***Ex. 4 (b). Show that the planes which cut $ax^2 + by^2 + cz^2 = 0$ in perpendicular generators, touch the cone**

$$\Sigma\,[x^2/(b+c)] = 0$$

(*Meerut* 90)

Sol. As in Ex. 3 (a) P. 24 Ch. VIII we can prove that if the plane $ux + vy + wz = 0$ cuts the cone $ax^2 + by^2 + cz^2 = 0$ in perpendicular generators, then

$$(b+c)\,u^2 + (c+a)\,v^2 + (a+b)\,w^2 = 0. \qquad \text{...(i)}$$

(To be proved in the exam.)

Again as in Ex. 4 (a) above we can prove (to be proved in the exam.) that if the plane $ux + vy + wz = 0$ touches the cone

$$\frac{x^2}{(b+c)} + \frac{y^2}{(c+a)} + \frac{z^2}{(a+b)} = 0$$

then we must have $$\frac{u^2}{1/(b+c)} + \frac{v^2}{1/(c+a)} + \frac{w^2}{1/(a+b)} = 0 \qquad \textbf{(Note)}$$

or $$(b+c)\,u^2 + (c+a)\,v^2 + (a+b)\,w^2 = 0,$$

which is the same as (i) Hence proved.

****Ex. 5. Show that the locus of the line of intersection of perpendicular tangent planes to the cone $ax^2 + by^2 + cz^2 = 0$ is the cone**

$$\mathbf{a\,(b+c)\,x^2 + b\,(c+a)\,y^2 + c\,(a+b)\,z^2 = 0.}$$

(*Lucknow* 92; *Purvanchal* 96)

Sol. Let $ux + vy + wz = 0$ be any tangent plane to the cone

$$ax^2 + by^2 + cz^2 = 0.$$

Then the normal to this plane through the vertex of the cone viz. $x/u = y/v = z/w$ will be a generator of the reciprocal cone

$$(x^2/a) + (y^2/b) + (z^2/c) = 0 \quad \text{...See Ex. 1 (a) Page 41}$$

$\therefore$ We have $$(u^2/a) + (v^2/b) + (w^2/c) = 0$$

or $$bcu^2 + cav^2 + abw^2 = 0. \qquad \text{...(i)}$$

This equation being quadratic in u, v, w shows that there will be two tangent planes like $$ux + vy + wz = 0.$$

Let the line of the intersection of these two tangent planes be

$$x/l = y/m = z/n \qquad \text{...(ii)}$$

Since this line lies on the plane $ux + vy + wz = 0$

so we have $$ul + vm + wn = 0. \qquad \text{...(iii)}$$

Now the direction of the normal to the plane viz. u, v, w are given by the relations (i) and (iii). Again if the two planes of the form $ux + vy + wz = 0$ be perpendicular, then we have $$u_1u_2 + v_1v_2 + w_1w_2 = 0. \qquad \text{...(iv)}$$

Now eliminating w between (i) and (iii) we get

$$bcu^2 + cav^2 + ab\,[-(ul + vm)/n]^2 = 0$$

or $$bcn^2u^2 + can^2v^2 + ab\,(ul + vm)^2 = 0$$

or $$(bcn^2 + abl^2)\,u^2 + 2ablm\,uv + (can^2 + abm^2)\,v^2 = 0$$

or $$(bcn^2 + abl^2)\,(u/v)^2 + 2ablm\,(u/v) + (can^2 + abm^2) = 0,$$

which is a quadratic in u/v and if its roots are u_1/v_1 and u_2/v_2 then we get

$$\frac{u_1}{v_1}\cdot\frac{u_2}{v_2} = \frac{abm^2 + can^2}{ban^2 + abl^2}$$

or $$\frac{u_1u_2}{abm^2 + can^2} = \frac{v_1v_2}{bcn^2 + abl^2} = \frac{w_1w_2}{cal^2 + bcm^2}, \text{ by symmetry}$$

$\therefore$ From (iv) we have

$$(abm^2 + can^2) + (bcn^2 + abl^2) + (cal^2 + bcm^2) = 0$$

or $$a\,(b + c)\,l^2 + b\,(c + a)\,m^2 + c\,(a + b)\,n^2 = 0 \qquad \text{...(v)}$$

$\therefore$ The locus of the line of intersection (ii), obtained by eliminating l, m, n between (ii) and (v) is $a\,(b + c)\,x^2 + b\,(c + a)\,y^2 + c\,(a + b)\,z^2 = 0$, which evidently is a cone with vertex at origin, being homogeneous equation of second degree in x, y, z.

Hence proved.

***Ex. 6. A line OP is such that the two planes through OP, each of which cuts the cone $ax^2 + by^2 + cz^2 = 0$ in perpendicular generators are perpendicular, prove that the locus of OP is a cone and find it.**

Sol. Let the equations of the line OP be $\quad x/l = y/m = z/n \qquad$ ...(i)

The equation of any plane through (i) is $\quad ux + vy + wz = 0, \qquad$ ...(ii)

where $$ul + vm + wn = 0. \qquad \text{...(iii)}$$

Also from Ex. 3 (a) Page 24 we know that if the plane (ii) cuts the cone $ax^2 + by^2 + cz^2 = 0$ in perpendicular generators then

$$(b + c)\,u^2 + (c + a)\,v^2 + (a + b)\,w^2 = 0. \qquad \text{...(iv)}$$

Now the direction ratios of the normal to the plane (ii) viz. u, v, w are given by the relations (iii) and (iv). Eliminating w between (iii) and (iv) we have

$$(b + c)\,u^2 + (c + a)\,v^2 + (a + b)\,[-(ul + vm)/n]^2 = 0$$

or $$(b + c)\,n^2u^2 + (c + a)\,n^2v^2 + (a - b)\,(ul + vm)^2 = 0$$

or $$[(b + c)\,n^2 + (a + b)\,l^2]\,u^2 + 2\,(a + b)\,lmuv + [(c + a)\,n^2 + (a + b)\,m^2]\,v^2 = 0$$

or $$[(b + c)\,n^2 + (a + b)\,l^2]\,(u/v)^2 + 2\,(a + b)\,lm\,(u/v) + [(c + a)\,n^2 + (a + b)\,m^2] = 0$$

which is a quadratic in u/v and if its roots are u_1/v_1, u_2/v_2, then we have

$$\frac{u_1}{v_1}\cdot\frac{u_2}{v_2} = \text{product of the roots.}$$

or $$\frac{u_1u_2}{v_1v_2}=\frac{(c+a)\,n^2+(a+b)\,m^2}{(b+c)\,n^2+(a+b)\,l^2}$$

or $$\frac{u_1u_2}{(a+b)\,m^2+(c+a)\,n^2}=\frac{v_1v_2}{(b+c)\,n^2+(a+b)\,l^2}$$

$$=\frac{w_1w_2}{(c+a)\,l^2+(b+c)\,m^2}, \text{ by symmetry } \quad ...(v)$$

Again if the two planes

$u_1x+v_1y+w_1z=0$ and $u_2x+v_2y+w_2z=0$ are perpendicular (given) then

$$u_1u_2+v_1v_2+w_1w_2=0$$

or $$[(a+b)\,m^2+(c+a)\,n^2]+[(b+c)\,n^2+(a+b)\,l^2]$$
$$+[(c+a)\,l^2+(b+c)\,m^2]=0, \text{ from (v)}$$

or $$(2a+b+c)\,l^2+(2b+c+a)\,m^2+(2c+a+b)\,n^2=0. \quad ...(vi)$$

$\therefore$ The locus of OP *i.e.* the line (i) is obtained by eliminating l, m, n between (i) and (vi) and is given by

$$(2a+b+c)\,x^2+(2b+c+a)\,y^2+(2c+a+b)\,z^2=0,$$

which evidently is a cone with vertex at origin, being homogeneous equation of second degree in x, y, z. Ans.

Ex. 7. Show that the general equation to a cone which touches the coordinates planes is $a^2x^2+b^2y^2+c^2z^2-2bcyz-2cazx-2abxy=0$.

(*Garhwal* 91; *Kanpur* 97; *Meerut* 96)

Sol. The cone which touches the co-ordinate planes is reciprocal to the cone which contains the three co-ordinate axes as these axes are normal to the co-ordinate planes. **(Note)**

Now the equation of the cone which contains the three coordinate axes is

$$fyz+gzx+hxy=0. \quad ...(i)$$

...See § 8.03 Page 2 Ch. VIII

For this cone 'a' $=0$; 'b' $=0$; 'c' $=0$; 'f' $=\frac{1}{2}f$; 'g' $=\frac{1}{2}g$ and 'h' $=\frac{1}{2}h$.

$\therefore$ $$A=\text{“}bc-f^2\text{”}=0-\tfrac{1}{4}f^2\,;\ B=-\tfrac{1}{4}g^2\,;\ C=-\tfrac{1}{4}h^2$$

and $$F=gh-af=\tfrac{1}{4}gh\,;\ G=\tfrac{1}{4}hf\,;\ H=\tfrac{1}{4}fg.$$

$\therefore$ The required equation of the cone reciprocal to (i) is

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$$

or $$-\tfrac{1}{4}f^2x^2-\tfrac{1}{4}g^2y^2-\tfrac{1}{4}h^2z^2+2\left(\tfrac{1}{4}gh\right)yz+2\left(\tfrac{1}{4}hf\right)zx+2\left(\tfrac{1}{4}fg\right)xy=0$$

or $$f^2x^2+g^2y^2+h^2z^2-2ghyz-2hfzx-2fgxy=0,$$

which is of the form

$$a^2x^2+b^2y^2+c^2z^2-2bcyz-2cazx-2abxy=0.$$ Hence proved.

Ex. 8. Prove that the equation $\sqrt{(fx)} \pm \sqrt{(gy)} \pm \sqrt{(hz)} = 0$ represents a cone which touches the co-ordinate planes.

(*Garhwal* 96; *Kanpur* 95, 93; *Kumaun* 91)

And that the equation of the reciprocal cone is $fyz + gzx + hxy = 0$

(*Garhwal* 96; *Kanpur* 95)

Sol. The given equation can be written as $\sqrt{(fx)} \pm \sqrt{(gy)} = \mp \sqrt{(hz)}$

or $\quad fx + gy \pm 2\sqrt{(fgxy)} = hz$, squaring both sides

or $\quad (fx + gy - hz)^2 = [\pm 2\sqrt{(fgxy)}]^2 = 4fgxy$

or $$f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx - 2fgxy = 0. \qquad ...(i)$$

The equation is a homogeneous equation of second degree hence it represents a quadratic cone.

The co-ordinate plane $x = 0$ meets (i) where

$$g^2y^2 + h^2z^2 - 2ghyz = 0 \quad \text{or} \quad (gy - hz)^2 = 0$$

which being a perfect square it follows that the plane $x = 0$ touches it. Similarly we can show that $y = 0$, $z = 0$ also touch the cone (i). Hence proved.

Again for the cone (i), we have

$'a' = f^2$, $'b' = g^2$, $'c' = h^2$, $'f' = -gh$, $'g' = -hf$, $'h' = -fg$

$\therefore \quad A = bc - f^2 = g^2h^2 - (-gh)^2 = 0$, Similarly $B = 0 = C$

$$F = gh - af = (-hf)(-fg) - f^2(-gh) = 2f^2gh.$$

Similarly, $\quad G = 2g^2hf, H = 2h^2fg.$

$\therefore$ The required equation of the cone reciprocal to (i) is

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$$

or $$2f^2ghyz + 2g^2hfzx + 2h^2fgxy = 0$$

or $$fyz + gzx + hxy = 0.$$ Hence proved.

Exercises on § 8.08 — § 8.10

Ex. 1. Prove that the tangent planes to the cone $fyz + gzx + hxy = 0$ are perpendicular to the generators of the cone

$$f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2fhzx - 2fgxy = 0.$$

Ex. 2. Find the equations of the tangent planes to $2x^2 - 6y^2 + 3z^2 = 5$, which pass through the line $x + 9y - 3z = 0$, $3x - 3y + 6z = 5$.

Ex. 3. Find the locus of the lines through the vertex of the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ perpendicular to its tangent planes.

[**Hint.** See § 8.10 Page 40.] (*Kumaun* 94)

***§ 8.11. Envelopoing cone.**

Definition. The locus of tangent lines drawn from a given point to a given surface is known as the *enveloping cone* or the *tangent cone* of that surface with the given point as its vertex. (*Garhwal* 90, *Purvanchal* 97)

(a) To find the equation of the enveloping cone of the sphere $x^2 + y^2 + z^2 = a^2$ with vertex at the point (α, β, γ) *(Garhwal 90; Kumaun 92)*

The equations of a line through (α, β, γ) are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)} \qquad ...(i)$$

Any point on this line is $(\alpha + lr, \beta + mr, \gamma + nr)$. ...(ii)

If the line (i) meets the given sphere at a distance r from point (α, β, γ) then the point given by (ii) must lie on the given sphere and so we have

$$(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr)^2 = a^2$$

or $$r^2 (l^2 + m^2 + n^2) + 2r (l\alpha + m\beta + n\gamma) + (\alpha^2 + \beta^2 + \gamma^2 - a^2) = 0. \qquad ...(iii)$$

If the line (i) is a tangent to the given sphere, then the line (i) should meet the sphere in two coincident points, the condition for the same is that the roots of (iii) are equal, *i.e.*, "$B^2 = 4AC$"

i.e. $$4 (l\alpha + m\beta + n\gamma)^2 \; 4 (l^2 + m^2 + n^2) (\alpha^2 + \beta^2 + \gamma^2 - a^2) \qquad ...(iv)$$

$\therefore$ The locus of the line (i) which is tangent to the given sphere is obtained by eliminating l, m, n between (i) and (iv) and is

$$[\alpha (x - \alpha) + \beta (y - \beta) + \gamma (z - \gamma)]^2$$
$$= [(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2] (\alpha^2 + \beta^2 + \gamma^2) \qquad ...(v)$$

If $S \equiv x^2 + y^2 + z^2 - a^2$, $S_1 = \alpha^2 + \beta^2 + \gamma^2 - a^2$ and

$T = \alpha x + \beta y + \gamma z - a^2$, then the equation (v) can be written as

$$(T - S_1)^2 = (S + S_1 - 2T) S_1 \qquad \textbf{(Note)}$$

or $$T^2 - 2TS_1 + S_1^2 = SS_1 + S_1^2 - 2T_1 \quad \text{or} \quad \boxed{SS_1 = T^2}$$

or $$(x^2 + y^2 + z^2 - a^2) (\alpha^2 + \beta^2 + \gamma^2 - a^2) = (\alpha x + \beta y + \gamma z - a^2)^2$$

(b) To find the equation of the envelopiing cone for the conicoid $ax^2 + by^2 + cz^2 = 1$ from the point (α, β, γ). *(Gorakhpur 96)*

The equation of a line through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)} \qquad ...(i)$$

Any point on this line is $(\alpha + lr, \beta + mr, \gamma + nr)$ ...(ii)

If the line (i) meets the given conicoid $ax^2 + by^2 + cz^2 = 1$ at a distance r from point (α, β, γ), then the point given by (ii) must lie on the given conicoid and so we have $\quad a (\alpha + lr)^2 + b (\beta + mr)^2 + c (\gamma + nr)^2 = 1$

or $$r^2 (al^2 + bm^2 + cn^2) + 2r (al\alpha + bm\beta + cn\gamma)$$
$$+ (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0 \quad ...(iii)$$

If the line (i) is a tangent to the given conicoid, then the line (i) should meet the conicoid in two coincident points, the condition for the same is that the roots of (iii) are equal, *i.e.* $$B^2 = 4AC$$

i.e. $$4\,(al\alpha + bm\beta + cn\gamma)^2 = 4\,(al^2 + bm^2 + cn^2)\,(a\alpha^2 + b\beta^2 + c\gamma^2 - 1) \quad \text{...(iv)}$$

$\therefore$ The locus of the line (i) which is tangent to the given conicoid is obtained by eliminating l, m, n between (i) and (iv) and is

$$[a\,(x-\alpha)\,\alpha + b\,(y-\beta)\,\beta + c\,(z-\gamma)\,\gamma]^2$$
$$= [a\,(x-\alpha)^2 + b\,(y-\beta)^2 + c\,(z-\gamma)^2]\,(a\alpha^2 + b\beta^2 + c\gamma^2 - 1) \quad \text{...(v)}$$

If $S \equiv ax^2 + by^2 + cz^2 - 1$; $S_1 = a\alpha^2 + b\beta^2 + c\gamma^2 - 1$ and

$T = a\alpha x + b\beta\gamma + c\,\gamma z - 1$, then the equation (v) can be written as

$$(T - S_1)^2 = (S + S_1 - 2T)\,S_1 \qquad \textbf{(Note)}$$

or $$T^2 - 2TS_1 + S_1^2 = SS_1 + S_1^2 - 2TS_1 \quad \text{or} \quad SS_1 = T^2 \quad \text{...(ii)}$$

or $$(ax^2 + by^2 + cz^2 - 1)\,(a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = (a\alpha x + b\beta y + c\gamma z - 1)^2$$

Exercise. Find the equation of the enveloping cone of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$ from the point (α, β, γ).

Proceed exactly as in § 8.11 above.

Solved Examples on § 8.11.

Ex. 1. Find the envelopoing cone of the sphere $x^2 + y^2 + z^2 + 2x - 2y - 2$ with the vertex at (1, 1, 1)

Sol. Here $S = x^2 + y^2 + z^2 + 2x - 2y - 2$.

$\therefore$ $$S_1 = 1^2 + 1^2 + 1^2 + 2.1 - 2.1 - 2 = 1$$

and $$T = x\,.\,1 + y\,.\,1 + z\,.\,1 + (x+1) - (y+1) - 2 \qquad \textbf{(Note)}$$
$$= 2x + z - 2.$$

$\therefore$ The required equation of the enveloping cone is $SS_1 = T^2$

...See § 8.11 Page 46 Ch. VIII

or $$(x^2 + y^2 + z^2 + 2x - 2y - 2)\,(1) = (2x + z - 2)^2$$

or $$x^2 + y^2 + z^2 + 2x - 2y - 2 = 4x^2 + z^2 + 4 + 4xz - 8x - 4z$$

or $$3x^2 - y^2 + 4zx - 10x + 2y - 4z + 6 = 0 \qquad \textbf{Ans.}$$

***Ex. 1 (b). Find the equation of the envelopoing cone of the sphere given below whose vertex is at the origin**

$$\mathbf{x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0} \qquad (\textit{Kanpur } 97, 94)$$

Sol. Here $S = x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d$

$\therefore$ $$S_1 = 0 + 0 + 0 + 2u\,(0) + 2v\,(0) + 2w\,(0) + d = d$$

and $$T = x\,.\,0 + y\,.\,0 + z\,.\,0 + u\,(x+0) + v\,(y+0) + w\,(z+0) + d$$
$$= ux + vy + wz + d$$

$\therefore$ Required equation of the enveloping cone is

$$SS_1 = T^2$$ ...See § 8.11 Page 46 Ch. VIII

or $$(x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d)\,(d) = (ux + vy + wz + d)^2$$

or $\quad d(x^2+y^2+z^2)+2uxd+2vyd+2wzd+d^2$

$$= u^2x^2+v^2y^2+w^2z^2+d^2+2uvxy+2uwxz$$
$$+2udx+2vwyz+2vyd+2wzd$$

or $\quad (u^2-d)x^2+(v^2-d)y^2+(w^2-d)z^2+2vwyz+2wuzx+2uvxy=0$ **Ans.**

*__Ex. 2. Prove that the plane z = 0 cuts the enveloping cone of the sphere $x^2+y^2+z^2=11$ which has the vertex at (2, 4, 1) in a rectangular hyperbola.__

Sol. Here $S=x^2+y^2+z^2-11,\ S_1=2^2+4^2+1^2-11=10$

and $\quad T=x(2)+y(4)+z(1)-11=2x+4y+z-11.$

$\therefore$ The equation of the enveloping cone of the given sphere with vertex at (2, 4, 1) is $(x^2+y^2+z^2-11)(10)=(2x+4y+z-11)^2$. ...using $SS_1=T^2$

The plane $z=0$ meets this cone in the conic

$$10(x^2+y^2-11)=(2x+4y-11)^2,\ z=0$$

or $\quad 6x^2-6y^2-16xy+44x-88y-231=0,\ z=0.$

Since $\Delta\neq 0$ and sum of the coefficients of x^2 and y^2 in the first condition is zero, so these represent rectangular hyperbola.

(See Author's Co-ordinate Geometry) Hence proved.

***Ex. 3 (a). Prove that the lines drawn from the origin so as to touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ lie on the cone**

$$d(x^2+y^2+z^2)=(ux+vy+wz)^2$$

Sol. Here we are to show that the given cone is the enveloping cone of the given sphere with vertex at (0, 0, 0) **(Note)**

$\therefore$ Here $S=x^2+y^2+z^2+2ux+2vy+2wz+d;\ S_1=d$

and $\quad T=x(0)+y(0)+z(0)+u(x+0)+v(y+0)+w(z+0)+d$

$$=ux+vy+wz+d$$

$\therefore$ The enveloping cone of the given sphere with vertex at (0, 0, 0) is

$$(x^2+y^2+z^2+2ux+2vy+2wz+d)\,d=(ux+vy+wz+d)^2$$

or $\quad (x^2+y^2+z^2)\,d+(2ux+2vy+2wz)\,d+d^2=(ux+vy+wz)^2+d^2$

$$+2d(ux+vy+wz) \quad \textbf{(Note)}$$

or $\quad (x^2+y^2+z^2)\,d=(ux+vy+wz)^2.$ Hence proved.

Ex. 3 (b). Prove that the tangent lines from the origin of coordinates to the sphere $(x-a)^2+(y-b)^2+(z-c)^2=k^2$ lie on the cone given by the equation $(a^2+b^2+c^2-k^2)(x^2+y^2+z^2)=(ax+by+cz)^2$.

Sol. Here we are to find the enveloping cone of the given sphere with vertex at (0, 0, 0).

Here $S\equiv(x-a)^2+(y-b)^2+(z-c)^2-k^2$

or $\quad S\equiv x^2+y^2+z^2-2ax-2by-2cz+(a^2+b^2+c^2-k^2)$.

$$S_1 = (a^2 + b^2 + c^2 - k^2)$$

and $T = x \,.\, 0 + y \,.\, 0 + z \,.\, 0 - a(x+0) - b(y+0) - c(z+0) + (a^2 + b^2 + c^2 - k^2)$

or $$T = -ax - by - cz + (a^2 + b^2 + c^2 - k^2)$$

$\therefore$ The equation of the enveloping cone of the given sphere with vertex at (0, 0, 0) is "$SS_1 = T^2$"

or $[x^2 + y^2 + z^2 - 2ax - 2by - 2cz + (a^2 + b^2 + c^2 - k^2)]\,(a^2 + b^2 + c^2 - k^2)$
$$= [-(ax + by + cz) + (a^2 + b^2 + c^2 - k^2)]^2$$

or $(x^2 + y^2 + z^2)(a^2 + b^2 + c^2 - k^2) - 2(ax + by + cz)(a^2 + b^2 + c^2 - k^2)$
$$+ (a^2 + b^2 + c^2 - k^2)^2 = (ax + by + cz)^2 + (a^2 + b^2 + c^2 - k^2)^2$$
$$- 2(ax + by + cz)(a^2 + b^2 + c^2 - k^2)$$

or $(x^2 + y^2 + z^2)(a^2 + b^2 + c^2 - k^2) = (ax + by + cz)^2$. Hence proved.

****Ex. 4. The sections of the enveloping cone fo the surface $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ whose vertex is $P(x_1, y_1, z_1)$ by the plane $z = 0$ is (i) rectangular hyperbola, (ii) a parabola and (iii) a circle. Find the locus of the vertex P.**

Sol. For the given surface we have

$$S = x^2/a^2 + y^2/b^2 + z^2/c^2 - 1\ ;\ S_1 = x_1^2/a^2 + y_1^2/b^2 + z_1^2/c^2 - 1.$$

and $T = (xx_1/a^2) + (yy_1/b^2) + (zz_1/c^2) - 1$

$\therefore$ The enveloping cone is "$SS_1 = T^2$"

i.e. $$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} - 1\right)^2$$

Its section by the plane $z = 0$ is given by

$$z = 0, \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2 \quad \text{...(A)}$$

(i) If the equations (A) represent a rectangular hyperbola then the sum of the coefficients of x^2 and y^2 should be zero.

i.e. $$\frac{1}{a^2}\left(\frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) + \frac{1}{b^2}\left(\frac{x_1^2}{a^2} + \frac{z_1^2}{c^2} - 1\right) = 0 \quad \textbf{(Note)}$$

or $$\frac{x_1^2 + y_1^2}{a^2 b^2} + \frac{1}{c^2}\left(\frac{1}{a^2} + \frac{1}{b^2}\right) z_1^2 = \frac{1}{a^2} + \frac{1}{b^2}$$

or $$\frac{x_1^2 + y_1^2}{a^2 + b^2} + \frac{z_1^2}{c^2} = 1, \text{ dividing each term by } a^2 + b^2.$$

$\therefore$ The required locus of $P(x_1, y_1, z_1)$ is $\dfrac{x^2+y^2}{a^2+b^2}+\dfrac{z^2}{c^2}=1$.

(ii) If the equations (A) represent a parabola, then we should have

"$h^2 = ab$" **(See Author's Co-ordinate Geometry)**

Here 'a' = coeff. of $x^2 = \dfrac{1}{a^2}\left(\dfrac{y_1^2}{b^2}+\dfrac{z_1^2}{c^2}-1\right)$

'b' = coeff. of $y^2 = \dfrac{1}{b^2}\left(\dfrac{x_1^2}{a^2}+\dfrac{z_1^2}{c^2}-1\right)$

and 'h' = coeff. of $2xy = x_1y_1/a^2b^2$.

$\therefore$ If the equations (A) represent a parabola, then $h^2 = ab$

i.e. $$\frac{x_1^2y_1^2}{a^4b^4}=\frac{1}{a^2b^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)$$

or $$\frac{x_1^2}{a^2}\left(\frac{z_1^2}{c^2}-1\right)+\frac{y_1^2}{b^2}\left(\frac{z_1^2}{c^2}-1\right)+\left(\frac{z_1^2}{c^2}-1\right)^2=0. \qquad \textbf{(Note)}$$

or $$\left(\frac{z_1^2}{c^2}-1\right)\left[\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right]=0$$

or $(z_1^2/c^2)-1=0$, since $x_1^2/a^2+y_1^2/b^2+z_1^2/c^2-1\neq 0$ as $P(x_1, y_1, z_1)$ does not lie on the given surface

or $z_1^2 = c^2$ or $z_1 = \pm c$ **Ans.**

$\therefore$ The locus of $P(x_1, y_1, z_1)$ is $z = \pm c$

(iii) If (A) represents a circle then the coefficients of x^2 and y^2 should be equal and coefficient of xy should be zero.

i.e. $$\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right) \qquad ...(B)$$

and $$x_1y_1/a^2b^2=0 \qquad ...(C)$$

From (C) either $x_1 = 0$ or $y_1 = 0$

If $x_1 = 0$, then from (B) we have $\dfrac{1}{a^2}\left(\dfrac{y_1^2}{b^2}+\dfrac{z_1^2}{c^2}-1\right)=\dfrac{1}{b^2}\left(\dfrac{z_1^2}{c^2}-1\right)$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ is $x = 0$, $\dfrac{y^2}{b^2-a^2}+\dfrac{z^2}{c^2}=1$ **Ans.**

If $y_1 = 0$, then from (B) we have $\dfrac{1}{a^2}\left(\dfrac{z_1^2}{c^2}-1\right)=\dfrac{1}{b^2}\left(\dfrac{x_1^2}{a^2}+\dfrac{z_1^2}{c^2}-1\right)$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ is $y = 0, \dfrac{x^2}{a^2 - b^2} + \dfrac{z^2}{c^2} = 1.$ **Ans.**

Ex. 5. Find the locus of the point from which three mutually perpendicular tangent lines can be drawn to the surface

$$\mathbf{ax^2 + by^2 + 2cz = 0.}$$

Sol. Let $P(x_1, y_1, z_1)$ be the point, then the three mutually perpendicular tangent lines drawn from it to the given surface are the three mutually perpendicular generators of the enveloping cone of the given surface with $P(x_1, y_1, z_1)$ as its vertex.

The equation of enveloping cone is $SS_1 = T^2$.

i.e. $(ax^2 + by^2 + 2cz)(ax_1^2 + by_1^2 + 2cz_1) = [axx_1 + byy_1 + c(z + z_1)]^2$ **(Note)**

If this cone has three mutually perpendicular generators, then sum of the coefficients of x^2, y^2 and z^2 is zero.

i.e. $$a(by_1^2 + 2cz_1) + b(ax_1^2 + 2cz_1) + (-c^2) = 0$$

$\therefore$ On generalising, we have the required locus of $P(x_1, y_1, z_1)$ as

$$ab(x^2 + y^2) + 2c(a + b)z - c^2 = 0$$ **Ans.**

Ex. 6. Show that the three mutually perpendicular tangent lines can be drawn to the sphere $\mathbf{x^2 + y^2 + z^2 = r^2}$ from any point on the sphere

$$\mathbf{x^2 + y^2 + z^2 = (3/2)\, r^2}$$

Sol. The given spheres are $$x^2 + y^2 + z^2 = r^2 \qquad \text{...(i)}$$

and $$x^2 + y^2 + z^2 = (3/2)\, r^2 \qquad \text{...(ii)}$$

Let $P(x_1, y_1, z_1)$ be any point on the sphere (ii), then we have

$$x_1^2 + y_1^2 + z_1^2 = (3/2)\, r^2. \qquad \text{...(iii)}$$

Now the three mutually perpendicular tangent lines drawn from $P(x_1, y_1, z_1)$ to the sphere (i) are the three mutually perpendicular generators of the enveloping cone of the sphere (i) with $P(x_1, y_1, z_1)$ as the vertex.

The equation of this enveloping cone is "$SS_1 = T^2$"

i.e. $(x^2 + y^2 + z^2 - r^2)(x_1^2 + y_1^2 + z_1^2 - r^2) = (xx_1 + yy_1 + zz_1 - r^2)^2$

If this cone has three mutually perpendicular generators then the sum of the coefficients of x^2, y^2 and z^2 must be zero.

i.e. $$(y_1^2 + z_1^2 - r^2) + (x_1^2 + z_1^2 - r^2) + (x_1^2 + y_1^2 - r^2) = 0$$

or $x_1^2 + y_1^2 + z_1^2 - (3/2)\, r^2 = 0$, which is true by virtue of (iii) Hence proved.

Exercises on § 8.11.

Ex. 1. Find the enveloping cone of the sphere $x^2 + y^2 + z^2 - 2x + 4z = 1$ with its vertex at (1, 1, 1). **Ans.** $4x^2 + 3y^2 - 5z^2 - 6zy - 8x + 16z = 4$.

Ex. 2. Find the locus of a luminous point which moves so that the sphere $x^2 + y^2 + z^2 = 2az$ casts a parabolic shadow on the plane $z = 0$.

Ans. $z^2 (x^2 + y^2 + z^2 - 4az + 4a^2) = 2a (x^2 + y^2)$

[**Hint :** Take the luminous point as the vertex of the enveloping cone].

Ex. 3. Find the locus of luminous point of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ which casts a cricular shadow on the plane $z = 0$.

Ex. 4. Find the locus of the points from which three mutually perpendicular tangent lines can be drawn to the surface $ax^2 + by^2 + cz^2 = 0$.

Ans. $a (b + c) x^2 + b (c + a) y^2 + c (a + b) z^2 = a + b + c.$

Ex. 5. Show that the plane $z = a$ meets any enveloping cone of the sphere $x^2 + y^2 + z^2 = a^2$ in a conic which has a focus at the point (0, 0, 0).

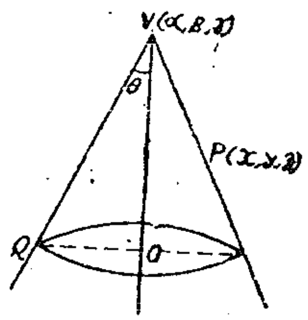
****§ 8.12. Right Circular Cone.**

Definition. *The surface generated by a line passing through a fixed point (called* **vertex***) and making a constant angle with a fixed line through the vertex is known as* **the right circular cone**. *The fixed line is called the* **axis of the cone** *and the constant angle is called the* **semi-vertical angle** *of the cone.*

(*Kanpur* 95; *Kumaun* 96, 93; *Purvanchal* 96)

To prove that the section of a right cone by a plane perpendicular to its axis is a circle.

Let V be the vertex and VO the axis of the right curcular cone. Let θ be its semi-vertical angle. Let any plane at right angles to the axis meet the axis in O. Let Q be any point on this section. Join OQ. Now OQ is a line on the plane section of the cone which is at right angles to the axis VO and so OQ is also at right angles to VO. Also $\angle OVQ = \theta$, so $OQ = VO \tan \theta$. But θ is constant and hence OQ is also of constant length for all positions of Q on the section of the cone through O at right angles to the axis VO. Hence the locus of Q is a circle with O as the centre and radius OQ.



(Fig. 36)

For this fact, the cone is called right circular cone.

Equation of the right circular cone. (*Kanpur* 95; *Kumaun* 96, 93)

Let the coordinates of the vertex V be (α, β, γ) and the equations of the axis VO be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}.$$

Let the semi-vertical angle of the cone be θ. (See fig. 36 above).

Let $P(x, y, z)$ be any point on the cone, then the generator VP makes on angle θ with axis VO. The direction ratios of the line VP are $x - \alpha, y - \beta, z - \gamma$

and those of the axis VO from (i) are l, m, n.

$$\therefore \cos\theta = \frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{(l^2+m^2+n^2)}\sqrt{\{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\}}}$$

Hence the required equation of the right circular cone is

$$[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2$$
$$=\cos^2\theta\,(l^2+m^2+n^2)\,[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2] \quad \text{...(ii)}$$

Cor. If the vertex is at the origin, then the above equation (ii) of the right circular cone reduces to

$$(lx+my+nz)^2=\cos^2\theta\,(l^2+m^2+n^2)(x^2+y^2+z^2)$$
$$=(1-\sin^2\theta)(l^2+m^2+n^2)(x^2+y^2+z^2)$$

or
$$(l^2+m^2+n^2)(x^2+y^2+z^2)\sin^2\theta$$
$$=(l^2+m^2+n^2)(x^2+y^2+z^2)-(lx+my+nz)^2$$
$$=(mz-ny)^2+(nx-lz)^2+(ly-mx)^2, \quad \text{...(iii)}$$

using Lagrange's Identity

Further if z-axis be the axis of the cone, then we have $l=0=m$ and then the equation (iii) reduces to $n^2(x^2+y^2+z^2)\sin^2\theta=n^2y^2+n^2x^2$

or $z^2\sin^2\theta=(x^2+y^2)(1-\sin^2\theta)$ or $\boxed{x^2+y^2=z^2\tan^2\theta}$ ...(iv)

Solved Examples on § 8.12.

Ex. 1 (a). Find the equation of the right circular cone whose axis is x = y = z, vertex is origin and whose semi-vertical angle is 45°.

Sol. The vertex of the cone is $V(0, 0, 0)$, its semi-vertical angle is 45° and its axis is given by $x=y=z$. ...(i)

Let $P(x, y, z)$ be any point on the cone. Then the direction ratios of the generator VP are $x-0, y-0, z-0$ *i.e.* x, y, z.

Also VP is making an angle of 45° with the line (i), so we get the equation of the cone as

$$\cos 45° = \frac{x.1+y.1+z.1}{\sqrt{(x^2+y^2+z^2)}\,.\,\sqrt{(1^2+1^2+1^2)}}$$

or
$$\frac{1}{\sqrt{2}}=\frac{x+y+z}{\sqrt{(x^2+y^2+z^2)}\sqrt{3}} \quad \text{or} \quad 3(x^2+y^2+z^2)=2(x+y+z)^2$$

or
$$x^2+y^2+z^2-4yz-4zx-4xy=0. \quad \textbf{Ans.}$$

Ex. 1 (b). Find the equation of the right circular cone whose vertex is the origin and whose axis is the line x = t, y = 2t, z = 3t and which has a vertical angle of 60°.

Or

Find the equation of the right circular cone, whose vertex is at the origin, whose axis is the line x/1 = y/2 = z/3 and whose semi-vertical angle is 60°.

Sol. The vertex of the cone is $V(0, 0, 0)$, its semi-vertical angle is 30° and its axis is given by
$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}=t \quad \text{...(i)}$$

Let $P(x, y, z)$ be any point of the cone. Then the direction ratios of the generator VP are $x-0, y-0, z-0$ *i.e.* x, y, z.

Also as VP makes an angle of 30° with the line (i), so the required equation of the cone is

$$\cos 30^\circ = \frac{x.1 + y.2 + z.3}{\sqrt{(x^2+y^2+z^2)} \cdot \sqrt{(1^2+2^2+3^2)}}$$

or $$\frac{\sqrt{3}}{2} = \frac{x+2y+3z}{\sqrt{(x^2+y^2+z^2)}\sqrt{(14)}}$$

or $$42(x^2+y^2+z^2) = 4(x+2y+3z)^2$$

or $$38x^2 + 26y^2 + 6z^2 - 16xy - 48yz - 24zx = 0$$ **Ans.**

Ex. 1 (c). Find the equation of the right circular cone whose vertex is (0, 0, 0), axis Ox (i.e. x-axis) and semi-vertical angle is α. (*Agra* 92, 90)

Sol. The vertex of the cone is $O(0, 0, 0)$, its semi-vertical angle is α and direction cosines of its axis are 1, 0, 0.

Let $P(x, y, z)$ be any point on the cone. Then the direction ratios of the generator OP are $x-0, y-0, z-0$ *i.e.* x, y, z.

Also OP is making an angle α with the axis of the cone, so we get the equation of the cone as

$$\cos\alpha = \frac{x.1 + y.0 + z.0}{\sqrt{(x^2+y^2+z^2)}\sqrt{(1+0+0)}} = \frac{x}{\sqrt{(x^2+y^2+z^2)}}$$

or $$(x^2+y^2+z^2)\cos^2\alpha = x^2 \quad \text{or} \quad x^2 = (y^2+z^2)\cot^2\alpha.$$ **Ans.**

Ex. 1 (d). Prove that $x^2 - y^2 + z^2 - 4x + 2y + 6z + 12 = 0$ represents a right circular cone whose vertex is the point (2, 1, −3), whose axis is parallel to Oy and whose semi-vertical angle is 45°.

Sol. The vertex of the cone is $V(2, 1, -3)$, its semi-vertical angle is 45° and its axis is parallel to Oy *i.e.* the direction cosines of its axis are 0, 1, 0.

∴ The equation of the axis of the cone is $\frac{x-2}{0} = \frac{y-1}{1} = \frac{z+3}{0}$ ...(i)

Let $P(x, y, z)$ be any point on the cone. Then the direction ratios of the generator VP are $x-2, y-1, z+3$.

Also VP is making on angle of 45° with the axis whose d.c.'s are 0, 1, 0 so the equation of the cone [from § 8.12 Page 53 Ch. VIII] is

$$\cos 45^\circ = \frac{(x-2).0 + (y-1).1 + (z+3).0}{\sqrt{(0^2+1^2+0^2)}\sqrt{[(x-2)^2+(y-1)^2+(z+3)^2]}}$$

or $$\frac{1}{\sqrt{2}} = \frac{(y-1)}{\sqrt{[(x-2)^2+(y-1)^2+(z+3)^2]}}$$

or $$(x-2)^2 + (y-1)^2 + (z+3)^2 = 2(y-1)^2,$$ squaring and cross multiplying

or $$(x-2)^2 - (y-1)^2 + (z+3)^2 = 0$$

or $$x^2 - y^2 + z^2 - 4x + 2y + 6z + 12 = 0.$$ Hence proved.

Ex. 1 (e). Find the equation to the right circular cone whose vertex is the origin, axis the axis of z and semi-vertical angle 30°. (*Gorakhpur* 92)

Sol. The vertex of the cone is $V(0, 0, 0)$, semi-vertical angle is 30° and the axis is z-axis, whose d. cosines are 0, 0, 1.

∴ The equation of the axis of the cone is

$$\frac{x-0}{0} = \frac{y-0}{0} = \frac{z-0}{1} \quad i.e. \quad \frac{x}{0} = \frac{y}{0} = \frac{z}{1} \qquad \text{...(i)}$$

Let $P(x, y, z)$ be any point on the cone. Then the direction ratios of the generator VP are $x-0, y-0, z-0$ *i.e.* x, y, z. Also VP makes an angle of 30° with the axis of the cone whose d.c.'s are 0, 0, 1, so the equation of the cone [from § 8.12 Page 53 Ch. VIII] is

$$\cos 30° = \frac{x.0 + y.0 + z.1}{\sqrt{(x^2+y^2+z^2)}.\sqrt{(0^2+0^2+1^2)}} = \frac{z}{\sqrt{(x^2+y^2+z^2)}}$$

or $$\frac{\sqrt{3}}{2} = \frac{z}{\sqrt{(x^2+y^2+z^2)}} \quad \text{or} \quad 3(x^2+y^2+z^2) = (2z)^2$$

or $$3x^2 + 3y^2 - z^2 = 0$$ **Ans.**

Ex. 1 (f). Obtain the equation of the right circular cone with vertex at (1, – 2, – 1)., semi vertical angle 60° and the axis

$$\frac{1}{3}(x-1) = -\frac{1}{4}(y+2) = \frac{1}{5}(z+1). \qquad (\textit{Rohilkhand } 96)$$

Sol. Do as Ex. 1 (d) above.

Ans. $7x^2 - 7y^2 - 25z^2 + 48xy + 80yz - 60zx + 22x + 4y + 17z + 78 = 0$

***Ex. 2. Find the equation to the right ciruclar cone whose vertex is (2, – 3, 5), axis makes equal angles with the coordinate axes and semi-vertical angle is 30°.**

Sol. Let $V(2, -3, 5)$ be the vertex of the cone. Also the direction cosines of the axis of the cone are $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ as it makes equal angles with the coordinate axes.

Let $P(x, y, z)$ be any point on the cone. Then the direction ratios of the generator VP are $x-2, y+3, z-5$.

Also VP is making an angle of 30° with axis of the cone, so we get the equation of the cone as

$$\cos 30° = \frac{(x-2).(1/\sqrt{3}) + (y+3)(1/\sqrt{3}) + (z-5)(1/\sqrt{3})}{\sqrt{[(x-2)^2+(y+3)^2+(z-5)^2]}\sqrt{[(1/\sqrt{3})^2+(1/\sqrt{3})^2+(1/\sqrt{3})^2]}}$$

or $$\frac{\sqrt{3}}{2} = \frac{(x-2)+(y+3)+(z-5)}{\sqrt{3}.\sqrt{[(x-2)^2+(y+3)^2+(z-5)^2]}}$$

or $$9[(x-2)^2+(y+3)^2+(z-5)^2] = 4[(x-2)+(y+3)+(z-5)]^2$$

or $$5(x^2+y^2+z^2) - 8(xy+yz+zx) - 4x + 86y - 58z + 278 = 0.$$ **Ans.**

Ex. 3 (a). Find the equation of the cone generated by rotating the line x/l = y/m = z/n about the line x/a = y/b = z/c as axis.

Sol. The axis of the cone is $x/a = y/b = z/c$ ...(i)

and that of any generator is $x/l = y/m = z/n$. ...(ii)

If θ be the semi-vertical angle of the cone, we have

$$\cos\theta = \frac{al + bm + cn}{\sqrt{(\Sigma a^2)}\sqrt{(\Sigma l^2)}} \quad \text{...(iii)}$$

Since both (i) and (ii) intersect at (0, 0, 0), so the vertex of the cone is $O(0, 0, 0)$. Let $P(x, y, z)$ be any point on the cone, then the direction ratios of the generator OP are $x - 0, y - 0, z - 0$ *i.e.* x, y, z.

Also OP is inclined to (i) at an angle θ, therefore we get

$$\cos\theta = \frac{ax + by + cz}{\sqrt{(\Sigma a^2)}\sqrt{(\Sigma x^2)}} \quad \text{...(iv)}$$

Equating the values of $\cos\theta$ from (iii) and (iv) we get the required equation of the cone as $\dfrac{al + bm + cn}{\sqrt{(\Sigma l^2)}} = \dfrac{ax + by + cz}{\sqrt{(\Sigma x^2)}}$

or $\quad (al + bm + cn)^2 (x^2 + y^2 + z^2) = (ax + by + cz)^2 (l^2 + m^2 + n^2)$,

squaring and cross multiplying. **Ans.**

Ex. 3 (b). Find the equation of the cone formed by rotating the line 2x + 3y = 6, z = 0 about the y-axis. (*Bundelkhand* 96, 90)

Sol. The axis of the cone is y-axis, whose d.c.'s are 0, 1, 0 and the equation of any generator is given as $2x + 3y = 6,\ z = 0$

i.e.
$$\frac{x}{3} = \frac{y-2}{-2} = \frac{z}{0} \quad \text{...(i)}$$

The point of intersection of y-axis (*i.e.* $x = 0,\ z = 0$) and the line (i) is (0, 2, 0), so the co-ordinates of the vertex of the cone is $V(0, 2, 0)$.

Let θ be the semi-vertical angle of the cone, then θ is the angle between y-axis and the line (i), so we have

$$\cos\theta = \frac{0.3 + 1.(-2) + 0.0}{\sqrt{(0 + 1 + 0)}\sqrt{\{3^2 + (-2)^2 + (0)^2\}}} = \frac{-2}{\sqrt{(13)}} \quad \text{...(ii)}$$

Also let $P(x, y, z)$ be any point on the cone, then VP is a generator of the cone and its direction ratios are $x - 0, y - 2, z - 0$ or $x, y - 2, z$.

Also θ is the angle between y-axis and VP, so we get

$$\cos\theta = \frac{0.x + 1.(y-2) + 0.z}{\sqrt{(0 + 1 + 0)}\sqrt{\{x^2 + (y-2)^2 + z^2\}}} \quad \text{...(iii)}$$

Equating the value of $\cos\theta$ from (ii) and (iii) we get the equation of the cone as
$$\frac{-2}{\sqrt{(13)}} = \frac{y-2}{\sqrt{\{x^2 + (y-2)^2 + z^2\}}}$$

or $\quad 4\{x^2 + (y-2)^2 + z^2\} = 13(y-2)^2$, squaring both sides.

or $\quad 4x^2 - 9(y-2)^2 + 4z^2 = 0.$ **Ans.**

***Ex. 4 (a). Find the equation to the right circular cone whose vertex is O, axis OZ and semi-vertical angle α.** (*Kumaun* 91)

Sol. Let $P(x, y, z)$ be any point on the cone. From P draw PN perpendicular to z-axis.

Then $PN = OM = \sqrt{(x^2 + y^2)}$

and $ON = z$. **(Note)**

From the adjoining figure it is evident that

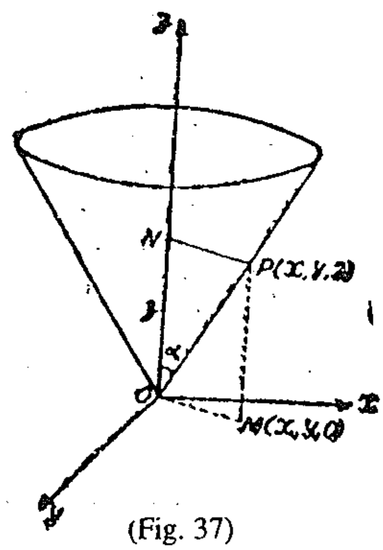
$$\tan \alpha = PN/ON$$

or $$\tan \alpha = \sqrt{(x^2 + y^2)}/z$$

or $$z \tan \alpha = \sqrt{(x^2 + y^2)}$$

or $$x^2 + y^2 = z^2 \tan^2 \alpha.$$

Hence proved.



(Fig. 37)

****Ex. 4 (b). If a right circular cone has three mutually perpendicular generators, show that the semi-vertical angle is $\tan^{-1} \sqrt{2}$.**

Sol. Let vertex of the cone be origin and its axis be z-axis then if θ be the semi-vertical angle of the cone, its equation can be proved as in Ex. 4 (a) above to be $$x^2 + y^2 - z^2 \tan^2 \theta = 0.$$

If it has three mutually perpendicular generators, then

'$a + b + c = 0$' (See § 8.07 Page 34)

i.e. $1 + 1 - \tan^2 \theta = 0$ or $\tan^2 \theta = 2$ or $\theta = \tan^{-1} \sqrt{2}$. Hence proved.

***Ex. 5. Show that the semi-vertical angle of the right circular cone which has three mutually perpendicular tangent planes is**

$$\tan^{-1} (1/\sqrt{2}) \text{ or } \cot^{-1} \sqrt{2}.$$

Sol. Let the right circular cone as in Ex. 4 (a) above be taken as

$$x^2 + y^2 - z^2 \tan^2 \theta = 0. \qquad \text{...(i)}$$

It has three mutually perpendicular tangent planes then its reciprocal cone

$$\frac{x^2}{1} + \frac{y^2}{1} - \frac{z^2}{\tan^2 \theta} = 0$$ ...See Ex. 1 (a) Page 41

must have three mutually perpendicular generators and the condition for the same is $$1 + 1 - (1/\tan^2 \theta) = 0. \qquad \text{...(ii)}$$

i.e. $\cot^2 \theta = 2$ or $\theta = \cos^{-1} (\sqrt{2})$. Hence proved.

Also from (ii) we can get $\tan^2 \theta = 1/2$ or $\theta = \tan^{-1} (1/\sqrt{2})$.

***Ex. 6 (a). Find the equation of the right circular cone which passes through the point (1, 1, 2) and has its vertex at the origin, axis the line $x/2 = y/(-4) = z/3$.** (*Gorakhpur* 91; *Rohilkhand* 95)

Sol. The direction ratios of the generator passing through (1, 1, 2) and the vertex $V(0, 0, 0)$ are $1-0, 1-0, 2-0$ *i.e.* 1, 1, 2

Also the direction ratios of the axis of the cone are $2, -4, 3$ (given).

$\therefore$ If θ be the semi-vertical angle of the cone, then we have

$$\cos\theta = \frac{1.2 + 1.(-4) + 2.3}{\sqrt{(1^2+1^2+2^2)}\sqrt{[2^2+(-4)^2+3^2]}} = \frac{2}{\sqrt{(29)}} \quad ...(i)$$

Also if $P(x, y, z)$ be any point on the cone, then the direction ratios of generator VP are $x-0, y-0, z-0$. *i.e.* x, y, z. This generator is also inclined to the axis of the cone at an angle θ, so

$$\cos\theta = \frac{x.2 + y.(-4) + z.3}{\sqrt{(x^2+y^2+z^2)}\sqrt{[2^2+(-4)^2+3^2]}} \quad ...(ii)$$

Equating (i) and (ii) we have the equation of the cone as

$$\frac{2x-4y+3z}{\sqrt{(x^2+y^2+z^2)}\sqrt{(29)}} = \frac{2}{\sqrt{(29)}}$$

or $$(2x-4y+3z)^2 = 4(x^2+y^2+z^2)$$

or $$12y^2 - 5z^2 - 16xy - 24yz + 12zx = 0.$$ **Ans.**

Ex. 6 (b). A right circular cone is passing through the point (1, 1, 1) and its vertex is at the point (1, 0, 1). The axis of the cone is equally inclined to the coordinate axes. Find the equation of the cone. (*Avadh* 91)

Sol. The direction ratios of the generator passing through (1, 1, 1) and the vertex $V(1, 0, 1)$ are $1-1, 1-0, 1-1$ *i.e.* 0, 1, 0

Also the direction ratios of the axis of the cone which is equally inclined to the coordinate axes is 1, 1, 1. **(Note)**

$\therefore$ If θ be the semi-vertical angle of the cone, then we have

$$\cos\theta = \frac{0.1 + 1.1 + 0.1}{\sqrt{(0^2+1^2+0^2)}\sqrt{(1^2+1^2+1^2)}} = \frac{1}{\sqrt{3}} \quad ...(i)$$

Also if $P(x, y, z)$ be any point on the cone, then direction ratios of the generator VP are $x-1, y-0, z-1$.

This generator is also inclined to the axis of the cone at an angle θ, so

$$\cos\theta = \frac{(x-1).1 + (y-0).1 + (z-1).1}{\sqrt{[(x-1)^2+(y-0)^2+(z-1)^2]}\sqrt{(1^2+1^2+1^2)}}$$

or $$\frac{1}{\sqrt{3}} = \frac{x+y+z-2}{\sqrt{[(x-1)^2+y^2+(z-1)^2]}.\sqrt{3}}, \text{ from (i)}$$

or $$\sqrt{[(x-1)^2+y^2+(z-1)^2]} = x+y-z-2$$

or $$(x-1)^2+y^2+(z-1)^2 = (x+y-z-2)^2, \text{ on squaring}$$

or $$yz + zx - xy + x + 2y - 3z - 1 = 0, \text{ on simplifying.}$$ **Ans.**

Ex. 7. The axis of a right circular cone, with origin O as vertex, makes equal angles with the co-ordinate axes and the cone passes through the line drawn from O with direction cosines proportional to 1, – 2, 3. Find its equation.

Sol. The vertex of the cone is $O(0,0,0)$ and since its axis makes equal angles with the co-ordinates axes, so the equations of its axis can be taken as

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{1} \quad (\because l=m=n) \qquad \text{(Note)}$$

Also the d.r.'s of its generators are given as $1, -2, 2$.

$\therefore$ If θ be the semi-vertical angle of the cone, then

$$\cos\theta=\frac{1.1+1.(-2)+1.2}{\sqrt{(1+1+1)}\sqrt{\{1^2+(-2)^2+(2)^2\}}}=\frac{1}{3\sqrt{3}} \qquad \text{...(i)}$$

Also if $P(x, y, z)$ be any point on the cone, then OP is a generator and the d.r.'s of OP are $x-0, y-0, z-0$, *i.e.* x, y, z. Also OP makes an angle θ with the axis of the cone.

$$\therefore \cos\theta=\frac{x.1+y.1+z.1}{\sqrt{(1+1+1)}\sqrt{(x^2+y^2+z^2)}} \qquad \text{...(ii)}$$

Equating (i) and (ii), the required equation of the cone is

$$\frac{x+y+z}{\sqrt{3}\sqrt{(x^2+y^2+z^2)}}=\frac{1}{3\sqrt{3}} \quad \text{or} \quad 9(x+y+z)^2=x^2+y^2+z^2$$

or $$4(x^2+y^2+z^2)+9(yz+zx+xy)=0. \qquad \textbf{Ans.}$$

***Ex. 8. Lines are drawn from O with direction cosines proportional to (1, 2, 2), (2, 3, 6), (3, 4, 12). Prove that the axis of the right circular cone through them has direction cosines $(-1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ and that the semi-vertical angle of the cone is $\cos^{-1}(1/\sqrt{3})$.** (*Kanpur* 91)

Sol. The vertex of the cone is $O(0,0,0)$ and so let the equation of its axis be $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$, where l, m, n are direction cosines. ...(i)

Let θ be the semi-vertical angle of the cone. Then the given lines with direction ratios (1, 2, 2), (2, 3, 6), (3, 4, 12) or direction cosines (1/3, 1/3, 2/3), (2/7, 3/7, 6/7), (3/13, 4/13, 12/13) makes angle θ with (i).

$$\therefore \text{ We have } \cos\theta=l(1/3)+m(2/3)+n(2/3) \qquad \text{...(ii)}$$
$$=l(2/7)+m(3/7)+n(6/7)$$
$$=l(3/13)+m(4/13)+n(12/13).$$

From these we have

$$(1/3)l+(2/3)m+(2/3)n=(2/7)l+(3/7)m+(6/7)n$$

and $$(1/3)l+(2/3)m+(2/3)n=(3/13)l+(4/13)m+(12/13)n$$

or $$[(1/3)-(2/7)]l+[(2/3)-(3/7)]m+[(2/3)-(6/7)]n=0$$

and $$[(1/3)-(3/13)]l+[(2/3)-(4/13)]m+[(2/3)-(12/13)]n=0$$

or $$(1/21)l+(5/21)m-(4/21)n=0$$

and $$(4/39)l+(14/39)m-(10/39)n=0$$

or $$l+5m-4n=0 \quad \text{and} \quad 2l+7m-5n=0$$

Solving these simultaneously we get

$$\frac{l}{-25+28}=\frac{m}{-8+5}=\frac{n}{7-10} \quad \text{or} \quad \frac{l}{-1}=\frac{m}{1}=\frac{n}{1}$$

$\therefore$ Direction cosines of axis of the cone, from (i), are proportional to $-1, 1, 1$ or the direction cosines are

$$\frac{-1 \,.\, 1 \,.\, 1}{\sqrt{[(-1)^2+1^2+1^2]}} \quad i.e. \quad \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$ Hence proved.

Also from (ii) we have

$$\cos\theta = -\frac{1}{\sqrt{3}}(1/3) + \frac{1}{\sqrt{3}}(2/3) + \frac{1}{\sqrt{3}}(2/3)$$

or $$\cos\theta = \frac{1}{3\sqrt{3}}(-1+2+2) = \frac{1}{\sqrt{3}} \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Hence proved.

Exercises on § 8.12.

Ex. 1. Find the equation of the right circular cone whose vertex is origin, whose axis is the line $x = \frac{1}{2} y = \frac{1}{3} z$ and whose vertical angle is 60°.

Ans. $19x^2 + 13y^2 + 3z^2 = 8xy + 24yz + 12zx$

Ex. 2. Find the equation of the circular cone whose axis is the line $x = y = z$ and semi-vertical angle is 30°, the vertex being at the origin.

Ans. $5(x^2 + y^2 + z^2) = 8(xy + yz + zx)$

Ex. 3. Find the equation of the right circular cone whose vertex is the origin, axis the x-axis and the semi-vertical angle is 60°. **Ans.** $3x^2 - y^2 - z^2 = 0.$

Ex. 4. Find the equation of the right circular cone whose vertex is the origin, axis the y-axis and the semi-vertical angle is 30°. **Ans.** $3x^2 - y^2 + 3z^2 = 0$

***Ex. 5.** Obtain the equation to a right circular cone whose vertex is the origin, axis the y-axis and the semi-vertical angle is α.

Ans. $x^2 - y^2 \tan^2 \alpha + z^2 = 0$

Ex. 6. Find the equation of the right ciruclar cone, whose vertex is $(3, 2, 1)$, axis is the line $\frac{1}{4}(x-3) = y - 2 = \frac{1}{3}(z-1)$ and semi-vertical angle is 30°. (*Garhwal* 93)

Ans. $7x^2 + 37y^2 + 21z^2 - 16xy - 12yz - 48zx + 38x - 98y + 126z - 32 = 0.$

***Ex. 7.** Find the equation to the right ciruclar cone having the semi-vertical angle equal to α and the straight line $x/l = y/m = z/n$ as its axis, l, m, n being the direction cosines.

[Hint : Take (0, 0, 0) as vertex]

Ans. $(x^2 + y^2 + z^2) = (lx + my + nz)^2 \sec^2 \alpha.$

***Ex. 8.** A right circular cone has its vertex at $(2, 3, -5)$, its axis passes through $(3, -2, 6)$ and its semi-vertical angle is 30°. Find the equation of the cone. **Ans.** $5x^2 + 5y^2 + 5z^2 + 8xy - 8yz + 8zx - 4x - 86y + 58z + 278 = 0$

Ex. 9. Find the equation of the right circular cone whose vertex is the origin, axis is the axis of y and the semi-vertical angle is $\pi/3$.

Ans. $x^2 + z^2 = 3y^2$

Ex. 10. Find the equation of a right circular cone whose vertex is $(2, -3, 5)$ and axis makes equal angles with the axes of co-ordinates and passes through $(1, -2, 3)$.

§ 8.13. Standard Equation of a cone.

The equation $\mathbf{ax^2 + by^2 + cz^2 = 0}$ is called the standard equation of a cone. It is a particular case of the general equaton of the cone whose vertex is origin.

We can prove as particular cases of § 8.01 to § 8.11 or independently by similar methods that

(i) the tangent planes to the cone at (x_1, y_1, z_1) is $aax_1 + byy_1 + czz_1 = 0$

(ii) the condition for the plane $ux + vy + wz = 0$ to touch this cone is

$$bcu^2 + cav^2 + abw^2 = 0.$$

(iii) the polar plane of (x_1, y_1, z_1) is $axx_1 + byy_1 + czz_1 = 0$.

(iv) The equation of the pair of tangent planes passing through the line OP, where O is $(0, 0, 0)$ and P is (x_1, y_1, z_1) is $SS_1 = T^2$

i.e. $$(ax^2 + by^2 + cz^2)(ax_1^2 + by_1^2 + cz_1^2) = (axx_1 + byy_1 + czz_1)^2$$

Exercise

Ex. Find the condition that the section of the surface $ax^2 + by^2 + cz^2 = 1$ by the plane $lx + my + nz = 0$ may be (i) a parabola, (ii) an ellipse, (iii) a hyperbola.

CYLINDER

§ 8.14. Cylinder.

Definition. *The surface generated by a variable straight line moving parallel to a fixed straight line and satisfying one more conditon (intersecting a given curve or touching a given surface) is called a* **cylinder.**

If the generating line is always at a constant distance from the fixed straight line then the cylinder so generated is called a **right ciruclar cylinder,** whose radius is this constant distance and axis is the fixed straight line.

****§ 8.15. Equation of a cylinder through a given conic.**

(*Kumaun* 94, 91; *Lucknow* 91, 90; *Rohilkhand* 95, 90)

Let the generators of the cylinder be parallel to the line

$$x/l = y/m = z/n \qquad \text{...(i)}$$

and the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \; z = 0 \qquad \text{...(ii)}$$

Let $P(x_1, y_1, z_1)$ be any point on the cylinder then the equations of a generator through P with the help of (i) are given by

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \text{...(iii)}$$

This generator meets the plane $z=0$ in the point given by

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{0-z_1}{n} \text{ i.e. } \left(x_1-\frac{l z_1}{n},\ y_1-\frac{m z_1}{n},\ 0\right)$$

$\therefore$ The generator (iii) meets the conic (ii) if

$$a\left(x_1-\frac{l z_1}{n}\right)^2+2h\left(x_1-\frac{l z_1}{n}\right)\left(y_1-\frac{m z_1}{n}\right)+b\left(y_1-\frac{m z_1}{n}\right)^2$$
$$+2g\left(x_1-\frac{l z_1}{n}\right)+2f\left(y_1-\frac{m z_1}{n}\right)+c=0$$

or
$$a(nx_1-lz_1)^2+2h(nx_1-lz_1)(ny_1-mz_1)+b(ny_1-mz_1)^2$$
$$+2gn(nx_1-lz_1)+2fn(ny_1-mz_1)+cn^2=0$$

$\therefore$ The required equation of the cylinder on the locus of $P(x_1, y_1, z_1)$ is

$$a(nx-lz)^2+2h(nx-lz)(ny-mz)+b(ny-mz)^2$$
$$+2gn(nx-lz)+2fn(ny-mz)+cn^2=0$$

Cor. 1. If the generator of the cylinder is parallel to z-axis whose d.c.'s are 0, 0, 1 *i.e.* $l=0=m$, then the above equation of the cylinder reduces to

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \textbf{(Note)}$$

[**Note.** Here students should note that this equation in two dimensional geometry represents a conic whereas in theree dimensions represents a cylinder.]

In general every equation of the form $f(x, y)=0$ represents a cylinder passing through the curve $f(x, y)=0$, $z=0$ and whose generators are parallel to the z-axis. **(Remember)**

Similarly the equtation $\phi(y, z)=0$ represents a cylinder through the curve $\phi(y, z)=0$, $x=0$ and generators parallel to x-axis and the equation $\phi(z, x)=0$ represents a cylinder through the curve $\phi(z, x)=0$, $y=0$ and generatos parallel to y-axis.

Cor. 2. The equation of the cylinder which intersects the curve $f_1(x, y, z)=0$, $f_2(x, y, z)=0$ and whose generators are parallel to x-axis is obtained by eliminating x between $f_1(x, y, z)=0$ and $f_2(x, y, z)=0$.

Similarly by eliminating y (or z) between these equations, the equation of the cylinder passing through the curve given by these equations and generators parallel to y (or z) axis can be obtained.

Solved Examples on § 8.15.

****Ex. 1 (a). Find the equations of the quadric cylinder which intersects the curve $ax^2+by^2+cz^2=1$, $lx+my+nz=p$ and whose generators are parallel to the axis of z.**

(*Garhwal* 95, 91; *Kanpur* 96; *Meerut* 94)

Sol. The guiding curve is given by the equations

$ax^2 + by^2 + cz^2 = 1$ ...(i), $lx + my + nz = p$ ...(ii)

Now as the generators of the cylinder are parallel to z-axis, so the equations of the cylinder will not contain terms of z. **(Note)**

Hence the required equations of the cylindr will be obtained by eliminating z between (i) and (ii).

From (ii) we get $z = (p - lx - my)/n$

Substituting in (i) we get $ax^2 + by^2 + c\,[(p - lx - my)/n]^2 = 1$

or $an^2x^2 + bn^2y^2 + c\,(p^2 + l^2x^2 + m^2y^2 - 2plx - 2pmy + 2lmxy) - n^2 = 0$

or $(an^2 + cl^2)\,x^2 + (bn^2 + cm^2)\,y^2 + 2lcmxy - 2cplx - 2cpmy + (cp^2 - n^2) = 0$

Ans.

***Ex. 1 (b). Find the equation of the cylinder with generators parallel to z-axis and passing through the curve $ax^2 + by^2 = 2cz$, $lx + my + nz = p$.**

Sol. The curve is given by $ax^2 + by^2 = 2cz$, ...(i)

$lx + my + nz = p$. ...(ii)

Since generators are parallel to z-axis, so eliminating z between (i) and (ii) we get the required equation (See Cor. 2 Page 63 Ch. VIII) as

$$ax^2 + by^2 = 2c\,(p - lx - my)/n$$

or $n\,(ax^2 + by^2) + 2\,c\,(lx + my) - 2pc = 0.$ **Ans.**

***Ex. 1 (c). Find the equation of the cylinder with generators parallel to z-axis and passing through the curve $ax^2 + by^2 = 2z$, $lx + my + nz = p$.**

(*Avadh* 93; *Bundelkhand* 95; *Rohilkhand* 92)

Sol. Do as Ex. 1 (b) above. Here $c = 1$.

Ans. $n\,(ax^2 + by^2) + 2\,(lx + my) = 2p$

Ex. 1 (d). Find the equation of the cylinder with generators parallel to the axis of x and passing through the curve

$$ax^2 + by^2 + cz^2 = 1,\ lx + my + nz = p.$$ (*Bundelkhand* 96)

Sol. The curve is given by

$ax^2 + by^2 + cz^2 = 1$ ...(i) and $lx + my + nz = p$ ...(ii)

Since the generators are parallel to x-axis, so eliminating x between (i) and (ii) we get the required equations as $a\,[(p - my - nz)/l]^2 + by^2 + cz^2 = 1$

or $a\,(p - my - nz)^2 + bl^2y^2 + cl^2z^2 = l^2$

or $(am^2 + bl^2)\,y^2 + (an^2 + cl^2)\,z^2 + 2amnyz - 2ampy$

$- 2anpz + (ap^2 - l^2) = 0.$ **Ans.**

****Ex. 1 (e). Find the equation of a cylinder with generators parallel to y-axis which pass through the curve of intersection of the surfaces**

$$x^2 + y^2 + 2z^2 = 12,\ x - y + z = 1$$ (*Kumaun* 93)

****Ex. 2. Find the equation of the surface generated by a stratight line which is parallel to the line $y = mx, z = nx$ and intersects the ellipse $x^2/a^2 + y^2/b^2 = 1,\ z = 0$** (*Lucknow* 92; *Meerut* 90)

Sol. Equations of the given line are $\dfrac{x}{1} = \dfrac{y}{m} = \dfrac{z}{n}$. ...(i)

Let $P(x_1, y_1, z_1)$ be any point on the surface (which is a cylinder by definition), then the equations of the generator through P, which is parallel to (i) are

$$(x - x_1)/l = (y - y_1)/m = (z - z_1)/n. \qquad ...(ii)$$

This generator meets the plane $z = 0$ in the point given by

$$\frac{x - x_1}{1} = \frac{y - y_1}{m} = \frac{0 - z_1}{n}$$

i.e. in the point $[x_1 - (z_1/n),\ y_1 - (mz_1/n),\ 0]$ **(Note)**

This point must lie on the given guiding curve and so we have

$$\frac{[x_1 - (z_1/n)]^2}{a^2} + \frac{[y_1 - (mz_1/n)]^2}{b^2} = 1, \text{ from } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or $$b^2 (nx_1 - z_1)^2 + a^2 (ny_1 - mz_1)^2 = a^2 b^2 n^2$$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ or the equation of required surface is

$$b^2 (nx - z)^2 + a^2 (ny - mz)^2 = a^2 b^2 n^2. \qquad \textbf{Ans.}$$

***Ex. 3. Find the equation of the circular cylinder, whose generating lines have the direction cosines l, m, n and which passes through the fixed circle $x^2 + y^2 = a^2$ in the zox-plane.**

Sol. Let $P(x_1, y_1, z_1)$ be any point on the cylinder, then the equations of the generator through P are $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$ ...(i) **(Note)**

This generator meets the zox-plane *i.e.* $y = 0$ in the point given by

$$\frac{x - x_1}{l} = \frac{0 - y_1}{m} = \frac{z - z_1}{n}$$

i.e. in the point $[x_1 - (ly_1/m),\ 0,\ z_1 - (ny/m)]$

$\therefore$ This generator (i) intersects the given circle if

$$[x_1 - (ly_1/m)]^2 + [z_1 - (ny_1/m)]^2 = 1$$

or $$(mx_1 - ly_1)^2 + (mz_1 - ny_1)^2 = m^2$$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ or the required equation of the cylinder is

$$(mx - ly)^2 + (mz - ny)^2 = m^2 \qquad \textbf{Ans.}$$

****Ex. 4 (a). Find the equation of the cylinder whose generators are parallel to the line $x/1 = y/(-2) = z/3$ and passing through the curve $x^2 + 2y^2 = 1,\ z = 0$.**

(*Bundelkhand* 95; *Gorakhpur* 91; *Kanpur* 90; *Kumaun* 95; *Rohilkhand* 94)

Sol. Let $P(x_1, y_1, z_1)$ be any point on the cylinder, then the equations of the generator through P are $\dfrac{x-x_1}{1}=\dfrac{y-y_1}{-2}=\dfrac{z-z_1}{3}$ ...(i) **(Note)**

This generator meets the plane $z=0$ in the point given by

$$\frac{x-x_1}{1}=\frac{y-y_1}{-2}=\frac{0-z_1}{3}$$

i.e. in the point $[x_1-\frac{1}{3}z_1, y_1+\frac{2}{3}z_1, 0]$

$\therefore$ This generator (i) intersects the given conic if

$$(x_1-\tfrac{1}{3}z_1)^2+2[y_1+\tfrac{2}{3}z_1]^2=1.$$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ or the required equation of the cylinder is

$$(x-\tfrac{1}{3}z)^2+2[y+\tfrac{2}{3}z]^2=1$$

or $$[x^2+(1/9)z^2-(2/3)xz]+2[y^2+(4/9)z^2+(4/3)yz]=1$$

or $$9x^2+z^2-6xz+18y^2+8z^2+24yz-9=0$$

or $$3x^2+6y^2+3z^2-2xz+8yz-3=0.$$ **Ans.**

Ex. 4 (b). Find the equation of the cylinder whose generators are parallel to the line x = y/2 = z/2 and whose guiding curve is the ellipse

$$\mathbf{x^2+2y^2=1,\ z=0.}$$

Sol. Do as Ex. 4 (a) above. **Ans.** $4x^2+8y^2+9z^2-4xz-16yz-4=0$

****Ex. 5. Find the equation to the cylinder whose generators are parallel to the line x/1 = y (– 2) = z/3 and the guiding curve is the ellipse**

$$\mathbf{x^2+2y^2=1,\ z=3}$$ *(Agra 91; Meerut 91 S)*

Sol. Let $P(x_1, y_1, z_1)$ be any point on the cylinder, then the equations of the generator through P are $\dfrac{x-x_1}{1}=\dfrac{y-y_1}{-2}=\dfrac{z-z_1}{3}$ ...(i) **(Note)**

This generator meets the plane $z=3$ in the point given by

$$\frac{x-x_1}{1}=\frac{y-y_1}{-2}=\frac{3-z_1}{3}$$

i.e. in the point $[x_1-\frac{1}{3}z_1+1,\ y_1+(2/3)z_1-2,\ 3]$. (Note)

$\therefore$ This generator intersects the given conic if

$$[x_1-(1/3)z_1+1]^2+2[y_1+(2/3)z_1-2]^2=1.$$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ or the required equation of the cylinder is

$$(x-\tfrac{1}{3}z+1)^2+2[y+(2/3)z-2]^2=1$$

or $$x^2+(1/9)z^2+1-(2/3)xz+2x-(2/3)z+2[y^2+(4/9)z^2+4+(4/3)yz-4y-(8/3)z]=1$$

or $\quad x^2+2y^2+z^2-(2/3)\,xz+(8/3)\,yz+2x-8y-6z+8=0.$ **Ans.**

Ex. 6. Find the equation of the cylinder whose generators are parallel to the line and $x/1=y/2=z/3$ passes through the curve

$x^2+y^2=16$ and $z=0$. (*Agra* 90; *Bundelkhand* 93, 91; *Kanpur* 91)

Sol. Let $P(x_1, y_1, z_1)$ be any point on the cylinder, then the equations of the generator through P are $\quad \dfrac{x-x_1}{1}=\dfrac{y-y_1}{2}=\dfrac{z-z_1}{3}$

This generator meets the plane $z=0$ in the point given by

$$\frac{x-x_1}{1}=\frac{y-y_1}{2}=\frac{0-z_1}{3}$$

i.e. $\quad$ in the point $[x_1-\frac{1}{3}z_1,\ y_1-(2/3)\,z_1,\ 0]$.

$\therefore$ The generator intersecs the given curve if

$$(x_1-\tfrac{1}{3}z_1)^2+[y_1-(2/3)\,z_1]^2=16$$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ or the required equation of the cylinder is

$$(x-\tfrac{1}{3}z)^2+[y-(2/3)\,z]^2=16$$

or $$x^2+(1/9)\,z^2-(2/3)\,xz+y^2+(4/9)\,z^2-(4/3)\,yz=16$$

or $$9x^2+9y^2+5z^2-12yz-6zx-144=0$$ **Ans.**

Exercises on § 8.15

Ex. 1. Obtain the equation of the cylinder which passes through $y^2=4ax$, $z=0$ and whose generators are parallel to the line $x=y=z$.

Ans. $(y-z)^2=4a\,(x-z)$

Ex. 2. Find the equation of the cylinder whose generating line is parallel to z-axis and the guiding curve is $x^2+y^2=z,\ x+y+z=0$.

Ex. 3. Find the equation of the cylinder which intersects the curve $ax^2+by^2+cz^2=1,\ lx+my+nz=p$ and whose generators are parallel to the axis of x.

Ans. $(bl^2+am^2)\,y^2+(cl^2+am^2)\,z^2+2amnyz-2ampy-2anpz+(ap^2-l^2)=0$

Ex. 4. Find the equation to the cylinder whose generators are parallel to the line $x/1=y/-2=z/3$ and guiding curve is the ellipse $x^2+4y^2=1,\ z=6$.

(*Bundelkhand* 90)

Ans. $9x^2+36y^2+25z^2-6zx+48yz+36x-288y-204z+603=0$

***Ex. 5.** Find the equation of the cylindr whose generators are parallel to the line $x=y/2=z/3$ and whose guiding curve is the ellipse $x^2+2y^2=1,\ z=0$.

(*Bundelkhand* 92)

Ans. $3x^2+6y^2+3z^2-2xz-8yz-3=0$

Ex. 6. Show that the generating line of the cylinder given below is parallel to x-axis : $az^2 + 2hyz + by^2 + 2gx + 2fy + k = 0$ (*Kanpur* 94)

[**Hint.** See § 8.15 Cor 1. Page 63 Ch. VIII]

§ 8.16. Equation of a right circular cylinder. (*Kanpur* 94; *Kumaun* 95)

Let r be the radius of the cylinder and the equations of its axis are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \text{...(i)}$$

If $P(x_1, y_1, z_1)$ be any point on the cylinder, then the length of perpendicular from P to (i) must be r and so from § 4.14 in the Chapter on Straight Line we have

$$r^2(l^2+m^2+n^2) = \{n(y_1-\beta) - m(z_1-\gamma)\}^2 + \{l(z_1-\gamma) - n(x_1-\alpha)\}^2 + \{m(x_1-\alpha) - l(y_1-\beta)\}^2$$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ or the required equation of the cylinder is given by

$$\{n(y-\beta) - m(z-\gamma)\}^2 + \{l(z-\gamma) - n(x-\alpha)\}^2 + \{m(x-\alpha) - l(y-\beta)\}^2 = r^2(l^2+m^2+n^2).$$

Solved Examples on § 8.16.

Ex. 1. Find the equation of a right circular cylinder whose axis is $x/l = y/m = z/n$.

Sol. Let r be the radius of the cylindr and the equations of its axis are given by

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}. \qquad \text{...(i)}$$

If $P(x_1, y_1, z_1)$ be any point on the cylinder, then the length of the perpendicular from P to (i) must be r and so from § 4.14 in the Chapter on Straight Lines we have

$$r^2(l^2+m^2+n^2) = \{ny_1 - mz_1\}^2 + \{lz_1 - nx_1\}^2 + \{mx_1 - ly_1\}^2$$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ or the required equation to the cylinder is given by $(ny-mz)^2 + (lz-nx)^2 + (mx-ly)^2 = r^2(l^2+m^2+n^2)$. **Ans.**

****Ex. 2 (a). Find the equation of the right circular cylinder of radius 2 and having as axis the line $\frac{1}{2}(x-1) = (y-2) = \frac{1}{2}(z-3)$.**

(*Avadh* 94; *Bundelkhand* 94; *Kumaun* 96, 92; *Lucknow* 91, 90; *Purvanchal* 94; *Rohilkhand* 93)

Sol. Let $P(x_1, y_1, z_1)$ be any point on the cylinder. Then length of the perpendicular from $P(x_1, y_1, z_1)$ to the given line

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$$

must be equal to the radius 2. So from § 4.14 in Straight Lines we get

$$2^2[2^2+1^2+2^2] = [2(y_1-2) - 1(z_1-3)]^2 + \{2(z_1-3) - 2(x_1-1)\}^2 + \{1(x_1-1) - 2(y_1-2)\}^2$$

or $$4\,[9] = (2y_1 - z_1 - 1)^2 + (2z_1 - 2x_1 - 4)^2 + (x_1 - 2y_1 + 3)^2$$

$\therefore$ The required equation or the locus of $P(x_1, y_1, z_1)$ is

$$(2y - z - 1)^2 + (2z - 2x - 4)^2 + (x - 2y + 3)^2 = 36$$

or $$5x^2 + 8y^2 + 5z^2 - 4xy - 4yz - 8xz + 22x - 16y - 14z - 10 = 0. \quad \textbf{Ans.}$$

***Ex. 2 (b). Find the equation of right circular cylinder whose axis is x = 2y = – z and radius is 4.** (*Agra* 90; *Bundelkhand* 96; *Meerut* 91, 90)

Sol. Let $P(x_1, y_1, z_1)$ be any point on the cylinder. Then the length of perpendicular from P to the given axis $\dfrac{x}{1} = \dfrac{y}{1/2} = \dfrac{z}{-1}$, must be equal to the radius 4.

So we have $4\,[(1)^2 + (1/2)^2 + (-1)^2]$

$$= [y_1(-1) - (1/2)\,z_1]^2 + [z_1(1) - (-1)\,x_1]^2 + [x_1(1/2) - y_1(1)]^2.$$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ or the required equation to the cylinder is given by

$$4^2\,[1 + (1/4) + 1] = (1/4)(2y + z)^2 + (z + x)^2 + (1/4)(x - 2y)^2$$

or $$144 = (2y + z)^2 + 4\,(x + z)^2 + (x - 2y)^2$$

or $$5x^2 + 8y^2 + 5z^2 + 4yz + 8xz - 4xy = 144. \quad \textbf{Ans.}$$

***Ex. 2 (c). In Ex. 2 (a) above find the area of the section of the cylinder by the plane z = 0.** (*Bundelkhand* 96; *Meerut* 90)

Sol. Equation of the cylinder in Ex. 2 (a) Page 68 Ch. VIII is

$$5x^2 + 8y^2 + 5z^2 + 4yz + 8xz - 4xy = 144$$

Putting $z = 0$ in it, the equations of the section are

$$5x^2 + 8y^2 - 4xy = 144, \quad z = 0,$$

Need to check

which is an ellipse as '$h^2 < ab$' **(See Author's Coordinate Geometry)**

Putting $x = r\cos\theta$, $y = r\sin\theta$ in it, we get

$$r^2(5\cos^2\theta + 8\sin^2\theta - 4\sin\theta\cos\theta) = 144 \text{ on } z = 0,$$

or $$r^2 = 144/(5\cos^2\theta + 8\sin^2\theta - 4\sin^2\theta\cos\theta)$$

or $$r^2 = \frac{144\,(1 + \tan^2\theta)}{5 + 8\tan^2\theta - 4\tan\theta}, \quad \because \sec^2\theta = 1 + \tan^2\theta \qquad \text{...(i)}$$

where $$\boxed{\tan 2\theta = \frac{2h}{a - b}} = \frac{-4}{5 - 8} = \frac{4}{3}$$

or $$\frac{2\tan\theta}{1 - \tan^2\theta} = \frac{4}{3} \quad \text{or} \quad 2\tan^2\theta + 3\tan\theta - 2 = 0$$

or $$(2\tan\theta - 1)(\tan\theta + 2) = 0 \quad \text{or} \quad \tan\theta = \tfrac{1}{2}, -2$$

Substituting these in (i), we find that the lengths r_1, r_2 of semi axes of the above ellipse are given by

$$r_1^2 = \frac{144\,[1 + (1/2)^2]}{5 + 8\,(1/2)^2 - 4\,(1/2)} = 9 \quad \text{or} \quad r_1 = 3$$

and $$r_2^2 = \frac{144\,[1+(-2)^2]}{5+8\,(-2)^2-4\,(-2)} = 16 \quad \text{or} \quad r_2 = 4$$

$\therefore$ Required area of the ellipse $= \pi\, r_1\, r_2$

$= \pi \times 3 \times 4 = 12\,\pi$ square units. **Ans.**

Ex. 2 (d). Find the equation of the right circular cylinder whose axis is $\frac{1}{2}(x-1) = y-2 = \frac{1}{2}(z-3)$ and radius 5. (*Gorakhpur* 92; *Purvanchal* 92)

Sol. Let $P(x_1, y_1, z_1)$ be any point on the cylinder. Then length of perpendicular from $P(x_1, y_1, z_1)$ to the given line must be equal to the radius 5. So from § 4.14 in the chapter on Straight Lines, we get

$$5^2\,[2^2+1^2+2^2] = \{2\,(y_1-2) - 1\,(z_1-3)\}^2 + \{2\,(z_1-3) - 2\,(x_1-1)\}^2 + \{1\,(x_1-1) - 2\,(y_1-2)\}^2$$

or $$25\,(9) = (2y_1 - z_1 - 1)^2 + (2z_1 - 2x_1 - 4)^2 + (x_1 - 2y_1 + 3)^2$$

$\therefore$ The required equation or the locus of $P(x_1, y_1, z_1)$ is

$$25\,(9) = (2y - z - 1)^2 + (2z - 2x - 4)^2 + (x - 2y + 3)^2$$

or $$5x^2 + 8y^2 + 5z^2 - 4xy - 4yz - 8zx + 22x - 16y - 14z = 199$$

****Ex. 2 (e). Find the equation of the right circular cylinder whose axis is $\frac{1}{2}x = \frac{1}{3}y = \frac{1}{6}z$ and radius 5.**

Sol. Do as Ex. 2 (d) above.

Ans. $45x^2 + 40y^2 + 13z^2 - 12xy - 36yz - 24zx - 1225 = 0$

Ex. 2 (f). Find the equation of a right circular cylinder with axis $\frac{x-1}{3} = \frac{y-2}{-1} = \frac{z-3}{2}$ and radius 5. (*Rohilkhand* 96)

Sol. Do as Ex. 2 (d) above.

Ans. $5x^2 + 13y^2 + 10z^2 + 4yz - 12zx + 6xy + 14x - 70y - 56z - 203 = 0$

***Ex. 2. (g). Find the equation of the right circular cylinder of radius 2 whose axis is the line $(x-1)/2 = y/3 = (z-3)/1$.** (*Meerut* 98)

Sol. Do as Ex. 2 (d) above.

Ans. $10x^2 + 5y^2 + 13z^2 - 12xy - 6yz - 4zx - 8x + 30y - 74z + 59 = 0$

****Ex. 2 (h). Find the equation of the right circular cylinder of radius 3 and axis $\frac{1}{2}(x-1) = \frac{1}{2}(y-3) = \frac{1}{7}(5-z)$.**

(*Agra* 92; *Kanpur* 90; *Purvanchal* 91)

Sol. Let $P(x_1, y_1, z_1)$ be any point on the cylinder. Then the length of perpendicular from P to the given line

$\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-7}$ must be equal to the radius 3.

$\therefore$ From § 4.14 Str. Lines we get

$$3^2\,[2^2+2^2+(-7)^2]=\{-7\,(y_1-3)-2\,(z_1-5)\}^2$$
$$+\{-7\,(x_1-1)-2\,(z_1-5)\}^2+\{2\,(x_1-1)-2\,(y_1-3)\}^2$$

or $$513=(7y_1+2z_1-31)^2+(7x_1+2z_1+17)^2+4\,(x_1-y_1+2)^2$$
$$=49y_1^2+4z_1^2+961+28y_1z_1-434y_1-124z_1+49x_1^2+4z_1^2$$
$$+289+28x_1z_1+238x_1+68z_1+4x_1^2+4y_1^2+16-8x_1y_1+16x_1-16y_1$$

or $$53x_1^2+53y_1^2+8z_1^2+28y_1z_1+28z_1x_1-8x_1y_1$$
$$+254x_1-450y_1-56z_1+753=0.$$

$\therefore$ Locus of $P\,(x_1, y_1, z_1)$ or the required equation of the curve is
$$53x^2+53y^2+8z^2+28yz+28xz-8xy+254x$$
$$-450y-56z+753=0.$$ **Ans.**

****Ex. 3 (a). Find the equaton of the right circular cylinder whose axis is x − 2 = z, y = 0 and passes through the point (3, 0, 0).**

(*Bundelkhand* 91; *Purvanchal* 93)

Sol. The equations of the axis of the cylinder are $\dfrac{x-2}{1}=\dfrac{y}{0}=\dfrac{z}{1}$ ...(i)

Also (radius of the cylinder)^2

= [length of perpendicular from (3, 0, 0) to the line (i)]^2.

$$=\frac{1}{(1^2+0^2+1^2)}\,[(0.1-0.0)^2+\{0.1-1.(3-2)\}^2+\{0.(3-2)-1.0\}^2]$$

(Note)

$= (1/2)\,[1] = 1/2$ *i.e.* radius of the cylinder $= 1/\sqrt{2}$

Now let $P\,(x_1, y_1, z_1)$ be any point on the cylinder.

Then the length of perpendicular from $P\,(x_1, y_1, z_1)$ to the axis (i) must be equal to radius $1/\sqrt{2}$ of the cylinder *i.e.*

$$(\tfrac{1}{2})\,[1^2+0+1^2]=\{1.y_1-0.z_1\}^2+\{1.z_1-1.(x_1-2)\}^2+\{0.(x_1-2)-1.y_1\}^2$$

or $$1=y_1^2+(z_1-x_1+2)^2+y_1^2$$

or $$x_1^2+2y_1^2+z_1^2-2z_1x_1-4x_1+2z_1+3=0$$

$\therefore$ The locus of $P\,(x_1, y_1, z_1)$ or the required equation of the cylinder is
$$x^2+2y^2+z^2-2zx-4x+2z+3=0.$$ **Ans.**

Ex. 3 (b). Prove that the equation of the right circular cylinder whose axis is (x − 2)/2 = (y − 1)/1 = z/3 and passes through the point (0, 0, 3) is

$$\mathbf{10x^2+13y^2+5z^2-6yz-12zx-4xy-36x-18y+30z-135=0.}$$

Sol. Do as Ex. 3 (a) above. (*Agra* 91)

****Ex. 4. Find the equation of the right circular cylinder of radius 2 whose axis passes through (1, 2, 3) and has direction cosines proportional to (2, − 3, 6).**

(*Avadh* 90; *Meerut* 92 *P*; *Purvanchal* 96)

Sol. Let $P(x_1, y_1, z_1)$ be any point on the cylinder. Then the length of the perpendicular from $P(x_1, y_1, z_1)$ to the given axis

$$\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z-3}{6}$$

must be equal to 2. So from § 8.16 Page 68 Ch. VIII, we have

$$2^2[2^2+(-3)^2+6^2]=\{6(y_1-2)-(-3)(z_1-3)\}^2$$
$$+\{2(z_1-3)-6(x_1-1)\}^2+\{(-3)(x_1-1)-2(y_1-2)\}^2$$

or $$4(49)=9(2y_1+z_1-7)^2+4(z_1-3x_1)^2+(3x_1+2y_1-7)^2$$

or $$45x_1^2+40y_1^2+13z_1^2+36y_1z_1-24z_1x_1+12x_1y_1-42x_1-280y_1-126z_1+294=0.$$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ or the required equation of the cylinder is

$$45x^2+40y^2+13z^2+36yz-24zx+12xy-42x-280y-126z+294=0.$$

Ans.

IP ****Ex. 5 (a). Find the equation of the right circular cylinder which passes through the circle $x^2+y^2+z^2=9,\ x-y+z=3$.**

(*Avadh* 92; *Garhwal* 94, 92, 90; *Kanpur* 97, 95, 93; *Meerut* 96 *P*, 95, 92; *Purvanchal* 90)

Sol. The direction ratios of the axis of the cylinder, which is perpendicular to the plane of the circle given by $x-y+z=3$ are $1, -1, 1$.

So let one of the generators of the cylinder passing through any point (α, β, γ) on the cylinder be $\quad \dfrac{x-\alpha}{1}=\dfrac{y-\beta}{-1}=\dfrac{z-\gamma}{1}$.

Any point on this generator at a distance r from (α, β, γ) is

$$(\alpha+r, \beta-r, \gamma+r)$$

If this point lies on the given circle, then we have

$$(\alpha+r)^2+(\beta-r)^2+(\gamma+r)^2=9,\ (\alpha+r)-(\beta-r)+(\gamma+r)=3$$

or $$\alpha^2+\beta^2+\gamma^2+2r(\alpha-\beta+\gamma)+3r^2=9,\ \alpha-\beta+\gamma+3r=3.$$

Eliminating r we get

$$\alpha^2+\beta^2+\gamma^2+2(\alpha-\beta+\gamma)[\tfrac{1}{3}(3-\alpha+\beta-\gamma)]+3[\tfrac{1}{3}(3-\alpha+\beta-\gamma)]^2=9$$

or $$3(\alpha^2+\beta^2+\gamma^2)+2(\alpha-\beta+\gamma)(3-\alpha+\beta-\gamma)+(3-\alpha+\beta-\gamma)^2=27$$

or $$3(\alpha^2+\beta^2+\gamma^2)+(3-\alpha+\beta-\gamma)(3+\alpha-\beta+\gamma)=27$$

or $$3(\alpha^2+\beta^2+\gamma^2)+9-(\alpha-\beta+\gamma)^2=27 \qquad \textbf{(Note)}$$

or $$\alpha^2+\beta^2+\gamma^2+\alpha\beta-\alpha\gamma+\beta\gamma-9=0.$$

$\therefore$ The equation of the cylinder *i.e.* the locus of $P(x_1, y_1, z_1)$ is

$$x^2+y^2+z^2+xy-xz+yz-9=0.$$

Ans.

Ex. 5 (b). Find the equation of the right circular cylinder through the circle of intersection of $x^2+y^2+z^2=1$ and $x+y+z=1$. (*Lucknow* 92)

Sol. The d. ratios of the axis of cylinder, which is perpendicular to the plane of circle given by $x+y+z=1$ are 1, 1, 1.

So let one of the generators of the cylinder passing through any point $P(\alpha, \beta, \gamma)$ on the cylinder be $(x-\alpha)/1=(y-\beta)/1=(z-\gamma)/1$

Any point on this generator at a distance r from (α, β, γ) is $(\alpha+r, \beta+r, \gamma+r)$ and if ti lies on the given circle, we have

$$(\alpha+r)^2+(\beta+r)^2+(\gamma+r)^2=1, \ (\alpha+r)+(\beta+r)+(\gamma+r)=1$$

or $$(\alpha+r)^2+(\beta+r)^2+(\gamma+r)^2=1, \ 3r=1-(\alpha+\beta+\gamma)$$

Eliminating r, we get

$$\left[\alpha+\frac{1-(\alpha+\beta+\gamma)}{3}\right]^2+\left[\beta+\frac{1-(\alpha+\beta+\gamma)}{3}\right]^2+\left[\gamma+\frac{1-(\alpha+\beta+\gamma)}{3}\right]^2=1$$

or $$\alpha^2+\beta^2+\gamma^2-\beta\gamma-\gamma\alpha-\alpha\beta=1, \text{ on simplifying}$$

$\therefore$ Locus of $P(\alpha, \beta, \gamma)$ or the required equation of the cylinder is

$$x^2+y^2+z^2-yz-zx-xy=1.$$ **Ans.**

Ex. 5 (c). Prove that the equation of the right circular cylinder whose one section is the circle $x^2+y^2+z^2-x-y-z=0$, $x+y+z=1$ is

$$\mathbf{x^2+y^2+z^2-yz-zx-xy=1.}$$

[**Hint :** Do exactly as Ex. 5 (b) above.

Also if the point (α, β, γ) lies on the given circle, we have

$$(\alpha+r)^2+(\beta+r)^2+(\gamma+r)^2-(\alpha+r)-(\beta+r)-(\gamma+r)=0,$$
$$(\alpha+r)+(\beta+r)+(\gamma+r)=1$$

or $$(\alpha+r)^2+(\beta+r)^2+(\gamma+r)^2-1=0, \ (\alpha+\beta+\gamma)+3r=1.$$ **(Note)**]

Ans. $x^2+y^2+z^2-yz-zx-xy=1.$

Ex. 6. Show that the equation of the right circular cylinder described on the circle through the three points A (1, 0, 0), B (0, 1, 0) and C (0, 0, 1) as the guiding curve is $x^2+y^2+z^2-yz-zx-xy=1$.

(*Avadh* 91; *Garhwal* 93; *Meerut* 97; *Rohilkhand* 97, 91)

Sol. The equation of the plane ABC is $x+y+z=1$. ...(i)

Let the sphere through $A(1, 0, 0)$, $B(0, 1, 0)$ and $C(0, 0, 1)$ pass through the origin $O(0, 0, 0)$ also. Then the equation of the sphere $OABC$ is

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}=0$$

which reduces to $$x^2+y^2+z^2-x-y-z=0.$$ ...(ii)

$\therefore$ The equations of guiding circle of the cylinder are

$$x^2+y^2+z^2-x-y-z=0; \ x+y+z=1$$ ...(iii)

Now the d.r.'s of the axis of the cylinder which is perpendicular to the plane of the circle given by $x+y+z=1$ are 1, 1, 1.

So let one of the generators of the cylinder passing through any point $P(\alpha, \beta, \gamma)$ on the cylinder be $\dfrac{x-\alpha}{1}=\dfrac{y-\beta}{1}=\dfrac{z-\gamma}{1}$

Any point on the generator is $(\alpha+r, \beta+r, \gamma+r)$

If the point lies on the circle (iii), we have

$$(\alpha+r)^2+(\beta+r)^2+(\gamma+r)^2-(\alpha+r)-(\beta+r)-(\gamma+r)=0$$

and $$(\alpha+r)+(\beta+r)+(\gamma+r)=1$$

or $$(\alpha^2+\beta^2+\gamma^2-\alpha-\beta-\gamma)+r(2\alpha+2\beta+2\gamma-3)+3r^2=0 \qquad \text{...(iv)}$$

and $$r=(1/3)(1-\alpha-\beta-\gamma) \qquad \text{...(v)}$$

Eliminating r between (iv) and (v) we get

$$(\alpha^2+\beta^2+\gamma^2-\alpha-\beta-\gamma)+(1/3)(2\alpha+2\beta+2\gamma-3)(1-\alpha-\beta-\gamma)$$
$$+3(1/9)(1-\alpha-\beta-\gamma)^2=0$$

or $$3\alpha^2+3\beta^2+3\gamma^2-3\alpha-3\beta-3\gamma+(2\alpha+2\beta+2\gamma-3)(1-\alpha-\beta-\gamma)$$
$$+(1-\alpha-\beta-\gamma)^2=0$$

or $$(\alpha^2+\beta^2+\gamma^2)-(\alpha\beta+\beta\gamma+\gamma\alpha)-1=0, \text{ on simplifying.}$$

$\therefore$ The required equation of the cylinder or the lcous of $P(\alpha, \beta, \gamma)$ is

$$x^2+y^2+z^2-xy-yz-zx=1. \qquad \text{Hence proved.}$$

Exercises on § 8.16.

Ex. 1. Find the equation of the right circular cylinder whose axis is the line $(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n$ and radius 3.

[Hint : See § 8.16 Page 68 Ch. VIII]. *(Kanpur 92)*

Ex. 2. The radius of a normal section of right circular cylinder is 2 units; the axis lies along the straight line $\frac{1}{2}(x-1)=-(y+3)=\frac{1}{5}(z-2)$. Find the equation.

Ex. 3. Find the equation of the right circular cylinder whose radius is 1 and axis is z-axis.

Ex. 4. Find the equation of the right circular cylinder whose axis is $\frac{1}{2}(x-1)=-\frac{1}{3}(y-2)=\frac{1}{6}(z+3)$ and radius 2.

***Ex. 5.** Find the equation of the right circular cylinder whose axis is $x/2=y/3=z/6$ and radius 4. *(Gorakhpur 90; Purvanchal 95)*

Ans. $45x^2+40y^2+13z^2-36yz-24zx-12xy-441=0.$

Ex. 6. Find the equation of a righ circular cylinder of radius 2, whose axis passes through $(1, -2, 4)$ and has direction ratios 2, 3, 6.

Ans. $45x^2+40y^2+13z^2-36yz-24zx-12xy-18x+316y-152z+433=0$

Ex. 7. Find the equation of the circular cylinder whose guiding circle is $x^2+y^2+z^2=9,\ x-y+z=0$ **Ans. $x^2+y^2+z^2+xy+yz-zx-(27/2)=0$.

** § 8.17. Equation of Tangent Plane to a cylinder.

To find the equation of the tangent plane to the cylinder $ax^2+2hxy+by^2+2gx+2fy+c=0$ at the point (x_1, y_1, z_1) and to show that this tangent plane touches the cylinder along a generator.

The equation of the given cylinder is

$$ax^2+2hxy+by+2gx+2fy+c=0 \qquad \text{...(i)}$$

Let P be the point (x_1, y_1, z_1) which lies on the cylinder (i).

Therefore $\quad ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0 \qquad$ ...(ii)

Equations of any line through $P(x_1, y_1, z_1)$ are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \qquad \text{...(iii)}$$

Any point on this line is $(x_1+lr,\ y_1+mr,\ z_1+nr)$.

If this point lies on the cylinder (i), then we have

$$a(x_1+lr)^2+2h(x_1+lr)(y_1+mr)+b(y_1+mr)^2 + 2g(x_1+lr)+2f(y_1+mr)+c=0$$

or $\quad r^2(al^2+2hlm+bm^2)+2r[l.(ax_1+hy_1+g)+m(hx_1+by_1+f)] + (ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c)=0$

or $\quad r^2(al^2+2hlm+bm^2)+2r[l(ax_1+hy_1+g) + m(hx_1+by_1+f)]=0$, from (ii) ...(iv)

This is a quadratic in r and one value of r is zero. Now if the line (iii) is a tangent line to the cylinder (i), then this line touches the cylinder *i.e.* this line meets the cylinder only in one point (rather two coincident points) and so both the values of r given by (iv) must be zero.

And from (iv) we find that if the other value of r given by (iv) is zero then

$$l(ax_1+hy_1+g)+m(hx_1+by_1+f)=0 \qquad \text{...(v)}$$

∴ If the line (iii) is a tangent line to the cylinder (i) at $P(x_1, y_1, z_1)$ the d.c.'s of the line (iii) must satisfy the relation (v). Also the locus of this tangent line to the cylinder (i) at $P(x_1, y_1, z_1)$ is the required tangent plane.

∴ Eliminating l, m, n between (iii) and (v), the required equation of the tangent plane to the cylinder (i) at $P(x_1, y_1, z_1)$ is given by

$$(x-x_1)(ax_1+hy_1+g)+(y-y_1)(hx_1+by_1+f)=0$$

or $\quad x(ax_1+hy_1+g)+y(hx_1+by_1+f)=ax_1^2+2hx_1y_1+by_1^2+gx_1+fy_1$

Adding (gx_1+fy_1+c) to both sides, we get

$$x(ax_1+hy_1+g)+y(hx_1+by_1+f)+(gx_1+fy_1+c) = ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c$$

$= 0$, from (ii).

or $\quad \mathbf{axx_1 + h\,(xy_1 + x_1y) + byy_1 + g\,(x + x_1) + f\,(y + y_1) + c = 0.}$ ...(vi)

Again we observe from (i) there is no term containing z in it, hence the axis of the cylinder and consequently the generators of the cylinder are parallel to z-axis whose d.c.'s are 0, 0, 1. **(Note)**

$\therefore$ The equations of the generator through $P\,(x_1, y_1, z_1)$ of the cylinder (i) are

$$\frac{x - x_1}{0} = \frac{y - y_1}{0} = \frac{z - z_1}{1} \qquad \text{...(vii)}$$

Any point on this generator is $(x_1, y_1, r + z_1)$.

$\therefore$ Equation of tangent plane to (i) at this point $(x_1, y_1, z_1 + r)$ is

$$axx_1 + h\,(xy_1 + x_1y) + byy_1 + g\,(x + x_1) + f\,(y + y_1) + c = 0, \qquad \text{...(viii)}$$

obtained by replacingg x_1, y_1, z_1 by $x_1, y_1, z_1 + r$ respectively in (vi).

Now we observe that (vi) and (viii) are the same and also (viii) being free from r is the same for all values of r.

i.e. the equations of the tangent plane at every point of (vii) is the same. Hence the tangent plane to (i) at $P\,(x_1, y_1, z_1)$ touches the cylinder (i) along the generator (vii) through the point P. Hence proved.

***§ 8.18. Envelopoing Cylinder.**

Definition. *The locus of the tangents to a surface drawn in a given direction is called the enveloping cylinder of the surface i.e. the enveloping cylinder is that cylinder whose generators touch a given surface and are parallel to a given straight line.*

(a) Equation of the enveloping cylinder of sphere $x^2 + y^2 + z^2 = a^2$ and whose generators are parallel to the line $x/l = y/m = z/n$.

(*Kanpur* 97, 93; *Kumaun* 96, 92)

Let $P\,(\alpha, \beta, \gamma)$ be a point on the cylinder. Then equations of the generator of the cylinder through P drawn parallel to the given line are

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} = r \text{ (say)} \qquad \text{...(i)}$$

Any point on this generator is $(\alpha + lr, \beta + mr, \gamma + nr)$. If this point lies on the given sphere, we have $(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr) = a^2$

or $\quad r^2\,(l^2 + m^2 + n^2) + 2r\,(l\alpha + m\beta + n\gamma) + (\alpha^2 + \beta^2 + \gamma^2 - a^2) = 0$ ...(ii)

Since the generator (i) is a tangent line to the given sphere, so the two values of r given by (ii) must be equal and the condition for the same is

$$[2\,(l\alpha + m\beta - n\gamma)]^2 = 4\,(l^2 + m^2 + n^2)\,(\alpha^2 + \beta^2 + \gamma^2 - a^2)$$

$\therefore$ The equation of the enveloping cylinder or the locus of $P\,(\alpha, \beta, \gamma)$ is

$$(lx + my + nz)^2 = (l^2 + m^2 + n^2)\,(x^2 + y^2 + z^2 - a^2) \qquad \text{...(iii)}$$

Note. In § 8.11 (a) Page 47 we have proved that the equation of the envelopoing cone of the sphere $x^2 + y^2 + z^2 = a^2$ with vertex at (α, β, γ) is

$$(\alpha x+\beta y+\gamma z-a^2)^2=(x^2+y^2+z^2-a^2)(\alpha^2+\beta^2+\gamma^2-a^2)$$

If the vertex of the above cone be (lr, mr, nr), then above equation reduces to $r^2\left(lx+my+nz-\dfrac{a^2}{r}\right)^2=(x^2+y^2+z^2-a^2)(l^2r^2+m^2r^2+n^2r^2-a^2)$

or $$\left(lx+my+nz-\frac{a^2}{r}\right)^2=(x^2+y^2+z^2-a^2)\left(l^2+m^2+n^2-\frac{a^2}{r^2}\right)$$

Now if $r\to\infty$ *i.e.* if the vertex of the cone is at an infinite distance, then the above equation reduces to

$$(lx+my+nz)^2=(x^2+y^2+z^2-a^2)(l^2+m^2+n^2),$$

which is the same as (ii) above.

Hence the enveloping cylinder of the sphere $x^2+y^2+z^2=a^2$ can be taken as a limiting case of its (sphere's) enveloping cone whose vertex is at an infinite distance.

(b) Equation of the enveloping cylinder of the sphere $\mathbf{x^2+y^2+z^2+2ux+2vy+2wz+d=0}$ whose generators are parallel to the line $\mathbf{x/l=y/m=z/n}$. (*Agra* 91)

Sol. Let $P(\alpha,\beta,\gamma)$ be a point of the cylinder. Then the equations of the generator of the cylinder through P drawn parallel to the given line are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r\text{ (say)} \quad \text{...(i)}$$

Any point on this generator is $(\alpha+lr, \beta+mr, \gamma+nr)$. If this point lies on he given sphere, we have

$$(\alpha+lr)^2+(\beta+mr)^2+(\gamma+nr)^2+2u(\alpha+lr)+2v(\beta+mr)+2w(\gamma+nr)+d=0$$

or $$r^2(l^2+m^2+n^2)+2r(l\alpha+m\beta+n\gamma+ul+vm+wn)+(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d)=0 \quad \text{...(ii)}$$

Since the generator (i) is a tangent line to the given sphere, so the two values of r given by (ii) must be equal and the condition for the same is

'$B^2=4AC$'

i.e. $$[2(l\alpha+m\beta+n\gamma+ul+vm+wn)]^2=4(l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d)$$

$\therefore$ The equation of the enveloping cylinder or the locus of $P(\alpha,\beta,\gamma)$ is

$$(lx+my+nz+ul+vm+wn)^2=(l^2+m^2+n^2)(x^2+y^2+z^2+2ux+2vy+2wz+d)$$

or $$[l(x+u)+m(y+v)+n(z+w)]^2$$

$$= (l^2 + m^2 + n^2)(x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d) \quad \text{...(iii)}$$

Solved Examples on § 8.18

***Ex. 1. Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - 2x + 4y = 1$ having its generators parallel to the line $x = y = z$.**

(*Bundelkhand* 92; *Garhwal* 96; *Kanpur* 92, 91; *Kumaun* 94; *Meerut* 96, 93)

Sol. Let $P(\alpha, \beta, \gamma)$ be any point on the enveloping cylinder then the equaions of the generator through $P(\alpha, \beta, \gamma)$ are

$$\frac{x-\alpha}{1} = \frac{y-\beta}{1} = \frac{z-\gamma}{1} = r \text{ (say)}$$

Any point on this generator is $(\alpha + r, \ \beta + r, \ \gamma + r)$

If this point lies on the given sphere, then we get

$$(\alpha + r)^2 + (\beta + r)^2 + (\gamma + r)^2 - 2(\alpha + r) + 4(\beta + r) = 1$$

or $$3r^2 + 2r(\alpha + \beta + \gamma + 1) + (\alpha^2 + \beta^2 + \gamma^2 - 2\alpha + 4\beta - 1) = 0 \quad \text{...(i)}$$

Since this generator is a tangent to the given sphere, so the values of r obtained from (i) are equal and the condition for the same is

$$[2(\alpha + \beta + \gamma + 1)]^2 = 4^2.3(\alpha^2 + \beta^2 + \gamma^2 - 2\alpha + 4\beta - 1)$$

or $$\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta - 4\alpha + 5\beta - \gamma - 2 = 0$$

$\therefore$ The locus of $P(\alpha, \beta, \gamma)$ *i.e.* the required equation of the enveloping cylinder is $x^2 + y^2 + z^2 - yz - zx - xy - 4x + 5y - z - 2 = 0$. **Ans.**

Ex. 2 (a). Find the equation of a right circular cylinder which envelopes a sphere with centre (a, b, c) and radius r and has the generators parallel to the direction (l, m, n). (*Kumaun* 93)

Sol. The equation of the given sphere is $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

or $$x^2 + y^2 + z^2 - 2ax - 2by - 2cz + (a^2 + b^2 + c^2 - r^2) = 0 \quad \text{...(i)}$$

Let $P(\alpha, \beta, \gamma)$ be any point on the enveloping cylinder.

Then the equations of the generator through $P(\alpha, \beta, \gamma)$ are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = k \text{ (say)},$$

the generators are given parallel to the direction (l, m, n)

Any point on this generator is $(\alpha + lk, \beta + mk \ \gamma + nk)$.

If this point lies on the sphere (i), then we get

$$(\alpha + lk)^2 + (\beta + mk)^2 + (\gamma + nk)^2 - 2a(\alpha + lk) - 2b(\beta + mk)$$
$$- 2c(\gamma + nk) + (a^2 + b^2 + c^2 - r^2) = 0$$

or $$(l^2 + m^2 + n^2)k^2 + 2k(l\alpha + m\beta + n\gamma - al - bm - cn)$$
$$+ (\alpha^2 + \beta^2 + \gamma^2 - 2a\alpha - 2b\beta - 2c\gamma + a^2 + b^2 + c^2 - r^2) = 0$$

or $$k^2(l^2 + m^2 + n^2) - 2k[l(a - \alpha) + m(b - \beta) + n(c - \gamma)]$$
$$+ [(a-\alpha)^2 + (b-\beta)^2 + (c-\gamma)^2 - r^2] = 0 \quad \text{...(ii)}$$

Since this generator is a tangent to the given sphere (i), so the two values of k obtained from (ii) must be equal and the condition for the same is

$$[l(a-\alpha)+m(b-\beta)+n(c-\gamma)]^2=(l^2+m^2+n^2)[(a-\alpha)^2+(b-\beta)^2+(c-\gamma)^2-r^2]$$

$\therefore$ The locus of $P(\alpha, \beta, \gamma)$ or the required equation of the cylinder is

$$[l(a-x)+m(b-y)+n(c-z)]^2=(l^2+m^2+n^2)[(a-x)^2+(b-y)^2+(c-z)^2-r^2] \quad \textbf{Ans.}$$

Ex. 2 (b). Find the equation of the enveloping cylinder of the conicoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ whose generators are parallel to the line

$x/l=y/m=z/n$. (*Kanpur* 96, 92)

Sol. Let $P(\alpha, \beta, \gamma)$ be any point on the enveloping cylinder.
Then the equations of the generator through $P(\alpha, \beta, \gamma)$ are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \text{ (say)}$$

Any point on this generator is $(\alpha+lr, \ \beta+mr, \ \gamma+nr)$.
If this point lies on the given conicoid, then we get

$$\frac{(\alpha+lr)^2}{a^2}+\frac{(\beta+mr)^2}{b^2}+\frac{(\gamma+nr)^2}{c^2}=1$$

or $$r^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)+2r\left(\frac{l\alpha}{a^2}+\frac{m\beta}{b^2}+\frac{n\gamma}{c^2}\right)+\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)=0. \quad \text{...(i)}$$

Since this generator is a tangent to the given conicoid, so the two values of r obtained from (i) must be equal and the condition for same is

$$\left(\frac{l\alpha}{a^2}+\frac{m\beta}{b^2}+\frac{n\gamma}{c^2}\right)^2=\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)$$

$\therefore$ The locus of $P(\alpha, \beta, \gamma)$ or the required equation of the cylinder is

$$\left(\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}\right)^2=\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right) \quad \textbf{Ans.}$$

Ex. 2 (c). Find the equation of the enveloping cylinder to the surface $ax^2+by^2+cz^2=1$, whose generators are parallel to the line

$x/l=y/m=z/n$. (*Kanpur* 95)

Sol. Do as Ex. 2 (b) above.

Ans. $(alx+bmy+cnz)^2=(al^2+bm^2+cn^2)(ax^2+by^2+cz^2-1)$

Ex. 3. Prove that the enveloping cylinder of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ whose generators are parallel to the line

$$\frac{x}{0}=\frac{y}{\pm\sqrt{(a^2-b^2)}}=\frac{z}{c}$$

meet the plane $z=0$ in circles. (*Gorakhpur* 97)

Sol. Let $P(\alpha, \beta, \gamma)$ be any point on the enveloping cylinder. Then the equations of the generator through $P(\alpha, \beta, \gamma)$ are

$$\frac{x-\alpha}{0}=\frac{y-\beta}{\pm\sqrt{(a^2-b^2)}}=\frac{z-\gamma}{c}=r \text{ (say)}$$

Any point on this generator is $(\alpha, \beta \pm r\sqrt{(a^2-b^2)}, \gamma+cr)$

If this point lies on the given ellipsoid, then we get

$$\frac{\alpha^2}{a^2}+\frac{\{\beta \pm r\sqrt{(a^2-b^2)}\}}{b^2}+\frac{(\gamma+cr)^2}{c^2}=1$$

or $$r^2\left[\frac{a^2-b^2}{b^2}+\frac{c^2}{c^2}\right]+2r\left[\frac{c\gamma}{c^2}\pm\frac{\beta\sqrt{(a^2-b^2)}}{b^2}\right]+\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)=0 \quad \text{...(i)}$$

Since this generator is a tangent to the given ellipsoid, so the two values of r obtained from (i) must be equal and the condition for the same is

"$b^2=4ac$" *i.e.*

$$\left[\frac{\gamma}{c}\pm\frac{\beta\sqrt{(a^2-b^2)}}{b^2}\right]^2=\left[\frac{a^2-b^2}{b^2}+\frac{c^2}{c^2}\right]\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right]$$

or $$\frac{\gamma^2}{c^2}+\frac{\beta^2(a^2-b^2)}{b^4}\pm\frac{2\beta\gamma\sqrt{(a^2-b^2)}}{b^2c}=\left(\frac{a^2}{b^2}\right)\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right]$$

$\therefore$ The locus of $P(\alpha, \beta, \gamma)$ or the equation of the cylinder is

$$\frac{z^2}{c^2}+\frac{y^2(a^2-b^2)}{b^4}\pm\frac{2yz\sqrt{(a^2-b^2)}}{b^2c}=\frac{a^2}{b^2}\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right]$$

This meets the plane $z=0$ in the curve

$$\frac{y^2(a^2-b^2)}{b^4}=\frac{a^2}{b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right), \quad z=0 \qquad \textbf{(Note)}$$

i.e. $$\frac{x^2}{b^2}+\frac{y^2}{b^2}=\frac{a^2}{b^2}, z=0 \quad \textit{i.e.} \quad x^2+y^2=a^2, \; z=0$$

which is a circle of radius a on the plane $z=0$. Hence proved.

***Ex. 4. Show that the enveloping cylinder of the conicoid $ax^2+by^2+cz^2=1$ with generators perpendicular to x-axis meets the plane $z=0$ in parabolas.**

Sol. The d.c.'s of the z-axis are 0, 0, 1.

$\therefore$ The d.r.'s of the line perpendicular to z-axis are $l, m, 0$.

Let $P(\alpha, \beta, \gamma)$ be a point on the enveloping cylinder.

Then the equations of the generator through $P(\alpha, \beta, \gamma)$ are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{0}=r \text{ (say)}$$

Any point on it is $(\alpha+lr, \beta+mr, \gamma)$

If this point lies on the given conicoid, we get

$$a(\alpha+lr)^2+b(\beta+mr)^2+c(\gamma)^2=1$$

or $$r^2(al^2+bm^2)+2r(a\alpha l+b\beta m)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0. \quad \text{...(i)}$$

Since this generator is tangent to the given conicoid so the two values of r obtained from (i) must be equal and the condition for the same is

$$(a\alpha l + b\beta m)^2 = (al^2 + bm^2)(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)$$

$\therefore$ The equation of the enveloping cylinder of the given conicoid *i.e.* the locus of $P(\alpha, \beta, \gamma)$ is $(alx + bmy)^2 = (al^2 + bm^2)(ax^2 + by^2 + cz^2 - 1)$

Its section by the plane $z = 0$ is

$$(alx + bmy)^2 = (al^2 + bm^2)(ax^2 + by^2 - 1), \quad z = 0$$

or $$a^2l^2x^2 + b^2m^2y^2 + 2ablmxy = a^2l^2x^2 + abl^2y^2 - al^2 + abm^2x^2 + b^2m^2y^2 - bm^2; \quad z = 0$$

or $$ab(m^2x^2 + l^2y^2 - 2lmxy) = al^2 + bm^2, \quad z = 0$$

or $$ab(mx - ly)^2 = al^2 + bm^2, \quad z = 0,$$

which represents a parabola as the second degree terms form a perfect square.

Ex. 5. Find the equation of the enveloping cone of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and deduce from it the equation of the enveloping cylinder whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

Sol. As in Ex. 4 Page 50 of this chapter we can find that the equation of the enveloping cone of ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ with vertex at (x_1, y_1, z_1) is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} - 1\right)^2 \quad ...(i)$$

Also any point on the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ can be taken as (lr, mr, nr)

$\therefore$ If the vertex of the cone (i) is (lr, mr, nr), the above equation reduces to

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\left(\frac{l^2r^2}{a^2} + \frac{m^2r^2}{b^2} + \frac{n^2r^2}{c^2} - 1\right) = \left(\frac{xlr}{a^2} + \frac{ymr}{b^2} + \frac{nzr}{c^2} - 1\right)^2,$$

or $$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} - \frac{1}{r^2}\right) = \left(\frac{xl}{a^2} + \frac{my}{b^2} + \frac{nz}{c^2} - \frac{1}{r}\right)^2,$$

dividing both sides by r^2. **(Note)**

Now if $r \to \infty$ *i.e.* the vertex of cone is at an infinite distance then the above equation reduces to

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) = \left(\frac{lx}{a^2} + \frac{my}{b^2} + \frac{nz}{c^2}\right)^2,$$

which is the required equation of the **enveloping cylinder.** Ans.

****Ex. 6 (a). Find the equation of the enveloping cylinder of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, whose axis is $x=y=z$ or whose generators are parallel to the line $x=y=z$.** *(Agra 92)*

Sol. Let $P(\alpha,\beta,\gamma)$ be any point on the enveloping cylinder.

Then the equations of the generator through $P(\alpha,\beta,\gamma)$ are

$$\frac{x-\alpha}{1}=\frac{y-\beta}{1}=\frac{z-\gamma}{1}=r \text{ (say)}$$

[Note that the generators are parallel to $x/1=y/1=z/1$]

Any point on this generator is $(\alpha+r,\ \beta+r,\ \gamma+r)$.

If this point $(\alpha+r,\ \beta+r,\ \gamma+r)$ lies on the given ellipsoid, then we have

$$\frac{(\alpha+r)^2}{a^2}+\frac{(\beta+r)^2}{b^2}+\frac{(\gamma+r)^2}{c^2}=1$$

or $$r^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+2r\left(\frac{\alpha}{a^2}+\frac{\beta}{b^2}+\frac{\gamma}{c^2}\right)+\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)=0 \quad \text{...(i)}$$

Since this generator through $P(\alpha,\beta,\gamma)$ is a tangent to the given ellipsoid, so the two values of r obtained from (i) must be equal and the condition for the same is "$B^2=4AC$".

i.e. $$4\left(\frac{\alpha}{a^2}+\frac{\beta}{b^2}+\frac{\gamma}{c^2}\right)^2=4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)$$

$\therefore$ The locus of $P(\alpha,\beta,\gamma)$ or the required equation of the enveloping cylinder of the given ellipsoid is

$$\left(\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}\right)^2=4\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right).$$

or $(xa^{-2}+yb^{-2}+zc^{-2})^2=(a^{-2}+b^{-2}+c^{-2})(x^2a^{-2}+y^2b^{-2}+z^2c^{-2}-1)$. **Ans.**

***Ex. 6 (b). Find the equation of the enveloping cylinder of the conicoid $ax^2+by^2+cz^2=1$ whose generators are parallel to the line $x=y=z$.**

Sol. Let $P(\alpha,\beta,\gamma)$ be any point on the enveloping cylinder.

Then the equations of the generator through $P(\alpha,\beta,\gamma)$ are

$$\frac{x-\alpha}{1}=\frac{y-\beta}{1}=\frac{z-\gamma}{1}=r \text{ (say)}$$

here the generators being parallel to t line $x/1=y/1=z/1$.

Any point $(\alpha+r,\ \beta+r,\ \gamma+r)$ on the given conicoid, then we have

$$a(\alpha+r)^2+b(\beta \quad)^2+c(\gamma+r)^2=1$$

or $$r^2(a+b+c)+2r(a\alpha+b\beta+ \quad)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0 \quad \text{...(i)}$$

Since this generator through $P(\alpha, \beta, \gamma)$ is a tangent to the given conicoid, so the two values of r obtained from (i) must be equal and the condition for the same is "$B^2 = 4AC$"

i.e. $$4(a\alpha + b\beta + c\gamma)^2 = 4(a+b+c)(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)$$

$\therefore$ The locus of $P(\alpha, \beta, \gamma)$ or the required equation of the enveloping cylinder of the given ellipsoid is

$$(ax + by + cz)^2 = (a+b+c)(ax^2 + by^2 + cz^2 - 1)$$

or $$(b+c)x^2 + (c+a)y^2 + (a+b)z^2 - 2abxy - 2bc\,yz - 2ca\,zx - (a+b+c) = 0 \quad \textbf{Ans.}$$

EXERCISES ON CHAPTER VIII

CONE

Ex. 1. Find the equation of the cone which passes through three co-ordinate axes and the straight lines

$$x = -\frac{1}{2}y = \frac{1}{3}z\,;\ \frac{1}{3}x = \frac{1}{2}y = -z.$$

Ex. 2. Show that the locus of the mid-points of chords of the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ drawn parallel to the line $x/l = y/m = z/n$ is the plane

$$x(al + hm + gn) + y(hl + bm + fn) + z(gl + fm + cn) = 0.$$

Ex. 3. Prove that the equation of the right circular cone which passes through the line $x = 3y = -5z$ and has $x = y = z$ as its axis also passes through the axes of co-ordinates.

[**Hint.** See § 8.12 Pages 53-54 of this chapter]

Ans. Cone is $xy + yz + zx = 0$.

Ex. 4. The plane $x - y - z = 1$ meets the co-ordinate axes in A, B, C. Prove that the equaton of the cone generated by lines through O, the origin, to meet the circle ABC is $yz + zx + xy = 0$

Ex. 5. Find the equation of the generator of the cone $x^2 + y^2 - z^2 = 0$ through the point (3, 4, 5).

Ex. 6. Find the equation to a cone which passes through the co-ordinates axes and the two lines $x = 3y = 2z$ and $x = -3y = 3z$.

Ex. 7. A variable plane is parallel to the given plane $(x/a) + (y/b) + (z/c) = 0$ and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the curve

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{a}{c} + \frac{c}{a}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0.$$

[**Hint :** See Ex. 6 Page 5 of this Chapter] *(Meerut 90)*

Ex. 8. Prove that the line $\frac{1}{2}x = -y = \frac{1}{2}z$ is the generator of the cone

$$2x^2 + 4y^2 - 3z^2 = 0.$$

Ex. 9. Find the equation of the cone through the co ordinate axes and the lines in which $lx + my + nz = 0$ cuts the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

Ex. 10. Show that the equation of pair of tangent planes to the cone $ax^2 + by^2 + cz^2 = 0$ passing through the line $x/l = y/m = z/n$ is

$$(al^2 + bm^2 + cn^2)(ax^2 + by^2 + cz^2) = (alx + bmy + cnz)^2$$

Ex. 11. Obtain the general equation of cone passing through the coordinate axes. (*Bundelkhand* 94)

Ex. 12. A is a point on axis OX and B on axis OY, so that angle OAB is constant and equal to α. Taking AB as diameter a circle is drawn whose plane is parallel to OZ. Prove that as AB varies, the circle generates the cone

$$2xy - z^2 \sin 2\alpha = 0 \qquad (\textit{Garhwal } 91)$$

Ex. 13. Show that $f(l, m, n) = 0$, where l, m, n are the direction cosines of the generator of the cone $f(x, y, z) = 0$ (*Kanpur* 97, 94)

[**Hint.** See § 8.02 Cor. Page 2 of this Chapter]

Ex. 14. Find the locus of the vertices of enveloping cones of the surface $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$, if sections of cones by the plane $z = 0$ are circles. (*Kanpur* 96)

Ex. 15. Cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ has three perpendicular generators if

(i) $(1/a) + (1/b) + (1/c) = 0$; (ii) $f + g + h = 0$;

(iii) $a + b + c = 0$; (iv) $a^2 + b^2 + c^2 = 0$ **Ans.** (iii).

CLYINDER

Ex. 1. Write down the equation of the generator of the cylinder $x^2 + y^2 = a^2$ which passes through the point $(a, 0, 0)$.

Ex. 2. Find the equation of the right circular cylinder whose axis is $x = 2y = -z$ and radius is 4. Prove that area of section of this cylinder by the plane $z = 0$ is 24π.

Ex. 3. Find the equation of the cylinder whose axis are parallel to the co-ordinate axes.

[**Hint.** See § 8.15 Cor 1 Page 63 of this chaper].

Ex. 4. Find the equation of the right circular cylinder described on the circle through the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ as the guiding curve.

CHAPTER IX

Conicoids

§ 9.01. Conicoids.

Definition. The conicoids are those surfaces the section of which by planes parallel to the co-ordinates planes are conics. (*Kumaun 93; Lucknow 91*)

The following surfaces are known as conicoids :

(i) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$ (Ellipsoid)

(ii) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$ (Hyperboloid of one sheet)

(iii) $$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$ (Hyperboloid of two sheets)

(iv) $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$$ (Elliptic paraboloid)

(v) $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}$$ (Hyperbolic paraboloid).

In the articles to follow in this chapter the first three conicoids (known as **Central conicoids**) will be discussed individually. The last two (viz. paraboloids) will be discussed in the next chapter on Paraboloids.

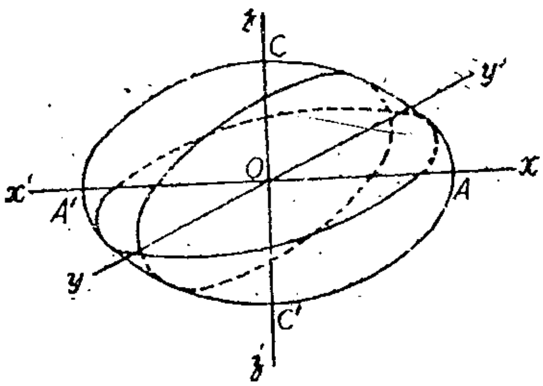
§ 9.02. The Ellipsoid.

The equation of the ellipsoid is $\mathbf{x^2/a^2+y^2/b^2+z^2/c^2=1}$ ...(i)

I. If $P(x_1, y_1, z_1)$ is a point on the surface, then we have

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c_1^2}=1$$

or $$\frac{(-x_1)^2}{a^2}+\frac{(-y_1)^2}{b^2}+\frac{(-z_1)^2}{c^2}=1.$$



(Fig. 1)

This shows that the point $Q(-x_1, -y_1, -z_1)$ also lies on the surface. But the middle point of the chord PQ is origin.

Hence origin bisects every chord of the ellipsoid passing through it. Therefore, origin is called the **centre** of ellipsoid.

II. It is evident that if $P(x_1, y_1, z_1)$ lies on the surface then the point $P_1(x_1, y_1, -z_1)$ also lies on it and the midle point of the chord PP_1 lies on the plane $z=0$ and chord PP_1 is also perpendicular to the plane $z=0$. Hence the plane $z=0$ bisects every chord perpendicular to it. In a similar way we can show that the other coordinate planes $x=0$ and $y=0$ also bisect the chords perpendicular to them.

Hence the surface represented by (i) is symmetrical with respect to the coordinate planes which are also known as the **principal planes of the ellipsoid.** These three principal planes taken in pairs intersect in three lines which are known as the **principal axes of the ellipsoid.** Hence these principal axes are the coordinate axes.

III. The equation (i) can be rewritten as

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{z^2}{c^2}$, which shows that the values of x^2 and y^2 would be negative *i.e.* x and y would be imaginary provided $z^2>c^2$ or in other words the values of x and y would be real only if z lies between $+c$ and $-c$ *i.e.* the surface lies between the planes $z=c$ and $z=-c$. Similarly the surface lies between the planes $x=a$, $x=-a$ and $y=b$, $y=-b$ and hence it is **a closed surface.**

IV. The axis of x given by $y=0, z=0$, meets the surface in points $A(a, 0, 0)$ and $A'(-a, 0, 0)$ which shows that the surface intercepts a length $2a$ on x-axis. Similarly we can show that the surface intercepts lengths $2b$ and $2c$ from y and z-axes respectively. These intercepts $2a, 2b, 2c$ are known as the lengths of the axes of the ellipsoid.

V. The section of the surface of the ellipsoid given by (i) by the plane $z=\lambda$ is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{\lambda^2}{c^2},\ z=\lambda. \qquad \text{...(ii)}$$

Its semi-axes are $a\sqrt{\left(1-\frac{\lambda^2}{c^2}\right)}$ and $b\sqrt{\left(1-\frac{\lambda^2}{c^2}\right)}$ and its centre lies on the z-axis at a distance λ from the origin.

Now the value of λ must lie between $+c$ and $-c$ because the surface is closed, bounded by planes $z=c$, $z=-c$ parallel to xy-plane. Hence for $\lambda>c$ the section of the ellipsoid is not real.

If however $\lambda=c$ or $-c$, then from (ii) we observe that both the axes of the ellipse are zero and so these sections reduce to the points $C(0, 0, c)$ and $C'(0, 0, -c)$ respectively *i.e.* the planes $z=c$ and $z=-c$ touch the ellipsoid at C and C'.

If $\lambda=0$, then the section is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z=0$ on the xy-plane.

From above we conclude that λ increases from 0 to c and decreases from 0 to $-c$, the sections of the ellipsoid (which are ellipses) go on decreasing and ultimately reduce to points $C(0, 0, c)$ and $C'(0, 0, -c)$.

In a similar way we can consider the sections of the ellipsoid given by (i) by planes parallel to yz or zx-planes.

§ 9.03. Hyperboloid of one sheet.

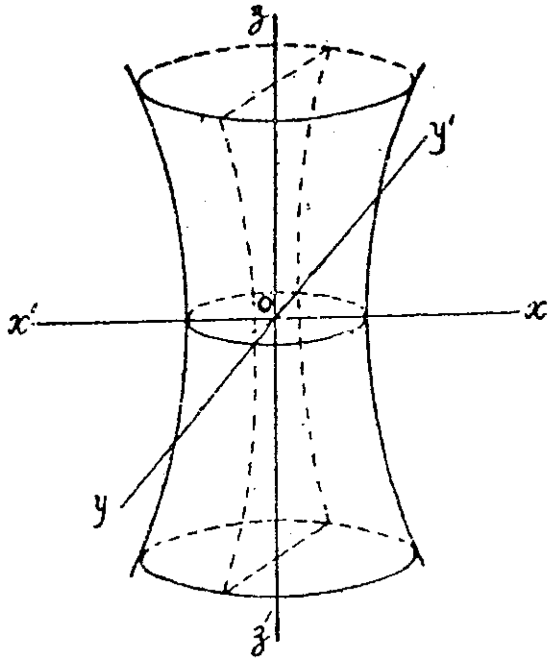
The equation of the hyperboloid of one sheet is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1. \qquad ...(i)$$

As in the case of the ellipsoid we can prove that —

I. The origin bisects every chord of the surface passing through it and hence origin is the centre of this surface too.

II. The coordinate planes bisect every chord perpendicular to them, hence the surface represented by (i) is symmetrical with respect to the coordinate planes which are also known as the principal planes of the surface. These three principal planes taken in pairs intersect in three lines viz. the coordinate axes which are known as principal axes.



(Fig. 2)

III. The axis of x given by $y=0$, $z=0$ meets the surface (ii) on points $(a, 0, 0)$ and $(-a, 0, 0)$ which shows that the surface intersects a length $2a$ on x-axis. Similarly the length intercepted on y-axis is $2b$ whereas the z-axis (*i.e.* $x=0$, $y=0$) does not meet the surface in real points.

IV. The section of the surface given by (i) by the plane $z=\lambda$ which is parallel to the xy-plane is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{\lambda^2}{c^2}, \quad z=\lambda. \qquad ...(ii)$$

Its semi-axis are $a\sqrt{\left(1+\frac{\lambda^2}{c^2}\right)}$ and $b\sqrt{\left(1+\frac{\lambda^2}{c^2}\right)}$ and the centre lies on the z-axis at a distance λ from the origin.

Also as λ increases we conclude from (ii) that the elliptic sections go on increasing in size as lengths of their semi-axes increase. Hence λ varies from $-\infty$ to ∞ for the whole surface.

V. The section of the surface given by (ii) the plane $y=k$ which is parallel to the zx-plane, is the hyperbola

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{k^2}{b^2},\ y=k.$$

In a similar way we can show that the section of the surface given by (ii) by the planes parallel to yz-plane are also hyperbolas.

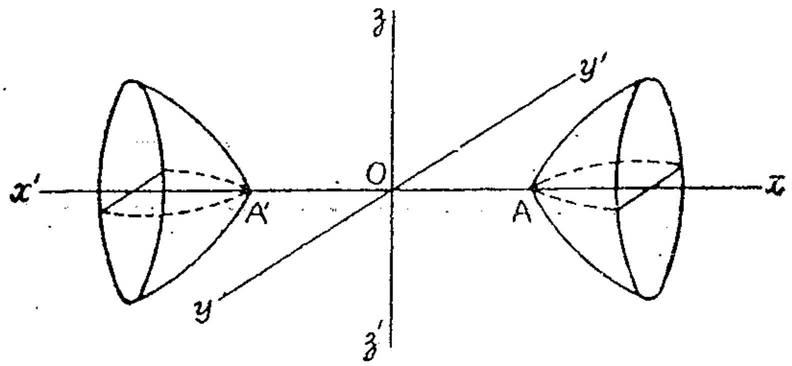
§ 9.04. Hyperboloid of two sheets.

The equation of the hyperboloid of two sheets is

$$\frac{\mathbf{x}^2}{\mathbf{a}^2}-\frac{\mathbf{y}^2}{\mathbf{b}^2}-\frac{\mathbf{z}^2}{\mathbf{c}^2}=\mathbf{1}. \qquad ...(i)$$

As in the case of the ellipsoid, we can prove that

I. The origin bisects every chord of the surface passing through it and hence origin is the centroid of this surface too.



(Fig. 3)

II. The coordinate planes bisect every chord perpendicular to them, hence the surface represented by (i) is symmetrical with respect to the coordinate planes which are also known as the principal planes of the surface. These three principal planes taken in pairs intersect in three lines *viz.* the coordinate axes which are known as principal axes.

III. The axis of x given by $y=0, z=0$ meets the surface (i) in points $(a,0,0)$ and $(-a,0,0)$ which shows that the surface intercepts a length $2a$ on x-axis whereas y and z-axes do not meet the surface in real points.

IV. The section of the surface (i) by the plane $z=\lambda$ is the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1+\frac{\lambda^2}{c^2},\ z=\lambda.$$

Similarly the section of the surface by the plane $y=k$ which is parallel to zx-plane is the hyperbola

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 + \frac{k^2}{b^2}, \quad y = k.$$

V. The section of the surface given by (i) by the plane $x = \mu$, which is parallel to yz-plane, is the ellipse

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{\mu^2}{a^2} - 1, \quad x = \mu. \qquad \text{...(ii)}$$

Its semi-axis are $b\sqrt{[(\mu^2/a^2) - 1]}$ and $c\sqrt{[(\mu^2/a^2) - 1]}$ and the centre lies on x-axis at a distance μ from the origin.

Now the ellipse given by (ii) is real provided $\mu^2 > a^2$ *i.e.* μ is numerically greater then a.

Also the elliptic sections of the surface given by (i) by planes parallel to yz-plane will go on increasingg in-size as μ increases beyond a. And the ellipse given by (ii) is imaginary if $\mu < a$ numerically *i.e.* the surface does not exist for values of $\mu < a$ numerically *i.e.* no part of the surface exists between the plane $x = a$ and $x = -a$.

Note : A conicoid whose all chords through the origin are bisected at the origin is called a **central conicoid.**

The first three conicoids given in § 9.01 are central conicoids.

Exercises on § 9.01 — § 9.04.

Ex. Classify central conicoids.

(Hint : See § 9.01 Page 1 of this chapter).

§ 9.05. Standard Equation of the Conicoid.

The standard equation of the conicoid is $\mathbf{ax^2 + by^2 + cz^2 = 1}$. ...(i)

It represents an ellipsoid if a, b and c are all positive ; a hyperboloid of one sheet if one of these a, b, c is negative and the other two positive ; a hyperboloid of two sheets if two of them are negative and the third positive and an imaginary ellipsoid if all the three are negative.

**§ 9.06. Tangent plane.

To find the equation of the tangent plane to the conicoid $ax^2 + by^2 + cz^2 = 1$ at the point (α, β, γ).

(*Agra* 91; *Allahabad* 92; *Kanpur* 91; *Kumaun* 96, 94, 92, 91; *Rohilkhand* 97)

The equations of any line through (α, β, γ) are

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \qquad \text{...(i)}$$

Any point on this line is $(\alpha + lr, \beta + mr, \gamma + nr)$.

If the line (i) cuts the conicoid at this point, then we have

$$a(\alpha + lr)^2 + b(\beta + mr)^2 + c(\gamma + nr)^2 = 1.$$

or $$r^2(al^2 + bm^2 + cn^2) + 2r(al\alpha + bm\beta + cn\gamma) + a\alpha^2 + b\beta^2 + c\gamma^2 = 1. \qquad \text{...(ii)}$$

Also as (α, β, γ) lies on the conicoid so we have

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 1. \qquad \text{...(iii)}$$

$\therefore$ From (ii) and (iii), we have

$$r^2 (al^2 + bm^2 + cn^2) - 2r\,(al\alpha + bm\beta + cn\gamma) = 0 \qquad \text{...(iv)}$$

This line (i) will touch the conicoid if both the values of r are zero.

$\therefore$ From (iv), we have $\quad al\alpha + bm\beta + cn\gamma = 0. \qquad$ ...(v)

Hence (v) is the condition for the line (i) to be a tangent plane to the conicoid at (α, β, γ).

The locus of all such lines through (α, β, γ) is the tangent line to the conicoid at (α, β, γ) and its equation is obtained by eliminating l, m, n between (i) and (v).

$\therefore$ The required equation of the tangent plane is

$$a\alpha(x-\alpha) + b\beta(y-\beta) + c\gamma(z-\gamma) = 0$$

or
$$a\alpha x + b\beta y + c\gamma z = a\alpha^2 + b\beta^2 + c\gamma^2 = 1, \text{ by (iii)}$$

or
$$\mathbf{a\alpha x + b\beta y + c\gamma z = 1.}$$

****§ 9.07. Condition of tangency.**

(a) *To find the condition that the plane $lx + my + nz = p$ may touch the conicoid* $\quad ax^2 + by^2 + cz^2 = 1.$

(*Agra* 92, 90; *Gorakhpur* 95; *Kumaun* 95, 93; *Rohilkhand* 96)

Let the plane $\quad lx + my + nz = p \qquad$ ...(i)

touch the conicoid $\quad ax^2 + by^2 + cz^2 = 1 \qquad$ ...(ii)

at the point (α, β, γ).

Now the equation of the tangent plane to (i) at (α, β, γ) is

$$a\alpha x + b\beta y + c\gamma z = 1. \qquad \text{...(iii)}$$

If (i) touches (ii) at (α, β, γ), then (i) and (iii) represent the same plane and hence comparing (i) and (ii), we get

$$\frac{a\alpha}{l} = \frac{b\beta}{m} = \frac{c\gamma}{n} = \frac{1}{p} \quad \text{or} \quad \alpha = \frac{l}{ap},\ \beta = \frac{m}{bp},\ \gamma = \frac{n}{cp}, \qquad \text{...(iv)}$$

Also as (α, β, γ) lies on (ii), so we have

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 1$$

or
$$a\left(\frac{l}{ap}\right)^2 + b\left(\frac{m}{bp}\right)^2 + c\left(\frac{n}{cp}\right)^2 = 1, \text{ from (iv)}$$

or
$$\boxed{\frac{\mathbf{l^2}}{\mathbf{a}} + \frac{\mathbf{m^2}}{\mathbf{b}} + \frac{\mathbf{n^2}}{\mathbf{c}} = \mathbf{p^2},} \qquad \text{...(v)}$$

which is the required condition.

The coordinates of the point where (i) touches (ii) are given by (iv) *i.e.*

$$\boxed{\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)}$$

Also from (i) and (v) we conclude that the planes

$\boxed{lx+my+nz=\pm\sqrt{\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)}}$ always touch the conicoid (ii) and are the tangent planes parallel to the plane $lx+my+nz=p$.

(b) *To find the condition that the plane $lx+my+nz=p$ may touch the ellipsoid* $x^2/a^2+y^2/b^2+z^2/c^2=1.$ *(Rohilkhand 90)*

Let the plane $lx+my+nz=p$ ...(i)

touch the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ ...(ii)

at the point (α, β, γ).

Now the equation of the tangent plane to (ii) at (α, β, γ) is

$$\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}+\frac{z\gamma}{c^2}=1. \quad \text{...(iii)}$$

If (i) touches (ii) at (α, β, γ), then (i) and (iii) represent the same plane and hence comparing (i) and (iii) we get

$$\frac{\alpha/a^2}{l}=\frac{\beta/b^2}{m}=\frac{\gamma/c^2}{n}=\frac{1}{p} \text{ or } \alpha=\frac{a^2l}{p},\ \beta=\frac{b^2m}{p},\ \gamma=\frac{c^2n}{p} \quad \text{...(iv)}$$

Also as (α, β, γ) lies on (ii), so we have

$$(\alpha^2/a^2)+(\beta^2/b^2)+(\gamma^2/c^2)=1$$

or $$(a^4l^2/p^2a^2)+(b^4m^2/p^2b^2)+(c^4n^2/p^2c^2)=1$$

or $$\mathbf{a^2l^2+b^2m^2+c^2n^2=p^2}, \quad \text{...(v)}$$

which is the required condition.

Also from (iv) the point of contact is $\left(\frac{a^2l}{p}, \frac{b^2m}{p}, \frac{c^2n}{p}\right)$.

From (i) and (v) we conclude that the planes

$\mathbf{lx+my+nz=\pm\sqrt{(a^2l^2+b^2m^2+c^2n^2)}}$ always touch the ellipsoid (ii) and are the tangent planes parallel to the plane $lx+my+nz=p$.

Cor. If the plane $x\cos\alpha+y\cos\beta+z\cos\gamma=p$ touches the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, then from part (b) above we get

$$a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma=p^2$$

***§ 9.08 Director Sphere.**

Definition. The locus of the point of intersection of three mutually perpendicular tangent planes to a concoid is defined as its director sphere.

To find the equation of the director sphere (from above definition) of the conicoid $ax^2+by^2+cz^2=1.$

(Avadh 95; Garhwal 96; Gorakhpur 96; Rohilkhand 92)

The equation of the conicoid is $ax^2+by^2+cz^2=1$ ...(i)

The equation of the tangent planes to (i) are

$$l_1x+m_1y+n_1z=\sqrt{[(l_1^2/a)+(m_1^2/b)+(n_1^2/c)]} \quad \text{...(ii)}$$

$$l_2x+m_2y+n_2z=\sqrt{[(l_2^2/a)+(m_2^2/b)+(n_2^2/c)]} \quad \text{...(iii)}$$

and $$l_3x+m_3y+n_3z=\sqrt{[(l_3^2/a)+(m_3^2/b)+(n_3^2/c)]} \quad \text{...(iv)}$$

If the above three planes are mutually at right angles, we get

$$l_1^2+l_2^2+l_3^2=1,\ m_1^2+m_2^2+m_3^2=1,\ n_1^2+n_2^2+n_3^2=1$$

and $$l_1m_1+l_2m_2+l_3m_3=0,\ m_1n_1+m_2n_2+m_3n_3=0,$$
$$n_1l_1+n_2l_2+n_3l_3=0. \qquad ...(v)$$

To find the equation of the director sphere of (i) we are to find the locus of the point of intersection of the three tangent planes *i.e.* we are to eliminate l_1, m_1, n_1 ; l_2, m_2, n_2 and l_3, m_3, n_3 from (ii), (iii), (iv) with the help of (v).

Squaring and adding (ii), (iii) and (iv), we get

$$(l_1x+m_1y+n_1z)^2+(l_2x+m_2y+n_2z)^2+(l_3x+m_3y+n_3z)^2$$
$$=\frac{1}{a}(l_1^2+l_2^2+l_3^2)+\frac{1}{b}(m_1^2+m_2^2+m_3^2)+\frac{1}{c}(n_1^2+n_2^2+n_3^2)$$

or $$x^2\Sigma l_1^2+y^2\Sigma m_1^2+z^2\Sigma n_1^2+2xy\Sigma l_1m_1+2yz\Sigma m_1n_1$$
$$+2zx\Sigma n_1l_1=(1/a)+(1/b)+(1/c),\ \because\ \Sigma l_1^2=1 \text{ etc.}$$

or $$\mathbf{x^2+y^2+z^2=(1/a)+(1/b)+(1/c)},$$

$\because \Sigma l_1^2=1$ etc. and $\Sigma l_1m_1=0$ etc.

****Cor.** The equation of the director sphere of the ellipsoid
$$x^2/a^2+y^2/b^2+z^2/c^2=1 \text{ is } \mathbf{x^2+y^2+z^2=a^2+b^2+c^2}.$$

Solved Examples on § 9.01 to § 9.08.

Ex. 1 (a). Find the equation of the tangent plane to the conicoid $2x^2-6y^2+3z^2=5$ at the point (1, 0, – 1).

Sol. The equation of the tangent plane to the given conicoid at (α, β, γ) is

$$2x\alpha-6y\beta+3z\gamma=5. \qquad \text{...See § 9.06 Page 5}$$

$\therefore$ The required tangent plane at (1, 0, – 1) is

$$2x(1)-6y(0)+3z(-1)=5,\ \text{putting } \alpha=1, \beta=0, \gamma=-1$$

or $$2x-3z=5. \qquad \textbf{Ans.}$$

Ex. 1 (b). Show that the plane $3x+12y-6z-17=0$ touches the conicoid $3x^2-6y^2+9z^2+17=0$ and find the point of contact.

(*Avadh* 95; *Kumaun* 90; *Rohilkhand* 92)

Sol. Let the plane $\quad 3x+12y-6z=17 \qquad ...(i)$

touch the conicoid $\quad 3x^2-6y^2+9z^2=-17 \qquad ...(ii)$

at (x_1, y_1, z_1)

The tangent plane at (x_1, y_1, z_1) to (ii) is

$$3xx_1-6yy_1+9zz_1=-17 \qquad ...(iii)$$

As (i) and (iii) represent the same plane, so comparing these we have

$$\frac{3x_1}{3}=\frac{-6y_1}{12}=\frac{9z_1}{-6}=\frac{-17}{17}$$

which gives $x_1=-1, y_1=2, z_1=2/3$

Since these values of x_1, y_1 and z_1 satisfy the equation of the given conicoid, so the point (– 1, 2, 2/3) lies on the given conicoid and hence the plane touches the conicoid (ii) at the point (– 1, 2, 2/3) **Ans.**

Ex. 1 (c). Find the coordinates of the point of contact of the plane $4x-6y+3z=5$ and the conicoid $2x^2-6y^2+3z^2=5$.

Sol. Do as Ex. 1 (b) above. Ans. (2, 1, 1)

Ex. 1 (d). Obtain the tangent planes to the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ which are parallel to the plane

$$lx + my + nz = p$$ (*Kanpur* 92)

Sol. Any plane parallel to the given plane is

$$lx+my+nz=k \qquad ...(i)$$

Now as in § 9.07 (b) Page 7 Ch. IX, we can prove that the condition for the plane (i) to touch the given ellipsoid is

$$a^2l^2+b^2m^2+c^2n^2=k^2. \text{ (To be proved in examination)}$$

or $$k=\pm\sqrt{(a^2l^2+b^2m^2+c^2n^2)}$$

Subsituting these values of k in (i), the required tangent planes are

$$lx+my+nz=\pm\sqrt{(a^2l^2+b^2m^2+c^2n^2)}.$$ **Ans.**

***Ex. 1 (e). Find the equations of two tangent planes of the surface $ax^2+by^2+cz^2=1$ which are parallel to the plane $lx+my+nz=0$.**

(*Meerut* 90)

Hint : Do as Ex. 1 (d) above.

Ans. $lx+my+nz=\pm\sqrt{[(l^2/a)+(m^2/b)+(n^2/c)]}$

Ex. 1 (f). Find the equations of the tangent planes to the surface $3x^2-6y^2+9z^2+17=0$ parallel to the plane $x+4y-2z=0$.

Sol. Any plane parallel to the given plane is $x+4y-2z=p$ ...(i)

If this plane touches the ellipsoid $3x^2-6y^2+9z^2=-17$

or $$(-3/17)x^2+(6/17)y^2+(-9/17)z^2=1$$

then the condition of tangency is

"$(l^2/a)+(m^2/b)+(n^2/c)=p^2$" ...See § 9.07 (a) Page 6

or $$[1^2/(-3/17)]+[4^2/(6/17)]+[2^2/(-9/17)]=p^2$$

or $$p^2=(-17/3)+(136/3)-(68/9)$$

or $$9p^2=-51+408-68=289$$

or $$p=\pm\sqrt{(289/9)}=\pm(17/3)$$

$\therefore$ From (i) the required tangent planes are

$$x+4y-2z=\pm(17/3) \quad \text{or} \quad 3x+12y-6z=\pm 17.$$ **Ans.**

****Ex. 2 (a). Find the equations of the tangent planes to $7x^2+5y^2+3z^2=60$ which pass through the line $7x+10y=30,\ 5y-3z=0$.**

Sol. Any plane through the line $7x+10y-30=0,\ 5y-3z=0$ is

$$(7x+10y-30)+\lambda(5y-3z)=0$$

or $$7x+(10+5\lambda)y-3\lambda z=30 \qquad ...(i)$$

If it touches the conicoid $7x^2+5y^2+3z^2=60$

or $$\frac{7}{60}x^2+\frac{5}{60}y^2+\frac{3}{60}z^2=1$$

then $$\text{“}\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2\text{”}$$ ...See § 9.07 (a) Page 6

or $$\frac{60\,(7)^2}{2}+\frac{60\,(10+5\lambda)^2}{5}+\frac{60\,(-3\lambda)^2}{3}=30^2.$$

or $$420+12\,(100+100\lambda+25\lambda^2)+20\,(9\lambda^2)=900$$

or $2\lambda^2-5\lambda+3=0$ or $(\lambda+1)(2\lambda+3)=0$ or $\lambda=-1,-(3/2)$

$\therefore$ From (i) the required tangent planes are

$$7x+5y+3z=30;\quad 14x+10y+9z=50.$$ **Ans.**

Ex. 2 (b). Find the equations to the tangent planes to the hyperbolid $2x^2-6y^2+3z^2=5$ which pass through the line

$$x+9y-3z=0=3x-3y+6z-5.$$

(*Garhwal* 95, 94, 93, 92; *Lucknow* 92; *Meerut* 90 *S*)

Sol. Any plane through the given line is

$$(x+9y-3z)+\lambda\,(3x-3y+6z-5)=0$$

or $$(1+3\lambda)\,x+(9-3\lambda)\,y+(6\lambda-3)\,z=5\lambda \quad \text{...(i)}$$

If this plane touches the given hyperboloid

$$2x^2-6y^2+3z^2=5 \quad \text{or} \quad \frac{x^2}{(5/2)}+\frac{y^2}{(-5/6)}+\frac{z^2}{(5/3)}=1$$

then $$\text{“}a^2l^2+b^2m^2+c^2n^2=p^2\text{”}$$ ...See § 9.07 (b) Page 7

or $(5/2)\,(1+3\lambda)^2+(-5/6)\,(9-3\lambda)^2+(5/3)\,(6\lambda-3)^2=(5\lambda)^2$ **(Note)**

or $15\,(9\lambda^2+6\lambda+1)-5\,(9\lambda^2-54\lambda+81)+10\,(36\lambda^2-36\lambda+9)=150\lambda^2$

or $300\lambda^2-300=0$ or $\lambda^2=1$ or $\lambda=\pm 1.$

$\therefore$ From (i), the required tangent planes are

$$(1\pm 3)\,x+(9\mp 3)\,y+(\pm 6-3)\,z=\pm 5$$

or $$4x+6y+3z=5 \quad \text{and} \quad 2x-12y+9z=5.$$ **Ans.**

***Ex. 3 (a). Tangent planes are drawn to the conicoid $ax^2+by^2+cz^2=1$ through (α,β,γ). Show that the perpendicular from the centre of the conicoid to these planes generate the cone**

$$(\alpha x+\beta y+\gamma z)^2=x^2/a+y^2/b+z^2/c$$

Sol. Any plane through (α,β,γ) is $l\,(x-\alpha)+m\,(y-\beta)+n\,(z-\gamma)=0$

or $$lx+my+nz=l\alpha+m\beta+n\gamma \quad \text{...(i)}$$

If it is a tangent plane to the conicoid, then

$$l^2/a+m^2/b+n^2/c=(l\alpha+m\beta+n\gamma)^2$$ ....See § 9.07 (a) Page 6

Also the equations of the line through the centre (0, 0, 0) of the conicoid perpendicular to (i) are $x/l=y/m=z/n.$ ...(iii)

Its locus is obtained by eliminating l, m, n between (ii) and (iii) and is

$$x^2/a+y^2/b+z^2/c=(x\alpha+y\beta+z\gamma)^2$$ Hence proved.

***Ex. 3 (b). Tangent planes are drawn to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ through the point (α, β, γ). Prove that the perpendiculars to them from the origin generate the cone**

$$(\alpha x + \beta y + \gamma z)^2 = a^2x^2 + b^2y^2 + c^2z^2$$

Sol. Any plane through (α, β, γ) is $l(x-\alpha) + m(y-\beta) + n(z-\gamma) = 0$

or $$lx + my + nz = l\alpha + m\beta + n\gamma \quad \text{...(i)}$$

If it is a tangent plane to the given ellipsoid, then

$$a^2l^2 + b^2m^2 + c^2n^2 = (l\alpha + m\beta + n\gamma)^2 \quad \text{...See § 9.07 (b) – (v) Page 7}$$

Also the equation of the line through the centre (0, 0, 0) of the given ellipsoid perpendicular to (i) is $x/l = y/m = z/n$ ...(iii)

Its locus is obtained by eliminating l, m, n between (i) and (iii) and is

$$a^2x^2 + b^2y^2 + c^2z^2 = (x\alpha + y\beta + z\gamma)^2.$$ Hence proved.

***Ex. 4. Prove that the locus of points from which three mutually perpendicular tangent planes can be drawn to touch the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, $z = 0$ is the sphere $x^2 + y^2 + z^2 = a^2 + b^2$.**

Sol. Let the tangent plane to the given ellipse

$$(x^2/a^2) + (y^2/b^2) = 1, \quad z = 0 \quad \text{...(i)}$$

be $$lx + my + nz = p. \quad \text{...(ii)}$$

The plane (ii) meets the plane $z = 0$ in the line

$$lx + my = p, \ z = 0.$$

i.e. The line $lx + my = p$, $z = 0$ is in the xy-plane.

If the plane (ii) touches the ellipse (i) then the line $lx + my = p$, $z = 0$ touches the ellipse $x^2/a^2 + y^2/b^2 = 1$ in the xy-plane and the condition for the same is $p^2 = a^2l^2 + b^2m^2$ **(See Author's Coordinate Geometry)**

Hence from (ii) the equation of any tangent plane to the ellipse (i) is

$$lx + my + nz = \sqrt{(a^2l^2 + b^2m^2)} \quad \text{...(iii)}$$

With the help of (iii), let the three mutually perpendicular tangent planes to the ellipse (i) be $l_1x + m_1y + n_1z = \sqrt{(a^2l_1^2 + b^2m_1^2)}$,

$$l_2x + m_2y + n_2z = \sqrt{(a^2l_2^2 + b^2m_2^2)}$$

and $$l_3x + m_3y + n_3z = \sqrt{(a^2l_3^2 + b^2m_3^2)}$$

Squaring and adding these we have

$$(l_1x + m_1y + n_1z)^2 + (l_2x + m_2y + n_2z)^2 + (l_3x + m_3y + n_3z)^2$$
$$= (a^2l_1^2 + b^2m_1^2) + (a^2l_2^2 + b^2m_2^2) + (a^2l_3^2 + b^2m_3^2)$$

or $$(l_1^2 + l_2^2 + l_3^2)x^2 + (m_1^2 + m_2^2 + m_3^2)y^2 + (n_1^2 + n_2^2 + n_3^2)z^2$$
$$+ 2(l_1m_1 + l_2m_2 + l_3m_3)xy + 2(m_1n_1 + m_2n_2 + m_3n_3)yz$$
$$+ 2(n_1l_1 + n_2l_2 + n_3l_3)zx = a^2(l_1^2 + l_2^2 + l_3^2) + b^2(m_1^2 + m_2^2 + m_3^2)$$

or $$(1)x^2+(1)y^2+(1)z^2+2(0)xy+2(0)yz+2(0)zx$$
$$=a^2(1)+b^2(1) \quad \because \quad \Sigma l_1^2=1,\ \Sigma l_1 m_1=0 \text{ etc.}$$

or $$x^2+y^2+z^2=a^2+b^2$$ Hence proved.

Ex. 5 (a). The tangent plane to the surface $x^2+12y^2+4z^2=8$ at the point (1, 1/2, 1) meets the coordinate axes at A, B, C. Find the centroid of Δ ABC.

Sol. The tangent plane to the given surface at (1, 1/2, 1) is
$$x(1)+12y(1/2)+4z(1)=8$$
or $x+6y+4z=8$, which meets the coordinate axes at $A(8,0,0)$ $B(0,4/3,0)$ and $C(0,0,2)$.

∴ Centroid of ΔABC is
$$\frac{8+0+0}{3},\ \frac{0+(4/3)+0}{3},\ \frac{0+0+2}{3} \quad i.e. \left(\frac{8}{3},\frac{4}{9},\frac{2}{3}\right)$$ **Ans.**

***Ex. 5 (b). A tangent plane to the conicoid $ax^2+by^2+cz^2=1$ meets the coordinate axes in P, Q and R. Find the locus of the centroid of the Δ PQR.**

(*Kanpur* 91)

Sol. Any tangent plane to the given conicoid is
$$lx+my+nz=\sqrt{(l^2/a+m^2/b+n^2/c)} \quad \text{...(i)}$$
This plane meets the x-axis at P, so the coordinates of P are $[(1/l)\sqrt{(l^2/a+m^2/b+n^2/c)},0,0]$, putting $y=0=z$ in (i).

Similarly Q and R are
$$\left[0,\frac{1}{m}\sqrt{\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)},0\right] \text{ and } \left[0,0,\frac{1}{n}\sqrt{\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)}\right]$$

∴ If (x_1, y_1, z_1) be the centroid of ΔPQR, then
$$x_1=\frac{1}{3}\left[\frac{1}{l}\sqrt{\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)}+0+0\right]=\frac{1}{3l}\left[\sqrt{\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)}\right]$$
$$y_1=\frac{1}{3m}\sqrt{\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)} \text{ and } z_1=\frac{1}{3n}\sqrt{\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)}$$
$$\therefore (3lx_1)^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c} \text{ or } \frac{9l^2}{a}=\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)\cdot\frac{1}{ax_1^2}$$
Similarly $$\frac{9m^2}{b}=\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)\frac{1}{by_1^2};\ \frac{9n^2}{c}=\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)\frac{1}{cz_1^2}$$

Adding $9\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)=\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)\left(\frac{1}{ax_1^2}+\frac{1}{by_1^2}+\frac{1}{cz_1^2}\right)$

or $$\frac{1}{ax_1^2}+\frac{1}{by_1^2}+\frac{1}{cz_1^2}=9$$

$\therefore$ The required locus is $(1/ax^2)+(1/by^2)+(1/cz^2)=9$. **Ans.**

****Ex. 5 (c). A tangent plane to the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ meets the co-ordinate axes in the points P, Q, and R. Find the locus of the centroid of the triangle PQR.** *(Gorakhpur 97)*

Sol. Any tangent plane to the given ellipsoid is

$$lx+my+nz=\sqrt{(a^2l^2+b^2m^2+c^2n^2)} \quad ...(i)$$

This plane meets the x-axis at P, So the coordinates P are $[(1/l)\sqrt{(a^2l^2+b^2m^2+c^2n^2)}, 0, 0]$ putting $y=0=z$ in (i).

Similarly Q and R are

$$\left[0, \frac{1}{m}\sqrt{(a^2l^2+b^2m^2+c^2n^2)}, 0\right] \text{ and } \left[0, 0, \frac{1}{n}\sqrt{(a^2l^2+b^2m^2+c^2n^2)}\right]$$

$\therefore$ If (x_1, y_1, z_1) be the centroid of ΔPQR, then

$$x_1=\frac{1}{3}\left[\frac{1}{l}\sqrt{(a^2l^2+b^2m^2+c^2n^2)}+0+0\right]=\frac{1}{3l}\sqrt{(a^2l^2+b^2m^2+c^2n^2)}$$

Similarly $\quad y_1=(1/3m)\sqrt{(a^2l^2+b^2m^2+c^2n^2)}$

and $\quad z_1=(1/3n)\sqrt{(a^2l^2+b^2m^2+c^2n^2)}$.

$\therefore (3lx_1)^2=a^2l^2+b^2m^2+c^2n^2$ or $9l^2x_1^2=a^2l^2+b^2m^2+c^2n^2$

or $$9a^2l^2/(a^2l^2+b^2m^2+c^2n^2)=a^2/x_1^2$$

Similarly $$9b^2m^2/(a^2l^2+b^2m^2+c^2n^2)=b^2/y_1^2$$
$$9c^2n^2/(a^2l^2+b^2m^2+c^2n^2)=c^2/z_1^2$$

Adding these we get

$$9(a^2l^2+b^2m^2+c^2n^2)/(a^2l^2+b^2m^2+c^2n^2)=(a^2/x_1^2)+(b^2/y_1^2)+(c^2/z_1^2)$$

or $$9=(a^2/x_1^2)+(b^2/y_1^2)+(c^2/z_1^2)$$

$\therefore$ The required locus of (x_1, y_1, z_1) is $(a^2/x^2)\ (b^2/y^2)+(c^2/z^2)=9$. **Ans.**

***Ex. 6 (a). Find the locus of the foot of the central perpendicular on varying tangent planes to the ellipsoid**

$$x^2/a^2+y^2/b^2+z^2/c^2=1. \quad \textit{(Meerut 90 S)}$$

Sol. Any tangent plane to the given ellipsoid is

$$lx+my+nz=\sqrt{(a^2l^2+b^2m^2+c^2n^2)} \quad ...(i)$$

The equations of perpendicular from the centre (0, 0, 0) of the given ellipsoid to the plane (i) [the d.c.'s of the normal to (i) are l, m, n] are

$$x/l=y/m=z/n=\lambda \text{ (say)} \quad ...(ii)$$

We are to find the locus of the foot of the perpendicular (ii) on the plane (i), which can be obtained by eliminating l, m, n between (i) and (ii). Hence the required equation is

$$(x/\lambda)\, x + (y/\lambda)\, y + (z/\lambda)\, z = \sqrt{[a^2 (x/\lambda)^2 + b^2 (y/\lambda)^2 + c^2 (z/\lambda)^2]}$$

or $$(x^2 + y^2 + z^2)^2 = a^2 x^2 + b^2 y^2 + c^2 z^2 \qquad \textbf{Ans.}$$

***Ex. 6 (b). Find the locus of the perpendicular from the origin to the tangent planes to the conicoid $ax^2 + by^2 + cz^2 = 1$ which cuts off from its axes intercepts, the sum of whose reciprocals is equal to a constant k.**

Sol. Any tangent plane to the given conicoid is

$$lx + my + nz = \sqrt{[(l^2/a) + (m^2/b) + (n^2/c)]}. \qquad \text{...(i)}$$

The intercept made by this plane on x-axis *i.e.* $y = 0 = z$ is

$$(1/l)\sqrt{[(l^2/a) + (m^2/b) + (n^2/c)]}$$

Similarly the intercepts made by plane (i) on y and z-axes are $(1/m)\sqrt{[(l^2/a) + (m^2/b) + (n^2/c)]}$ and $(1/n)\sqrt{[(l^2/a) + (m^2/b) + (n^2/c)]}$.

Given the sum of the reciprocals of these intercepts $= k$.

i.e. $$\frac{l + m + n}{\sqrt{[(l^2/a) + (m^2/b) + (n^2/c)]}} = k$$

or $$(l + m + n)^2 = k^2 [(l^2/a) + (m^2/b) + (n^2/c)]. \qquad \text{...(ii)}$$

Also the equations of the perpendicular from the origin (0, 0, 0) to (i) are

$$x/l = y/m = z/n = 1/\lambda \text{ (say)} \qquad \text{...(iii)}$$

Eliminating l, m, n between (ii) and (iii), we get

$$[(\lambda x) + (\lambda y) + (\lambda z)]^2 = k^2 [(\lambda^2 x^2/a) + (\lambda^2 y^2/b) + (\lambda^2 z^2/c)]$$

or $$(x + y + z)^2 = k^2 [(x^2/a) + (y^2/b) + (z^2/c)] \qquad \textbf{Ans.}$$

Ex. 7 (a). Tangent planes are drawn to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ through the point (α, β, γ). Prove that perpendiculars to them drawn through (α, β, γ) generate the cone

$$[\alpha (x - \alpha) + \beta (y - \beta) + \gamma (z - \gamma)]^2 = a^2 (x - \alpha)^2 + b^2 (y - \beta)^2 + c^2 (z - \gamma)^2.$$

Sol. The equation of any plane through (α, β, γ) is

$$l (x - \alpha) + m (y - \beta) + n (z - \gamma) = 0 \qquad \textbf{(Note)}$$

or $$lx + my + nz = l\alpha + m\beta + n\gamma \qquad \text{...(i)}$$

Also we know that the equation of any tangent plane to the given ellipsoid is $$lx + my + nz = \sqrt{(a^2 l^2 + b^2 m^2 + c^2 n^2)}. \qquad \text{...(ii)}$$

$\therefore$ If (i) is a tangent plane to the given ellipsoid, then comparing (i) and (ii) we get $$(l\alpha + m\beta + n\gamma)^2 = a^2 l^2 + b^2 m^2 + c^2 n^2 \qquad \text{...(iii)}$$

Also the equations of any perpendicular to the plane (ii) from (α, β, γ) are

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} = 1/\lambda \text{ (say)}. \qquad \text{...(iv)}$$

Eliminating l, m, n between (iii) and (iv) we get the equation of the cone generated by (iv) as

$$[\lambda (x-\alpha)\alpha + \lambda (y-\beta)\beta + \lambda (z-\gamma)\gamma]^2$$
$$= a^2\lambda^2 (x-\alpha)^2 + b^2\lambda^2 (y-\beta)^2 + c^2\lambda^2 (z-\gamma)^2$$

or $\quad [\alpha (x-\alpha) + \beta (y-\beta) + \gamma (z-\gamma)] = a^2 (x-\alpha)^2 + b^2 (y-\beta)^2 + c^2 (z-\gamma)^2.$

Hence proved.

Ex. 7 (b). Tangent planes are drawn to the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$ through the point (α, β, γ). Prove that the perpendiculars to them from the origin generate the cone

$$(\alpha x + \beta y + \gamma z)^2 = a^2x^2 + b^2y^2 + c^2z^2.$$

Sol. The equation of any plane through (α, β, γ) is

$$l (x-\alpha) + m (y-\beta) + n (z-\gamma) = 0$$

or $$lx + my + nz = l\alpha + m\beta + n\gamma \quad \text{...(i)}$$

Also the equation of any tangent plane to the given ellipsoid is

$$lx + my + nz = \sqrt{(a^2l^2 + b^2m^2 + c^2n^2)} \quad \text{...(ii)}$$

If (i) is a tangent plane to the given elliposid, then comparing (i) and (ii) we get $$(l\alpha + m\beta + n\gamma)^2 = a^2l^2 + b^2m^2 + c^2n^2 \quad \text{...(iii)}$$

Also equations of the perpendicular line from (0, 0, 0) to the plane (i) are

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n} = \frac{1}{\lambda} \text{ (say)} \quad \text{...(iv)}$$

Eliminating l, m, n between (iii) and (iv) we get the equation of the cone generated by (iv) as $\quad (\lambda x\alpha + \lambda y\beta + \lambda z\gamma)^2 = a^2\lambda^2x^2 + b^2\lambda^2y^2 + c^2\lambda^2z^2$

or $\quad (\alpha x + \beta y + \gamma z)^2 = a^2x^2 + b^2y^2 + c^2z^2$ Hence proved.

***Ex. 8. If 2r is the distance between the parallel tangent planes to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, prove that a line through the origin perpendicular to the planes lies on the cone**

$$x^2 (a^2 - r^2) + y^2 (b^2 - r^2) + z^2 (c^2 - r^2) = 0$$

Sol. We know that the equations of two tangent planes to the ellipsoid are

$lx + my + nz = \pm \sqrt{(a^2l^2 + b^2m^2 + c^2n^2)}$ ...See § 9.07 (b) Page 7

The distance between the above two planes

$$= \frac{2\sqrt{(a^2l^2 + b^2m^2 + c^2n^2)}}{\sqrt{(l^2 + m^2 + n^2)}} \quad \text{(Note)}$$

$$= 2r \text{ (given)}.$$

$$\therefore \quad r^2 (l^2 + m^2 + n^2) = a^2l^2 + b^2m^2 + c^2n^2$$

or $$l^2 (a^2 - r^2) + m^2 (b^2 - r^2) + n^2 (c^2 - r^2) = 0 \quad \text{...(i)}$$

Also the equations of perpendicular from (0, 0, 0) to the above plane are

$$x/l = y/m = z/n = 1/\lambda \text{ (say)} \quad \text{...(ii)}$$

Eliminating l, m, n between (i) and (ii), we get the locus of (ii) as

$$(\lambda x)^2 (a^2 - r^2) + (\lambda y)^2 (b^2 - r^2) + (\lambda z)^2 (c^2 - r^2) = 0$$

or $$x^2 (a^2 - r^2) + y^2 (b^2 - r^2) + z^2 (c^2 - r^2) = 0$$ Hence proved.

***Ex. 9. Prove that the equation of the tangent planes to the conicoid $ax^2 + by^2 + cz^2 = 1$, which passes through the line**

$$u \equiv lx + my + nz - p = 0,\ u' \equiv l'x + m'y + n'z - p' = 0 \text{ is}$$

$$u^2 \left(\frac{l'^2}{a} + \frac{m'^2}{b} + \frac{n'^2}{c} - p'^2\right) - 2uu'\left(\frac{ll'}{a} + \frac{mm'}{b} + \frac{nn'}{c} - pp'\right) + u'^2 \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} - p\right) = 0$$

Sol. The equation of any plane through the given line is

$$u + \lambda u' = 0 \quad \text{...(i)}$$

i.e. $$(lx + my + nz - p) + \lambda (l'x + m'y + n'z - p') = 0$$

or $$(l + \lambda l')x + (m + \lambda m')y + (n + \lambda n')z = p + \lambda p' \quad \text{...(ii)}$$

If this plane (ii) touches the given conicoid then

"$(l^2/a) + (m^2/b) + (n^2/c) = p^2$" ...See § 9.07 (a)-(v) Page 6

or $$[(l + \lambda l')^2/a] + [(m + \lambda m')^2/b] + [(n + \lambda n')^2/c] = (p + \lambda p')^2$$

or $$\Sigma\left[\left(l - \frac{ul'}{u'}\right)^2 / a\right] = \left(p - \frac{up'}{u'}\right)^2, \text{ putting } \lambda = -\frac{u}{u'} \text{ from (i)}$$

or $$\Sigma\,[(lu' - ul')^2/a] = (pu' - up')^2$$

or $$u^2\left(\frac{l'^2}{a} + \frac{m'^2}{b} + \frac{n'^2}{c} - p'^2\right) + u'^2\left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} - p^2\right) - 2uu'\left(\frac{ll'}{a} + \frac{mm'}{b} + \frac{nn'}{c} - pp'\right) = 0.$$ Hence proved.

***Ex. 10. Show that the tangent planes at the extremities of any diameter of an ellipsoid are parallel.**

Sol. Let the equation of the ellipsoid be

$$(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1 \quad \text{...(i)}$$

As its centre is (0, 0, 0) so any diameter of this ellipsoid is a line thorugh (0, 0, 0) and its equations are given by $x/l = y/m = z/n$ ...(ii)

Any point on this diameter is (lr, mr, nr). If this point lies on (i), then

$$(l^2r^2/a^2) + (m^2r^2/b^2) + (n^2r^2/c^2) = 1$$

or $$r^2\,[(l^2/a^2) + (m^2/b^2) + (n^2/c^2)] = 1$$

or $$r = \pm 1/\sqrt{[(l/a)^2 + (m/b)^2 + (n/c)^2]} = \pm k \quad \text{...(iii)}$$

$\therefore$ The extremities of the diameter (ii) are

$$(lk, mk, nk) \quad \text{and} \quad (-lk, -mk, -nk),$$

where k is given by (iii).

Now the equation of the tangent plane to the ellipsoid (i) at (lk, mk, nk) is given by $\frac{xlk}{a^2}+\frac{ymk}{b^2}+\frac{znk}{c^2}=1$ or $\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}=\frac{1}{k}$ ...(iv)

Similarly the equation of the tangent plane to (i) at the other extremity $(-lk, -mk, -nk)$ of (ii) is $\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}=-\frac{1}{k}$ ...(v)

Since the equations (iv) and (v) differ only in the constant terms, so these represent parallel planes (each being a linear equation in x, y, z). Hence proved.

***Ex. 11. Through a fixed point (k, 0, 0) pairs of perpendicular lines are drawn to the conicoid $ax^2+by^2+cz^2=1$. Show that the plane through any pair touches the cone**

$$\frac{(x-k)^2}{(b+c)(ak^2-1)}+\frac{y^2}{c(ak^2-1)-a}+\frac{z^2}{b(ak^2-1)-a}=0.$$

Sol. Any line through the point $(k, 0, 0)$ is

$$\frac{x-k}{l}=\frac{y-0}{m}=\frac{z-0}{n} \quad ...(i)$$

Any point on (i) is $(k+lr, mr, nr)$, which is at a distance r from $(k, 0, 0)$.

$\therefore$ The distances of the points where the line (i) meets the given conicoid

$$ax^2+by^2+cz^2=1 \quad ...(ii)$$

are given by $a(k+lr)^2+b(mr)^2+c(nr)^2=1$

or $r^2(al^2+bm^2+cn^2)+2aklr+(ak^2-1)=0$ ...(iii)

If the line (i) touches (ii) at $(k, 0, 0)$, then the two values of r given by (iii) must be coincident and the condition for the same is "$B^2=4AC$"

i.e. $(2akl)^2=4(al^2+bm^2+cn^2)(ak^2-1)$

or $(al^2+bm^2+cn^2)(ak^2-1)=a^2k^2l^2$ ...(iv)

Now let the two perpendicular tangent lines through $(k, 0, 0)$ be

$$\frac{x-k}{l_1}=\frac{y}{m_1}=\frac{z}{n_1} \text{ and } \frac{x-k}{l_2}=\frac{y}{m_2}=\frac{z}{n_2} \quad ...(v)$$

Then from (iv) we get $(al_1^2+bm_1^2+cn_1^2)(ak^2-1)=a^2k^2l_1^2$

and $(al_2^2+bm_2^2+cn_2^2)(ak^2-1)=a^2k^2l_2^2$

Adding these we get

$$[a(l_1^2+l_2^2)+b(m_1^2+m_2^2)+c(n_1^2+n_2^2)](ak^2-1)=a^2k^2(l_1^2+l_2^2) \quad ...(vi)$$

If the line $(x-k)/l_3=(y-0)/m_3=(z-0)/n_3$ be the normal to the plane containing the tangent lines given by (v), then we obtain a set of three mutually perpendicular lines for which we have the relations

$$l_1^2 + l_2^2 + l_3^2 = 1 = l_3^2 + m_3^2 + n_3^2, \text{ etc.} \qquad \textbf{(Note)}$$

i.e. $$l_1^2 + l_2^2 = m_3^2 + n_3^2$$

Similarly $$m_1^2 + m_2^2 = n_3^2 + l_3^2,\ n_1^2 + n_2^2 = l_3^2 + m_3^2$$

Substituting these values in (vi) we get

$$[a\,(m_3^2 + n_3^2) + b\,(n_3^2 + l_3^2) + c\,(l_3^2 + m_3^2)]\,(ak^2 - 1) = a^2k^2\,(m_3^2 + n_3^2)$$

or $$l_3^2\,[(b+c)\,(ak^2 - 1)] + m_3^2\,[(a+c)\,(ak^2 - 1) - a^2k^2] + n_3^2\,[(a+b)\,(ak^2 - 1) - a^2k^2] = 0$$

or $$l_3^2\,(b+c)\,(ak^2 - 1) + m_3^2\,[c\,(ak^2 - 1) - a] + n_3^2\,[b\,(ak^2 - 1) - a] = 0 \quad \text{...(viii)}$$

Eliminating l_3, m_3, n_3 between (viii) and $\dfrac{x-k}{l_3} = \dfrac{y}{m_3} = \dfrac{z}{n_3}$ we find that the normal to the plane containing the lines given by (v) generators the cone

$$(x-k)^2\,(b+c)\,(ak^2 - 1) + y^2\,[c\,(ak^2 - 1) - a] + z^2\,[b\,(ak^2 - 1) - a] = 0$$

and the plane itself touches the reciprocal cone

$$\frac{(x-k)^2}{(b+c)\,(ak^2 - 1)} + \frac{y^2}{c\,(ak^2 - 1) - a} + \frac{z^2}{b\,(ak^2 - 1) - a} = 0 \qquad \text{Hence proved.}$$

[**Note :** See the section on reciprocal cones in the chapter VIII].

***Ex. 12. If the line of intersection of perpendicular tangent planes to the ellipsoid whose equation referred to rectangular axes is $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ passes through the fixed point (0, 0, k), show that it lies on the cone**

$$\mathbf{x^2\,(b^2 + c^2 - k^2) + y^2\,(c^2 + a^2 - k^2) + (z-k)^2\,(a^2 + b^2) = 0} \qquad \textit{(Meerut 90)}$$

Sol. The equation of any plane through $(0, 0, k)$ is

$$l\,(x-0) + m\,(y-0) + n\,(z-k) = 0 \text{ or } lx + my + nz = nk. \quad \text{...(i)}$$

If the plane (i) is a tangent plane to the given ellipsoid, then

$$a^2l^2 + m^2b^2 + c^2n^2 = n^2k^2 \qquad \text{...See § 9.07 (b) Page 7}$$

or $$a^2l^2 + m^2b^2 + n^2\,(c^2 - k^2) = 0. \quad \text{...(ii)}$$

Again the equations of any line through $(0, 0, k)$ are

$$\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z-k}{\nu} \quad \text{...(iii)}$$

Since this line (iii) lines on the plane (i), so we have

$$l\lambda + m\mu + n\nu = 0. \quad \text{...(iv)}$$

From (ii) and (iv) we find that there are two sets of values of the d.c.'s l, m, n of the normal to (i) and hence there will be two tangent planes to the given ellipsoid.

Eliminating n between (ii) and (iv) we get

$$a^2l^2 + b^2m^2 + (c^2 - k^2)\left(-\frac{l\lambda + m\mu}{\nu}\right)^2 = 0$$

or $\quad l^2(a^2\nu^2+\lambda^2c^2-\lambda^2k^2)+m^2(b^2\nu^2+\mu^2c^2-\mu^2k^2)+2lm\lambda\mu(c^2-k^2)=0$

or $\quad (a^2\nu^2+\lambda^2c^2-\lambda^2k^2)(l/m)^2+2\lambda\mu(c^2-k^2)(l/m)+(b^2\nu^2+\mu^2c^2-\mu^2k^2)=0$

If its roots are l_1/m_1 and l_2/m_2, then

$$\frac{l_1l_2}{m_1m_2}=\text{product of the roots}=\frac{b^2\nu^2+\mu^2c^2-\mu^2k^2}{a^2\nu^2+\lambda^2c^2-\lambda^2k^2}$$

or
$$\frac{l_1l_2}{b^2\nu^2+\mu^2(c^2-k^2)}=\frac{m_1m_2}{a^2\nu^2+\lambda^2(c^2-k^2)}=\frac{n_1n_2}{b^2\lambda+a^2\mu^2},\quad \text{...(v)}$$

by symmetry. (Note)

Also as the planes are perpendicular, so we have

$$l_1l_2+m_1m_2+n_1n_2=0$$

or $\quad [b^2\nu^2+\mu^2(c^2-k^2)]+[a^2\nu^2+\lambda^2(c^2-k^2)]+[b^2\lambda^2+a^2\mu^2]=0$, from (v)

or $$\lambda^2(b^2+c^2-k^2)+\mu^2(a^2+c^2-k^2)+\nu^2(a^2+b^2)=0. \quad \text{...(vi)}$$

Eliminating λ, μ, ν between (iii) and (vi) ; we find that the cone generated by the line (ii) is

$$x^2(b^2+c^2-k^2)+y^2(a^2+c^2-k^2)+(z-k)^2(a^2+b^2)=0$$

Hence proved.

Exercises on § 9.06—§ 9.08

Ex. 1. Write down the equation of director sphere of

$$3x^2+5y^2+z^2+2=0.$$

[Hint : See § 9.08 Page 7] **Ans.** $15(x^2+y^2+z^2)+34=0$

Ex. 2. Find the condition under which the plane $\alpha x+\beta y+\gamma z=1$ may touch the conicoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$.

[Hint : See § 9.07 (b) Page 7] **Ans.** $a^2\alpha^2+b^2\beta^2+c^2\gamma^2=1$

Ex. 3. Find the locus of the perpendicular from the origin to the tangent planes to the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ which cuts off from its axes intercepts ; the sum of whose reciprocals is equal to a constant $(1/k)$.

[Hint : See Ex. 6 (b) Page 14] **Ans.** $k^2(x+y+z)^2=a^2x^2+b^2y^2+c^2z^2$.

*__Ex. 4.__ Find the equations to the tangent planes to $7x^2-8y^2-z^2+21=0$ which pass through the line $7x-6y+6=0$, $z=3$.

Ex. 5. Tangent planes are drawn to the conicoid $ax^2+by^2+cz^2=1$ through (α, β, γ). Prove that the perpendiculars to them from the origin generate the cone $(\alpha x+\beta y+\gamma z)^2=(x^2/a)+(y^2/b)+(z^2/c)$

[Hint : See Ex. 7 (b) Page 15]

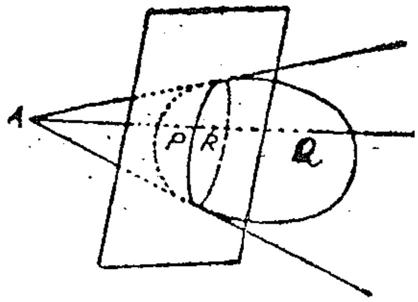
Ex. 6. Prove that the equation of the two tangent planes to the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ which passes through the line $u\equiv lx+my+nz-p=0$, $u'\equiv l'x+m'y+n'z-p'=0$ is

$$u^2(a^2l'^2 + b^2m'^2 + c^2n'^2 - p'^2) + u'^2(a^2l^2 + b^2m^2 + c^2n^2 - p^2)$$
$$- 2uu'(a^2ll' + b^2mm' + c^2nn' - pp') = 0.$$

[Hint : See Ex. 9 Page 16]

***§ 9.09 The Polar Plane.** (*Agra* 91; *Rohilkhand* 93)

Definition. If through a point $A(\alpha, \beta, \gamma)$ a line APQ be drawn to meet the conicoid $ax^2 + by^2 + cz^2 = 1$ in P and Q, then the locus of R, the harmonic conjugate of A with respect to P and Q (*i.e.* AP, AR and AQ are in harmonic progression), is defined as the polar of A with respect to the conicoid.



(Fig. 4)

Any line through $A(\alpha, \beta, \gamma)$ is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)} \quad \text{...(i)}$$

Its intersection with the conicoid is given by

$$r^2(al^2 + bm^2 + cn^2) + 2r(al\alpha + bm\beta + cn\gamma)$$
$$+ (a\alpha^2 + b\beta^2 + c\gamma^2 - 1), \quad \text{...(ii)}$$

putting $x = \alpha + lr$, $y = \beta + mr$, $z = \gamma + nr$ in the equation of the conicoid.

Here r stands for AP and AQ. Also as AP, AR and AQ are in H.P.

$$\therefore \quad \frac{2}{AR} = \frac{1}{AP} + \frac{1}{AQ} = \frac{1}{r_1} + \frac{1}{r_2}, \quad \text{...(iii)}$$

where r_1 and r_2 are the roots of (ii).

From (ii) we have $r_1 + r_2 = -(al\alpha + bm\beta + cn\gamma)/(al^2 + bm^2 + cn^2)$

and $\quad r_1r_2 = (a\alpha^2 + b\beta^2 + c\gamma^2 - 1)/(al^2 + bm^2 + cn^2)$.

From (iii) we get $\dfrac{2}{AR} = \dfrac{r_1 + r_2}{r_1r_2} = \dfrac{-2(al\alpha + bm\beta + cn\gamma)}{(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)}$

or $\quad (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = -[a\alpha(l \cdot AR) + b\beta(m \cdot AR) + c\gamma(n \cdot AR)] \quad$ ...(iv)

(Note)

Now if R be (x, y, z) and its distance from $A(\alpha, \beta, \gamma)$ on the line AQ be AR, then from (i), we have $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n} = AR$ **(Note)**

or $\quad x - \alpha = l \cdot AR,\ y - \beta = m \cdot AR$ and $z - \gamma = n \cdot AR$.

Substituting these values in (iv) we are able to eliminate l, m, n and thus we get the equation of the locus of R as

$$a\alpha^2 + b\beta^2 + c\gamma^2 - 1 = -[a\alpha(x-\alpha) + b\beta(y-\beta) + c\gamma(z-\gamma)]$$

or $\quad a\alpha x + b\beta y + c\gamma z = 1$ is the polar plane of $A(\alpha, \beta, \gamma)$ with respect to the given conicoid $\quad ax^2 + by^2 + cz^2 = 1$.

Note 1. The equation of the polar plane of A (α, β, γ) with respect to the conicoid $ax^2+by^2+cz^2=1$ is the same as that of the tangent plane of this conicoid at (α, β, γ).

Note 2. The polar plane of A is the tangent plane at A when A is on the surface.

***§ 9.10. Pole of a given plane.** (*Rohilkhand* 90)

Let (α, β, γ) be the pole of the plane $\quad lx+my+nz=p \quad$ ...(i)

with respect to the conicoid $\quad ax^2+by^2+cz^2=1. \quad$ ...(ii)

The polar plane of (α, β, γ) with respect to (ii) is

$$a\alpha x+b\beta y+c\gamma z=1. \quad \text{...(iii)}$$

Comparing (i) and (iii) which represent the same plane, we get

$$\frac{a\alpha}{l}=\frac{b\beta}{m}=\frac{c\gamma}{n}=\frac{1}{p}$$

whence we get $\alpha=l/pa,\ \beta=m/pb,\ \gamma=n/pc.$

$\therefore$ The pole of the plane (i) with respect to the conicoid (ii) is

$$\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$$

***§ 9.11. Conjugate points, conjugate planes and polar lines.**

(a) We can easily prove that if the polar plane of a point $P\,(\alpha_1, \beta_1, \gamma_1)$ passes through $Q\,(\alpha_2, \beta_2, \gamma_2)$ then the polar of Q passes through P and two such points are called **conjugate points.**

(b) We can also prove that if the pole of the plane 'α' (say) lies on another plane 'β' (say) then the pole of the plane 'β' lies on the plane 'α' and two such planes are called **conjugate planes.**

(c) If the polar plane of any point on a line AB passes through a line PQ, then the polar plane of any point on PQ passes through that point on AB and therefore passes through AB and such two lines AB and PQ are called **polar lines with respect to the conicoid.** (*Kanpur* 90)

Let the line AB be $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}=r$ (say) and the equation of the conicoid be $\quad ax^2+by^2+cz^2=1.$

Any point on this line is $(\alpha+lr,\ \beta+mr, \gamma+nr)$. The polar plane of this point with respect to the conicoid is $ax\,(\alpha+lr)+by\,(\beta+mr)+cz\,(\gamma+nr)=1$

or $$(a\alpha x+b\beta y+c\gamma z-1)+r\,(alx+bmy+cnz)=0.$$

Evidently this plane, for all values of r, passes through the line

$$a\alpha x+b\beta y+c\gamma z-1=0=alx+bmy+cnz,$$

which are the equations of the polar line PQ of AB with respect to the given conicoid.

Solved Examples on § 9.09 to § 9.11

Ex. 1. Find the ploar plane of the point (2, – 3, 4) with respect to the conicoid $x^2+2y^2+z^2=4$.

Sol. Required polar plane is

$$x(2)+2y(-3)+z(4)=4 \qquad \text{...See § 9.09 Page 20}$$

or
$$x-3y+2z=2. \qquad \textbf{Ans.}$$

Ex. 2. If P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are any two points then find the equation of the polar of PQ with respect to the conicoid

$$ax^2+by^2+cz^2=1.$$

Sol. The equations of the line PQ are

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}=r \text{ (say)}$$

Any point on this line is

$$[x_1+r(x_2-x_1),\ y_1+r(y_2-y_1),\ z_1+r(z_2-z_1)]$$

The polar plane of this point with respect to the given conicoid is

$$ax[x_1+r(x_2-x_1)]+by[y_1+r(y_2-y_1)]+cz[z_1+r(z_2-z_1)]=1$$

or
$$(axx_1+byy_1+czz_1-1)+r[ax(x_2-x_1)+by(y_2-y_1)+cz(z_2-z_1)]=0$$

Evidently this plane for all values of r, passes through the line given by

$$axx_1+byy_1+czz_1-1=0 \qquad \text{...(i)}$$

and
$$ax(x_2-x_1)+by(y_2-y_1)+cz(z_2-z_1)=0 \qquad \text{...(ii)}$$

From (ii), we get $axx_2+byy_2+czz_2=axx_1+byy_1+czz_1=1$, from (i)

$\therefore$ The equations of the polar of PQ with respect to the conicoid are

$$axx_1+byy_1+czz_1=1 \quad \text{and} \quad axx_2+byy_2+czz_2=1. \qquad \textbf{Ans.}$$

[**Note.** From above example it is evident that if P and Q lie on the given conicoid, then the polar of the line PQ is the line of intersection of the tangent planes at P and Q].

****Ex. 3. Find the surface generated by (or find the locus of) straight lines drawn through a fixed point (α, β, γ) at right angles to their polar with respect to the conicoid $ax^2+by^2+cz^2=1$.**

(*Garhwal* 95; *Gorakhpur* 97; *Kumaun* 91; *Lucknow* 90; *Rohilkhand* 91)

Sol. Any line through (α, β, γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \text{ (say)} \qquad \text{...(i)}$$

Any point on (i) is $(\alpha+lr,\ \beta+mr,\ \gamma+nr)$.

The polar plane of this point with respect to the given conicoid is

$$ax(\alpha+lr)+by(\beta+mr)+cz(\gamma+nr)=1$$

or
$$(a\alpha x+b\beta y+c\gamma z-1)+r(alx+bmy+cnz)=0,$$

Evidently this plane, for all values of r passes through the line

$$a\alpha x+b\beta y+c\gamma z=1\ ;\ alx+bmy+cnz=0$$

which is the polar of the line (i) with respect to the given conicoid.

The direction cosines of this line are proportional to

$$bc\,(n\beta - m\gamma),\ ca\,(l\gamma - n\alpha),\ ab\,(m\alpha - l\beta)\,. \qquad \textbf{(Note)}$$

Since this line is perpendicular to the line (i) so we have

$$l\,.\,bc\,(n\beta - m\gamma) + m\,.\,ca\,(l\gamma - n\alpha) + n\,.\,ab\,(m\alpha - l\beta) = 0$$

or
$$\frac{\beta}{ma} - \frac{\gamma}{na} + \frac{\gamma}{nb} - \frac{\alpha}{lb} + \frac{\alpha}{lc} - \frac{\beta}{mc} = 0,$$

dividing each term by $lmn\,abc$

or
$$\frac{\alpha}{l}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{\beta}{m}\left(\frac{1}{c} - \frac{1}{a}\right) + \frac{\gamma}{n}\left(\frac{1}{a} - \frac{1}{b}\right) = 0. \qquad ...(ii)$$

$\therefore$ By eliminating l, m, n between (i) and (ii), we get the locus of the line

(i) as
$$\frac{\alpha}{x-\alpha}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{\beta}{y-\beta}\left(\frac{1}{c} - \frac{1}{a}\right) + \frac{\gamma}{z-\gamma}\left(\frac{1}{a} - \frac{1}{b}\right) = 0. \qquad \textbf{Ans.}$$

bf **Ex. 4 (a). Show that the equation of the polar of the line $\frac{1}{2}(x-1) = \frac{1}{3}(y-2) = \frac{1}{4}(z-3)$ with respect to $x^2 - 2y^2 + 3z^2 = 4$ are**

$$\frac{1}{3}(x+6) = \frac{1}{3}(y-2) = z - 2.$$

Sol. The given line is
$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}. \qquad ...(i)$$

The equations of polar line of (i) w.r. to the given conicoid

$$(\tfrac{1}{4})\,x^2 - (\tfrac{1}{2})\,y^2 + (\tfrac{3}{4})\,z^2 = 1$$

are $\quad (1/4)\,(1)\,x + (-1/2)\,(2)\,y + (3/4)\,(3)\,z = 1$

and $\quad (1/4)\,(2)\,x + (-1/2)\,(3)\,y + (3/4)\,(4)\,z = 0$

or $\quad x - 4y + 9z = 4 \quad$ and $\quad x - 3y + 6z = 0$

or $\quad (x+6) - 4\,(y-2) + 9\,(z-2) = 0$

and $\quad (x+6) - 3\,(y-2) + 6\,(z-2) = 0 \qquad$ **(Note)**

Solving these simutaneously, we get

$$\frac{x+6}{-24+27} = \frac{y-2}{9-6} = \frac{z-2}{-3+4}$$

or
$$\frac{1}{3}(x+6) = \frac{1}{3}(y-2) = (z-2) \qquad \textbf{Ans.}$$

bf **Ex. 4 (b). Find the locus of straight lines through a fixed point (α, β, γ) whose polar lines with respect to the quadratics**

$ax^2 + by^2 + cz^2 = 1$ and $a'x^2 + b'y^2 + c'z^2 = 1$ are coplanar.

Sol. Any line through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)} \qquad ...(i)$$

The equations of the polar line of (i) w.r. to $ax^2 + by^2 + cz^2 = 1$ are

$a\alpha x + b\beta y + c\gamma z = 1 \qquad ...(ii)\ ; \qquad alx + bmy + cnz = 0. \qquad ...(iii)$

And the equations of polar line of (i) w.r. to $a' x^2 + b' y^2 + c' z^2 = 1$ are

$$a' \alpha x + b' \beta y + c' \gamma z = 1 \quad \text{...(iv)}$$

and

$$a' lx + b' my + c' nz = 0 \quad \text{...(v)}$$

From (ii) and (iv), we have

$$(a - a') \alpha x + (b - b') \beta y + (c - c') \gamma z = 0 \quad \text{...(vi)}$$

From (iii) and (v), solving simultaneously, we have

$$\frac{lx}{bc' - b'c} = \frac{my}{ca' - c'a} = \frac{nz}{ab' - a'b} \quad \text{...(vii)}$$

Eliminating x, y, z between (vi) and (vii) we get

$$\frac{(a - a') \alpha (bc' - b'c)}{l} = \frac{(b - b') (ca' - c'a)}{m} + \frac{(c - c') \gamma (ab' - a'b)}{n} = 0 \text{....(viii)}$$

Eliminating l, m, n between (i) and (viii), we get the locus of the line as

$$\sum \frac{(a - a') \alpha (bc' - b'c)}{(x - \alpha)} = 0. \qquad \textbf{Ans.}$$

Ex. 5 (a). Prove that the locus of the poles of the tangent planes of $ax^2 + by^2 + cz^2 = 1$ with respect to $a' x^2 + b' y^2 + c' z^2 = 1$ is the conicoid

$$\frac{(a' x)^2}{a} + \frac{(b' y)^2}{b} + \frac{(c' z)^2}{c} = 1.$$

Sol. Let
$$lx + my + nz = p \quad \text{...(i)}$$
be a tangent plane to the conicoid $ax^2 + by^2 + cz^2 = 1$.

Then
$$(l^2/a) + (m^2/b) + (n^2/c) = p^2 \quad \text{...(ii)}$$

Let (x_1, y_1, z_1) be the pole of the plane (i) w.r. to $a' x^2 + b' y^2 + c' z^2 = 1$.

Then we have
$$a' x_1 x + b' y_1 y + c' z_1 z = 1 \quad \text{...(iii)}$$

Comparing (i) and (iii), we get $\dfrac{a' x_1}{l} = \dfrac{b' y_1}{m} = \dfrac{c' z_1}{n} = \dfrac{1}{p}$ ...(iv)

Eliminating l, m, n between (ii) and (iv), we get

$$\frac{(a' x_1 p)^2}{a} + \frac{(b' y_1 p)^2}{b} + \frac{(c' z_1 p)^2}{c} = p^2$$

or

$$\frac{(a' x_1)^2}{a} + \frac{(b' y_1)^2}{b} + \frac{(c' z_1)^2}{c} = 1.$$

$\therefore$ The required locus of is (x_1, y_1, z_1) is

$$\frac{(a' x)^2}{a} + \frac{(b' y)^2}{b} + \frac{(c' z)^2}{c} = 1. \qquad \text{Hence proved.}$$

Ex. 5 (b). Prove that the locus of the poles of the tanget planes of $a^2 x^2 + b^2 y^2 - c^2 z^2 = 1$ with respect to $\alpha^2 x^2 + \beta^2 y^2 + \gamma^2 z^2 = 1$ is hyperbolid of one sheet. *(Lucknow 91)*

Sol. Proceeding exactly as in Ex. 5 (a) above we can find that the required locus is

$$\frac{\alpha^4 x^2}{a^2}+\frac{\beta^4 y^2}{b^2}+\frac{\gamma^4 z^2}{(-c^2)}=1 \quad \text{or} \quad \frac{x^2}{(a^2/\alpha^4)}+\frac{y^2}{(b^2/\beta^4)}-\frac{z^2}{(c^2/\gamma^4)}=1.$$

which is a hyperbolid of one sheet. Hence proved.

***Ex. 6. Show that the locus of the pole of the plane $lx+my+nz=p$ with respect to the system of conicoids $\Sigma\,[x^2/(a^2+k)]=1$ is a straight line perpendicular to the given plane, where k is a parameter.** (*Garhwal* 96)

Sol. Let (α, β, γ) be the pole of the plane $\quad lx+my+nz=p$ ...(i)

with respect to the conicoid $\dfrac{x^2}{(a^2+k)}+\dfrac{y^2}{(b^2+k)}+\dfrac{z^2}{(c^2+k)}=1.$

The polar plane of (α, β, γ) w.r. to this conicoid is

$$\frac{\alpha x}{(a^2+k)}+\frac{\beta y}{(b^2+k)}+\frac{\gamma z}{(c^2+k)}=1. \qquad \text{...(ii)}$$

Since (i) and (ii) represent the same plane, therefore, comparing them, we get

$$\frac{\alpha/(a^2+k)}{l}=\frac{\beta/(b^2+k)}{m}=\frac{\gamma/(c^2+k)}{n}=\frac{1}{p}$$

or $$\alpha=(a^2+k)\frac{l}{p},\ \beta=(b^2+k)\frac{m}{p},\ \gamma=(c^2+k)\frac{n}{p} \qquad \text{(Note)}$$

or $$\frac{\alpha-(a^2l/p)}{l}=\frac{k}{p}=\frac{\beta-(b^2m/p)}{m}=\frac{\gamma-(c^2n/p)}{n}$$

$\therefore$ The locus of (α, β, γ) is $\dfrac{x-(a^2l/p)}{l}=\dfrac{y-(b^2m/p)}{m}=\dfrac{z-(c^2n/p)}{n}$,

which is a straight line and its direction cosines being l, m, n is perpendicular to the plane (i). **Ans.**

***Ex. 7. Prove that the lines through (α, β, γ) at right angles to their polars with respect to $\dfrac{x^2}{a+b}+\dfrac{y^2}{2a}+\dfrac{z^2}{2b}=1$ generate the cone**

$$(y-\beta)(\alpha z-\gamma x)+(z-\gamma)(\alpha y-\beta x)=0.$$

Sol. The equation of the given conicoid be

$$a'x^2+b'y^2+c'z^2=1, \qquad \text{...(i)}$$

where $\quad a'=1/(a+b),\ b'=1/(2a),\ c'=1/(2b).$ ...(ii)

Then as in Ex. 3 Page 22 we can prove that required locus is

$$\frac{\alpha}{x-\alpha}\left(\frac{1}{b'}-\frac{1}{c'}\right)+\frac{\beta}{y-\beta}\left(\frac{1}{c'}-\frac{1}{a'}\right)+\frac{\gamma}{z-\gamma}\left(\frac{1}{a'}-\frac{1}{b'}\right)=0$$

or $$\frac{\alpha}{x-\alpha}(2a-2b)+\frac{\beta}{y-\beta}[2b-(a+b)]+\frac{\gamma}{z-\gamma}(a+b-2a)=0, \text{ from (ii)}$$

or $$\frac{2\alpha}{x-\alpha}-\frac{\beta}{y-\beta}-\frac{\gamma}{z-\gamma}=0$$

or $$2x(y-\beta)(z-\gamma)-\beta(x-\alpha)(z-\gamma)-\gamma(x-\alpha)(y-\beta)=0$$

or $$[\alpha(y-\beta)(z-\gamma)-\beta(x-\alpha)(z-\gamma)]+[\alpha(y-\beta)(z-\gamma)-\gamma(x-\alpha)(y-\beta)]=0,\quad \textbf{(Note)}$$

breaking the first term into two parts.

or $$(z-\gamma)[\alpha y-\alpha\beta-\beta x+\beta\alpha]+(y-\beta)[\alpha z-\alpha\gamma-\gamma x+\gamma\alpha]=0$$

or $$(z-\gamma)(\alpha y-\beta x)+(y-\beta)(\alpha z-\gamma x)=0.\qquad \text{Hence proved.}$$

****Ex. 8. Find the conditions that the lines AB and PQ given by**

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}\quad\text{and}\quad\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$$

should be polar with respect to the conicoid $ax^2+by^2+cz^2=1$

(*Gorakhpur* 95; *Rohilkhand* 94)

Sol. We know that the equations of the polar line of AB with respect to the given conicoid are $a\alpha x+b\beta y+c\gamma z-1=0$ ...(i)

and $$alx+bmy+cnz=0. \quad \text{...(ii)}$$

But if the line PQ given by $(x-\alpha')/l'=(y-\beta')/m'=(z-\gamma')/n'$ is the polar line of AB, then the line PQ lies on both the planes given by (i) and (ii). The conditions for the same are that (a) the point (α',β',γ') lies on both the planes (i) and (ii), and (b) the line PQ is perpendicular to the normals to the planes (i) and (ii). **(Note)**

Hence the required conditions are

$$a\alpha\alpha'+b\beta\beta'+c\gamma\gamma'-1=0,\quad al\alpha'+bm\beta'+cn\gamma'=0$$

and $$a\alpha l'+b\beta m'+c\gamma n'=0,\quad all'+bmm'+cnn'=0.$$

***Ex. 9. Find the condition that the line $(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n$ should intersect the polar of the line**

$$(x-\alpha')/l'=(y-\beta')/m'=(z-\gamma')/n'$$

with respect to the conicoid $ax^2+by^2+cz^2=1$. (*Rohilkhand* 95)

Sol. Let the lines be AB and PQ given by

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}\quad\text{and}\quad\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}.$$

The equations of the polar line AB with respect to given conicoid are

$$a\alpha x+b\beta y+c\gamma z-1=0 \quad \text{...(i)}$$

and $$alx+bmy+cnz=0. \quad \text{...(ii)}$$

But if the line PQ intersects the polar of line AB given by (i), and (ii), then any point on PQ say $(\alpha'+l'r,\ \beta'+m'r,\ \gamma'+n'r)$ will satisfy both the planes (i) and (ii) for the same value of r.

$$\therefore\ a\alpha(\alpha'+l'r)+b\beta(\beta'+m'r)+c\gamma(\gamma'+n'r)-1=0$$

and $$al(\alpha'+l'r)+bm(\beta'+m'r)+cn(\gamma'+n'r)=0$$

or $$(a\alpha\alpha'+b\beta\beta'+c\gamma\gamma'-1)+r(a\alpha l'+b\beta m'+c\gamma n')=0 \quad \text{...(iii)}$$

and $$(al\alpha'+bm\beta'+cn\gamma')+r(all'+bmm'+cnn')=0 \quad \text{...(iv)}$$

Eliminating r between (iii) and (iv), we get the required conditions as

$$\frac{a\alpha\alpha'+b\beta\beta'+c\gamma\gamma'-1}{a\alpha l'+b\beta m'+c\gamma n'}=\frac{al\alpha'+bm\beta'+cn\gamma'}{all'+bmm'+cnn'}$$

or $\quad (a\alpha l' + b\beta m' + c\gamma n')(al\alpha' + bm\beta + cn\gamma')$

$$= (all' + bmm' + cnn')(a\alpha\alpha' + b\beta\beta' + c\gamma\gamma' - 1). \qquad \textbf{Ans.}$$

Exercises on § 9.09—§ 9.11.

Ex. 1. Find the condition that the line

$$(x-\alpha)/l = (y-\beta)/m = (z-\gamma)/n$$

intersects its polar line with respect to the surface $ax^2 + by^2 + cz^2 = 1$.

(Hint : See Ex. 9 Page 26)

Ex. 2. Find the equations to the polar of the line $-2x = 25y - 1 = 2z$ with respect to the conicoid $2x^2 - 25y^2 + 2z^2 = 1$

[Hint : See Ex. 4 (a) Page 23]

****§ 9.12. Locus of chords with a given mid-point.** (*Rohilkhand* 96)

Let the mid point of the chord be (α, β, γ) and its equations be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)} \qquad \text{...(i)}$$

Then any point on this line is $(\alpha + lr, \beta + mr, \gamma + nr)$.

If this point lies on $ax^2 + by^2 + cz^2 = 1$, then we have

$$a(\alpha + lr)^2 + b(\beta + mr)^2 + c(\gamma + nr)^2 = 1$$

or $\quad r^2(al^2 + bm^2 + cn^2) + 2r(al\alpha + bm\beta + cn\gamma) + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0$

$$\text{...(ii)}$$

which gives the distances of the points of intersection of the line (i) and the above conicoid from the point (α, β, γ).

$\because$ (α, β, γ) is the mid point of the chord given by (i), so its distance from the two points of intersection of (i) and the conicoid should be equal but opposite *i.e.* the sum of the values of r given by (ii) should be zero and the conditon for the same is $\quad al\alpha + bm\beta + cn\gamma = 0. \quad$ ...(iii)

The required locus of the chords with (α, β, γ) as mid-point is obtained by eliminating l, m, n between (i) and (iii) and the same is

$$a\alpha(x-\alpha) + b\beta(y-\beta) + c\gamma(z-\gamma) = 0 \qquad \text{...(iv)}$$

(*Rohilkhand* 96)

or $$a\alpha x + b\beta y + c\gamma z = a\alpha^2 + b\beta^2 + c\gamma^2$$

or $$a\alpha x + b\beta y + c\gamma z - 1 = a\alpha^2 + b\beta^2 + c\gamma^2 - 1$$

or $$\boxed{T = S_1} \left.\begin{array}{l}\text{where } T = a\alpha x + b\beta y + c\gamma z - 1 \\ \text{and } S_1 = a\alpha^2 + b\beta^2 + c\gamma^2 - 1\end{array}\right\} \qquad \text{...(v)}$$

This plane given by (iv) or (v) meets the conicoid $ax^2 + by^2 + cz^2 = 1$ in a conic of which (α, β, γ) is the centre **(Note)**

Solved Examples on § 9.12.

Ex. 1. Find the equation to the plane which cuts $2x^2 - 3y^2 + 5z^2 = 1$ in a conic whose centre is at the point (2, 1, 3). (*Kumaun* 90)

Sol. Here $S = 2x^2 - 3y^2 + 5z^2 - 1 = 0$ and centre of the section is $(2, 1, 3)$.

$\therefore \quad S_1 = 2(2)^2 - 3(1)^2 + 5(3)^2 - 1 = 8 - 3 + 45 - 1 = 49$

and $\quad T = 2x(2) - 3y(1) + 5z(3) - 1 = 4x - 3y + 15z - 1.$

$\therefore$ The required equation is $T = S_1$ ...See § 9.12 above

or $\quad 4x - 3y + 15z - 1 = 49 \quad$ or $\quad 4x - 3y + 15z = 50.$ **Ans.**

Ex. 2. Find the centre of the conic

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} = 1, \quad 2x + 2y - z = 3.$$

Sol. Let (α, β, γ) be the centre of the conic *i.e.* the section of the ellipsoid

$$(x^2/9) + (y^2/16) + (z^2/4) = 1 \qquad \text{...(i)}$$

by the plane

$$2x + 2y - z = 3. \qquad \text{...(ii)}$$

Then the equation of the plane section whose centre is (α, β, γ) is

$$T = S_1 \;\; i.e. \;\; \frac{x\alpha}{2} + \frac{y\beta}{16} + \frac{z\gamma}{4} - 1 = \frac{\alpha^2}{9} + \frac{\beta^2}{16} + \frac{\gamma^2}{4} - 1.$$

or

$$\frac{\alpha x}{9} + \frac{\beta y}{16} + \frac{\gamma z}{4} = \frac{\alpha^2}{9} + \frac{\beta^2}{16} + \frac{\gamma^2}{4}. \qquad \text{...(iii)}$$

Now (ii) and (iii) represent the same plane, so comparing them we get

$$\frac{(\alpha/9)}{2} = \frac{(\beta/16)}{2} = \frac{(\gamma/4)}{-1} = \frac{[(\alpha^2/9) + (\beta^2/16) + (\gamma^2/4)]}{3} = \lambda \text{ (say)}$$

whence, we get $\dfrac{\alpha}{18} = \dfrac{\beta}{32} = \dfrac{\gamma}{-4} = \lambda \quad$ or $\quad \alpha = 18\lambda, \;\; \beta = 32\lambda, \;\; \gamma = -4\lambda$

Also (α, β, γ) lies on (iii), so $2\alpha + 2\beta - \gamma = 3$

or $\quad 2(18\lambda) + 2(32\lambda) - (-4\lambda) = 3$, from (iv)

or $\quad (36 + 64 + 4)\lambda = 3 \quad$ or $\quad \lambda = 3/104.$

$\therefore$ From (iv), the required centre is,

$$\left(\frac{54}{104}, \frac{96}{104}, \frac{-12}{104}\right) \text{ or } \left(\frac{27}{52}, \frac{12}{13}, \frac{-3}{26}\right) \qquad \textbf{Ans.}$$

Ex. 3. Prove that the centres of sections of $ax^2 + by^2 + cz^2 = 1$ by the planes which are at a constant distance p from the origin lie on the surface

$$(ax^2 + by^2 + cz^2)^2 = p^2 (a^2x^2 + b^2y^2 + c^2z^2).$$

Sol. If (α, β, γ) be the centre of the section of the given ellipsoid then equation of this section of the sphere is "$T = S_1$"

i.e. $\quad (ax\alpha + by\beta + cz\gamma - 1) = (a\alpha^2 + b\beta^2 + c\gamma^2 - 1)$

or

$$-ax\alpha - by\beta - c\gamma z + (a\alpha^2 + b\beta^2 + c\gamma^2) = 0 \qquad \text{...(i)}$$

The distance of this plane (i) from the origin $(0, 0, 0)$ is given as p.

$$\therefore p = \frac{a\alpha^2 + b\beta^2 + c\gamma^2}{\sqrt{[(a\alpha)^2 + (b\beta)^2 + (c\gamma)^2]}}$$

or

$$p^2 (a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2) = (a\alpha^2 + b\beta^2 + c\gamma^2)^2.$$

$\therefore$ The locus of the centre (α, β, γ) is

$$p^2 (a^2x^2 + b^2y^2 + c^2z^2) = (ax^2 + by^2 + cz^2)^2$$

Ex. 4. Prove that the centre of the section of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ by the plane ABC whose equation is $x/a + y/b + z/c = 1$ is the centroid of the triangle ABC.

Sol. The equation of the ellipsoid is

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0 \quad \text{...(i)}$$

and the equation of the plane ABC is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$. ...(ii)

Let (α, β, γ) be the centre of the section of (i) by the plane (ii) then the equation of this section is "$T = S_1$"

i.e. $$\frac{x\alpha}{a^2} + \frac{y\beta}{b^2} + \frac{z\gamma}{c^2} - 1 = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1$$

or $$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}. \quad \text{...(iii)}$$

Also as the point (α, β, γ) lies on (ii), so $\Sigma (\alpha/a) = 1$ ...(iv)

The equation (ii) and (iii) represent the same plane, so comparing them,

we get $$\frac{(\alpha/a^2)}{(1/a)} = \frac{(\beta/b^2)}{(1/b)} = \frac{(\gamma/c^2)}{(1/c)} = \frac{(\alpha^2/a^2) + (\beta^2/b^2) + (\gamma^2/c^2)}{1} = k \text{ (say)},$$

from (iv)

$\therefore \alpha = ak,\ \beta = bk,\ \gamma = ck$ and $(\alpha^2/a^2) + (\beta^2/a^2) + (\gamma^2/c^2) = k$

$\therefore (a^2k^2/a^2) + (b^2k^2/b^2) + (c^2k^2/c^2) = k$ or $3k^2 = k$ or $k = 1/3$

$\therefore \alpha = ak = a/3,\ \beta = bk = b/3,\ \gamma = ck = c/3.$

$\therefore$ The centre of the section of (i) by the plane (ii) is (α, β, γ)

or $$(a/3, b/3, c/3).$$

Also the coordinates of the vertices of ΔABC are

$$A(a, 0, 0),\ B(0, b, 0), \text{ and } C(0, 0, c)$$

$\therefore$ The coordinates of the centroid of ΔABC are $(a/3, b/3, c/3)$.

Hence the centre of the section of (i) by (ii) is the centroid of ΔABC.

****Ex. 5 (a). Find the locus of the mid points of the chords of the conicoid $ax^2 + by^2 + cz^2 = 1$ which passes through (α, β, γ).**

(Rohilkhand 94, 90)

Sol. Let (x_1, y_1, z_1) be the mid-point of the chord of the given conicoid. Then the locus of the chords of the given conicoid with (x_1, y_1, z_1) as mid-point is

"$T = S_1$",

where $T = axx_1 + byy_1 + czz_1 - 1$ and $S_1 = ax_1^2 + by_1^2 + cz_1^2 - 1$

i.e. $$axx_1 + byy_1 + czz_1 - 1 = ax_1^2 + by_1^2 + cz_1^2 - 1$$

or $$axx_1 + byy_1 + czz_1 = ax_1^2 + by_1^2 + cz_1^2$$

If it passes through (α, β, γ), we have

$$a\alpha x_1 + b\beta y_1 + c\gamma z_1 = ax_1^2 + by_1^2 + cz_1^2$$

$\therefore$ The required locus of the mid-point (x_1, y_1, z_1) of the chords of the given conicoid is $ax^2 + by^2 + cz^2 = a\alpha x + b\beta y + c\gamma z$

or $$ax(x-\alpha) + by(y-\beta) + cz(z-\gamma) = 0 \qquad \textbf{Ans.}$$

Ex. 5 (b). Find the locus of the centre of all sections of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ by planes which pass through a fixed point (x', y', z').

Sol. Let (α, β, γ) be the centre of the above section of the ellipsoid

$$S \equiv x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 = 0.$$

Then the equation of this section is "$T = S_1$"

or $$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} - 1 = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1$$

or $$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}.$$

If this plane passes through (x', y', z'), then we have

$$\frac{\alpha x'}{a^2} + \frac{\beta y'}{b^2} + \frac{\gamma z'}{c^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}$$

$\therefore$ The locus of the centre (α, β, γ) is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

or $$\frac{x(x-x')}{a^2} + \frac{y(y-y')}{b^2} + \frac{z(z-z')}{c^2} = 0 \qquad \textbf{Ans.}$$

Ex. 6. Show that the centre of the conic $lx + my + nz = p$, $ax^2 + by^2 + cz^2 = 1$ is the point $\left(\frac{lp}{ap_0^2}, \frac{mp}{bp_0^2}, \frac{np}{cp_0^2}\right)$,

where $\quad l^2 + m^2 + n^2 = 1 \quad$ and $\quad p_0^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}$.

Sol. If (α, β, γ) be the centre of the conic which is the section of the ellipsoid $\qquad S \equiv ax^2 + by^2 + cz^2 - 1 = 0 \qquad$ ...(i)

by the plane $\qquad lx + my + nz = p. \qquad$ ...(ii)

Then the equation of the plane section with (α, β, γ) as centre is

"$T = S_1$" or $ax\alpha + by\beta + c\gamma z - 1 = a\alpha^2 + b\beta^2 + c\gamma^2 - 1$

$$a\alpha x + b\beta y + c\gamma z = a\alpha^2 + b\beta^2 + c\gamma^2. \qquad \text{...(iii)}$$

$\because$ The equations (ii) and (iii) represent the same plane.

$\therefore$ we have $\dfrac{a\alpha}{l}=\dfrac{b\beta}{m}=\dfrac{c\gamma}{n}=\dfrac{a\alpha^2+b\beta^2+c\gamma^2}{p}=k$ (say)

$\therefore\ \alpha=\dfrac{lk}{a},\ \beta=\dfrac{mk}{b},\ \gamma=\dfrac{nk}{c}$ and $a\alpha^2+b\beta^2+c\gamma^2=pk$

or $$a\,(lk/a)^2+b\,(mk/b)^2+c\,(nk/c)^2=pk$$

or $$k\,[(l^2/a)+(m^2/b)+(n^2/c)]=p$$

or $$kp_0^2=p,\quad \because\ p_0^2=l^2/a+m^2/b+n^2/c \quad \text{(given)}$$

or $$k=p/p_0^2.$$

$\therefore$ The centre of the conic, given by (i) and (ii) is (α,β,γ)

or $$\left(\frac{lk}{a},\frac{mk}{b},\frac{nk}{c}\right)\ \text{or}\ \left(\frac{lp}{ap_0^2},\frac{mp}{bp_0^2},\frac{np}{cp_0^2}\right).$$ Hence proved.

***Ex. 7. Show that the line joining a point P to the centre of a conicoid $ax^2+by^2+cz^2=1$ passes through the centre of the section of the conicoid by the polar plane of P.**

Sol. Let (x',y',z') be the coordinates of the point P

Then the polar plane of $P\,(x',y',z')$ with respect to the given conicoid is

$$axx'+byy'+czz'=1 \qquad \text{...(i)}$$

Let (α,β,γ) be the centre of the section of the given conicoid by the plane (i), then equation of this plane section can also be written as

$$\text{“}T=S_1\text{”}\ \text{or}\ a\alpha x+b\beta y+c\gamma z-1=a\alpha^2+b\beta^2+c\gamma^2-1$$

or $$a\alpha x+b\beta y+c\gamma z=a\alpha^2+b\beta^2+c\gamma^2. \qquad \text{...(ii)}$$

Since the equations (i) and (ii) represent the same plane, so comparing them, we get $$\frac{\alpha}{x'}=\frac{\beta}{y'}=\frac{\gamma}{z'}. \qquad \text{...(iii)}$$

Also the equations of the line joining the point $P\,(x',y',z')$ to the centre $(0,0,0)$ of the given conicoid are $$\frac{x}{x'}=\frac{y}{y'}=\frac{z}{z'}.$$

If this line passes through the centre (α,β,γ) of the section of given conicoid by the plane (i), then $\alpha/x'=\beta/y'=\gamma/z'$ which is true by virtue of (iii).

Hence the line joining $P\,(x',y',z')$ to the centre of the given conicoid pass through the centre (α,β,γ) of the section of the conicoid by the polar plane (i) of P.

Ex. 8. Prove that the section of the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

whose centre is at the point $(\frac{1}{3}a,\frac{1}{3}b,\frac{1}{3}c)$ passes through the extremities of the axes.

Sol. The ellipsoid is $S = x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 = 0.$ ...(i)

The equation of the section of this ellipsoid with $(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c)$ as its centre is "$T = S_1$"

i.e. $$\frac{x.\frac{1}{3}a}{a^2} + \frac{y.\frac{1}{3}b}{b^2} + \frac{z.\frac{1}{3}c}{c^2} - 1 = \frac{(\frac{1}{3}a)^2}{a^2} + \frac{(\frac{1}{3}b)^2}{b^2} + \frac{(\frac{1}{3}c)^2}{c^2} - 1$$

or $$\frac{x}{3a} + \frac{y}{3b} + \frac{z}{3c} = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3} \quad \text{or} \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

which is a plane evidently passing through $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, the three extremities to the axes of the ellipsoid given by (i) above.

(See § 9.02 Page 1)

Ex. 9. Show that the centres of the section of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ that are (I) parallel to a given line lie on a fixed plane and (II) that pass through a given line lie on a conic.

Sol. Let (α, β, γ) be the centre of the section of the given ellipsoid.

Then the equation of this plane section is "$T = S_1$"

or $$\frac{x\alpha}{a^2} + \frac{y\beta}{b^2} + \frac{z\gamma}{c^2} - 1 = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1$$

or $$\frac{x\alpha}{a^2} + \frac{y\beta}{b^2} + \frac{z\gamma}{c^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}. \quad \text{...(i)}$$

(I) If the plane is parallel to a line

$$\frac{x - x'}{l} = \frac{y - y'}{m} = \frac{z - z'}{n} \quad \text{...(ii)}$$

then the normal to the plane (i) must be perpendicular to line (ii)

$$\therefore \quad \frac{\alpha}{a^2} \cdot l + \frac{\beta}{b^2} \cdot m + \frac{\gamma}{c^2} \cdot n = 0. \quad \text{...(iii)}$$

$\therefore$ The locus of the centre (α, β, γ) is the plane

$$\frac{lx}{a^2} + \frac{my}{b^2} + \frac{nz}{c^2} = 0. \quad \text{...(iv)}$$

(II) If the plane (i) passes through the line (ii), then the normal to the plane (i) will be perpendicular to the line (ii) and the condition for the same is (iii) which after generalising α, β, γ reducdes to (iv).

Also the point (x', y', z') on line (ii) must lie on the plane (i) and so we have

$$\frac{x'\alpha}{a^2} + \frac{y'\beta}{b^2} + \frac{z'\gamma}{c^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}$$

Generalising (α, β, γ) we have $\dfrac{xx'}{a^2} + \dfrac{yy'}{b^2} + \dfrac{zz'}{c^2} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$ ...(v)

which is a conicoid.

$\therefore$ The locus of (α, β, γ) is given by (iv) and (v) *i.e.* by the section of conicoid (v) by the plane (iv), hence a conic.

***Ex. 10. Find the locus of the centres of sections of $ax^2 + by^2 + cz^2 = 1$ which touch $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$.**

(*Gorakhpur* 96; *Lucknow* 90; *Rohilkhand* 95, 93)

Sol. Let (x_1, y_1, z_1) be the centre of the section of conicoid

$$ax^2 + by^2 + cz^2 = 1 \qquad \text{...(i)}$$

The equation of the section is $\quad T = S_1$

or $$axx_1 + byy_1 + czz_1 - 1 = ax_1^2 + by_1^2 + cz_1^2 - 1$$

or $$ax_1x + by_1y + cz_1z = (ax_1^2 + by_1^2 + cz_1^2). \qquad \text{...(i)}$$

If the plane (i) touches the conicoid $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$, then we must have

$$\text{“}\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2\text{”} \qquad \text{...§ 9.07 (a) Page 6}$$

$$\therefore \quad \frac{(ax_1)^2}{\alpha} + \frac{(by_1)^2}{\beta} + \frac{(cz_1)^2}{\gamma} = (ax_1^2 + by_1^2 + cz_1^2)^2.$$

$\therefore$ The required locus of (x_1, y_1, z_1) is

$$\frac{a^2x^2}{\alpha} + \frac{b^2y^2}{\beta} + \frac{c^2z^2}{\gamma} = (ax^2 + by^2 + cz^2)^2. \qquad \textbf{Ans.}$$

****Ex. 11. Prove that the middle points of the chords of $ax^2 + by^2 + cz^2 = 1$ which are parallel to $x = 0$ and touch $x^2 + y^2 + z^2 = r^2$ lie on the surface $by^2 (bx^2 + by^2 + cz^2 - br^2) + cz^2 (cx^2 + by^2 + cz^2 - cr^2) = 0$**

(*Gorakhpur* 95; *Rohilkhand* 91)

Sol. The equations of any line having (α, β, γ) as mid-point and parallel to the plane $x = 0$ is $$\frac{x - \alpha}{0} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} = \lambda \text{ (say)} \qquad \text{...(i)}$$

where m and n are variables.

Any point on this line is $(\alpha, \beta + m\lambda, \gamma + n\lambda)$. If this point lies on the conicoid $ax^2 + by^2 + cz^2 = 1$, then we have $a\alpha^2 + b(\beta + m\lambda)^2 + c(\gamma + n\lambda)^2 = 1$

or $$\lambda^2 (bm^2 + cn^2) + 2\lambda (b\beta m + c\gamma n) + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0. \qquad \text{...(ii)}$$

$\therefore$ (α, β, γ) is the mid-point of the chord (i) of the given conicoid, so that sum of the roots of equation (ii), which is a quadratic in λ, must be zero. **(Note)**

i.e. $$bm\beta - cn\gamma = 0 \qquad \text{...(iii)}$$

Also the line (i) touches the sphere $x^2 + y^2 + z^2 = r^2$.

$\therefore$ The length of perpendicular from the centre (0, 0, 0) of the sphere to (i) must be equal to the radius r of the sphere

i.e. $$\left[\begin{vmatrix} -\alpha & -\beta \\ 0 & m \end{vmatrix}^2 + \begin{vmatrix} -\beta & -\gamma \\ m & n \end{vmatrix}^2 + \begin{vmatrix} -\gamma & -\alpha \\ n & 0 \end{vmatrix}^2\right] \div (m^2 + n^2) = r^2$$

...See chapter on Straight Lines

or $$m^2\alpha^2 + (n\beta - m\gamma)^2 + \alpha^2 n^2 = r^2 (m^2 + n^2)$$

or $$(r^2 - \alpha^2)(m^2 + n^2) = (n\beta - m\gamma)^2$$

or $$(r^2 - \alpha^2)[(m/n)^2 + 1] = [\beta - (m/n)\gamma]^2 \quad \text{...(iv)}$$

Also from (iii), we have $m/n = (-c\gamma)/(b\beta)$

Substituting this value in (iv), we get

$$(r^2 - \alpha^2)[(c^2\gamma^2/b^2\beta^2) + 1] = [\beta + (c\gamma/b\beta)\gamma]^2$$

or $$(r^2 - \alpha^2)[c^2\gamma^2 + b^2\beta^2] = [b\beta^2 + c\gamma^2]^2$$

$\therefore$ The required locus of (α, β, γ) is

$$(r^2 - x^2)(c^2z^2 + b^2y^2) = (by^2 + cz^2)^2$$

or $$c^2r^2z^2 + b^2r^2y^2 - c^2x^2z^2 - b^2y^2x^2 = b^2y^4 + c^2z^4 + 2bcy^2z^2$$

or $$by^2(bx^2 + by^2 + cz^2 - br^2) + cz^2(cz^2 + by^2 + cx^2 - cr^2) = 0$$

Hence proved.

Exercises on § 9.12

Ex. 1. Find the centre of the conic given by the equations :—

$$3x^2 + 2y^2 - 15,\ z^2 = 4,\ 2x - 2y - 5z = -5. \quad \textbf{Ans. } (-2, 3, -1)$$

Ex. 2. Find the equation to the plane which cuts the surface $x^2 + 4y^2 - 5z^2 = 1$ in a conic whose centre is at the point (2, 3, 4).

Ans. $x + 6y - 10z + 20 = 0$

Ex. 3. Find the equation of the plane which cuts the surface $2x^2 + 3y^2 + 5z^2 = 4$ in a conic whose centre is the point (1, 2, 3).

****§ 9.13. Normal.**

To find the equations of the normal to the conicoid

$$ax^2 + by^2 + cz^2 = 1 \text{ at } (\alpha, \beta, \gamma). \quad (Agra\ 91)$$

The equation of the tangent plane to the conicoid at (α, β, γ) is

$$a\alpha x + b\beta y + c\gamma z = 1 \quad \text{...(i)}$$

The required normal is a line through (α, β, γ) at right angles to the tangent plane given by (i) hence the required equations of the normal are

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{c\gamma}, \quad \text{...(ii)}$$

whose direction ratios are $a\alpha$, $b\beta$, $c\gamma$.

Also if p be the length of the perpendicular from the origin to the tangent plane given by (i), then $$p = \frac{1}{\sqrt{[(a\alpha)^2 + (b\beta)^2 + (c\gamma)^2]}}$$

or $$(a\alpha p)^2 + (b\beta p)^2 + (c\gamma p)^2 = 1 \quad \text{...(iii)}$$

From (ii) and (iii), we conclude that the direction cosines of the normal to the conicoid at (α, β, γ) are $a\alpha p, b\beta p, c\gamma p$.

Also if the normal at (α, β, γ) given by (ii) passes through the fixed point (x_1, y_1, z_1), then we have

$$\frac{x_1-\alpha}{a\alpha}=\frac{y_1-\beta}{b\beta}=\frac{z_1-\gamma}{c\gamma}=r \text{ (say).}$$

From $\dfrac{x_1-\alpha}{a\alpha}=r$, we get $x_1=\alpha+a\alpha r=\alpha(1+ar)$ or $\alpha=\dfrac{x_1}{1+ar}$

Similarly we can get $\beta=\dfrac{y_1}{1+br}$, $\gamma=\dfrac{z_1}{1+cr}$

Also as (α, β, γ) is a point on the conicoid, so we get $a\alpha^2+b\beta^2+c\gamma^2=1$

or $$a\left(\frac{x_1}{1+ar}\right)^2+b\left(\frac{y_1}{1+br}\right)^2+c\left(\frac{z_1}{1+cr}\right)^2=1,$$

substituting the values of α, β, γ calculated above.

This equation is of sixth degree in r, hence six normals can be drawn to a conicoid from any point. **(Note)**

***Note.** If the conicoid is the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, then the equation of the normal to it at (α, β, γ) are

$$\frac{x-\alpha}{(\alpha/a^2)}=\frac{y-\beta}{(\beta/b^2)}=\frac{z-\gamma}{(\gamma/c^2)} \qquad \text{...(iv)}$$

(*Rohilkhand* 90)

The actual direction cosines of this normal are

$$p\alpha/a^2,\ p\beta/b^2,\ p\gamma/c^2 \text{ where } (p\alpha/a^2)^2+(p\beta/b^2)^2+(p\gamma/c^2)^2=1$$

or $$p^2\,[(\alpha^2/a^4)+(\beta^2/b^4)+(\gamma^2/c^4)]=1$$

or $$p=1/\sqrt{\left(\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4}\right)}$$

= length of the perpendicular from (0, 0, 0) to the tangent plane

$(\alpha x/a^2)+(\beta y/b^2)+(\gamma z/c^2)=1$ of the ellipsoid at (α, β, γ)

Solved Examples on § 9.13.

****Ex. 1. Find the distance of the points of intersection of the coordinate planes and the normal at P (α, β, γ) to the ellipsoid**

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

or The normal at any point P (α, β, γ) to the conicoid meets the three principal planes at G_1, G_2, G_3; show that $PG_1 : PG_2 : PG_3 = a^2 : b^2 : c^2$.

(*Agra* 90; *Avadh* 94)

Sol. We know that equations of normals to the ellipsoid at $P(\alpha, \beta, \gamma)$ are

$$\frac{x-\alpha}{(p\alpha/a^2)}=\frac{y-\beta}{(p\beta/b^2)}=\frac{z-\gamma}{(p\gamma/c^2)}=r \text{ (say),} \qquad \textbf{(Note)}$$

where r denotes the distance of any point on the normal from $P(\alpha, \beta, \gamma)$.

Let the normal at $P(\alpha, \beta, \gamma)$ meet the coordinate planes *viz.* yz, zx and xy planes at G_1, G_2 and G_3, then putting $x=0$, $y=0$ and $z=0$ in succession in the

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above equation of the normal we have respectively

$$PG_1 = -a^2/p,\ PG_2 = -b^2/p \text{ and } PG_3 = -c^2/p \qquad \text{...(i)}$$

where $$(1/p^2) = (\alpha^2/a^4) + (\beta^2/b^4) + (\gamma^2/c^4) \qquad \text{...(ii)}$$

or $$PG_1 : PG_2 : PG_3 = a^2 : b^2 : c^2.$$

****Ex. 2. In the above example, if $PG_1^2 + PG_2^2 + PG_3^2 = k^2$, then find the locus of P.**

Sol. Given $$PG_1^2 + PG_2^2 + PG_3^2 = k^2$$

or $(-a^2/p)^2 + (-b^2/p)^2 + (-c^2/p)^2 = k^2$, from (i) of Ex. 1 above

or $$a^4 + b^4 + c^4 = p^2k^2$$

or $$\frac{1}{p^2} = \frac{k^2}{a^4 + b^4 + c^4} \quad \text{or} \quad \frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4} + \frac{\gamma^2}{c^4} = \frac{k^2}{a^4 + b^4 + c^4},$$

from (ii) of Ex. 1 above.

$\therefore P(\alpha, \beta, \gamma)$ lies on $$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = \frac{k^2}{a^4 + b^4 + c^4} \qquad \text{...(ii)}$$

Also $P(\alpha, \beta, \gamma)$ lies on the ellipsoid and hence the locus of $P(\alpha, \beta, \gamma)$ is the curve of intersection of the ellipsoid and (i). **Ans.**

***Ex. 3. Find the length of the normal chord through P of the ellipsoid $\Sigma(x^2/a^2) = 1$ and prove that if it is equal to $4PG_3$, where G_3 is the point where the normal chord through P meets the xy-plane, then P lies on the cone** $$\frac{x^2}{a^6}(2c^2 - a^2) + \frac{y^2}{b^6}(2c^2 - b^2) + \frac{z^2}{c^4} = 0.$$

Sol. Let P be (α, β, γ), then the equations of the normal to the given ellipsoid at $P(\alpha, \beta, \gamma)$ are $$\frac{x - \alpha}{(p\alpha/a^2)} = \frac{y - \beta}{(p\beta/b^2)} = \frac{z - \gamma}{(p\gamma/c^2)} = r \text{ (say)} \qquad \text{...(i)}$$

where $$\frac{1}{p^2} = \frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4} + \frac{\gamma^2}{c^4} \qquad \text{...(ii)}$$

$\therefore$ The coordinates of any point Q on the normal (i) are

$$\left(\alpha + \frac{p\alpha}{a^2} r, \beta + \frac{p\beta}{b^2} r, \gamma + \frac{p\gamma}{c^2} r\right),$$

where r is the distance of Q from P.

If Q lies on the given ellipsoid *i.e.* PQ is the normal chord, then

$$\frac{1}{a^2}\left(\alpha + \frac{p\alpha}{a^2} r\right)^2 + \frac{1}{b^2}\left(\beta + \frac{p\beta}{b^2} r\right)^2 + \frac{1}{c^2}\left(\gamma + \frac{p\gamma}{c^2} r\right)^2 = 1$$

or $$r^2p^2\left(\frac{\alpha^2}{a^6} + \frac{\beta^2}{b^6} + \frac{\gamma^2}{c^6}\right) + 2rp\left(\frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4} + \frac{\gamma^2}{c^4}\right) + \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}\right) = 1$$

or $$r^2p^2\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)+2rp\left(\frac{1}{p^2}\right)=0, \text{ from (ii) and } \Sigma\frac{\alpha^2}{a^2}=1$$

as $P\,(\alpha, \beta, \gamma)$ lies on the given conicoid.

or $$r=\frac{-2}{p^3\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)}=\text{length of normal chord } PQ \qquad \text{...(iii)}$$

Also as in Ex. 1 Page 35 we can show that $PG_3=-c^2/p$ ...(iv)

Given $PQ=4PG_3$ or $$\frac{-2}{p^3\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)}=4\left(-\frac{c^2}{p}\right)$$

or $$2c^2\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)=\frac{1}{p^2}=\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4}, \text{ from (ii)}$$

or $$\frac{\alpha^2}{a^6}(2c^2-a^2)+\frac{\beta^2}{b^6}(2c^2-b^2)+\frac{\gamma^2}{c^6}(2c^2-c^2)=0.$$

$\therefore$ The locus of $P\,(\alpha, \beta, \gamma)$ is $\dfrac{x^2}{a^6}(2c^2-a^2)+\dfrac{y^2}{b^6}(2c^2-b^2)+\dfrac{z^2}{c^4}=0.$

Hence proved.

****Ex. 4. If Q is a point on the normal to the ellipsoid $\Sigma\,(x^2/a^2)=1$ at the point P, such that $3PQ=PG_1+PG_2+PG_3$, where G_1, G_2, G_3 are the points where the normal at P meets the yz, zx and xy planes, then the locus of Q is**

$$\frac{a^2x^2}{(2a^2-b^2-c^2)^2}+\frac{b^2y^2}{(2b^2-c^2-a^2)^2}+\frac{c^2z^2}{(2c^2-a^2-b^2)^2}=\frac{1}{9}$$

(*Rohilkhand* 91)

Sol. Let Q be (x_1, y_1, z_1), then as in Ex. 3 above we can prove that

$$x_1=\alpha+\frac{p\alpha}{a^2}r,\ y_1=\beta+\frac{p\beta}{b^2}r,\ z_1=\gamma+\frac{p\gamma}{c^2}r. \qquad \text{...(i)}$$

and $$r\equiv-\frac{2}{p^3\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)}=PQ \qquad \text{...See (iii) of Ex. 3 above ...(ii)}$$

Given $$PQ=\tfrac{1}{3}(PG_1+PG_2+PG_3)$$

or $$r=\frac{1}{3}\left(-\frac{a^2}{p}-\frac{b^2}{p}-\frac{c^2}{p}\right), \text{ from (i) of Ex. 1 Page 35}$$

or $$pr=-\tfrac{1}{3}(a^2+b^2+c^2) \qquad \text{...(iii)}$$

$\therefore$ From (i) and (iii), we have

$$x_1 = \alpha + \frac{\alpha}{a^2}[-\tfrac{1}{3}(a^2+b^2+c^2)] = \frac{\alpha(2a^2-b^2-c^2)}{3a^2}$$

or
$$\frac{\alpha}{a} = \frac{3ax_1}{2a^2-b^2-c^2} \quad \text{...(iv)}$$

Similarly from (i) and (iii), we can get

$$\frac{\beta}{b} = \frac{3by_1}{2b^2-c^2-a^2}, \ \frac{\gamma}{c} = \frac{3cz_1}{2c^2-a^2-b^2} \quad \text{...(v)}$$

Also as $P(\alpha, \beta, \gamma)$ lies on the ellipsoid $\Sigma(x^2/a^2) = 1$, so we have

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = 1 \text{ or } \left(\frac{\alpha}{a}\right)^2 + \left(\frac{\beta}{b}\right)^2 + \left(\frac{\gamma}{c}\right)^2 = 1$$

or
$$\left(\frac{3ax_1}{2a^2-b^2-c^2}\right)^2 + \left(\frac{3by_1}{2b^2-c^2-a^2}\right)^2 + \left(\frac{3cz_1}{2c^2-a^2-b^2}\right)^2 = 1$$

$\therefore$ The locus of $Q(x_1, y_1, z_1)$ is

$$\frac{a^2x^2}{(2a^2-b^2-c^2)^2} + \frac{b^2y^2}{(2b^2-c^2-a^2)^2} + \frac{c^2z^2}{(2c^2-a^2-b^2)^2} = \frac{1}{9}. \quad \text{Ans.}$$

****Ex. 5. The normal at a variable point P of the ellipsoid $\Sigma(x^2/a^2) = 1$ meets the xy-plane in G_3 and G_3Q is drawn parallel to z-axis and equal to G_3P. Prove that the locus of Q is given by**

$$x^2/(a^2-c^2) + y^2/(b^2-c^2) + z^2/c^2 = 1.$$

Find the locus of R, if OR is drawn from the centre equal and parallel to G_3P.

Sol. As in Ex. 3 Page 36 we can show that the coordinates of any point Q, on the normal at $P(\alpha, \beta, \gamma)$ to the given ellipsoid, at a distance r from P is

$$\left(\alpha + \frac{p\alpha}{a^2}r, \ \beta + \frac{p\beta}{b^2}r, \ \gamma + \frac{p\gamma}{c^2}r\right).$$

Also from Ex. 1 Page 35 we know that $PG = -c^2/p$, so substituting $-c^2/p$ for r, the coordintaes of G_3 are

$$\left(\alpha - \frac{c^2\alpha}{a^2}, \ \beta - \frac{c^2\beta}{b^2}, \ 0\right)$$

$\therefore$ The equations of the line G_3Q, which passes through G_3 and is parallel to z-axis are

$$\frac{x - \{\alpha - (c^2\alpha/a^2)\}}{0} = \frac{y - \{\beta - (c^2\beta/b^2)\}}{0} = \frac{z-0}{1} = r_1 \text{ (say)},$$

where r_1 denotes the distance of any point on this line G_3Q from G_3.

$\therefore$ $G_3Q = G_3P$, so putting $r_1 = G_3P = -c^2/p$ we have the coordinates of Q as

$$\left[\alpha - \frac{c^2\alpha}{a^2},\ \beta - \frac{c^2\beta}{b^2},\ -\frac{c^2}{p}\right]$$

$\therefore$ If Q is (x_1, y_1, z_1), then $x_1 = \alpha - \dfrac{c^2\alpha}{a^2}$, $y_1 = \beta - \dfrac{c^2\beta}{b^2}$, $z_1 = -\dfrac{c^2}{p}$

or $$\alpha = \frac{a^2x_1}{a^2 - c^2},\ \beta = \frac{b^2y_1}{b^2 - c^2},\ \frac{z_1}{c} = -\frac{c_1}{p}. \quad \text{...(i)}$$

Also as $P(\alpha, \beta, \gamma)$ lies on the given ellipsoid, so we have

$$(\alpha^2/a^2) + (\beta^2/b^2) + (\gamma^2/c^2) = 1. \quad \text{...(ii)}$$

The required locus of Q is obtained by eliminating α, β, γ between the relations given in (i) and (ii) and generalising x_1, y_1, z_1.

Now from (i), we get

$$\frac{z_1^2}{c^2} = \frac{c^2}{p} = c^2\left[\frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4} + \frac{\gamma^2}{c^4}\right],$$ from (ii) of Ex. 1 Page 35.

or $$\frac{z_1^2}{c^2} = \frac{c^2\alpha^2}{a^4} + \frac{c^2\beta^2}{b^4} + \frac{\gamma^2}{c^2} \quad \text{or} \quad \frac{\gamma^2}{c^2} = \frac{z_1^2}{c^2} - \frac{c^2\alpha^2}{a^4} - \frac{c^2\beta^2}{b^4} \quad \text{...(iii)}$$

Substituting the values of α, β and γ from (i) and (iii) in (ii), we get

$$\frac{a^2x_1^2}{(a^2 - c^2)^2} + \frac{b^2y_1^2}{(b^2 - c^2)^2} + \left(\frac{z_1^2}{c^2} - \frac{c^2\alpha^2}{a^4} - \frac{c^2\beta^2}{b^4}\right) = 1$$

or $$\frac{a^2x_1^2}{(a^2 - c^2)^2} + \frac{b^2y_1^2}{(b^2 - c^2)^2} + \frac{z_1^2}{c^2} - \frac{c^2x_1^2}{(a^2 - c^2)^2} - \frac{c^2y_1^2}{(b^2 - c^2)^2} = 1, \text{ from (i)}$$

or $$\frac{x_1^2}{(a^2 - c^2)^2}(a^2 - c^2) + \frac{y_1^2}{(b^2 - c^2)^2}(b^2 - c^2) + \frac{z_1^2}{c^2} = 1$$

or $$\frac{x_1^2}{(a^2 - c^2)} + \frac{y_1^2}{(b^2 - c^2)} + \frac{z_1^2}{c^2} = 1$$

$\therefore$ The locus of $Q(x_1, y_1, z_1)$ is $\dfrac{x^2}{(a^2 - c^2)} + \dfrac{y^2}{(b^2 - c^2)} + \dfrac{z^2}{c^2} = 1.$

Hence proved.

Again the equations of the line OR, drawn through the centre $O(0, 0, 0)$ of the given ellipsoid and parallel to the normal through P and equal to $PG_3 = -c^2/p$ are $$\frac{x - 0}{(p\alpha/a^2)} = \frac{y - 0}{(p\beta/b^2)} = \frac{z - 0}{(p\gamma/c^2)} = -\frac{c^2}{p}.$$

$\therefore$ The coordinates of R are $\left(-\dfrac{c^2\alpha}{a^2},\ -\dfrac{c^2\beta}{b^2},\ -\gamma\right)$

If R is (x_2, y_2, z_2) then $x_2 = -c^2\alpha/a^2,\ y_2 = -c^2\beta/b^2,\ z_2 = -\gamma$

or $$\alpha = -a^2x_2/c^2,\ \beta = -b^2y_2/c^2,\ \gamma = -z_2$$

Substituting these values in (ii) and generalising (x_2, y_2, z_2), we get

$$\frac{a^2x^2}{c^4} + \frac{b^2y^2}{c^4} + \frac{z^2}{c^2} = 1 \quad \text{or} \quad a^2x^2 + b^2y^2 + c^2z^2 = c^4,$$

which is the required locus of R. **Ans.**

Ex. 6. If a length PQ be taken on the normal at any point P of the ellipsoid $\Sigma (x^2/a^2) = 1$ such that $PQ = \lambda^2/p$, where λ is constant and p is the length of the perpendicular from the origin to the tangent plane at P, the locus of Q is $$\frac{a^2x^2}{(a^2+\lambda^2)^2} + \frac{b^2y^2}{(b^2+\lambda^2)^2} + \frac{c^2z^2}{(c^2+\lambda^2)^2} = 1.$$

Sol. Let P be (α, β, γ), then the equations of the normal to the given ellipsoid at $P(\alpha, \beta, \gamma)$ are

$$\frac{x-\alpha}{(p\alpha/a^2)} = \frac{y-\beta}{(p\beta/b^2)} = \frac{z-\gamma}{(p\gamma/c^2)} = r = PQ = \frac{\lambda^2}{p} \text{ (given)}$$

Then the coordinates of Q are $\left(\alpha + \frac{p\alpha}{a^2}\cdot\frac{\lambda^2}{p},\ \beta + \frac{p\beta}{b^2}\cdot\frac{\lambda^2}{p},\ \gamma + \frac{p\gamma}{c^2}\cdot\frac{\lambda^2}{p}\right)$

or $$\left(\alpha + \frac{\lambda^2\alpha}{a^2},\ \beta + \frac{\lambda^2\beta}{b^2},\ \gamma + \frac{\lambda^2\gamma}{c^2}\right)$$

$\therefore$ If Q is (x_1, y_1, z_1), then we have

$$x_1 = \alpha\left(1 + \frac{\lambda^2}{a^2}\right),\ y_1 = \beta\left(1 + \frac{\lambda^2}{b^2}\right),\ z_1 = \gamma\left(1 + \frac{\lambda^2}{c^2}\right).$$

whence $$\alpha = \frac{a^2x_1}{a^2+\lambda^2},\ \beta = \frac{b^2y_1}{b^2+\lambda^2},\ \gamma = \frac{c^2z_1}{c^2+\lambda^2} \quad \text{...(i)}$$

Also as $P(\alpha, \beta, \gamma)$ lies on the given ellipsoid, so we have

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{cs} = 1 \quad \text{or} \quad \frac{a^2x_1^2}{(a^2+\lambda^2)^2} + \frac{b^2y_1^2}{(b^2+\lambda^2)^2} + \frac{c^2z_1^2}{(c^2+\lambda^2)^2} = 1.$$

$\therefore$ The locus of $Q(x_1, y_1, z_1)$ is

$$\frac{a^2x^2}{(a^2+\lambda^2)^2} + \frac{b^2y^2}{(b^2+\lambda^2)^2} + \frac{c^2z^2}{(c^2+\lambda^2)^2} = 1. \quad \text{Hence proved.}$$

Ex. 7. If the normals at P and Q, points on the ellipsoid, intersect then PQ is at right angles to its polar with respect to the ellipsoid.

Sol. Let $P(\alpha_1, \beta_1, \gamma_1)$ and $Q(\alpha_2, \beta_2, \gamma_2)$ be the points on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad \text{...(i)}$$

Then the equations of the normals at P and Q to (i) are

$$\frac{x-\alpha_1}{\alpha/a^2}=\frac{y-\beta_1}{\beta/b^2}=\frac{z-\gamma_1}{\gamma_1/c^2} \quad \text{and} \quad \frac{x-\alpha_2}{\alpha_2/a^2}=\frac{y-\beta_2}{\beta_2/b^2}=\frac{z-\gamma_2}{\gamma_2/c^2}$$

If these two normals intersect, then we have

$$\begin{vmatrix} \alpha_1-\alpha_2 & \beta_1-\beta_2 & \gamma_1-\gamma_2 \\ \alpha_1/a^2 & \beta_1/b^2 & \gamma_1/c^2 \\ \alpha_2/a^2 & \beta_2/b^2 & \gamma_2/c^2 \end{vmatrix}=0 \qquad \text{...See Chapter IV}$$

or
$$\Sigma\left[(\alpha_1-\alpha_2)\left(\frac{\beta_1\gamma_2-\gamma_1\beta_2}{b^2c^2}\right)\right]=0. \qquad \text{...(ii)}$$

Also the equations of the line PQ are

$$\frac{x-\alpha_1}{\alpha_2-\alpha_1}=\frac{y-\beta_1}{\beta_2-\beta_1}=\frac{z-\gamma_1}{\gamma_2-\gamma_1} \qquad \text{...(iii)}$$

and the polar of this line w.r. to (i) is given by

$$\frac{\alpha_1 x}{a^2}+\frac{\beta_1 y}{b^2}+\frac{\gamma_1 z}{c^2}=1 \qquad \text{...(iv)}$$

and
$$\Sigma\left[(\alpha_1-\alpha_2)\frac{x}{a^2}\right]=0 \qquad \text{...(v)}$$

or (iv) and $\dfrac{\alpha_2 x}{a^2}+\dfrac{\beta_2 y}{b^2}+\dfrac{\gamma_2 z}{c^2}=1$, from (iv) and (v) ...(vi)

If l, m, n be d.c.'s of the line given by (iv) and (vi), we have

$$\frac{l}{\left(\dfrac{\beta_1\gamma_2-\beta_2\gamma_1}{b^2c^2}\right)}=\frac{m}{\left(\dfrac{\gamma_1\alpha_2-\gamma_2\alpha_1}{c^2a^2}\right)}=\frac{n}{\left(\dfrac{\alpha_1\beta_2-\alpha_2\beta_1}{b^2c^2}\right)} \qquad \text{...(vii)}$$

$\therefore$ If PQ and this polar line of PQ w.r. to (i) are at right angles, then

$$\Sigma\, l\,(\alpha_2-\alpha_1)=0 \quad \text{or} \quad \Sigma\,(\alpha_1-\alpha_2)\left(\frac{\beta_1\gamma_2-\beta_2\gamma_1}{b^2c^2}\right)=0,$$

which is true by virtue of (ii). Hence proved.

***Ex. 8. Normals at P and P′, points of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$, meet the xy-plane in G_3 and G'_3 and make angles θ and θ' with PP′. Prove that $PG_3\cos\theta+P'G'_3\cos\theta'=0$.** (*Rohilkhand* 91)

Sol. Let the points P and P' be α, β, γ and $(\alpha', \beta', \gamma')$ respectively, then as in Ex. 5 Page 38 Ch. IX $PG=-c^2/p$ and $P'G'_3=-c^2/p'$.

Now as direction cosines of the normals at P and P' are respectively $p\alpha/a^2, p\beta/b^2, p\gamma/c^2$ and $p'\alpha'/a^2, p'\beta'/b^2, p'\gamma'/c^2$, so the direction cosines of

PP' are $\qquad \dfrac{\alpha'-\alpha}{PP'}, \dfrac{\beta'-\beta}{PP'}, \dfrac{\gamma'-\gamma}{PP'}$ **(Note)**

$$\therefore\ PG_3 \cos\theta = -\frac{c^2}{p}\left[\frac{p\alpha}{a^2}\cdot\frac{\alpha'-\alpha}{PP'} + \frac{p\beta}{b^2}\cdot\frac{\beta'-\beta}{PP'} + \frac{p\gamma}{c^2}\cdot\frac{\gamma'-\gamma}{PP'}\right]$$

$$= -\frac{c^2}{PP'}\left[\frac{\alpha\alpha'}{a^2} + \frac{\beta\beta'}{b^2} + \frac{\gamma\gamma'}{c^2} - \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}\right)\right]$$

$$= -\frac{c^2}{PP'}\left[\frac{\alpha\alpha'}{a^2} + \frac{\beta\beta'}{b^2} + \frac{\gamma\gamma'}{c^2} - 1\right], \qquad \text{...(i)}$$

$\because$ (α, β, γ) lies on the given ellipsoid

Similarly $P'G'_3 \cos\theta = -\dfrac{c^2}{PP'}\left[1 - \left(\dfrac{\alpha\alpha'}{a^2} + \dfrac{\beta\beta'}{b^2} + \dfrac{\gamma\gamma'}{c^2}\right)\right]$

$= -PG_3 \cos\theta$, from (i)

$\therefore$ $\qquad PG_3 \cos\theta + P'G'_3 \cos\theta = 0.$ Hence proved.

Exercises on § 9.13.

Ex. 1. Find the normal to the conicoid

$(x^2/4) + (y^2/9) - (z^2/36) = 1$ at the point (2, 3, 6). (*Kumaun* 91)

**§ 9.14. Number of normals from a given point to an ellipsoid.

(*Meerut* 90 *S*; *Rohilkhand* 90)

The equations of the normal at $P(\alpha, \beta, \gamma)$ to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad \text{...(i)}$$

are
$$\frac{x-\alpha}{\alpha/a^2} = \frac{y-\beta}{\beta/b^2} = \frac{z-\gamma}{\gamma/c^2} = \lambda \text{ (say)} \qquad \text{...(ii)}$$

If this normal passes through a given point (x_1, y_1, z_1), then we have

$$\frac{x_1-\alpha}{(\alpha/a^2)} = \frac{y_1-\beta}{(\beta/b^2)} = \frac{z_1-\gamma}{(\gamma/c^2)} = \lambda$$

whence we have $x_1 = \alpha + \dfrac{\alpha\lambda}{a^2} = \alpha\left(\dfrac{a^2+\lambda}{a^2}\right)$ etc.

or
$$\alpha = \frac{a^2 x_1}{a^2+\lambda},\ \beta = \frac{b^2 y_1}{b^2+\lambda},\ \gamma = \frac{c^2 z_1}{c^2+\lambda} \qquad \text{...(iii)}$$

But $P(\alpha, \beta, \gamma)$ lies on (i), so we have $\dfrac{\alpha^2}{a^2} + \dfrac{\beta^2}{b^2} + \dfrac{\gamma^2}{c^2} = 1$

or $$\frac{\alpha^2 x_1^2}{(a^2+\lambda)^2}+\frac{b^2 y_1^2}{(b^2+\lambda)^2}+\frac{c^2 z_1^2}{(c^2+\lambda)^2}=1, \text{ from (iii).} \qquad \text{...(iv)}$$

This equation is of sixth degree in λ and so gives six values of λ correspoinding to each of which we get a point (α, β, γ) on (i) the normal at which passes through the given point (x_1, y_1, z_1). Hence there are six points on an ellipsoid the normals at which pass through a given point or **six normals can be drawn to an ellipsoid from a given point.**

§ 9.15. Cubic curve through the feet of the normals.

From § 9.14 above we know that if the normal at $P(\alpha, \beta, \gamma)$ to the ellipsoid $\Sigma(x^2/a^2)=1$ passes through the point (x_1, y_1, z_1), then

$$\alpha=\frac{a^2 x_1}{a^2+\lambda},\ \beta=\frac{b^2 y_1}{b^2+\lambda},\ \gamma=\frac{c^2 z_1}{c^2+\lambda}$$

Now consider the curve whose parametric equations are

$$x=\frac{a^2 x_1}{a^2+\lambda},\ y=\frac{b^2 y_1}{b^2+\lambda},\ z=\frac{c^2 z_1}{c^2+\lambda}, \qquad \text{...(i)}$$

where λ is a parameter, having six values given by (iv) of § 9.14 above corresponding to each of which we get a point on the curve (i).

The points of intersection of the curve (i) with any plane $Ax+By+Cz+D=0$ are given by

$$A\left(\frac{a^2 x_1}{a^2+\lambda}\right)+B\left(\frac{b^2 y_1}{b^2+\lambda}\right)+C\left(\frac{c^2 z_1}{c^2+\lambda}\right)+D=0 \qquad \text{...(ii)}$$

This equation (ii) is a cubic in λ and therefore the curve (i) meets the plane given above in three points and therefore the curve (i) is a cubic curve.

Hence the six feet of the normals that can be drawn to an ellipsoid from a given point are the points of intersection of a certain cubic curve with the ellipsoid.

****§ 9.16. Cone thrugh six concurrent normals.**

(*Allahabad* 92; *Gorakhpur* 96)

In § 9.14 Page 42 we have proved that from any point (x_1, y_1, z_1) six normals can be drawn to an ellipsoid $\Sigma(x^2/a^2)=1$.

Now the equations of any line through (x_1, y_1, z_1) are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}. \qquad \text{...(i)}$$

If this line is a normal to the given ellipsoid, then from § 9.13 Pages 34 – 35 we know that $l=p\alpha/a^2,\ m=p\beta/b^2,\ n=p\gamma/c^2$. ...(ii)

Also from result (iii) of § 9.14 Page 42, we know that

$$\alpha = \frac{a^2 x_1}{(a^2 + \lambda)}, \ \beta = \frac{b^2 y_1}{(b^2 + \lambda)}, \ \gamma = \frac{c^2 z_1}{(c^2 + \gamma)} \quad \text{...(iii)}$$

$\therefore$ From (ii) and (iii) we have

$$l = \frac{p}{a^2} \cdot \frac{a^2 x_1}{(a^2 + \lambda)} = \frac{p x_1}{(a^2 + \lambda)} \quad \text{or} \quad \frac{p x_1}{l} = a^2 + \lambda.$$

Similarly $p y_1 / m = b^2 + \lambda$ and $p z_1 / n = c^2 + \lambda$.

$$\therefore \frac{p x_1}{l}(b^2 - c^2) + \frac{p y_1}{m}(c^2 - a^2) + \frac{p z_1}{n}(a^2 - b^2)$$

$$= (a^2 + \lambda)(b^2 - c^2) + (b^2 + \lambda)(c^2 - a^2) + (c^2 + \lambda)(a^2 - b^2) \quad \textbf{(Note)}$$

$$= 0. \quad \text{...(iv)}$$

From (i) and (iv) we conclude that normals (i) lies on the cone

$$\frac{p x_1 (b^2 - c^2)}{x - x_1} + \frac{p y_1 (c^2 - a^2)}{y - y_1} + \frac{p z_1 (a^2 - b^2)}{z - z_1} = 0$$

or

$$\frac{x_1 (b^2 - c^2)}{x - x_1} + \frac{y_1 (c^2 - a^2)}{y - y_1} + \frac{z_1 (a^2 - b^2)}{z - z_1} = 0 \quad \text{...(v)}$$

Hence the six normals to the given ellipsoid from the point (x_1, y_1, z_1) lie on the cone (v) which is of second degree.

****Note.** From § 9.14 Page 42 we have the parametric equations of the curve through the feet of the six normals as

$$x = \frac{a^2 x_1}{a^2 + \lambda}, \ y = \frac{b^2 y_1}{b^2 + \lambda}, \ z = \frac{c^2 z_1}{c^2 + \lambda},$$

where λ is a parameter.

Evidently these satisfy the equation (v) above of the cone through the above six concurrent normals [students can substitute these values in equation (v) above and see for themselves] for all values of λ. Hence the cubic curve through the feet of the six normals from a given point to an ellipsoid lies on the cone through the above six concurrent normals.

Solved Examples on § 9.14 to § 9.16.

***Ex. 1 (a). Prove that the feet of the six normals from (x_1, y_1, z_1) to the ellipsoid $\Sigma (x^2/a^2) = 1$ lie on the curve of intersection of the ellipsoid and the cone** $\dfrac{a^2 (b^2 - c^2) x_1}{x} + \dfrac{b^2 (c^2 - a^2) y_1}{y} + \dfrac{c^2 (a^2 - b^2) z_1}{z} = 0.$ *(Avadh 94)*

Sol. From § 9.14 Page 42 we know that the coordinates of the six feet of the normals drawn to the given ellipsoid from the point (x_1, y_1, z_1) are given by $x = \dfrac{a^2 x_1}{a^2 + \lambda}, \ y = \dfrac{b^2 y_1}{b^2 + \lambda}, \ z = \dfrac{c^2 z_1}{c^2 + \lambda}$ where λ is given by an equation of sixth degree. (Note)

$$\therefore \quad \lambda = \frac{a^2 x_1}{x} - a^2 = \frac{b^2 y_1}{y} - b^2 = \frac{c^2 z_1}{z} - c^2 \qquad \textbf{(Note)}$$

$$\therefore \left(\frac{a^2 x_1}{x} - a^2\right)(b^2 - c^2) + \left(\frac{b^2 y_1}{y} - b^2\right)(c^2 - a^2) + \left(\frac{c^2 z_1}{z} - c^2\right)(a^2 - b^2)$$

$$= \lambda (b^2 - c^2) + \lambda (c^2 - a^2) + \lambda (a^2 - b^2) \qquad \textbf{(Note)}$$

or $\quad \Sigma [(a^2 x_1/x)(b^2 - c^2)] - [\Sigma a^2 (b^2 - c^2)]$

$$= \lambda [(b^2 - c^2) + (c^2 - a^2) + (a^2 + b^2)] = \lambda (0) = 0$$

or $$\Sigma [(a^2 x_1/x)(b^2 - c^2)] = 0. \qquad \text{...(i)}$$

This equation represents a cone and is the locus of the feet of the normals but these feet of the normals also lie on the given ellipsoid, therefore the feet of these normals lie on the curve of intersection of the cone (i) and the given ellipsoid. Hence proved.

Ex. 1 (b). Show that the feet of the six normals drawn to the ellipsoid $ax^2 + by^2 + cz^2 = 1$ from any point (x_1, y_1, z_1) lies on the curve of intersection of the given ellipsoid and the cone

$$\frac{a(b-c)x_1}{x} + \frac{b(c-a)y_1}{y} + \frac{c(a-b)z_1}{z} = 0.$$

Sol. Do as Ex 1 (a). above

***Ex. 2. A is a fixed point and P a variable point such that its polar plane with respect to the ellipsoid $\Sigma (x^2/a^2) = 1$ is at right angles to AP, show that the locus of P is the cubic curve through feet of the normals from A.**

Sol. Let A and P be the points (x_1, y_1, z_1) and (α, β, γ) respectively.

Then the polar plane of P w.r to the given ellipsoid is

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = 1. \qquad \text{...(i)}$$

Also the direction ratios of AP are $x_1 - \alpha, y_1 - \beta, z_1 - \gamma$.

Since AP is at right angles to polar plane (i), so we have

$$\frac{x_1 - \alpha}{(\alpha/a^2)} = \frac{y_1 - \beta}{(\beta/b^2)} = \frac{z_1 - \gamma}{(\gamma/c^2)} = \lambda \text{ (say)} \qquad \textbf{(Note)}$$

or $$x_1 = \alpha\left(1 + \frac{\lambda}{a^2}\right) \quad \text{or} \quad \alpha = \frac{a^2 x_1}{a^2 + \lambda},$$

Similarly $$\beta = \frac{b^2 y_1}{b^2 + \lambda} \quad \text{and} \quad \gamma = \frac{c^2 z_1}{c^2 + \lambda}.$$

Hence the locus of $P(\alpha, \beta, \gamma)$ is the curve given by

$$x=a^2x_1/(a^2+\lambda),\ y=b^2y_1/(b^2+\lambda),\ z=c^2z_1/(c^2+\lambda),$$

which we know is the cubic curve through the feet of the six normals to the given ellipsoid from the point A. (See § 9.15 Page 43)

Hence proved.

Ex. 3. Prove that for all values of λ, the normals to the conicoid $\Sigma[(x^2/(a^2+\lambda))]=1$ which pass through a point (x_1, y_1, z_1) meet the plane $z=0$ in points on the conic

$$(c^2-a^2)\frac{x_1}{x}+(b^2-c^2)\frac{y_1}{y}+(a^2-b^2)=0,\ z=0.$$

Sol. From § 9.16 Page 43 we know that the equation of the cone through the six concurrent normals from (x_1, y_1, z_1) to the ellipsoid $\Sigma(x^2/a^2)=1$ is

$$\frac{x_1(b^2-c^2)}{x-x_1}+\frac{y_1(c^2-a^2)}{y-y_1}+\frac{z_1(a^2-b^2)}{z-z_1}=0.$$

$\therefore$ Here the equation of the cone through the six concurrent normals from the point (x_1, y_1, z_1) to the given conicoid

$$\Sigma\frac{x^2}{(a^2+\lambda)}=1 \quad \text{is} \quad \Sigma\frac{x_1[(b^2+\lambda)-(c^2+\lambda)]}{x-x_1}=0$$

or
$$\frac{x_1(b^2-c^2)}{x-x_1}+\frac{y_1(c^2-a^2)}{y-y_1}+\frac{z_1(a^2-b^2)}{z-z_1}=0. \qquad \text{...(i)}$$

It meets the plane $z=0$, where

$$\frac{x_1(b^2-c^2)}{(x-x_1)}+\frac{y_1(c^2-a^2)}{(y-y_1)}+\frac{z_1(a^2-b^2)}{0-z_1}=0.$$

or $\quad x_1(b^2-c^2)(y-y_1)+y_1(c^2-a^2)(x-x_1)=(a^2-b^2)(x-x_1)(y-y_1)$

or $\quad x_1y(b^2-c^2+a^2-b^2)+xy_1(c^2-a^2+a^2-b^2)$

$\qquad -x_1y_1(b^2-c^2+c^2-a^2+a^2-b^2)+xy(b^2-a^2)=0$, on expanding

or $\quad x_1y(a^2-c^2)+xy_1(c^2-b^2)+xy(b^2-a^2)=0$

or
$$\frac{x_1(c^2-a^2)}{x}+\frac{y_1(b^2-c^2)}{y}+(a^2-b^2)=0,$$

dividing each term by $-xy$.

$\therefore$ The plane $z=0$ meets the cone (i) in the conic given by

$$\frac{x_1(c^2-a^2)}{x}+\frac{y_1(b^2-c^2)}{y}+(a^2-b^2)=0,\ z=0. \quad \text{Hence proved.}$$

***Ex. 4. Prove that the normals to the ellipsoid $\Sigma(x^2/a^2)=1$ at all points of intersection with $lyz+mzx+nxy=0$ intersect the line**

$$\frac{a^2x}{l(a^2-b^2)(c^2-a^2)}=\frac{b^2y}{m(b^2-c^2)(a^2-b^2)}=\frac{c^2z}{n(c^2-a^2)(b^2-c^2)}.$$

(*Rohilkhand* 93)

Sol. The equations of the normal at any point (α, β, γ) to the given ellipsoid are
$$\frac{x-\alpha}{(\alpha/a^2)} = \frac{y-\beta}{(\beta/b^2)} = \frac{z-\gamma}{(\gamma/c^2)}. \quad \text{...(i)}$$

Also as (α, β, γ) lies on the plane $lyz + mzx + nxy = 0$, so we get
$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0 \quad \text{...(ii)}$$

If the line (i) intersects the given line, then we must have
$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha/a^2 & \beta/b^2 & \gamma/c^2 \\ \frac{l(a^2-b^2)(c^2-a^2)}{a^2} & \frac{m(b^2-c^2)(a^2-b^2)}{b^2} & \frac{n(c^2-a^2)(b^2-c^2)}{c^2} \end{vmatrix} = 0$$

or $$\begin{vmatrix} \alpha a^2(b^2-c^2) & \beta b^2(c^2-a^2) & \gamma c^2(a^2-b^2) \\ \alpha(b^2-c^2) & \beta(c^2-a^2) & \gamma(a^2-b^2) \\ l & m & n \end{vmatrix} = 0,$$

multiplying 1st, 2nd, 3rd columns by $a^2(b^2-c^2)$, $b^2(c^2-a^2)$ and $c^2(a^2-b^2)$ respectively and cancelling $(a^2-b^2)(b^2-c^2)(c^2-a^2)$ from 3rd row.

or $$\begin{vmatrix} \alpha a^2(b^2-c^2) & \beta b^2(c^2-a^2) & \gamma c^2(a^2-b^2) \\ \alpha(b^2-c^2) & \beta(c^2-a^2) & \gamma(a^2-b^2) \\ l\alpha\beta\gamma & m\alpha\beta\gamma & n\alpha\beta\gamma \end{vmatrix} = 0,$$

multiplying 3rd row by $\alpha\beta\gamma$

or $$\begin{vmatrix} a^2(b^2-c^2) & b^2(c^2-a^2) & c^2(a^2-b^2) \\ (b^2-c^2) & (c^2-a^2) & (a^2-b^2) \\ l\beta\gamma & m\alpha\gamma & n\alpha\beta \end{vmatrix} = 0,$$

taking out α, β and γ common from 1st, 2nd and 3rd columns respectively

or $$\begin{vmatrix} 0 & b^2(c^2-a^2) & c^2(a^2-b^2) \\ 0 & (c^2-a^2) & (a^2-b^2) \\ 0 & m\alpha\gamma & n\alpha\beta \end{vmatrix} = 0,$$

adding all the columns to 1st and using (ii).

This is evidently true. Hence proved.

***Ex. 5. Prove that the lines drawn from the origin parallel to the normal $ax^2 + by^2 + cz^2 = 1$ at its points of intersection with the plane $lx + my + nz = p$ generate the cone**
$$p^2\left(\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c}\right) = \left(\frac{lx}{a} + \frac{my}{b} + \frac{nz}{c}\right)^2$$

Sol. Let (α, β, γ) be the point of intersection of the given conicoid and the given plane, then we have $a\alpha^2 + b\beta^2 + c\gamma^2 = 1$. ...(i) and $$l\alpha + m\beta + n\gamma = p \qquad \text{...(ii)}$$

Also the equations of the normals to the given conicoid at (α, β, γ) are

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{c\gamma}$$ ...See § 9.13 Pages 34-35

$\therefore$ The equations of the line through the origin parallel to this line are

$$x/(a\alpha) = y/(b\beta) = z/(c\gamma) \qquad \text{...(iii)}$$

From (i) and (iii), we have $a\alpha^2 + b\beta^2 + c\gamma^2 = \left(\dfrac{l\alpha + m\beta + n\gamma}{p}\right)^2$

or $$p^2(a\alpha^2 + b\beta^2 + c\gamma^2) = (l\alpha + m\beta + n\gamma)^2$$

or $$p^2\left[\frac{(a\alpha)^2}{a} + \frac{(b\beta)^2}{b} + \frac{(c\gamma)^2}{c}\right] = \left[\frac{l(a\alpha)}{a} + \frac{m(b\beta)}{b} + \frac{n(c\gamma)}{c}\right]^2 \qquad \textbf{(Note)}$$

or $$p^2\left[\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c}\right] = \left(\frac{lx}{a} + \frac{my}{b} + \frac{nz}{c}\right)^2,$$

from (iii) eliminating α, β, γ.

Hence the line (iii) generates the above cone. Hence proved.

****Ex. 6 (a). If P, Q, R, P′, Q′, R′ are the feet of the six normals from a point to the ellipsoid $\Sigma(x^2/a^2) = 1$, and the plane PQR is given by $lx + my + nz = p$, prove that the plane P′ Q′ R′ is given by**

$$\frac{x}{a^2 l} + \frac{y}{b^2 m} + \frac{z}{c^2 n} + \frac{1}{p} = 0.$$

Sol. Let (x_1, y_1, z_1) be the given point, then the feet of the six normals from it to the given ellipsoid (see § 9.14 Page 42) are given by

$$\alpha = \frac{a^2 x_1}{a^2 + \lambda}, \quad \beta = \frac{b^2 y_1}{b^2 + \lambda}, \quad \gamma = \frac{c^2 z_1}{c^2 + \lambda} \qquad \text{...(i)}$$

and the six values of λ are given by the equation

$$\frac{a^2 x_1^2}{(a^2+\lambda)^2} + \frac{b^2 y_1^2}{(b^2+\lambda)^2} + \frac{c^2 z_1^2}{(c^2+\lambda)^2} - 1 = 0 \qquad \text{...(ii)}$$

Now the equation of the plane PQR is given as $lx + my + nz = p$ and the three of the six feet of the normals lie on it, so we have

$$\frac{la^2 x_1}{a^2+\lambda} + \frac{mb^2 y_1}{b^2+\lambda} + \frac{nc^2 z_1}{(c^2+\lambda)} - p = 0, \qquad \text{...(iii)}$$

substituting the values from (i) in the equation of the plane PQR.

Let the equation of the plane $P'Q'R'$ be

$$l'x + m'y + n'z = p' \qquad \text{...(iv)}$$

Since the remaining three feet of the normals lie on it so we have

$$\frac{l' a^2 x_1}{a^2+\lambda}+\frac{m' b^2 y_1}{b^2+\lambda}+\frac{n' c^2 z_1}{c^2+\lambda}-p'=0 \quad \text{...(v)}$$

Hence we find that equation (ii) is the product of (iii) and (v), so comparing them we have

$$\frac{ll' a^4 x_1^2}{(a^2+\lambda)^2}=\frac{a^2 x_1^2}{(a^2+\lambda)^2} \quad \text{or} \quad ll' a^2=1 \quad \text{or} \quad l'=\frac{1}{a^2 l}$$

Similarly $\quad m'=\frac{1}{b^2 m},\ n'=\frac{1}{c^2 n}$ and $p'=-\frac{1}{p}$

$\therefore$ From (iv) the required equation of the plane $P'Q'R'$ is

$$\frac{x}{a^2 l}+\frac{y}{b^2 m}+\frac{z}{c^2 n}+\frac{1}{p}=0. \qquad \text{Hence proved.}$$

***Ex. 6 (b). If P, Q, R and P′, Q′, R′ are the feet of the six normals drawn from an external point to the conicoid $ax^2+by^2+cz^2=1$ and the plane PQR is given by $lx+my+nz=p$, then prove that the plane P′, Q′, R′ is $(ax/l)+(by/m)+(cz/n)+(1/p)=0$.** *(Allahabad 91)*

Sol. Do as Ex. 6 (a) above.

***Ex. 7. If the feet of the three normals from P to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ lie on the plane $x/a+y/b+z/c=1$ prove that the feet of the other three lie on the plane $x/a+y/b+z/c+1=0$ and P lies on the line $a(b^2-c^2)x=b(c^2-a^2)y=c(a^2-b^2)z$.**

Sol. Do first part as Ex. 6 (a) Page 48 Ch. IX.

[Here $l=1/a, m=1/b, n=1/c, p=1$.

$\therefore l'=1/a^2 l=1/a,\ m'=1/b,\ n'=1/c,\ p'=-1/p=-1$

The six feet of the normals, therefore lie on

$$\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1\right)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1\right)=0 \quad \text{or} \quad \left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)^2-1=0$$

or
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)+2\left(\frac{yz}{bc}+\frac{zx}{ac}+\frac{xy}{ab}\right)=1$$

or
$$\frac{yz}{bc}+\frac{zx}{ac}+\frac{xy}{ab}=0, \ \because \text{ the feet also lie on } \Sigma(x^2/a^2)=1$$

Hence the feet of the normals lie on $\quad \frac{yz}{bc}+\frac{zx}{ca}+\frac{xy}{ab}=0. \quad$ ...(i)

Also from Ex. 1 (a) Page 44 we know that if the normals be drawn from the point $P(x_1, y_1, z_1)$ then the feet of the normals lie on the cone

$$\frac{a^2 x_1(b^2-c^2)}{x}+\frac{b^2 y_1(c^2-a^2)}{y}+\frac{c^2 z_1(a^2-b^2)}{z}=0$$

or $$a^2 x_1 (b^2 - c^2) yz + b^2 y_1 (c^2 - a^2) zx + c^2 z_1 (a^2 - b^2) xy = 0 \quad ...(iii)$$

Comparing (i) and (ii), we get

$$\frac{a^2 (b^2 - c^2) x_1}{1/(bc)} = \frac{b^2 (c^2 - a^2) y_1}{1/(ca)} = \frac{c^2 (a^2 - b^2) z_1}{1/(ab)}$$

or $$a (b^2 - c^2) x_1 = b (c^2 - a^2) y_1 = c (a^2 - b^2) z_1.$$

$\therefore$ The required locus of $P (x_1, y_1, z_1)$ is the line

$$a (b^2 - c^2) x = b (c^2 - a^2) y = c (a^2 - b^2) z. \qquad \text{Hence proved.}$$

Ex. 8. If A, A′ are the poles of the planes PQR, P′ Q′ R′ of Ex. 6 (a) Page 48 then prove that $AA'^2 - OA^2 - OA'^2 = 2 (a^2 + b^2 + c^2)$.

Sol. Let A and A' be (x_2, y_2, z_2) and (x_3, y_3, z_3).

Then the polars of A and A' w.r.t. to the given ellipsoid are

$$\frac{xx_2}{a^2} + \frac{yy_1}{b^2} + \frac{zz_2}{c^2} - 1 = 0 \quad \text{and} \quad \frac{xx_3}{a^2} + \frac{yy_3}{b^2} + \frac{zz_3}{c^2} - 1 = 0.$$

Since the feet of three normals lie on each of these planes therefore as in Ex. 6 (a) Page 48, we have

$$\Sigma \left(\frac{a^2 x_1}{a^2 + \lambda} \cdot \frac{x_2}{a^2} \right) - 1 = 0 \quad \text{and} \quad \Sigma \left(\frac{a^2 x_1}{a^2 + \lambda} \cdot \frac{x_3}{a^2} \right) - 1 = 0$$

or $$\frac{x_1 x_2}{a^2 + \lambda} + \frac{y_1 y_2}{b^2 + \lambda} + \frac{z_1 z_2}{c^2 + \lambda} - 1 = 0 \quad ...(i)$$

and $$\frac{x_1 x_3}{a^2 + \lambda} + \frac{y_1 y_3}{b^2 + \lambda} + \frac{z_1 z_3}{c^2 + \lambda} - 1 = 0 \quad ...(ii)$$

Also we know from § 9.14 Page 42 that the feet of the six normals are given by

$$\frac{a^2 x_1^2}{(a^2 + \lambda)^2} + \frac{b^2 y_1^2}{(b^2 + \lambda)^2} + \frac{c^2 z_1^2}{(c^2 + \lambda)^2} - 1 = 0 \quad ...(iii)$$

Hence the product of (i) and (ii) is (iii) so comparing them we have

$$\frac{x_2 x_3}{a^2} = \frac{y_2 y_3}{b^2} = \frac{z_2 z_3}{c^2} = \frac{1}{-1} \qquad \textbf{(Note)}$$

or $$x_2 x_3 = -a^2, \; y_2 y_3 = -b^2, \; z_2 z_3 = -c^2 \quad ...(iv)$$

Now $AA'^2 - OA^2 - OA'^2 = [(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2]$

$$- (x_2^2 + y_2^2 + z_2^2) - (x_3^2 + y_3^2 + z_3^2)$$

$= -2 (x_2 x_3 + y_2 y_3 + z_2 z_3) = -2 (-a^2 - b^2 - c^2)$, from (iv)

$= 2 (a^2 + b^2 + c^2)$.

Hence proved.

***Ex. 9. Prove that four normals to the ellipsoid $\Sigma (x^2/a^2) = 1$ pass through any point on the curve of intersection of the ellipsoid and the conicoid $x^2 (b^2 + c^2) + y^2 (c^2 + a^2) + z^2 (a^2 + b^2) = b^2 c^2 + c^2 a^2 + a^2 b^2$.**

Sol. Let (x_1, y_1, z_1) be the point from which the normals are drawn to the given ellipsoid. If this point lies on the curve of intersection of the given ellipsoid and conicoid, then we have

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1 \qquad ...(i)$$

and $$x_1^2(b^2+c^2)+y_1^2(c^2+a^2)+z_1^2(a^2+b^2)=b^2c^2+c^2a^2+a^2b^2 \qquad ...(ii)$$

Also we know from § 9.14 Page 42 that the six values of λ correspoiding to each of which we get a point on the given ellipsoid the normals at which passes through (x_1, y_1, z_1) is given by the equation

$$\frac{a^2x_1^2}{(a^2+\lambda)^2}+\frac{b^2y_1^2}{(b^2+\lambda)^2}+\frac{c^2z_1^2}{(c^2+\lambda)^2}=1$$

or $$\Sigma\,[a^2x_1^2(b^2+\lambda)^2(c^2+\lambda)^2]=(a^2+\lambda)^2(b^2+\lambda)^2(c^2+\lambda)^2$$

or $$\lambda^6+2(a^2+b^2+c^2)\lambda^5+\ldots$$

$$\ldots+2a^2b^2c^2\,[\Sigma\,x_1^2(b^2+c^2)+(a^2b^2+b^2c^2+c^2a^2)]\,\lambda$$

$$+[\Sigma\,(a^2x_1^2b^4c^4)-a^4b^4c^4]=0 \qquad ...(iii)$$

If only four normals are drawn, then two roos of (iii) should be zero and therefore, the constant term and the coefficient of λ in (ii) should be zero.

i.e. $$a^2x_1^2b^4c^4+b^2y_1^2c^4a^4+c^2z_1^2a^4b^4-a^4b^4c^4=0 \qquad ...(iv)$$

and $$x_1^2(b^2+c^2)+y_1^2(c^2+a^2)+z_1^2(a^2+b^2)=a^2b^2+b^2c^2+c^2a^2 \qquad ...(v)$$

From (iv) we have $\quad (x_1^2/a^2)+(y_1^2/b^2)+(z_1^2/c^2)=1 \qquad ...(vi)$

Now (v) and (vi) are true by virtue of (i) and (ii). Hence the equation (iii) gives only four values of λ and therefore only four normals can be drawn.

Exercise on § 9.14 to 9.16

Ex. Prove that the six normals drawn from any point to a central conicoid meet a principal plane in six points which lie on a rectangular hyperbola. (*Kumaun* 95)

[**Hint :** See Ex. 3. Page 46 Ch. IX]

§ **9.17. Dimaetral Plane** *i.e. ~~the locus of the mid-points of a system of~~ parallel chords.*

Let the conicoid be $\quad ax^2+by^2+cz^2=1 \qquad ...(i)$

Consider a line $\quad \dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}. \qquad ...(ii)$

Any point on this line is $(\alpha+lr,\ \beta+mr,\ \gamma+nr)$. Therefore the distance of the points of intersection of (i) and (ii) from the point (α, β, γ) are the roots of the equation $\quad a(\alpha+lr)^2+b(\beta+mr)^2+c(\gamma+nr)^2=1$

or $\quad r^2(al^2+bm^2+cn^2)+2r(al\alpha+bm\beta+cn\gamma)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0$ ...(iii)

If (α, β, γ) is the mid-point of the chord, then as in § 9.12 Page 27, we have

$$al\alpha+bm\beta+cn\gamma=0 \quad \text{...(iv)}$$

Since the chords are parallel, so l, m, n are fixed and therefore the locus of the mid-point (α, β, γ) from (iv) is the plane $alx+bmy+cnz=0$

$\therefore$ The equation of the diametral plane or the locus of the mid-points of a system of parallel chords with direction cosines l, m, n is

$$\mathbf{alx+bmy+cnz=0} \quad \text{...(v)}$$

which evidently passes through the centre (0, 0, 0) of the conicoid.

Note : For the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, the equation of a diametral plane is

$$\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}=0, \quad \text{...(vi)}$$

which bisects a system of parallel chords, of the ellipsoid, with direction cosines l, m, n.

Example. *Show that every plane through the centre is a diametral plane of the conicoid.*

Sol. Let $\quad Ax+By+Cz=0$ ...(i)

be any plane through the centre of the conicoid

$$ax^2+by^2+cz^2=1 \quad \text{...(ii)}$$

If (i) is a diametral plane of (ii) corresponding to the direction l, m, n (say), then (i) is the same as $\quad alx+bmy+cnz=0$ ...(iii)

Comparing (i) and (iii), we have

$$\frac{al}{A}=\frac{bm}{B}=\frac{cn}{C} \quad \text{or} \quad \frac{l}{A/a}=\frac{m}{B/b}=\frac{n}{C/c}$$

$\therefore$ The plane (i) is a diametral plane corresponding to the direction

$$A/a, B/b, C/C$$

Hence proved.

Exercise on §9.17.

Ex. Find the equation of the diametral plane of the line $x/l=y/m=z/n$ with respect to the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$.

Prove further that diametral plane is parallel to tangent planes at the extremities of that diameter. (*Kanpur* 90)

**§ 9.18. Conjugate diameter and conjugate diametral planes.

If $P(x_1, y_1, z_1)$ be any point on the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ whose centre is $O(0, 0, 0)$, then the direction ratios of the line OP are x_1, y_1, z_1 and the equation of the diametral plane of the line OP from (vi) of § 9.17 above is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0. \qquad ...(i)$$

Let $Q(x_2, y_2, z_2)$ be any point on the plane and on the ellipsoid, then

$$\frac{x_2x_1}{a^2}+\frac{y_2y_1}{b^2}+\frac{z_2z_1}{c^2}=0 \qquad ...(ii)$$

The relation is the condition that the diametral plane of line OP should pass through the point Q on the ellipsoid. ~~But~~ from the symmetry of the relation (ii) we also conclude that the diametral plane of the line OQ passes through the point P on the ellipsoid.

Now if the diametral planes of OP and OQ intersect in the line which cuts the surface of the ellipsoid in the point $R(x_3, y_3, z_3)$ then R lies on the diametral planes of OP and OQ and therefore P and Q are on the diametral plane of the line OR i.e. on plane $\frac{xx_3}{a^2}+\frac{yy_3}{b^2}+\frac{zz_3}{c^2}=0.$ ...(iii)

$\therefore$ **The diametral plane of OR is the plane OPQ. Similarly the diametral plane of OP and OQ are QOR and ROP respectively.**

Definitions :

(a) Conjugate semi-diameters. The three semi-diameters OP, OQ and OR are called the conjugate semi-diameters of the ellipsoid with centre O, provided the plane containing any two of them is the diametral plane of the third.

(b) Conjugate diametral planes. The three planes POQ, QOR and ROP are called conjugate diametral planes provided each is the diametral plane of the line of intersection of the other two.

****Relation between the coordinates of P, Q and R.**

Since P, Q and R lie on the ellipsoid, so we have

$$\left.\begin{aligned}(x_1^2/a^2)+(y_1^2/b^2)+(z_1^2/c^2)=1\,;\\(x_2^2/a^2)+(y_2^2/b^2)+(z_2^2/c^2)=1\,;\\(x_3^2/a^2)+(y_3^2/b^2)+(z_3^2/c^2)=1\,;\end{aligned}\right\} \qquad ...(I)$$

Also as the diametral plane of any one of OP, OQ and OR passes through the extremities of the other two, so we have

$$\left.\begin{aligned}\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0,\\ \frac{x_2x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}=0,\\ \text{and}\quad \frac{x_3x_1}{a^2}+\frac{y_3y_1}{b^2}+\frac{z_3z_1}{c^2}=0,\end{aligned}\right\} \qquad ...(II)$$

Also from relation (I), we observe that

$$x_1/a, y_1/b, z_1/c : x_2/a, y_2/b, z_2/c \text{ and } x_3/a, y_3/b, z_3/c \quad ...(III)$$

are the direction cosines of some three lines, since $l^2 + m^2 + n^2 = 1$ if l, m, n are the d.c.'s of any line.

Also if two lines with d.c.'s l_1, m_1, n_1, and l_2, m_2, n_2 are at right angles to each other, then $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ and therefore from relations (II) we conclude that the three lines with d.c.'s as given in (III), are mutually perpendicular.

Also we know that if three lines with d.c.'s $l_1, m_1, n_1 : l_2, m_2, n_2$ and l_3, m_3, n_3 are mutually perpendicular, then we have

$$l_1^2 + l_2^2 + l_3^2 = m_1^2 + m_2^2 + m_3^2 = n_1^2 + n_2^2 + n_3^2 = 1$$

and $\quad l_1 m_1 + l_2 m_2 + l_3 m_3 = 0 = m_1 n_1 + m_2 n_2 + m_3 n_3 = n_1 l_1 + n_2 l_2 + n_3 l_3$

(See § 6.04 also)

Therefore from (iii), we have $\dfrac{x_1^2}{a^2} + \dfrac{x_2^2}{a^2} + \dfrac{x_3^2}{a^2} = 1$, etc.

and $\quad \dfrac{x_1}{a} \cdot \dfrac{y_1}{b} + \dfrac{x_2}{a} \cdot \dfrac{y_2}{b} + \dfrac{x_3}{a} \cdot \dfrac{y_3}{b} = 0$, etc.

i.e. $\quad \mathbf{x_1^2 + x_2^2 + x_3^2 = a^2,\ y_1^2 + y_2^2 + y_3^2 = b^2,\ z_1^2 + z_2^2 + z_3^2 = c^2} \quad ...(IV)$

and $\quad \mathbf{x_1 y_1 + x_2 y_2 + x_3 y_3 = 0,\ y_1 z_1 + y_2 z_2 + y_3 z_3 = 0,\ z_1 x_1 + z_2 x_2 + z_3 x_3 = 0.}$

...(V)

Solving first and last equations of (II), we have

$$\frac{x_1/a}{\left(\dfrac{y_2 z_3 - y_3 z_2}{bc}\right)} = \frac{y_1/b}{\left(\dfrac{z_2 x_3 - z_3 x_2}{ca}\right)} = \frac{z_1/c}{\left(\dfrac{x_2 y_3 - x_3 y_2}{ab}\right)}$$

$$= \frac{\sqrt{[\Sigma (x_1/a)^2]}}{\sqrt{[\{(y_2 z_3 - y_3 z_2)/bc\}^2]}} = \pm 1, \text{ from (I) and (II)}$$

$$\therefore \frac{x_1}{a} = \pm \frac{y_2 z_3 - y_3 z_2}{bc};\ \frac{y_1}{b} = \pm \frac{z_2 x_3 - z_3 x_2}{ca};\ \frac{z_1}{c} = \pm \frac{x_2 y_3 - x_3 y_2}{ab} \quad ...(VI)$$

****§ 9.19. Properties of conjugate semi-diameters of ellipsoid.**

In the properties below OP, OQ, OR are as defined in § 9.18 above.

***Property I.** *Sum of the squares of any three conjugate semi-diameters of an ellipsoid is constant.* (*Kumaun* 96)

Proof. From § 9.18 Pages 52-54 we have

$$OP^2 + OQ^2 + OR^2 = (x_1^2 + y_1^2 + z_1^2) + (x_2^2 + y_2^2 + z_2^2) + (x_3^2 + y_3^2 + z_3^2)$$

$$= (x_1^2 + x_2^2 + x_3^2) + (y_1^2 + y_2^2 + y_3^2) + (z_1^2 + z_2^2 + z_3^2)$$

$= a^2 + b^2 + c^2$, from (IV) of § 9.18 Page 54. Hence proved.

Property II. *Sum of the squares of the projections of three conjugate semi-diameters on any line is constant.*

Proof. If l, m, n be the d.c.'s of the given line, then the projections of the semi-diameter OP (as defined in § 9.18 Pages 52-54) on this line

$$= (x_1 - 0)\, l + (y_1 - 0)\, m + (z_1 - 0)\, n = lx_1 + my_1 + nz_1$$

Similarly the projections of the semi-diameters OP and OR on this line are $lx_2 + my_2 + nz_2$ and $lx_3 + my_3 + nz_3$ respectively.

$\therefore$ Sum of the squares of these projections

$$= (lx_1 + my_1 + nz_1)^2 + (lx_2 + my_2 + nz_2)^2 + (lx_3 + my_3 + nz_3)^2$$

$$= l^2 (x_1^2 + x_2^2 + x_3^2) + m^2 (y_1^2 + y_2^2 + y_3^2) + n^2 (z_1^2 + z_2^2 + z_3^2)$$
$$+ 2lm\, (x_1y_1 + x_2y_2 + x_3y_3) + 2mn\, (y_1z_1 + y_2z_2 + y_3z_3)$$
$$+ 2nl\, (z_1x_1 + z_2x_2 + z_3x_3)$$

$= l^2a^2 + m^2b^2 + n^2c^2$, from (IV) and (V) of § 9.18 Page 54.

$=$ constant, as l, m, n are constants for the given line. Hence proved.

Property III. *Sum of the squares of projections of three conjugate semi-diameters on any plane is constant.*

Let l, m, n be the d.c.'s of the normal to any given plane. Then the projection of OP on the given plane $= OP^2 - (lx_1 + my_1 + nz_1)^2$ **(Note)**

Similarly the squares of projections of OQ and OR on the given plane are $OQ^2 - (lx_2 + my_2 + nz_2)^2$ and $OR^2 - (lx_3 + my_3 + nz_3)^2$ respectively.

$\therefore$ Sum of the squares of projections of OP, OQ, OR on the given plane

$$= [OP^2 - (lx_1 + my_1 + nz_1)^2] + [OQ^2 - (lx_2 + my_2 + nz_2)^2]$$
$$+ [OR^2 - (lx_3 + my_3 + nz_3)^2]$$

$$= (OP^2 + OQ^2 + OR^2) - [(lx_1 + my_1 + nz_1)^2 + (lx_2 + my_2 + nz_2)^2$$
$$+ (lx_3 + my_3 + nz_3)^2]$$

$= a^2 + b^2 + c^2 - [l^2a^2 + m^2b^2 + n^2c^2]$, from Prop. I and II above

$=$ constant, as l, m, n are constants for the given plane.

***Property IV.** *Volume of the paralleopiped having three conjugate semi-diameters of an ellipsoid as coterminus edges is constant.*

Proof. The volume of the parallelopiped having OP, OQ, OR as coterminus edges $= 6 \times$ volume of the tetrahedron (O, PQR). **(Note)**

$$= 6 \times \tfrac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}, \text{ See § 5.01 (Particular case)} \qquad \ldots(i)$$

$= V$ (say)

Then $V^2 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}^2 = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}^2$ **(Note)**

$$= \begin{vmatrix} x_1^2+x_2^2+x_3^2 & x_1y_1+x_2y_2+x_3y_3 & x_1z_1+x_2z_2+x_3z_3 \\ x_1y_1+x_2y_2+x_3y_3 & y_1^2+y_2^2+y_3^2 & y_1z_1+y_2z_2+y_3z_3 \\ x_1z_1+x_2z_2+x_3z_3 & y_1z_1+y_2z_2+y_3z_3 & z_1^2+z_2^2+z_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{vmatrix}, \text{ from (iv), (v) of § 9.18 P. 54}$$

or $$V^2 = a^2b^2c^2 \quad \text{or} \quad V = abc.$$

Hence from (i), the volume of the parallelopiped having OP, OQ, OR as coterminus edges is abc, which is constant. Hence proved.

R **Property V.** *If OP, OQ and OR are the conjugate semidiameters of an ellipsoid, then the sum of the squares of the areas QOR, ROP and POQ is constant.*

Proof. Let A_1, A_2, A_3 denote the areas QOR, ROP and POQ respectively and let $l_r, m_r, n_r, (r = 1, 2, 3)$ be the d.c.'s of the normals to the planes QOR, ROP and POQ respectively.

Projecting the area QOR on the plane $x = 0$, we get a triangle whose vertices are $(0, y_2, z_2)$, $(0, 0, 0)$ and $(0, y_3, z_3)$ and its area

$= \frac{1}{2}(y_2z_2 - y_3z_3) = \pm\frac{1}{2} bcx_1/a$, from § 9.18 (VI) Page 54

or $$A_1l_1 = \pm\frac{bcx_1}{2a} \quad \text{...(i)}$$

Similarly projecting area QOR on the planes $y = 0$ and $z = 0$ succesively,

we get $$A_1m_1 = \pm\frac{cay_1}{2b} \quad \text{and} \quad A_1n_1 = \pm\frac{abz_1}{2c} \quad \text{...(ii)}$$

Squaring and adding (i) and (ii), we have

$$A_1^2(l_1^2+m_1^2+n_1^2) = \frac{1}{4}[(bcx_1/a)^2 (cay_1/b)^2 + (abz_1/c)^2]$$

or $$A_1^2 = \frac{1}{4}\left[\frac{b^2c^2x_1^2}{a^2} + \frac{c^2a^2y_1^2}{b^2} + \frac{a^2b^2z_1^2}{c^2}\right] \quad \text{...(iii)}$$

Similarly from the areas ROP and POQ, we can get

$$A_2^2 = \frac{1}{4}\left[\frac{b^2c^2x_2^2}{a^2} + \frac{c^2a^2y_2^2}{b^2} + \frac{a^2b^2z_2^2}{c^2}\right] \quad \text{...(iv)}$$

and $$A_3^2 = \frac{1}{4}\left[\frac{b^2c^2x_3^2}{a^2} + \frac{c^2a^2y_3^2}{b^2} + \frac{a^2b^2z_3^2}{c^2}\right] \quad ...(v)$$

Adding (iii), (iv) and (v), we get

$$A_1^2 + A_2^2 + A_3^2 = \frac{1}{4}\left[\frac{b^2c^2}{a^2}(\Sigma x_1^2) + \frac{c^2a^2}{b^2}(\Sigma y_1^2) + \frac{a^2b^2}{c^2}(\Sigma z_1^2)\right]$$

$$= \frac{1}{4}[b^2c^2 + c^2a^2 + a^2b^2], \quad \text{...See § 9.18 (IV) P. 54}$$

$$= \text{constant.}$$

Solved Examples on § 9.18 and § 9.19.

IF OP, OQ and OR be the conjugate semi-diameters of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ and P, Q, R be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively, then

****Ex. 1 (a). Find the equation of the plane PQR.**

Sol. Let the equation of the plane PQR be $lx + my + nz = p$. ...(i)

Since $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ lie on it so we get

$lx_1 + my_1 + nz_1 = p$ ...(ii) ; $lx_2 + my_2 + nz_2 = p$ ...(iii)

and $lx_3 + my_3 + nz_3 = p$ ...(iv)

Multiplying (ii), (iii) and (iv) by x_1, x_2 and x_3 respectively and adding we have

$$l(x_1^2 + x_2^2 + x_3^2) + m(x_1y_1 + x_2y_2 + x_3y_3) + n(x_1z_1 + x_2z_2 + x_3z_3) = p(x_1 + x_2 + x_3)$$

or $la^2 = p(x_1 + x_2 + x_3)$, from (IV) and (V) of § 9.18 Page 54.

or $l = p(x_1 + x_2 + x_3)/a^2$.

Similarly $m = p(y_1 + y_2 + y_3)/b^2$ and $n = p(z_1 + z_2 + z_3)/c^2$.

Substituting these values of l, m, n in (i) we get the required equation of the plane PQR as

$$\frac{x}{a^2}(x_1 + x_2 + x_3) + \frac{y}{b^2}(y_1 + y_2 + y_3) + \frac{z}{c^2}(z_1 + z_2 + z_3) = 1. \quad \textbf{Ans.}$$

***Ex. 1 (b). Prove that if the plane $lx + my + nz = p$, passes through the points P, Q, R, then $a^2l^2 + b^2m^2 + c^2n^2 = 3p^2$.** *(Gorakhpur 97)*

Solution. Proceeding as in Ex. 1 (a) above we get the relations (ii), (iii) and (iv), so $(lx_1 + my_1 + nz_1)^2 + (lx_2 + my_2 + nz_2)^2 + (lx_3 + my_3 + nz_3)^2 = 3p^2$

or $$l^2(\Sigma x_1^2) + m^2(\Sigma y_1^2) + n^2(\Sigma z_1^2) + 2lm(\Sigma x_1y_1) + 2mn(\Sigma y_1z_1) + 2nl(\Sigma z_1z_2) = 3p^2$$

or $l^2a^2 + m^2b^2 + n^2c^2 = 3p^2$, from (IV), and (V) of § 9.18 Page 54.

Hence proved.

***Ex. 2. Prove that the pole of the plane PQR lies on the ellipsoid**

$$x^2/a^2+y^2/b^2+z^2/c^2=3.$$ (*Gorakhpur* 96)

Solution. As in Ex. 1 (a) above we can prove that equation of the plane

PQR is $$\frac{x}{a^2}(\Sigma x_1)+\frac{y}{b^2}(\Sigma y_1)+\frac{z}{c^2}(\Sigma z_1)=1. \qquad ...(i)$$

If (α, β, γ) be the pole of this plane, then the polar plane of (α, β, γ) w.r. to the given ellipsoid is $$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=1. \qquad ...(ii)$$

Now (i) and (ii) represent the same plane, so comparing them we have

$$\frac{\alpha}{x_1+x_2+x_3}=\frac{\beta}{y_1+y_2+y_3}=\frac{\gamma}{z_1+z_2+z_3}=1 \qquad ...(iii)$$

From (iii), we have

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=\frac{(x_1+x_2+x_3)^2}{a^2}+\frac{(y_1+y_2+y_3)^2}{b^2}+\frac{(z_1+z_2+z_3)^2}{c^2}$$

$$=\frac{\Sigma x_1^2}{a^2}+\frac{\Sigma y_1^2}{b^2}+\frac{\Sigma z_1^2}{c^2}+2\left(\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}\right)$$

$$+2\left(\frac{x_2x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}\right)+2\left(\frac{x_3x_1}{a^2}+\frac{y_3y_1}{b^2}+\frac{z_3z_1}{c^2}\right)$$

$$=\frac{a^2}{a^2}+\frac{b^2}{b^2}+\frac{c^2}{c^2}+2\,(0)+2\,(0)+2\,(0),$$

from (II), (IV) of § 9.18 Pages 52-54

or $$(\alpha^2/a^2)+(\beta^2/b^2)+(\gamma^2/c^2)=3.$$

$\therefore$ The locus of the pole (α, β, γ) of the plane PQR is

$$x^2/a^2+y^2/b^2+z^2/c^2=3.$$ Hence proved.

Ex. 3. Prove that the plane PQR touches the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=\frac{1}{3}$ at the centroid of the triangle PQR.

Sol. The coordinates of the centroid G (say) of the triangles PQR is

$$[\tfrac{1}{3}(x_1+x_2+x_3), \tfrac{1}{3}(y_1+y_2+y_3), \tfrac{1}{3}(z_1+z_2+z_3)].$$

$\therefore$ The equation of the tangent plane to the given ellipsoid at G is

$$\frac{x(x_1+x_2+x_3)}{3a^2}+\frac{y(y_1+y_2+y_3)}{3b^2}+\frac{z(z_1+z_2+z_3)}{3c^2}=\frac{1}{3} \qquad \textbf{(Note)}$$

or $$\frac{x(x_1+x_2+x_3)}{a^2}+\frac{y(y_1+y_2+y_3)}{b^2}+\frac{z(z_1+z_2+z_3)}{c^2}=1,$$

which from Ex. 1 (a) Page 57 is the equation of the plane PQR.

Hence the plane PQR touches the given ellipsoid at the centroid of $\Delta\, PQR$.

***Ex. 4. Prove that the locus of the foot of the perpendicular from the centre of the ellipsoid $\Sigma\,(x^2/a^2) = 1$ to the plane PQR is**

$$a^2x^2 + b^2y^2 + c^2z^2 = 3\,(x^2 + y^2 + z^2)^2. \qquad (\textit{Gorakhpur } 95)$$

Solution. Let $N\,(\alpha, \beta, \gamma)$ be the foot of the perpendicular from the centre $O\,(0, 0, 0)$ of the given ellipsoid to the plane PQR, then

$$ON = \sqrt{(\alpha^2 + \beta^2 + \gamma^2)} = \lambda \text{ (say)} \qquad ...(i)$$

Also the d.r.'s of the normal to the plane PQR is the same as those of the line ON *i.e.* $\dfrac{\alpha, \beta, \gamma}{\sqrt{(\alpha^2 + \beta^2 + \gamma^2)}}$ or $\dfrac{\alpha, \beta, \gamma}{\lambda}$ ...(ii)

$\therefore$ The equation of the plane PQR, the d.c.'s of whose normal are given by (ii) and the length of perpendicular to it from (0, 0, 0) given by (i) is

$$\frac{\alpha}{\lambda}x + \frac{\beta}{\lambda}y + \frac{\gamma}{\lambda}z = \lambda \qquad \textbf{(Note)}$$

or
$$\alpha x + \beta y + \gamma z = \lambda^2 \qquad ...(iii)$$

Also as in Ex. 1 (a) Page 57 we can prove that the equation of the plane PQR is $\dfrac{x}{a^2}(x_1 + x_2 + x_3) + \dfrac{y}{b^2}(y_1 + y_2 + y_3) + \dfrac{z}{c^2}(z_1 + z_2 + z_3) = 1$ ...(iv)

Since (iii) and (iv) represent the same plane, so comparing we get

$$\frac{x_1 + x_2 + x_3}{a^2\alpha} = \frac{y_1 + y_2 + y_3}{b^2\beta} = \frac{z_1 + z_2 + z_3}{c^2\gamma} = \frac{1}{\lambda^2} \qquad ...(v)$$

whence we get $\dfrac{(x_1 + x_2 + x_3)^2}{a^2} = \dfrac{a^2\alpha^2}{\lambda^4}$, $\dfrac{(y_1 + y_2 + y_3)^2}{b^2} = \dfrac{b^2\beta^2}{\lambda^4}$

and
$$\frac{(z_1 + z_2 + z_3)^2}{c^2} = \frac{c^2\gamma^2}{\lambda^4}. \qquad \textbf{(Note)}$$

Adding these and simplifying, we get

$$\frac{\Sigma x_1^2}{a^2} + \frac{\Sigma y_1^2}{b^2} + \frac{\Sigma z_1^2}{c^2} + 2\left[\Sigma\frac{x_1x_2}{a^2} + \Sigma\frac{x_2x_3}{a^2} + \Sigma\frac{x_3x_1}{a^2}\right]$$
$$= (l/\lambda^4)\,(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)$$

or $\lambda^4\,(1 + 1 + 1) = a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2$, using (II) and (IV) of § 9.18 Pages 52-54

or $3\,(\alpha^2 + \beta^2 + \gamma^2)^2 = a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2$, from (i).

$\therefore$ The required locus of $N\,(\alpha, \beta, \gamma)$ is

$$3\,(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2. \qquad \text{Hence proved.}$$

Ex. 5. If the axes are rectangular, find the locus of the equal conjugate diameters of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

(*Garhwal* 93; *Kanpur* 91; *Rohilkhand* 92)

Solution. Let r be the length of each semi-conjugate diameter.

Then $OP = OQ = r$, where P, Q, R are the extremities of the semi-conjugate diameters and their coordinates are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively.

Also we know $OP^2 + OQ^2 + OR^2 = a^2 + b^2 + c^2$

...See § 9.19 (Prop I) Page 54

or $$3r^2 = a^2 + b^2 + c^2 \quad \text{or} \quad r^2 = \tfrac{1}{3}(a^2 + b^2 + c^2) \qquad \text{...(i)}$$

Let l, m, n be the direction cosines of OP, then the coordinates of P can be taken as (lr, mr, nr),

$\because$ $P(lr, mr, nr)$ lies on the given ellipsoid, so we have

$$\frac{l^2r^2}{a^2} + \frac{m^2r^2}{b^2} + \frac{n^2r^2}{c^2} = 1 \quad \text{or} \quad r^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) = 1 = l^2 + m^2 + n^2 \quad \textbf{(Note)}$$

or $$\tfrac{1}{3}(a^2 + b^2 + c^2)\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) = l^2 + m^2 + n^2, \text{ from (i)}$$

or $$\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} = 3\,\frac{(l^2 + m^2 + n^2)}{(a^2 + b^2 + c^2)} \qquad \text{...(ii)}$$

Also the equations of the line OP are $$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \qquad \text{...(iii)}$$

Eliminating l, m, n between (ii) and (iii) we get the locus of OP as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{3(x^2 + y^2 + z^2)}{(a^2 + b^2 + c^2)}$$

or $$x^2\left[\frac{1}{a^2} - \frac{3}{a^2 + b^2 + c^2}\right] + y^2\left[\frac{1}{b^2} - \frac{3}{a^2 + b^2 + c^2}\right] + z^2\left[\frac{1}{c^2} - \frac{3}{a^2 + b^2 + c^2}\right] = 0$$

or $$\frac{x^2}{a^2}(2a^2 - b^2 - c^2) + \frac{y^2}{b^2}(2b^2 - c^2 - a^2) + \frac{z^2}{c^2}(2c^2 - a^2 - b^2) = 0,$$

which is a cone.

***Ex. 6.** Prove that the locus of the section of the ellipsoid $\Sigma(x^2/a^2) = 1$ by the plane PQR is the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = \frac{1}{3}$.

Solution. As in Ex. 1 (a) Page 57 we can prove that the equation of the plane PQR is

$$\left(\frac{x_1+x_2+x_3}{a^2}\right)x+\left(\frac{y_1+y_2+y_3}{b^2}\right)y+\left(\frac{z_1+z_2+z_3}{c^2}\right)z=1. \quad ...(i)$$

If (α, β, γ) be the centre of the section of the given ellipsoid by the plane PQR then the equation of PQR can be written as

$$\text{"}T=S_1\text{"} \quad i.e. \quad \frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2} \quad ...(ii)$$

$\therefore$ Equations (i) and (ii) represent the same plane, therefore comparing them we get

$$\frac{x_1+x_2+x_3}{\alpha}=\frac{y_1+y_2+y_3}{\beta}=\frac{z_1+z_2+z_3}{\gamma}=\frac{1}{\alpha^2/a^2+\beta^2/b^2+\gamma^2/c^2}$$

where $$\frac{\alpha}{a}=\left(\frac{x_1+x_2+x_3}{a}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)$$

Similarly $$\frac{\beta}{b}=\left(\frac{y_1+y_2+y_3}{b}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right).$$

$$\frac{\gamma}{c}=\left(\frac{z_1+z_2+z_3}{c}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)$$

Squaring and adding we get

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)^2\left[\left(\frac{x_1+x_2+x_3}{a}\right)^2+\left(\frac{y_1+y_2+y_3}{b}\right)^2+\left(\frac{z_1+z_2+z_3}{c}\right)^2\right]$$

or $$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=3\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)^2, \text{ as in Ex. 2 Page 58.}$$

or $$(\alpha^2/a^2)+(\beta^2/b^2)+(\gamma^2/c^2)=1/3.$$

$\therefore$ The required locus of (α, β, γ) is

$$(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1/3. \qquad \text{Hence proved.}$$

Ex. 7. Prove that the planes through a pair of equal conjugate diameters touches the cone

$$\frac{x^2}{a^2(2a^2-b^2-c^2)}+\frac{y^2}{b^2(2b^2-c^2-a^2)}+\frac{z^2}{c^2(2c^2-a^2-b^2)}=0$$

Sol. If $r=OP=OQ=OR$, then as in Ex. 5 Page 60 we can prove that

$$r^2 = \frac{1}{3}(a^2+b^2+c^2) \qquad ...(i)$$

Let us consider the plane through OQ and OR which is also the diametral plane to OP and therefore its equaion is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0 \qquad ...(ii)$$

The d.c.'s of the normal to the plane (ii) are $x_1/a^2, y_1/b^2, z_1/c^2$ ...(iii)

Also we know that the equation of the cone reciprocal to

$$x^2/\alpha+y^2/\beta+z^2/\gamma=0 \quad \text{is} \quad \alpha x^2+\beta y^2+\gamma z^2=0$$

$\therefore$ The equaiton of the cone reciprocal to the given cone is

$$a^2(2a^2-b^2-c^2)x^2+b^2(2b^2-c^2-a^2)y^2+c^2(2c^2-a^2-b^2)z^2=0 \quad ...(iv)$$

Now the plane (ii) will touch the given cone if the normal to plane (ii) whose d.c.'s are given by (iii), is a generator of the reciprocal cone (iv) ; and the condition for the same is $\Sigma[a^2(2a^2-b^2-c^2)(x_1/a^2)^2]=0$ **(Note)**

or $$2(x_1^2+y_1^2+z_1^2)-\frac{x_1^2}{a^2}(b^2+c^2)-\frac{y_1^2}{b^2}(c^2+a^2)-\frac{z_1^2}{c^2}(a^2+b^2)=0$$

or $$2(\Sigma x_1^2)=\frac{x_1^2}{a^2}(a^2+b^2+c^2-a^2)+\frac{y_1^2}{b^2}(a^2+b^2+c^2-b^2)+\frac{z_1^2}{c^2}\times(a^2+b^2+c^2-c^2) \quad \textbf{(Note)}$$

$$=(a^2+b^2+c^2)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}\right)-(x_1^2+y_1^2+z_1^2)$$

or $3(x_1^2+y_1^2+z_1^2)=(a^2+b^2+c^2)(1)$, $\because P(x_1,y_1,z_1)$ lies on the ellipsoid.

or $3(x_1^2+y_1^2+z_1^2)=3r^2$, from (i)

or $x_1^2+y_1^2+z_1^2=r^2$ or $OP^2=r^2$, which is true.

Hence the plane through OQ and OR touches the given cone.

Ex. 8. If λ, μ, ν are the angles between a set of equal conjugate diameters of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, then prove that

$$\cos^2\lambda+\cos^2\mu+\cos^2\nu=3\frac{\Sigma(b^2-c^2)^2}{2(a^2+b^2+c^2)^2}.$$

Sol. Let OP, OQ, OR be a set of conjugate semi-diameters of the given ellipsoid, such that $OP^2=OQ^2=OR^2=r^2=\frac{1}{3}(a^2+b^2+c^2)$ ...(i)

and the coordinates of P, Q and R be (x_1,y_1,z_1), (x_2,y_2,z_2) and (x_3,y_3,z_3).

Then the direction ratios of OQ and OR are x_2, y_2, z_2 and x_3, y_3, z_3 respectively.

$\therefore$ If λ be the angle between OQ and OR then we get

$$\cos\lambda = \frac{x_2x_3+y_2y_3+z_2z_3}{\sqrt{(x_2^2+y_2^2+z_2^2)}\sqrt{(x_3^2+y_3^2+z_3^2)}} = \frac{x_2x_3+y_2y_3+z_2z_3}{r^2}, \text{ from (i)}$$

or $$\cos^2\lambda = (x_2x_3+y_2y_3+z_2z_3)^2/r^4 \quad \text{...(ii)}$$

Also from Lagrange's identity, we have

$$(x_2^2+y_2^2+z_2^2)(x_3^2+y_3^2+z_3^2)-(x_2x_3+y_2y_3+z_2z_3)^2 = \Sigma(y_2z_3-y_3z_2)^2$$

or $$r^2r^2-(x_2x_3+y_2y_3+z_2z_3)^2 = \Sigma(bcx_1/a)^2,$$

from (VI) of § 9.18 Page 54

or $$(x_2x_3+y_2y_3+z_2z_3)^2 = r^4-\left[\frac{b^2c^2}{a^2}x_1^2+\frac{c^2a^2}{b^2}y_1^2+\frac{a^2b^2}{c^2}z_1^2\right]$$

$\therefore$ From (ii) we have $$\cos^2\lambda = 1-\frac{a^2b^2c^2}{r^4}\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right) \quad \text{...(iii)}$$

Similarly $$\cos^2\mu = 1-\frac{a^2b^2c^2}{r^4}\left(\frac{x_2^2}{a^4}+\frac{y_2^2}{b^4}+\frac{z_2^2}{c^4}\right) \quad \text{...(iv)}$$

and $$\cos^2\nu = 1-\frac{a^2b^2c^2}{r^4}\left(\frac{x_3^2}{a^4}+\frac{y_3^2}{b^4}+\frac{z_3^2}{c^4}\right) \quad \text{...(v)}$$

Adding (iii), (iv) and (v) we get $\cos^2\lambda+\cos^2\mu+\cos^2\nu$

$$= 3-\frac{a^2b^2c^2}{r^4}\left[\frac{x_1^2+x_2^2+x_3^2}{a^4}+\frac{y_1^2+y_2^2+y_3^2}{b^4}+\frac{z_1^2+z_2^2+z_3^2}{c^4}\right]$$

$$= 3-\frac{a^2b^2c^2}{r^4}\left[\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right], \quad \because \Sigma x_1^2 = a^2, \Sigma y_1^2 = b^2, \Sigma z_1^2 = c^2$$

$$= 3-\frac{1}{r^4}(b^2c^2+c^2a^2+a^2b^2) = 3-\frac{9(b^2c^2+c^2a^2+a^2b^2)}{(a^2+b^2+c^2)^2}, \text{ from (i)}$$

$$= 3[(a^2+b^2+c^2)^2-3(b^2c^2+c^2a^2+a^2b^2)/(a^2+b^2+c^2)^2]$$

$$= 3[a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2]/(a^2+b^2+c^2)^2$$

$$= (3/2)[2a^4+2b^4+2c^4-2b^2c^2-2c^2a^2-2a^2b^2]/(a^2+b^2+c^2)^2$$

$$= \frac{3}{2}\frac{[(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2]}{(a^2+b^2+c^2)^2}$$

Hence proved.

Ex. 9. Prove that the locus of the point of intersection of three tangent planes to $x^2/a^2+y^2/b^2+z^2/c^2=1$ which are parallel to conjugate diametral planes of $x^2/\alpha^2+y^2/\beta^2+z^2/\gamma^2=1$ is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}$$

Sol. Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ be the extremities of the conjugate semi-diameters of the ellipsoid

$$x^2/\alpha^2+y^2/\beta^2+z^2/\gamma^2=1 \qquad ...(i)$$

Then the diameteral plane of P w.r. to (i) is

$$\frac{xx_1}{\alpha^2}+\frac{yy_1}{\beta^2}+\frac{zz_1}{\gamma^2}=0 \qquad ...(ii)$$

Any plane parallel to (ii) is $\dfrac{xx_1}{\alpha^2}+\dfrac{yy_1}{\beta^2}+\dfrac{zz_1}{\gamma^2}=p_1$ ...(iii)

If (iii) is a tangent plane to $x^2/a^2+y^2/b^2+z^2/c^2=1$, then

"$p^2=a^2l^2+b^2m^2+c^2n^2$" ...See § 9.07 (b) Page 7

or $$p_1^2=a^2(x_1/\alpha^2)^2+b^2(y_1/\beta^2)^2+c^2(z_1/\gamma^2)^2 \qquad ...(iv)$$

Similarly the equation of other planes parallel to OQ and OR are

$$\frac{xx_2}{\alpha^2}+\frac{yy_2}{\beta^2}+\frac{zz_2}{\gamma^2}=p_2 \qquad ...(v)$$

and $$\frac{xx_3}{\alpha^2}+\frac{yy_3}{\beta^2}+\frac{zz_3}{\gamma^2}=p_3 \qquad ...(vi)$$

where $p_2^2=\Sigma\,[a^2(x_2/\alpha^2)^2]$ and $p_3^2=\Sigma\,[a^2(x_3/\alpha^2)^2]$ ...(vii)

Now for the locus of the point of intersection of (iii), (v) and (vi) we square and add (iii), (v) and (vi) and get

$$\Sigma\frac{x^2}{\alpha^4}(x_1^2+x_2^2+x_3^2)+\Sigma\frac{2xy}{\alpha^2\beta^2}(x_1y_1+x_2y_2+x_3y_3)=p_1^2+p_2^2+p_3^2$$

or $$\Sigma\left[\frac{x^2}{\alpha^4}\left(\alpha^2\right)\right]+\Sigma\frac{2xy}{\alpha^2\beta^2}(0)=\Sigma\frac{a^2}{\alpha^4}(x_1^2+x_2^2+x_3^2),$$ from (iv) and (vii) and using $\Sigma x_1^2=\alpha^2$ and $\Sigma x_1y_1=0$ etc.

or $$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^4}(\alpha^2)+\frac{b^2}{\beta^4}(\beta^2)+\frac{c^2}{\gamma^4}(\gamma^2),$$ since $\Sigma x_1^2=\alpha^2$ etc.

or $$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}$$ Hence proved.

Ex. 10. P is any point on the ellipsoid $\Sigma\,(x^2/a^2)=1$ and 2α, 2β are the principal axes of the section of the ellipsoid by the diametral plane of OP. Prove $OP^2=a^2+b^2+c^2-\alpha^2-\beta^2$ and $\alpha\beta p=abc$, where p is the perpendicular from O to the tangent plane at P.

Sol. Let OQ and OR be the semi-axes of the section of the ellipsoid by the diameteral plane of OP, then $OQ = \alpha$ and $OR = \beta$ (given). Hence OP, OQ and OR are the conjugate semi-diameters of the given ellipsoid.

We can prove, as in § 9.19 Prop. I Page 54, that

$$OP^2 + OQ^2 + OR^2 = a^2 + b^2 + c^2 \quad \text{or} \quad OP^2 = a^2 + b^2 + c^2 - OQ^2 - OR^2$$

or
$$OP^2 = a^2 + b^2 + c^2 - \alpha^2 - \beta^2$$

Also volume of the tetrahedron (O, PQR)

$= \frac{1}{3}$. area of $\Delta OQR \times$ perp. from P on $\Delta OQR = \frac{1}{3} (\frac{1}{2} \alpha\beta) p$, ...(i)

(Note)

where $p =$ length of perpendicular from P on ΔOQR

$=$ perp. from P on the tangent plane at P (Note)

Also volume of tetrahedron $(O, PQR) = (1/6)$ (volume of parallelopiped whose edges are OP, OQ and OR).

$= (1/6)(abc)$ ...See Prop. IV of § 9.19 Page 55

Hence from (i) and (ii) we get $(1/6)\, \alpha\beta p = (1/6)\, abc$

or
$$\alpha\beta p = abc.$$

Hence proved.

Ex. 11. Find locus of the asymptotic line drawn from origin to the concoid

$$\mathbf{ax^2 + by^2 + cz^2 = 1}$$

Sol. Any line drawn from the origin is $x/l = y/m = z/n$, ...(i)

Any point on it is (lr, mr, nr). If it lies on the given conicoid, then

$$a(lr)^2 + b(mr)^2 + c(nr)^2 = 1 \quad \text{or} \quad al^2 + bm^2 + cn^2 = 1/r^2 \qquad ...(ii)$$

Now r is infinite for the asymptotic line, so from (ii) we have

$$al^2 + bm^2 + cn^2 = 0 \qquad ...(iii)$$

Eliminating l, m, n between (i) and (ii) we get the required locus as

$$ax^2 + by^2 + cz^2 = 0$$

Ans.

Ex. 12. Prove that two asymptotic lines can be drawn from a point P to a conicoid $\mathbf{ax^2 + by^2 + cz^2 = 1}$ and they are at right angles if P lies on the cone $\mathbf{a^2(b+c)x^2 + b^2(c+a)y^2 + c^2(a+b)z^2 = 0}$.

Sol. Let P be (α, β, γ). Any line through P is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad ...(i)$$

Any point on this line is $(\alpha + lr,\ \beta + mr,\ \gamma + nr)$. If this point lies on the given conicoid we have

$$a(\alpha + lr)^2 + b(\beta + mr)^2 + c(\gamma + nr)^2 = 1$$

or
$$r^2(al^2 + bm^2 + cn^2) + 2r(al\alpha + bm\beta + cn\gamma) + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0$$

If both of its roots are infinite, then

$$al^2 + bm^2 + cn^2 = 0 \quad \text{...(i)} \qquad \text{and} \qquad al\alpha + bm\beta + cn\gamma = 0 \quad \text{...(iii)}$$

Let l_1, m_1, n_1 and l_2, m_2, n_2 be the d.c.'s of the liens whose d.c.'s are given by (ii) and (iii). Then as thse lines are at right angles, so we have

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \qquad \text{...(iv)}$$

Eliminating n between (ii) and (iii) we have

$$al^2 + bm^2 + c\left[-\frac{al\alpha + bm\beta}{c\gamma}\right]^2 = 0$$

or $$l^2 ac\,(c\gamma^2 + a\alpha^2) + 2lm\,(ac\alpha\beta) + m^2\,bc\,(c\gamma^2 + b\beta^2) = 0$$

or $$ac\,(c\gamma^2 + a\alpha^2)\,(l/m)^2 + 2ac\alpha\beta\,(l/m) + bc\,(c\gamma^2 + b\beta^2) = 0$$

$\therefore$ Its roots are l_1/m_1 and l_2/m_2 so we have

$$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \text{product of the roots} = \frac{bc\,(c\gamma^2 + b\beta^2)}{ac\,(c\gamma^2 + a\alpha^2)}$$

or $$\frac{l_1 l_2}{bc\,(b\beta^2 + c\gamma^2)} = \frac{m_1 m_2}{ca\,(c\gamma^2 + a\alpha^2)} = \frac{n_1 n_2}{ab\,(a\alpha^2 + b\beta^2)}, \text{ by symmetry.}$$

Substituting these values in (iv) we get

$$bc\,(b\beta^2 + c\gamma^2) + ca\,(c\gamma^2 + a\alpha^2) + ab\,(a\alpha^2 + b\beta^2) = 0$$

or $$a^2\,(b + c)\,\alpha^2 + b^2\,(c + a)\,\beta^2 + c^2\,(a + b)\,\gamma^2 = 0$$

$\therefore$ The locus of $P\,(\alpha, \beta, \gamma)$ is

$$a^2\,(b + c)\,x^2 + b^2\,(c + a)\,y^2 + c^2\,(a + b)\,z^2 = 0 \qquad \text{Hence proved.}$$

Exercises on § 9.19.

Ex. 1. Let $OP, OQ, OR,$ be three conjugate semi-diameters of $3x^2 + \lambda y^2 + z^2 = 1$. Find λ if $OP^2 + OQ^2 + OR^2 = 2$.

[**Hint :** Use $OP^2 + OQ^2 + OR^2 = \text{"}a^2 + b^2 + c^2\text{"} = \frac{1}{3} + \frac{1}{\lambda} + 1 = 2$

See § 9.19 Prop, I Page 54] **Ans.** $\lambda = 32$

Ex. 2. Show that the sum of the squares of the projections of three conjugate semi-diametrs of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$ on the axis of x is constant.

[**Hint :** See § 9.19 Prop II Page 55]

§ 9.20 Cone.

In the last chapter we have read the standard equation of a cone with vertex at the origin as $$ax^2 + by^2 + cz^2 = 0 \qquad \text{... (i)}$$

If (x_1, y_1, z_1) is a point on this cone, then we find that $(-x_1, -y_1, -z_1)$ also lies on the cone, and therefore the cone given by (i) above can be considered a central surface for which the vertex of the cone is the centre.

Here the coordinate planes are the conjugate diametral planes and the coordintae axes are the conjugate diameters. **(Note)**

We can find the different results obtained for the other central conicoids for the above cone also.

Solved Examples on § 9.20.

Ex. 1. Find the equation to the normal plane of the cone $ax^2 + by^2 + cz^2 = 0$, which passes through the generator

$$x/l = y/m = z/n.$$

Sol. Any point on the given generator is (lr, mr, nr) and the equation of the tangent plane to the cone at this point is

$$alx + bmy + cnz = 0. \qquad \text{...(i)}$$

Any plane through the given generator is $Ax + By + Cz = 0$ ...(ii)

where $$Al + Bm + Cn = 0. \qquad \text{...(iii)}$$

If this plane (ii) is a normal plane it must be perpendicular to (i) and the condition for the same is $A\,(al) + B\,(bm) + C\,(cn) = 0$ ...(iv)

Solving (iii) and (iv), we get

$$\frac{A}{mn\,(c-b)} = \frac{B}{nl\,(a-c)} = \frac{C}{lm\,(b-a)}$$

Substituting these proportionate values of A, B and C in (ii) the required equation is $m\,n\,(c-b)\,x + b l\,(a-c)\,y + l\,m\,(b-a)\,z = 0.$ **Ans.**

Ex. 2. Prove that the lines drawn through the origin at right angles to the normal planes of the cone $ax^2 + by^2 + cz^2 = 0$ generates the cone

$$\frac{a\,(b-c)^2}{x^2} + \frac{b\,(c-a)^2}{y^2} + \frac{c\,(a-b)^2}{z^2} = 0.$$

Sol. Let $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$ be a generator of the given cone.

Then $$al^2 + bm^2 + cn^2 = 0 \qquad \text{...(i)}$$

Also as in Ex. 1 above we can show that the normal plane of the given cone which passes through the above generator is

$$m\,n\,(c-b)\,x + n\,l\,(a-c)\,y + l\,m\,(b-a)\,z = 0$$

or $$\frac{(b-c)\,x}{l} + \frac{(c-a)\,y}{m} + \frac{(a-b)\,z}{n} = 0. \qquad \text{...(ii)}$$

Any line through origin at right angles to the plane (ii) is

$$\frac{x}{(b-c)/l} = \frac{y}{(c-a)/m} = \frac{z}{(a-b)/n} = 0. \qquad \text{...(iii)}$$

(Note)

Required locus of this line (iii) is obtained by eliminating l, m, n between (i) and (ii) and is

$$a\left(\frac{b-c}{x}\right)^2+b\left(\frac{c-a}{y}\right)^2+c\left(\frac{a-b}{z}\right)^2=0$$

or
$$\frac{a(b-c)^2}{x^2}+\frac{b(c-a)^2}{y^2}+\frac{c(a-b)^2}{z^2}=0$$
Hence proved.

Exercises on Chapter IX

Ex. 1. Name the surfaces or curves represented by the following equations :

(i) $(x^2/a^2)-(y^2/b^2)-(z^2/c^2)=1$;

(ii) $(x^2/a^2)+(y^2/b^2)=2(z/c)$;

(iii) $y^2=0$;

(iv) $x^2+y^2+z^2=9,\ 2x+3y+4z=5$;

(v) $x=2z$;

(vi) $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=0$

Ans. (i) Hyperbolid of two sheets ; (ii) elliptic paraboloid ; (iii) Pair of planes ; (iv) circle [section of the sphere $x^2+y^2+z^2=9$ by the plane $2x+3y+4z=5$] ; (v) plane ; (vi) cone, with vertex at origin.

Ex. 2. Show that the plane PQR touches a fixed sphere, where P, Q, R are the extremities of the conjugate semi-diameters of the ellipsoid $\Sigma(x^2/a^2)=1$.

Ex. 3. If the normal at the point P to the ellipsoid $\Sigma(x^2/a^2)=1$ meets the principal planes in points L, M, N, show that

$$PL/a^2=PM/b^2=PN/c^2.$$

Ex. 4. If the feet of six normals from the point (α, β, γ) to the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ are (x_r, y_r, z_r), $r=1, 2, 3, 4, 5, 6$ show that

$$a^2\alpha\,\Sigma(1/x_r)+b^2\beta\,\Sigma(1/y_r)+c^2\gamma\,\Sigma(1/z_r)=0,$$

where $\Sigma(1/x_r)=(1/x_1)+(1/x_2)+\ldots(1/x_6)$.

Ex. 5. Find the points of intersection of a line through a given point (α, β, γ) with the central conicoid $ax^2+by^2+cz^2=1$.

Chapter X

Paraboloids

§ 10.01 Elliptic Paraboloid and its tracing.

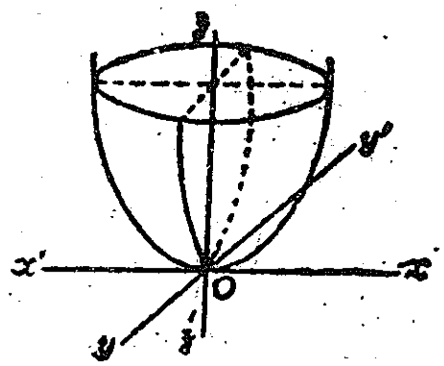
The equation of the elliptic paraboloid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c} \qquad \text{...(i)}$$

From this equation we observe the following facts :

I. It is evident that if $P(x_1, y_1, z_1)$ lies on it then the point $P_1(x_1, -y_1, z_1)$ also lies on it and the middle point of the chord PP_1 lies on the plane $y=0$ and chord PP_1 is also perpendicular to the plane $y=0$. Hence the plane $y=0$ *i.e.* xz-plane bisects every chord perpendicualr to it.

Similarly we can show that the plane $x=0$ *i.e.* yz-plane also bisects every chord perpendicular to it.



(Fig. 1)

Hence the surface represented by (i) is symmetrical with respect to xz and yz-planes.

II. From the equation (i) we find that z cannot be negative otherwise x^2 and y^2 would be negative, which is impossible. Hence no part of the surface extends to the negative side of xy-plane.

III. The section of the surface of the paraboloid given by (i) by the plane $z=\lambda$ is the ellipse $\quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{2\lambda}{c},\ z=\lambda.$ ...(ii)

Its semi-axes are $a\sqrt{(2\lambda/c)}$, $b\sqrt{(2\lambda/c)}$ and its centre lies on z-axis.

As λ increases, the elliptic section of the paraboloid goes on increasing in size as the lengths of their semi-axes increase when λ increases.

Hence the sections of the paraboloid given by (i) by the planes parallel to xy-plane are similar ellipses whose centres lie on the z-axis, which is the axis of paraboloid. **(Remember)**

IV. The section of the surface of the paraboloid given by (i) by the plane $x=k$ is the parabola

$$\frac{k^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c},\ x=k \qquad \text{or} \qquad y^2=b^2\left(\frac{2z}{c}-\frac{k^2}{a^2}\right),\ x=k \qquad \text{...(iii)}$$

The latus rectum of this parabola is $2b^2/c$ *i.e.* constant. The vertex of this parabola is $(k, 0, ck^2/2a^2)$ and lies on the parabola $x^2/a^2 = 2z/c$, $y = 0$, which is the section of (i) by the plane $y = 0$.

Similarly the section of the surface of the paraboloid by plane $y = \mu$ is also a parabola.

Hence the sections of the paraboloid by planes parallel to yz and xz-planes are parabolas.

§ 10.02 Hyperboloic Paraboloid and its tracing.

The equation of the hyperbolic paraboloid is :

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c} \qquad \text{...(i)}$$

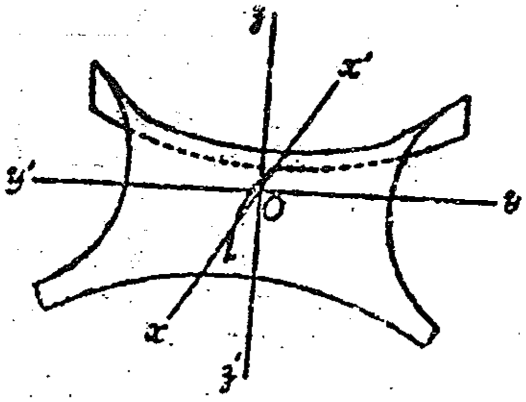
From this equation we observe the following facts :

I. It is evident (write as in § 10.01 above) that the surface represented by (i) is symmetrical with respect to xz and yz-planes.

II. The section of the surface of the paraboloid given by (i) by the plane $z = \lambda$ is the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2\lambda}{c}, \quad z = \lambda \quad \text{...(ii)}$$

Its centre is on z-axis at a distance λ from the origin.



(Fig. 2)

Its asymptotes are parallel to the lines $\frac{x}{a} + \frac{y}{b} = 0$, $z = 0$, which are the sections of the surface given by (i) by the plane $z = 0$.

Also if λ is positive, then the real axis (or transverse axis) of the hyperbola given by (ii) is parallel to the axis of x and if λ is negative then the real axis of the hyperbola given by (ii) is parallel to the axis of y.

III. The section of the surface of the paraboloid given by (i) by the plane $x = k$ is the parabola

$$\frac{k^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}, \quad x = k \qquad \text{or} \qquad y^2 = b^2\left(\frac{k^2}{a^2} - \frac{2z}{c}\right), \quad x = k \qquad \text{...(iii)}$$

Similarly the section of the surface of the given paraboloid by the plane $y = \mu$ is also a parabola.

Hence the sections of this paraboloid by planes parallel to yz or xz planes are parabolas.

§ 10.03 General Equation of the Paraboloid.

The general equation of the paraboloid is $\boxed{\mathbf{ax^2+by^2=2cz}}$ ...(i)

This represents an elliptic paraboloid if a and b are of the same sign and a hyperbolic paraboloid if a and b are of opposite signs.

§ 10.04. Tangent Plane.

***(a)** *To find the equation of the tangent plane to the paraboloid $ax^2+by^2=2cz$ at the point (α, β, γ).* (*Kanpur* 90; *Rohilkhand* 90)

The equations of any line through (α, β, γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{...(i)}$$

Any point on this line is $(\alpha+lr, \beta+mr, \gamma+nr)$.

If the line (i) cuts the given paraboloid at this point, then the co-ordinates of this point must satisfy the given equation of the paraboloid.

$$\therefore\ a(\alpha+lr)^2+b(\beta+mr)^2=2c(\gamma+nr)$$

or $$r^2(al^2+bm^2)+2r(al\alpha+bm\beta-cn)+(a\alpha^2+b\beta^2-2c\gamma)=0 \qquad \text{...(ii)}$$

Also as (α, β, γ) lies on the paraboloid, so we have

$$a\alpha^2+b\beta^2=2c\gamma \qquad \text{...(iii)}$$

$\therefore$ From (ii) and (iii) we have

$$r^2(al^2+bm^2)+2r(al\alpha+bm\beta-cn)=0. \qquad \text{...(iv)}$$

The line (i) will touch the given paraboloid if both the values of r given by (iv) are zero.

$\therefore$ From (iv) we have $\quad al\alpha+bm\beta-cn=0,$ ...(v)

which is the condition for the line (i) to be a tangent line to the paraboloid at (α, β, γ).

The locus of all such lines through (α, β, γ) is the tangent plane to the parabolid at (α, β, γ) and its equation, obtained by eliminating l, m, n between (i) and (v) is $\quad a\alpha(x-\alpha)+b\beta(y-\beta)-c(z-\gamma)=0$

or $$a\alpha x+b\beta y-cz=a\alpha^2+b\beta^2-c\gamma=c\gamma, \text{ from (iii)}$$

or $$\boxed{\mathbf{a\alpha x+b\beta y=c(z+\gamma)}}$$

***(b).** *To find the equation of the tangent plane to the paraboloid $(x^2/a^2)+(y^2/b^2)=2z/c$ at the point (α, β, γ).*

The equations of any line through (α, β, γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{...(i)}$$

Any point on this line is $(\alpha+lr, \beta+mr, \gamma+nr)$.

If the line (i) cuts the given paraboloid at this point, then the co-ordinates of the point must satisfy the given equation of the paraboloid.

$$\therefore \quad \frac{(\alpha+lr)^2}{a^2}+\frac{(\beta+mr)^2}{b^2}=\frac{2(\gamma+nr)}{c}$$

or $\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}\right)r^2+2\left(\frac{l\alpha}{a^2}+\frac{m\beta}{b^2}-\frac{n}{c}\right)r+\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-\frac{2\gamma}{c}\right)=0$ ...(ii)

Also as (α, β, γ) lies on the given paraboloid, so we have

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}=\frac{2\gamma}{c} \quad \text{or} \quad \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-\frac{2\gamma}{c}=0 \quad \text{...(iii)}$$

$\therefore$ From (ii) and (iii) we have

$$r^2\,[(l^2/a^2)+(m^2/b^2)]+2r\,[(l\alpha/a^2)+(m\beta/b^2)-(n/c)]=0 \quad \text{...(iv)}$$

The line (i) will touch the given paraboloid if both the values of r are zero. **(Note)**

$\therefore$ From (iv) we have $(\mathbf{l\alpha/a^2})+(\mathbf{m\beta/b^2})-(\mathbf{n/c})=\mathbf{0}$, ...(v)

which is the condition for the line (i) to be a tangent line to the parabolid at (α, β, γ).

The locus of all such lines through (α, β, γ) is the tangent plane to the paraboloid at (α, β, γ), and its equation, obtained by eliminating l, m, n between (i) and (v), is

$$\frac{\alpha}{a^2}(x-\alpha)+\frac{\beta}{b^2}(y-\beta)-\frac{1}{c}(z-\gamma)=0$$

or

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}-\frac{z}{c}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-\frac{\gamma}{c}$$

or

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}-\frac{z}{c}-\frac{\gamma}{c}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-\frac{2\gamma}{c}, \text{ adding } -\frac{\gamma}{c} \text{ to both sides,}$$

$$=0, \text{ from (iii)}$$

or

$$\frac{\alpha \mathbf{x}}{\mathbf{a}^2}+\frac{\beta \mathbf{y}}{\mathbf{b}^2}=\frac{(\mathbf{z}+\gamma)}{\mathbf{c}}$$

§ 10.05. Condition for tangency.

To find the condition that the plane $lx+my+nz=p$ *may touch the paraboloid* $ax^2+by^2=2cz$. (*Rohilkhand* 97)

Let the plane $lx+my+nz=p$ ...(i)

touch the paraboloid $ax^2+by^2=2cz$ ...(ii)

at the point (α, β, γ).

Now the equation of the tangent plane to (ii) at (α, β, γ) is

$$a\alpha x+b\beta y=c\,(z+\gamma) \quad \text{or} \quad a\alpha x+b\beta y-cz=c\gamma \quad \text{...(iii)}$$

If (i) touches (ii) at (α, β, γ) then (i) and (iii) represent the same plane and hence commparing (i) and (iii) we get

$$\frac{a\alpha}{l}=\frac{b\beta}{m}=\frac{-c}{n}=\frac{c\gamma}{p} \quad \text{or} \quad \alpha=-\frac{lc}{an},\ \beta=-\frac{mc}{bn},\ \gamma=-\frac{p}{n} \quad \text{...(iv)}$$

Also as (α, β, γ) lies on the paraboloid (ii), so we have $a\alpha^2+b\beta^2=2c\gamma$

or $$a\left(-\frac{lc}{an}\right)^2+b\left(-\frac{mc}{bn}\right)^2=2c\left(-\frac{p}{n}\right), \text{ from (iv).} \quad ...(v)$$

or $\boxed{(l^2/a)+(m^2/b)+(2np/c)=0}$ which is the required condition.

The coordinates of the point, where (i) touches (ii) are given by (iv)

$$\text{viz.} \left(-\frac{lc}{an}, -\frac{mc}{bn}, -\frac{p}{n}\right).$$

Cor. Substituting the values of p from (v) in (i) we conclude that the plane

$$2n(lx+my+nz)+c\left(\frac{l^2}{a}+\frac{m^2}{b}\right)=0$$

always touches the paraboloid (ii).

Solved Examples on § 10.04 and 10.05.

***Ex. 1 (a). Show that the plane $8x-6y-z=5$ touches the paraboloid $(x^2/2)-(y^2/3)=z$, and find the point of contact.** (*Agra* 92; *Garhwal* 94, 92)

Solution. Let the plane $8x-6y-z=5$ ...(i)

touch the paraboloid $(x^2/2)-(y^2/3)=z$ or $3x^2-2y^2=6z$ ...(ii)

at the point (α, β, γ)

The equation of the tangent plane to (ii) at (α, β, γ) is

$$3\alpha x-2\beta y=3(z+\gamma) \text{ or } 3\alpha x-2\beta y-3z=3\gamma \quad ...(iii)$$

If the plane (i) touches (ii) at (α, β, γ), then (i) and (iii) represent the same plane, and so comparing (i) and (iii), we get

$$\frac{3\alpha}{8}=\frac{-2\beta}{-6}=\frac{-3}{-1}=\frac{3\gamma}{5}$$

which gives $\alpha=8, \quad \beta=9, \quad \gamma=5$ ...(iv)

Also as (α, β, γ) lies on (ii), so we find $3\alpha^2-2\beta^2=6\gamma$ ...(v)

$\because$ values of α, β, γ given by (iv) satisfy (v), so the plane (i) touches the paraboloid (ii) at (α, β, γ).

Also from (iv), the coordinates of the point of contact are

(α, β, γ) *i.e.* $(8, 9, 5)$ **Ans.**

***Ex. 1 (b). Show that the plane $2x-4y-z+3=0$ touches the paraboloid $x^2-2y^2=3z$ and find the point of contact.** (*Rohilkhand* 96)

Solution. Do as Ex. 1 (a) above. **Ans.** $(3, 3, -3)$

Ex. 2. Find the condition that the plane $lx+my+nz=1$ may be a tangent plane to the paraboloid $x^2+y^2=2z$.

Solution. Let the plane $lx+my+nz=1$ ...(i)

touch the paraboloid $x^2+y^2=2z$ ...(ii)

at the point (α, β, γ).

The equation of the tangent plane to (ii) at (α, β, γ) is

$$\alpha x+\beta y=z+\gamma \text{ or } \alpha x+\beta y-z=\gamma \quad ...(iii)$$

If the plane (i) touches (ii) at (α, β, γ), then (i) and (ii) represent the same plane, and so comparing (i) and (iii) we get

$$\frac{\alpha}{l}=\frac{\beta}{m}=\frac{-1}{n}=\frac{\gamma}{1} \quad \text{or} \quad \alpha=-\frac{l}{n},\ \beta=-\frac{m}{n},\ \gamma=-\frac{1}{n} \qquad \text{...(iv)}$$

Also as (α, β, γ) lies on (ii), so we find $\alpha^2+\beta^2=2\gamma$

or $\quad (-l/n)^2+(-m/n)^2=2(-1/n)$, from (iv)

or $\quad l^2+m^2=-2n \quad$ or $\quad l^2+m^2+2n=0$ is the required condition. **Ans.**

****Ex. 3. Find the equation to the two tangent planes to the surface $ax^2+by^2=2z$ which pass through the line**

$u \equiv lx+my+nz-p=0$; $\quad u' \equiv l'x+m'y+n'z-p'=0$. (*Garhwal* 95).

Solution. The equation of any plane through the line $u=0$, $u'=0$ is

$$u+\lambda u'=0$$

or $$(l+\lambda l')x+(m+\lambda m')y+(n+\lambda n')z=p+\lambda p' \qquad \text{...(i)}$$

Let this plane touch the given surface $\quad ax^2+by^2=2z \qquad$ ...(ii)

at the point (α, β, γ).

Then (i) and the tangent plane to (ii) at (α, β, γ) viz

$$a\alpha x+b\beta y=z+\gamma \quad \text{or} \quad a\alpha x+b\beta y-z=\gamma \qquad \text{...(iii)}$$

must represent the same plane.

Comparing (i) and (iii) we get

$$\frac{a\alpha}{l+\lambda l'}=\frac{b\beta}{m+\lambda m'}=\frac{-1}{n+\lambda n'}=\frac{\gamma}{p+\lambda p'}$$

or $$\alpha=\frac{-(l+\lambda l')}{a(n+\lambda n')};\ \beta=\frac{-(m+\lambda m')}{b(n+\lambda n')};\ \gamma=\frac{-(p+\lambda p')}{(n+\lambda n')} \qquad \text{...(iv)}$$

Since (α, β, γ) lies on (ii), so we have $a\alpha^2+b\beta^2=2\gamma$

or $$a\left[\frac{-(l+\lambda l')}{a(n+\lambda n')}\right]^2+b\left[\frac{-(m+\lambda m')}{b(n+\lambda n')}\right]^2=2\left[\frac{-(p+\lambda p')}{(n+\lambda n')}\right]^2$$

or $$[(l+\lambda l')^2/a]+[(m+\lambda m')^2/b]=-2(p+\lambda p')(n+\lambda n')$$

or $$\left(\frac{l^2}{a}+\frac{m^2}{b}+2pn\right)+2\lambda\left(\frac{ll'}{a}+\frac{mm'}{b}+pn'+p'n\right)$$
$$+\lambda^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+2p'n'\right)=0 \qquad \text{...(v)}$$

Also λ is given by (i) $\quad$ or $\quad u+\lambda u'=0 \quad$ or $\quad \lambda=-u/u' \qquad$ ...(vi)

Eliminating λ between (v) and (vi) we get the required equation as

$$\left(\frac{l^2}{a}+\frac{m^2}{b}+2pn\right)+2\left(-\frac{u}{u'}\right)\left(\frac{ll'}{a}+\frac{mm'}{b}+pn'+p'n\right)$$
$$+\left(-\frac{u}{u'}\right)^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+2p'n'\right)=0$$

or $$u'^2\left(\frac{l^2}{a}+\frac{m^2}{b}+2pn\right)-2uu'\left(\frac{ll'}{a}+\frac{mm'}{b}+pn'+p'n\right)$$
$$+u^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+2p'n'\right)=0. \qquad \textbf{Ans.}$$

Ex. 4. Find the condition that the paraboloid

$$(x^2/a_1^2)+(y^2/b_1^2)=2\,(z/c_1),\ (x^2/a_2^2)+(y^2/b_2^2)=2\,(z/c_2)$$

$(x^2/a_3^2)+(y^2/b_3^2)=2\,(z/c_3)$ have a common tangent plane. (*Rohilkhand* 91)

Solution. Let the common tangent plane be $lx+my+nz=p$ ...(i)

Also we know (from § 10.05 Page 4) that the condition for the plane $lx+my+nz=p$ to touch the paraboloid $ax^2+by^2=2cz$ is

$$(l^2/a)+(m^2/b)+2\,(np/c)=0.$$

$\because$ If the plane (i) touches the paraboloid $(x^2/a_1^2)+(y^2/b_1^2)=2\,(z/c_1)$,

then we must have $$l^2a_1^2+m^2b_1^2+2npc_1=0; \qquad ...(ii)$$

as here $\quad$ 'a' $=1/a_1^2$, 'b' $=1/b_1^2$ and 'c' $=1/c_1$.

Similarly if the plane (i) touches the paraboloids

$$(x^2/a_2^2)+(y^2/b_2^2)=2\,(z/c_2),\ (x^2/a_3^2)+(y^2/b_3^2)=2\,(z/c_3)$$

we have $$l^2a_2^2+m^2b_2^2+2npc_2=0 \qquad ...(iii)$$

and $$l^2a_3^2+m^2b_3^2+2npc_3=0. \qquad ...(iv)$$

Eliminating l, m, n and $2np$ from (ii), (iii) and (iv) we get the required condition as

$$\begin{vmatrix} a_1^2 & b_1^2 & c_1 \\ a_2^2 & b_2^2 & c_2 \\ a_3^2 & b_3^2 & c_3 \end{vmatrix}=0 \qquad \textbf{Ans.}$$

Ex. 5. Two perpendicular tangent planes to the paraboloid $(x^2/a)+(y^2/b)=2z$ intersect in a line lying on the plane $x=0$. Prove that the line touches the parabola $x=0,\ y^2=(a+b)(2z+a)$.

Solution. Let the line of intersection of the two tangent planes be

$$my+nz=\lambda,\ x=0, \qquad ...(i)$$

since this lies on the plane $x=0$ (given).

$\therefore$ Equation of the plane through the line (i) is

$$(my+nz-\lambda)+kx=0 \quad \text{or} \quad kx+my+nz=\lambda \qquad ...(ii)$$

If the plane (ii) touches the paraboloid, then

$$\text{'}\frac{l^2}{a}+\frac{m^2}{b}+\frac{2pn}{c}=0\text{'} \qquad \text{...See § 10.05 Page 4}$$

or $$ak^2+bm^2+2\lambda n=0. \qquad ...(iii)\ \textbf{(Note)}$$

This being a quadratic in k, gives two values of k say k_1, k_2 such that

$$k_1k_2=(bm^2+2\lambda n)/a \qquad ...(iv)$$

Also from (ii) the direction ratios of the normals to the two tangent planes whose line of intersection is (ii) are k_1, m, n and k_2, m, n.

Also as these two tangent planes are perpendicular, so are their normals and consequently we have $k_1 . k_2 + m . m + n . n = 0$

or $$[(bm^2 + 2\lambda n)/a] + m^2 + n^2 = 0, \text{ from (iv)}$$

or $$(a + b) m^2 + an^2 + 2\lambda n = 0 \qquad ...(v)$$

Now we are to prove that the line (i) touches a parabola (given), so we are to find the envelope of (i) which satisfies the condition (v).

Eliminating λ between (i) and (v), the equations of the line of intersection of two tangent planes is

$$(a + b) m^2 + an^2 + 2 (my + nz) n = 0, x = 0.$$

or $$(a + b) (m/n)^2 + 2y (m/n) + (a + 2z) = 0, x = 0.$$

It is quadratic in (m/n) so its envelope is given by

$$\text{“}B^2 - 4AC = 0, x = 0 \qquad \text{...See Author's Diff. Calculus.}$$

or $$(2y)^2 - 4 (a + b) (a + 2z) = 0, x = 0$$

or $$y^2 = (a + b) (a + 2z), x = 0. \qquad \text{Hence proved.}$$

Ex. 6. Tangent planes at two points A and B of a paraboloid $ax^2 + by^2 = 2cz$ meet on the line CD. Prove that the plane through CD and mid-point of AB is parallel to the axis of the paraboloid.

Solution. The given paraboloid is $ax^2 + by^2 = 2cz$ ...(i)

Its axis is z-axis. (See § 10.01 Page 1 of this chapter)

Let the coordinates of A and B be $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$.

Since these points lie on (i) so we have

$$a\alpha_1^2 + b\beta_1^2 = 2c\gamma_1 \quad \text{and} \quad a\alpha_2^2 + b\beta_2^2 = 2c\gamma_2 \qquad ...(ii)$$

Also tangent planes at A and B to (i) are given by

$$a\alpha_1 x + b\beta_1 y - c (\gamma_1 + z) = 0 \quad \text{and} \quad a\alpha_2 x + b\beta_2 y - c (\gamma_2 + z) = 0 \qquad ...(iii)$$

These two tangent planes meet on the line CD, so the equation of any plane through the line CD is

$$[a\alpha_1 x + b\beta_1 y - c (\gamma_1 + z)] + \lambda [a\alpha_2 x + b\beta_2 y - c (\gamma_2 + z)] = 0 \qquad ...(iv)$$

If this plane passes through the mid-point of AB viz.

$[\frac{1}{2}(\alpha_1 + \alpha_2), \frac{1}{2}(\beta_1 + \beta_2), \frac{1}{2}(\gamma_1 + \gamma_2)]$ then we have

$$[a\alpha_1 . \tfrac{1}{2}(\alpha_1 + \alpha_2) + b\beta_1 \tfrac{1}{2}(\beta_1 + \beta_2) - c \{\gamma_1 + \tfrac{1}{2}(\gamma_1 + \gamma_2)\}]$$
$$+ \lambda [a\alpha_2 . \tfrac{1}{2}(\alpha_1 + \alpha_2) + b \beta_2 . \tfrac{1}{2}(\beta_1 + \beta_2) - c \{\gamma_2 + \tfrac{1}{2}(\gamma_1 + \gamma_2)\}] = 0$$

or $$[\{a\alpha_1^2 + b\beta_1^2 - 2c\gamma_1\} + \{a\alpha_1\alpha_2 + b\beta_1\beta_2 - c (\gamma_1 + \gamma_2)\}]$$
$$+ \lambda [\{a\alpha_2^2 + b\beta_2^2 - 2c\gamma_2\} + \{a\alpha_1\alpha_2 + b\beta_1\beta_2 - c (\gamma_1 + \gamma_2)\}] = 0$$

or $[a\alpha_1\alpha_2 + b\beta_1\beta_2 - c (\gamma_1 + \gamma_2)] (1 + \lambda) = 0$, with the help of (ii) or $$(1+\lambda)=0 \quad \text{or} \quad \lambda=-1.$$

Substituting this value of λ in (iv), we get the equation of the plane through CD and mid-point of AB as

$$[a\alpha_1 x+b\beta_1 y-c(\gamma_1+z)]-[a\alpha_2 x+b\beta_2 y-c(\gamma_2+z)]=0$$

or $$a(\alpha_1-\alpha_2)x+b(\beta_1-\beta_2)y-c(\gamma_1-\gamma_2)=0 \quad \text{...(v)}$$

The direction ratios of the normal to this plane are

$$a(\alpha_1-\alpha_2),\ b(\beta_1-\beta_2),\ 0.$$

Also the ratios of the axis of the paraboloid (i) viz. z-axis are (0, 0, 1)

Since $a(\alpha_1-\alpha_2).0+b(\beta_1-\beta_2).0+0.1=0$, so the plane (v) is parallel to the axis of the paraboloid.

Exercise on § 10.04—§10.05.

Ex. Find the equation of the tangent plane at (2, 0, 2) on the surface

$$x^2+y^2=2z.$$ **Ans.** $2x-z=2$

§ 10.06. Points of Intersection with a line.

Let a line $$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{...(i)}$$

intersect the paraboloid $$ax^2+by^2=2cz. \quad \text{...(ii)}$$

Any point on the line (i) is $(\alpha+lr,\ \beta+mr,\ \gamma+nr)$. If it lies on (ii) we have

$$a(\alpha+lr)^2+b(\beta+mr)^2=2c(\gamma+nr)$$

or $$(al^2+bm^2)r^2+2r(al\alpha+bm\beta-cn)+(a\alpha^2+b\beta^2-2c\gamma)=0 \quad \text{...(iii)}$$

This being a quadratic equation in r, we conclude that the line (i) meets the paraboloid (ii) in two points, whose distances from the point (α, β, γ) on the line (i) are given by two values of r given by the equation (iii).

In particular if the line (i) is parallel to the axis of the paraboloid (ii) *i.e.* z-axis, then $l=0$, $m=0$. This renders the coefficients of r^2 in (iii) as zero and so one value of r given by (iii) is infinite which shows that every line drawn parallel to the axis of the paraboloid (ii) meets it at an infinite distance from the point (α, β, γ) whereas the other is at a finite distance viz.

$-(a\alpha^2+b\beta^2-2c\gamma)/[2(al\alpha+bm\beta-cn)]$ from the point (α, β, γ).

Such lines are called **diameters** of the paraboloid.

§ 10.17. Locus of the point of intersection of three mutually perpendicular tangent planes. (*Gorakhpur* 97, 95; *Rohilkhand* 93, 91)

Let the paraboloid be $$ax^2+by^2=2cz \quad \text{...(i)}$$

The equations of any three tangent planes to (i) are

$$2n_1(l_1x+m_1y+n_1z)+c[(l_1^2/a)+(m_1^2/b)]=0 \quad \text{...(ii)}$$

$$2n_2(l_2x+m_2y+n_2z)+c[(l_2^2/a)+(m_2^2/b)]=0 \quad \text{...(iii)}$$

and $$2n_3(l_3x+m_3y+n_3z)+c[(l_3^2/a)+(m_3^2/b)]=0 \quad \text{...(iv)}$$

...See § 10.05 cor. Page 5 Ch. X

If the above three tangent planes are mutually perpendicular then we have $l_1^2 + l_2^2 + l_3^2 = 1$ etc. and $l_1 m_1 + l_2 m_2 + l_3 m_3 = 0$ etc. ...(v)

The locus of the point of intersection of these three tangent planes is obtained by eliminating l_1, m_1, n_1; l_2, m_2, n_2 and l_3, m_3, n_3 with the help of (v).

Adding (ii), (iii) and (iv) we get

$$2(n_1^2 + n_2^2 + n_3^2) z + c\left[\frac{1}{a}(l_1^2 + l_2^2 + l_3^2) + \frac{1}{b}(m_1^2 + m_2^2 + m_3^2)\right] = 0,$$

using $l_1 n_1 + l_2 n_2 + l_3 n_2 = 0 = m_1 n_1 + m_2 n_2 + m_3 n_3$

or $2z + c[(1/a) + (1/b)] = 0$, $\because l_1^2 + l_2^2 + l_3^2 = 1$ etc., from (v)

This is the locus of the point of intersection of three mutually perpendicular tangent planes.

Also from the equations of this locus (which is a plane) we observe the direction cosines of the normal to it are (0, 0, 1) *i.e.* this locus is perpendicular to z-axis.

§ 10.08. The Polar Plane.

Definition. If through a point $A(\alpha, \beta, \gamma)$ a line APQ be drawn to meet the paraboloid $ax^2 + by^2 = 2cz$ in P and Q, then the locus of R, the harmonic conjugate of A with respect to P and Q (*i.e.* AP, AR and AQ are in H. P.) is defined as the polar of A with respect to the paraboloid.

Any line through $A(\alpha, \beta, \gamma)$ is $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ ...(i)

Any point on this line is $(\alpha + lr, \beta + mr, \gamma + nr)$.

If this point lies on the paraboloid $ax^2 + by^2 = 2cz$ ...(ii)

then we have $a(\alpha + lr)^2 + b(\beta + mr)^2 = 2c(\gamma + nr)$

or $r^2(al^2 + bm^2) + 2r(al\alpha + bm\beta - cn) + (a\alpha^2 + b\beta^2 - 2c\gamma) = 0$ ...(iii)

The two values of r obtained from (iii) are AP and AQ. **(Note)**

Also as AP, AR and AQ are in H. P. (by def. of polar), so we have

$$\frac{2}{AR} = \frac{1}{AP} + \frac{1}{AQ} = \frac{1}{r_1} + \frac{1}{r_2},$$ ...(iv)

where r_1 and r_2 are roots of (iii).

From (iii) we have $r_1 + r_2 = -2(al\alpha + b\beta m - cn)/(al^2 + bm^2)$

and $r_1 r_2 = (a\alpha^2 + b\beta^2 - 2c\gamma)/(al^2 + bm^2)$

$$\therefore \quad \frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 + r_2}{r_1 r_2} = \frac{-2(al\alpha + bm\beta - cn)}{(a\alpha^2 + b\beta^2 - 2c\gamma)}$$

$\therefore$ From (iv) we have $\dfrac{2}{AR} = \dfrac{-2(al\alpha + b\beta m - cn)}{(a\alpha^2 + b\beta^2 - 2c\gamma)}$

or $(a\alpha^2 + b\beta^2 - 2c\gamma) = -a\alpha(l \, . \, AR) - b\beta(m \, . \, AR) + c(n \, . \, AR)$ ...(v) **(Note)**

Now if R be (x, y, z) and its distance from $A(\alpha, \beta, \gamma)$ be AR on the line AQ, then from (i) we get

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=AR.$$ **(Note)**

or $(x-\alpha)=l\,.\,AR,\; y-\beta=m\,.\,AR,\; z-\gamma=n\,.\,AR.$

Substituting these values in (v) we find that l, m, n are eliminated and we get the equation of the locus of R as

$$(a\alpha^2+b\beta^2-2c\gamma)=-a\alpha(x-\alpha)-b\beta(y-\beta)+c(z-\gamma)$$

or $\mathbf{a\alpha x+b\beta y=c\,(z+\gamma)}$ is the equation of the polar plane of $A\,(\alpha,\beta,\gamma)$ with respect to the paraboloid $ax^2+by^2=2cz$.

§ 10.09. Pole of a given plane.

Let (α,β,γ) be the pole of the plane $\quad lx+my+nz=p$ ...(i)

with respect to the paraboloid $\quad ax^2+by^2=2cz$ ...(ii)

The polar plane of (α,β,γ) with respect to (ii) is

$$a\alpha x+b\beta y=c\,(z+\gamma)\quad\text{or}\quad a\alpha x+b\beta y-cz=c\gamma \qquad ...(iii)$$

Comparing (i) and (iii) which represent the same plane we get

$$\frac{a\alpha}{l}=\frac{b\beta}{m}=\frac{-c}{n}=\frac{c\gamma}{p}\quad\text{or}\quad \alpha=-\frac{lc}{an},\; \beta=-\frac{mc}{bn},\; \gamma=-\frac{p}{n}$$

$\therefore$ The pole of the plane (i) with respect to the paraboloid (ii) is

$$\left(-\frac{lc}{an},-\frac{mc}{bn},-\frac{p}{n}\right)$$

***Solved Example. Find the locus of the perpendiculars from (α,β,γ) to its polar with respect to the paraboloid $x^2/a^2+y^2/b^2=2z$.**

Sol. The polar plane of (α,β,γ) with respect to the given paraboloid is

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}=(z+\gamma) \qquad ...(i)$$

The equations of any line through (α,β,γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad ...(ii)$$

If this line (ii) is perpendicular to the plane (i), then it must be parallel to the normal of (i) and therefore we have

$$\frac{(\alpha/a^2)}{l}=\frac{(\beta/b^2)}{m}=\frac{-1}{n}=\lambda\text{ (say)}$$ **(Note)**

which gives $\quad \dfrac{\alpha}{l}=a^2\lambda,\; \dfrac{\beta}{m}=b^2\lambda,$ where $\lambda=-\dfrac{1}{n}$

$$\therefore\ \frac{\alpha}{l}-\frac{\beta}{m}=(a^2-b^2)\,\lambda=\frac{-(a^2-b^2)}{n},\ \because\ \lambda=-\frac{1}{n}$$

or $$\frac{\alpha}{l}-\frac{\beta}{m}+\frac{(a^2-b^2)}{n}=0 \qquad ...(iii)$$

From (ii) and (iii) eliminating l, m, n we have the required locus as

$$\frac{\alpha}{(x-\alpha)}-\frac{\beta}{(y-\beta)}+\frac{(a^2-b^2)}{(z-\gamma)}=0.$$ **Ans.**

§ 10.10. Locus of chords with a given mid-point.

Let the paraboloid be $S \equiv ax^2 + by^2 - 2cz = 0$ ...(i)

Let (α, β, γ) be the mid-point of any chord to it and its equations be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)} \quad \text{...(ii)}$$

Any point on (ii) is $(\alpha + lr, \beta + mr, \gamma + nr)$. If this point lies on (i) then we have $a(\alpha + lr)^2 + b(\beta + mr)^2 - 2c(\gamma + nr) = 0$

or $r^2(al^2 + bm^2) + 2r(al\alpha + bm\beta - cn) + (a\alpha^2 + b\beta^2 - 2c\gamma) = 0,$ ...(iii)

which gives the distance of the point of intersection of the line (ii) and the paraboloid (i) from the point (α, β, γ).

Since (α, β, γ) is the mid-point of the chord given by (i), so its distance from the two points of intersection of (i) and (ii) should be equal but opposite. **(Note)**

i.e. the sum of the values of r given by (iii) must be zero

i.e. $$al\alpha + bm\beta - cn = 0 \quad \text{...(iv)}$$

The required locus of the chords with (α, β, γ) as mid-point is obtained by eliminating l, m, n between (ii) and (iii) and is

$$a\alpha(x-\alpha) + b\beta(y-\beta) + c(z-\gamma) = 0$$

or $$a\alpha x + b\beta y - cz = a\alpha^2 + b\beta^2 - c\gamma \quad \text{...(i)}$$

or $$a\alpha x + b\beta y - c(z+\gamma) = a\alpha^2 + b\beta - 2c\gamma \quad \textbf{(Note)}$$

or $$\mathbf{T = S_1}, \text{ where } \mathbf{S_1 = a\alpha^2 + b\beta^2 - 2c\gamma}$$

and $$\mathbf{T = a\alpha x + b\beta y - c(z+\gamma)}.$$

This plane meets the paraboloid (i) in a conic of which (α, β, γ) is the centre.

Solved Examples on § 10.10.

***Ex. 1. Show that the locus of centres of a system of parallel plane sections of a paraboloid is a diameter. Also show that the tangent plane at the extremity of the diameter is parallel to the plane sections.**

Solution. Let $C(x_1, y_1, z_1)$ be the centre of one of the plane sections of the parabolid $$ax^2 + by^2 = 2cz \text{ (say)} \quad \text{...(i)}$$
by planes parallel to a given plane $lx + my + nz = p$ ...(ii)

Then the equation of the section of (i) witth C as centre is "$S_1 = T$"

i.e. $$ax_1^2 + by_1^2 - 2cz_1 = axx_1 + byy_1 - c(z + z_1) \quad \textbf{(Note)}$$

or $$(ax_1)x + (by_1)y - (c)z = ax_1^2 + by_1^2 - cz_1 \quad \text{...(iii)}$$

Now the planes given by (ii) and (iii), being parallel, on comparing coefficients of x, y, z we have $\dfrac{ax_1}{l} = \dfrac{by_1}{m} = \dfrac{-c}{n}$ ...(iv)

$\therefore$ Locus of $C(x_1, y_1, z_1)$ from (iv) is the line

$$\frac{ax}{l} = \frac{by}{m} = \frac{-c}{n} \quad \text{...(v)}$$

whose equations can also be written as $anx + cl = 0,\ bny + cm = 0$ ...(vi)

$\therefore$ The locus of $C(x_1, y_1, z_1)$ is the line of intersection of the planes given by (v) which are evidently respectively parallel to the planes $x=0$ and $y=0$ i.e. z-axis, which we know is the axis of the paraboloid (i) and hence (v) is a diameter of (i).

Solving (v) and (i) we find $x=-lc/an$, $y=-mc/bn$

Substituting these values in (i) we get

$$a(l^2c^2/a^2n^2)+b(m^2c^2/b^2n^2)=2cz \quad \text{or} \quad z=\frac{1}{2}\frac{c}{n^2}\left(\frac{l^2}{a}+\frac{m^2}{b}\right)$$

$\therefore$ The coordinates of the extremity of diameter given by (iv) are given by

$$x=-\frac{lc}{an},\ y=-\frac{mc}{bn},\ z=\frac{c}{2n^2}\left(\frac{l^2}{a}+\frac{m^2}{b}\right)$$

$\therefore$ The equations of the tangent plane to (i) at the above extremity of the diameter is

$$ax\left(-\frac{lc}{an}\right)+by\left(-\frac{mc}{bn}\right)=c\left[z+\frac{c}{2n^2}\left(\frac{l^2}{a}+\frac{m^2}{b}\right)\right]$$

or $$2n\,[lx+my]=2n^2z+c\,[(l^2/a)+(m^2/b)]$$

or $$lx+my+nz=-\tfrac{1}{2}(c/n)\,[(l^2/a)+(m^2/b)],$$

which is evidently a plane parallel to the given plane (ii), hence parallel to the plane sections.

Hence proved.

***Ex. 2. Prove that the centre of the conic $ax^2+by^2=2z$, $lx+my+nz=p$ is the point**

$$\left(-\frac{l}{an}, -\frac{m}{bn}, \frac{k^2}{n^2}\right), \text{ where } k^2=\frac{l^2}{a}+\frac{m^2}{b}+np.$$

Solution. We know from § 10.10 Page 12 that the plane $T=S_1$ meets the paraboloid in a conic of which (α, β, γ) is the centre.

Here $$S\equiv ax^2+by^2-2z=0$$

$\therefore$ $S_1=a\alpha^2+b\beta^2-2\gamma$ and $T=a\alpha x+b\beta y-(z+\gamma)$

The equation of the plane which cuts the given paraboloid in a conic with centre (α, β, γ) is $T=S_1$

or $$a\alpha x+b\beta y-(z+\gamma)=a\alpha^2+b\beta^2-2\gamma$$

or $$a\alpha x+b\beta y-z=a\alpha^2+b\beta^2-\gamma. \qquad \text{...(i)}$$

But here the given plane is $lx+my+nz=p$. ...(ii)

Comparing (i) and (ii), we get $\dfrac{a\alpha}{l}=\dfrac{b\beta}{m}=-\dfrac{1}{n}=\dfrac{a\alpha^2+b\beta^2-\gamma}{p}$

These give $$\alpha=-\frac{l}{an},\ \beta=-\frac{m}{bn}$$

and $$\gamma=a\alpha^2+b\beta^2+\frac{p}{n}=a\left(-\frac{l}{an}\right)^2+b\left(-\frac{m}{bn}\right)^2+\frac{p}{n}$$

or $\gamma=\left(\dfrac{l^2}{a}+\dfrac{m^2}{b}+np\right)\Big/ n^2=\dfrac{k^2}{n^2}$, where $k^2=\dfrac{l^2}{a}+\dfrac{m^2}{b}+np$

Hence the required centre of the conic is (α, β, γ)

or $$\left(-\frac{l}{an}, -\frac{m}{bn}, \frac{k^2}{n^2}\right), \text{ where } k^2=\frac{l^2}{a}+\frac{m^2}{b}+np.$$

Ex. 3. Find the equation of the plane which cuts the paraboloid $x^2-2y^2=3z$ in the conic with centre (1, 2, 3).

Solution. Here the paraboloid is $S \equiv x^2-2y^2-3z=0$ ...(i)

The equation of the plane which cuts (i) in a conic with centre (α, β, γ) is

$$T=S_1$$

where $T=\alpha x-2\beta y-\frac{3}{2}(z+\gamma)$; $S_1=\alpha^2-2\beta^2-3\gamma$ **(Note)**

Here the centre is given as (1, 2, 3), so $\alpha=1,\ \beta=2,\ \gamma=3$.

$\therefore$ Here we have $T=x-4y-\frac{3}{2}(z+3)$ and $S_1=1-8-9=-16$.

$\therefore$ The required equation is $T=S_1$

or $x-4y-\frac{3}{2}(z+3)=-16$ or $2x-8y-3z+23=0$. **Ans.**

Exercise on § 10.10.

Ex. Find the equation of the plane which cuts the paraboloid $x^2-2y^2=z$ in a conic with its centre at (2, 3/2, 4). **Ans.** $4x-6y-z+5=0$.

§ 10.11. Normal to the paraboloid.

To find equation of the normal to the paraboloid $ax^2+by^2=2cz$ at the point (α, β, γ).

The equation of the tangent plane to the given paraboloid is

$$a\alpha x+b\beta y=c(z+\gamma) \quad \text{or} \quad a\alpha x+b\beta y-cz=c\gamma. \qquad \text{...(i)}$$

The required normal is a line through (α, β, γ) at right angles to the tangent plane given by (i), hence the equations of the normal are

$$\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{-c},$$

whose direction ratios are $a\alpha, b\beta, -c$.

*§ 10.12. Number of Normals from a point (α, β, γ).

(a) When the paraboloid is $ax^2+by^2=2cz$ ...(i)

The equations of the normal to (i) at the point (α, β, γ) are

$$\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{-c}$$

If this normal passes through the fixed point (x_1, y_1, z_1), then we have

$$\frac{x_1-\alpha}{a\alpha}=\frac{y_1-\beta}{b\beta}=\frac{z_1-\gamma}{-c}=r \text{ (say)}$$

From $\dfrac{x_1-\alpha}{a\alpha}=r$, we get $x_1=\alpha+a\alpha r=\alpha(1+ar)$ or $\alpha=\dfrac{x_1}{1+ar}$.

Similarly $\beta=\dfrac{y_1}{1+br}$. Also $\gamma=z_1+cr$.

Also (α, β, γ) being a point on the given paraboloid, we have

$$a\alpha^2+b\beta^2=2c\gamma$$

or
$$a\left(\frac{x_1}{1+ar}\right)^2+b\left(\frac{y_1}{1+br}\right)^2=2c(z_1+cr), \quad \text{...(ii)}$$

substituting the values of α, β, γ as calculated above.

This equation is of fifth degree in r, hence **five normals can be drawn to a paraboloid from any fixed point.**

*(b) When the paraboloid is $x^2/a^2+y^2/b^2=2z$. ...(i)

The equations of the normal to (iii) at the point (α, β, γ) are

$$\frac{x-\alpha}{(\alpha/a^2)}=\frac{y-\beta}{(\beta/b^2)}=\frac{z_1-\gamma}{-1}$$

If this normal passes through the fixed point (x_1, y_1, z_1), then we have

$$\frac{x_1-\alpha}{\alpha/a^2}=\frac{y_1-\beta}{(\beta/b^2)}=\frac{z_1-\gamma}{-1}=r \text{ (say).}$$

From $\dfrac{x_1-\alpha}{\alpha/a^2}=r$, we get $x_1=\alpha+(\alpha r/a^2)=\alpha[1+(r/a^2)]$

or $\alpha=\dfrac{x_1}{1+(r/a^2)}$. Similarly $\beta=\dfrac{y_1}{1+(r/b^2)}$. Also $\gamma=z_1+r$.

Also (α, β, γ) being a point on (iii), we have $(\alpha^2/a^2)+(\beta^2/b^2)=2\gamma$

or
$$\frac{1}{a^2}\left[\frac{x_1}{1+(r/a^2)}\right]^2+\frac{1}{b^2}\left[\frac{y_1}{1+(r/b^2)}\right]^2=2(z_1+r);$$

substituting the values of α, β, γ as calculated above.

As before this equation is of fifth degree in r, hence five normals can be drawn to a paraboloid from any fixed point. *(Rohilkhand 90)*

§ 10.13. Cubic through feet of the normals.

The five feet of the normals that can be drawn to a paraboloid from a given point are the intersection of a certain cubic curve with the paraboloid.

If the normal at the point (α, β, γ) to the paraboloid $ax^2+by^2=2cz$ passes through (x_1, y_1, z_1), then as in § 10.12 (a) Page 14, we can prove that

$$\alpha=\frac{x_1}{1+ar},\ \beta=\frac{y_1}{1+br},\ \gamma=z_1+cr. \quad \text{...(i)}$$

Now consider an equation whose parametric equations are

$$x=\frac{x_1}{1+ar},\ y=\frac{y_1}{1+br},\ z=z_1+cr. \quad \text{...(ii)}$$

where r is a parameter having five values given by (ii) of § 10.12 (a) Page 14 and corresponding to each of these five values we get a point on the curve (ii) above.

The points of intersection of the curve (ii) and any plane $a_1x+b_1y+c_1z+d_1=0$ are given by

$$a_1\left(\frac{x_1}{1+ar}\right)+b_1\left(\frac{y_1}{1+br}\right)+c_1(z_1+cr)+d_1=0 \qquad ...(iii)$$

which is a cubic equation in r and hence gives three values of r, showing that the curve (ii) meets the plane in three points and hence it is a curve.

§ 10.14. Cone through five concurrent normals. (*Rohilkhand* 90)

In § 10.12 (a) Page 14, we have proved that from any point (x_1, y_1, z_1) five normals can be drawn to the paraboloid $ax^2+by^2=2cz.$ ...(i)

We are now to prove that these five normals lie on a cone.

Now the equations of any line through (x_1, y_1, z_1) can be taken as

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \qquad ...(ii)$$

If it represents a normal to the paraboloid (i) at (α, β, γ), then

$l=a\alpha$, $m=b\beta$ and $n=-c$ (See § 10.12 (a) Page 14) ...(iii)

Also we know that there are five points such as (α, β, γ), the normals at which to (i) pass though (x_1, y_1, z_1) and as in § 10.12 (a) Page 14 we can prove that

$$\alpha=\frac{x_1}{1+ar},\ \beta=\frac{y_1}{1+br},\ \gamma=z_1+cr.$$

$\therefore$ From (iii) above, we get $l=\dfrac{ax_1}{1+ar}$, $m=\dfrac{by_1}{1+br}$, $n=-c$

or $$\frac{ax_1}{l}=1+ar,\ \frac{by_1}{m}=1+br,\ \frac{c}{n}=-1$$

Multiplying these by b, $-a$ and $b-a$ respectively and adding, r is eliminated and we get

$$\frac{ax_1}{l}b+\frac{by_1}{m}(-a)+\frac{c}{n}(b-a)=(1+ar)b+(1+br)(-a)-(b-a)$$

or $$\frac{abx_1}{l}-\frac{aby_1}{m}+\frac{c(b-a)}{n}=0. \qquad ...(iv)$$

Eliminating l, m, n between (iv) and (ii), we get the equation of the cone on which the normals lie as $\dfrac{ab\,x_1}{(x-x_1)}-\dfrac{ab\,y_1}{(y-y_1)}+\dfrac{c(b-a)}{(z-z_1)}=0.$

or $$\frac{x_1}{(x-x_1)}-\frac{y_1}{(y-y_1)}+\frac{c[(1/a)-(1/b)]}{(z-z_1)}=0 \qquad ...(v)$$

Solved Examples on § 10.11 to § 10.14.

Ex. 1. Find the equations of the normal at (4, 3, 5) on the paraboloid

$$\frac{1}{2}x^2 - \frac{1}{3}y^2 = z.$$

Solution. The equation of the paraboloid is

$$\frac{1}{2}x^2 - \frac{1}{3}y^2 = z \quad \text{or} \quad 3x^2 - 2y^2 = 6z \qquad \text{...(i)}$$

The equation of the tangent plane to (i) at (4, 3, 5) is

$$3x(4) - 2y(3) = 3(z+5) \quad \text{or} \quad 4x - 2y - z = 5. \qquad \text{...(ii)}$$

The required normal is the line through (4, 3, 5) at right angles to the tangent plane given by (ii). Hence the required equations are

$$\frac{x-4}{4} = \frac{y-3}{-2} = \frac{z-5}{-1} \qquad \textbf{Ans.}$$

****Ex. 2. Prove that the normals from (α, β, γ) to the paraboloid $(x^2/a^2) + (y^2/b^2) = 2z$ lies on the cone**

$$\frac{\alpha}{x-\alpha} - \frac{\beta}{y-\beta} + \frac{a^2-b^2}{z-\gamma} = 0$$

(*Gorakhpur* 97, 95; *Rohilkhand* 92)

Solution. The equations of any line through (α, β, γ) can be taken as

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \text{...(i)}$$

If it represents a normal to the given paraboloid

$$(x^2/a^2) + (y^2/b^2) = 2z \qquad \text{...(ii)}$$

at (x_1, y_1, z_1), then $\quad l = x_1/a^2,\ m = y_1/b^2,\ n = -1 \qquad$ ...(iii)

...See § 10.12 (b) Page 15 Ch. X

Also we know that there are five points such as (x_1, y_1, z_1), the normals at which to the paroboloid (ii) passes through the point (α, β, γ) and as in § 10.12 (b) Page 15 Ch. X we can prove that

$$x_1 = \frac{\alpha}{1+(r/a^2)},\ y_1 = \frac{\beta}{1+(r/b^2)},\ z_1 = \gamma + r$$

$\therefore$ From (iii) above, we get $l = \dfrac{\alpha/a^2}{1+(r/a^2)},\ m = \dfrac{\beta/b^2}{1+(r/b^2)}$

and $\quad n = -1 \quad$ or $\quad l = \dfrac{\alpha}{a^2+r},\ m = \dfrac{\beta}{b^2+r},\ n = -1$

or $\quad \dfrac{\alpha}{l} = a^2 + r,\ \dfrac{\beta}{m} = b^2 + r,\ \dfrac{1}{n} = -1$

Eliminating r between these we get $(\alpha/l) - (\beta/m) = -(a^2 - b^2)(1/n)$

or $\quad (\alpha/l) - (\beta/m) + [(a^2 - b^2)/n] = 0 \qquad$ ...(iv)

Eliminating l, m, n between (iv) and (i), we get the equation of the cone on which the normals to the given paraboloid from (α, β, γ) lie as

$$\frac{\alpha}{x-\alpha} - \frac{\beta}{y-\beta} + \frac{a^2-b^2}{z-\gamma} = 0 \qquad \text{Hence proved.}$$

Ex. 3. Prove that the cubic curve through the five feet of normals drawn to a paraboloid from any point (x_1, y_1, z_1) lies on the cone through the five concurrent normals.

Solution. From § 10.13 (ii) on Page 15 of this chapter we know that the parametric equations of the cubic curve through the feet of the normals from any point (x_1, y_1, z_1) are $\quad x = \dfrac{x_1}{1+ar},\ y = \dfrac{y_1}{1+br},\ z = z_1 + cr.$ ...(i)

Also the equation of the cone on which these five normals lie from § 10.14 Page 16 Ch. X, is
$$\frac{x_1}{x-x_1} - \frac{y_1}{y-y_1} + \frac{c\,[(1/a)-(1/b)]}{z-z_1} = 0. \qquad \text{...(ii)}$$

Substituting values of x, y, z from (i) in (ii) we find that
$$\frac{x_1}{x-x_1} - \frac{y_1}{y-y_1} + \frac{c\,[(1/a)-(1/b)]}{z-z_1}$$
$$= \frac{x_1}{x_1\,[\{1/(1+ar)\}-1]} - \frac{y_1}{y_1\,[\{1/(1+br)\}-1]} + \frac{c\,[(1/a)-(1/b)]}{(z_1+cr)-z_1}$$
$$= \frac{(1+ar)}{1-(1+ar)} - \frac{(1+br)}{1-(1+br)} + \frac{c\,[(1/a)-(1/b)]}{cr}$$
$$= \frac{(1+ar)}{-ar} - \frac{(1+br)}{-br} + \left[\frac{1}{ar} - \frac{1}{br}\right]$$
$$= -\frac{1}{ar} - 1 + \frac{1}{br} + 1 + \frac{1}{ar} - \frac{1}{br} = 0.$$

Thus (ii) is satisfied by the values of x, y, z given by (i).

Hence the cubic curve given by (i) lies on the cone given by (i).

****Ex. 4. Show that the feet of the normals from the point (α, β, γ) on the paraboloid $x^2 + y^2 = 2az$ lie on a sphere.** (*Rohilkhand* 94)

Solution. Let (x_1, y_1, z_1) be any point on the given paraboloid, then
$$x_1^2 + y_1^2 = 2az_1 \qquad \text{...(i)}$$

The tangent plane to this paraboloid at (x_1, y_1, z_1) is
$$xx_1 + yy_1 = a(z + z_1) \quad \text{or} \quad xx_1 + yy_1 - az = az_1$$

$\therefore$ The equation of the normal to the given paraboloid at (x_1, y_1, z_1) *i.e.* the line through (x_1, y_1, z_1) at right angles to the above tangent plane is
$$\frac{x-x_1}{x_1} = \frac{y-y_1}{y_1} = \frac{z-z_1}{-a}$$

If this normal passes through the fixed point (α, β, γ) then we have
$$\frac{\alpha-x_1}{x_1} = \frac{\beta-y_1}{y_1} = \frac{\gamma-z_1}{-a} \qquad \text{...(ii)}$$

or
$$\frac{\alpha-x_1}{x_1} = \frac{\beta-y_1}{y_1} = \frac{\gamma-z_1}{-a} = \frac{x_1(\alpha-x_1)+y_1(\beta-y_1)}{x_1(x_1)+y_1(y_1)}$$
$$= \frac{z_1(\gamma-z_1)}{z_1(-a)} \qquad \textbf{(Note)}$$

From the last two fractions we have $\dfrac{\alpha x_1 - x_1^2 + \beta y_1 - y_1^2}{x_1^2 + y_1^2} = \dfrac{\gamma z_1 - z_1^2}{-az_1}$

or $$\frac{(\alpha x_1 + \beta y_1) - (x_1^2 + y_1^2)}{2az_1} = \frac{2(\gamma z_1 - z_1^2)}{-2az_1}, \text{ from (i)}$$

or $$x_1^2 + y_1^2 - (\alpha x_1 + \beta y_1) - 2\gamma z_1 + 2z_1^2 = 0$$

or $$x_1^2 + y_1^2 + 2z_1^2 - 2\gamma z_1 = \alpha x_1 + \beta y_1 \quad \text{...(iii)}$$

Also from (ii) we have, $\dfrac{\alpha}{x_1} - 1 = \dfrac{\beta}{y_1} - 1$ **(Note)**

or $$\alpha/x_1 = \beta/y_1 \quad \text{or} \quad \alpha y_1 = \beta x_1 \quad \text{...(iv)}$$

Now from (iii) we have $x_1^2 + y_1^2 + 2z_1^2 - 2\gamma z_1 = (\alpha\beta x_1 + \beta^2 y_1)/\beta$

$$= (\alpha^2 y_1 + \beta^2 y_1)/\beta, \text{ from (iv)}$$

or $$x_1^2 + y_1^2 + 2z_1^2 - 2\gamma z_1 = (\alpha^2 + \beta^2)\, y_1/\beta \quad \text{...(v)}$$

Adding (i) and (v) we get

$$2x_1^2 + 2y_1^2 + 2z_1^2 - 2\gamma z_1 = 2az_1 + [\{2(\alpha^2 + \beta^2)\, y_1\}/(2\beta)]$$

or $$x_1^2 + y_1^2 + z_1^2 - (\gamma + a)\, z_1 - \{(\alpha^2 + \beta^2)\, y_1/(2\beta)\} = 0.$$

$\therefore$ The locus of the foot (x_1, y_1, z_1) of the normal from (α, β, γ) is

$$x^2 + y^2 + z^2 - (\gamma + a)\, z - \{(\alpha^2 + \beta^2)/(2\beta)\}\, y = 0.$$ Hence proved.

****Ex. 5. Prove that in general three normals can be drawn from a given point to the paraboloid of revolution $x^2 + y^2 = 2az$ but if the point lies on the surface $27a(x^2 + y^2) + 8(a - z)^3 = 0$, two of the three normals coincide.**

Solution. The equations of the normal at (x_1, y_1, z_1) to the paraboloid

$$x^2 + y^2 = 2az \quad \text{are} \quad \frac{x - x_1}{x_1} = \frac{y - y_1}{y_1} = \frac{z - z_1}{-a}$$

This passes through a given point (α, β, γ) if

$$\frac{\alpha - x_1}{x_1} = \frac{\beta - y_1}{y_1} = \frac{\gamma - z_1}{z_1} = \lambda \text{ (say)}$$

These gives $\alpha - x_1 = \lambda x_1$ or $x_1 = \alpha/(1 + \lambda)$
Similarly $y_1 = \beta/(1 + \lambda)$, $z_1 = \gamma + a\lambda$ ...(i)
Also (x_1, y_1, z_1) lies on the given paraboloid, so

$$x_1^2 + y_1^2 = 2az_1 \quad \text{or} \quad \left[\frac{\alpha}{1+\lambda}\right]^2 + \left[\frac{\beta}{1+\lambda}\right]^2 = 2a(\gamma + a\lambda), \text{ from (i)}$$

or $$\alpha^2 + \beta^2 = 2a(\gamma + a\lambda)(1 + \lambda)^2 \quad \text{...(ii)}$$

This being a cubic in λ gives three values of λ and so from (i) there are three points on the paraboloid normals at which pass through (α, β, γ).

The equation (ii) can be rewritten as

$$f(\lambda) \equiv 2a(1 + \lambda)^2(\gamma + a\lambda) - (\alpha^2 + \beta^2) = 0 \quad \text{...(iii)}$$

The condition that this equation has two equal roots is obtained by eliminating λ between $f(\lambda)=0$ and $f'(\lambda)=0$. **(Remember)**

From (iii) $f'(\lambda)=0$ means $2a(1+\lambda)^2(a)+4a(1+\lambda)(\gamma+a\lambda)=0$

or $\quad a(1+\lambda)+2(\gamma+a\lambda)=0, \because 1+\lambda\neq 0$

or $\quad (a+2\gamma)+\lambda(3a)=0$ or $\lambda=-(a+2\gamma)/(3a)$

Substituting this value of λ in (iii) we get

$$2a\left[1-\frac{a+2\gamma}{3a}\right]^2\left[\gamma-\frac{a(a+2\gamma)}{3a}\right]=\alpha^2+\beta^2$$

or $$2a[2(a-\gamma)]^2[a(\gamma-a)]=27a^3(\alpha^2+\beta^2)$$

or $$27a(\alpha^2+\beta^2)+8(a-\gamma)^3=0$$

$\therefore$ Locus of the point (α,β,γ) is

$$27a(x^2+y^2)+8(a-z)^3=0.$$ Hence proved.

Exercises on § 10.11 to § 10.14

Ex. 1. Find the equations of the normal at (0, 0, 2) on the surface

$$x^2+y^2=2z.$$ **Ans.** $\frac{1}{2}(x-2)=y/0=-(z-2)$

Ex. 2. Prove that from a given point (α,β,γ) five normals can be drawn to the paraboloid $ax^2+by^2=2z$.

Ex. 3. Find the equations of the normals to the paraboloid

$(x^2/a^2)+(y^2/b^2)=2z$ at the point (α,β,γ). **Ans.** $\dfrac{x-\alpha}{(\alpha/a^2)}=\dfrac{y-\beta}{(\beta/b^2)}=\dfrac{z-\gamma}{-1}$

§ 10.15. Dimateral Plane. *i.e. the locus of the mid-points of a system of parallel chords.*

Let the ellipsoid be $\quad ax^2+by^2=2cz$ ...(i)

Consider a line $\quad \dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ ...(ii)

Any point on this line is $(\alpha+lr,\ \beta+mr,\ \gamma+nr)$. Therefore the distances of the intersection of (i) and (ii) from the point (α,β,γ) are the roots of the equation $\quad a(\alpha+lr)^2+b(\beta+mr)^2=2c(\gamma+nr)$.

or $\quad r^2(al^2+bm^2)+2r(al\alpha+bm\beta-cn)+(a\alpha^2+b\beta^2-2c\gamma)=0$ ...(iii)

If (α,β,γ) is the mid-point of the chord, then the two values of r given by (iii) are equal and opposite *i.e.* sum of the two roots of (iii) is zero and as such we get $\quad al\alpha+bm\beta-cn=0.$ ...(iv)

Since the chords are parallel, so l, m, n are fixed and therefore from (iv) the locus of the mid-point (α,β,γ) is the plane.

$$\mathbf{alx+bmy-cn=0,}$$ ...(v)

which is the required equation of the diameteral plane on the locus of the mid-points of a system of parallel chords with direction cosines l, m, n.

The direction ratios of the normal to this plane being $al, bm, 0$ we find that this plane is perpendicular to z-axis whose d.c.'s are 0, 0, 1.

Hence the diametral plane given by (v) of the paraboloid (i) is parallel to the axis of the paraboloid (i).

COR. Any plane parallel to z-axis $lx+my+n=0$, say will be a diametral plane of the paraboloid $ax^2+by^2=2cz$ corresponding to the direction ratios $l/a, m/b, -n/c$. **(Remember)**

§ 10.16. Conjugate Diametral Planes.

Definition. The diametral planes are said to be conjugate if each bisects chords parallel to the other.

Let us take two diametral planes $l_1x+m_1y+n_1=0$ and $l_2x+m_2y+n_2=0$ of the paraboloid
$$ax^2+by^2=2cz. \qquad ...(i)$$

The first of these diametral plane viz. $l_1x+m_1y+n_1=0$, bisects the chords of the paraboloid (i), with direction-ratios $l_1/a, m_1/b, -n_1/c$.

...See COR. § 10.15 above.

If the chords with these direction ratios are parallel to the second diametral plane $l_2x+m_2y+n_2=0$, *i.e.* are perpendicular to the normal of the second plane, then we have

$$\frac{l_1}{a}\cdot l_2+\frac{m_1}{b}\cdot m_2+\left(-\frac{n_1}{c}\right)0=0 \quad \text{or} \quad \boxed{\frac{l_1l_2}{a}+\frac{m_1m_2}{b}=0,} \qquad ...(iii)$$

which is the condition for the plane $l_1x+m_1y+n_1=0$ and $l_2x+m_2y+n_2=0$ to be conjugate diametral planes of the paraboloid $ax^2+by^2=2cz$.

Solved Examples on § 10.15 and 10.16.

Ex. 1. Prove that the equations of the chord through the point (1, 2, 3) which is bisected by the diametral plane $10x-24y=21$ of the paraboloid $5x^2-6y^2=7z$ are $(x-1)=\frac{1}{2}(y-2)=\frac{1}{3}(z-3)$.

Solution. Let l, m, n be the direction cosines of the system of parallel chords, then the equation of the corresponding diametral plane of the paraboloid $ax^2+by^2=2cz$ is $alx+bmy-cn=0$

[See § 10.15 (v) Page 20 Ch. X]

$\therefore$ The equation of the diametral plane of the given paraboloid $5x^2-6y^2=7z$ corresponding to this system of parallel chords is

$$5lx-6my-(7/2)n=0 \quad \text{or} \quad 10lx-12my-7n=0 \qquad ...(i)$$

But the given diametral plane is $10x-24y-21=0$ ...(ii)

Comparing (i) and (ii) we get $\dfrac{10l}{10}=\dfrac{-12m}{-24}=\dfrac{-7n}{-21}$ or $\dfrac{l}{1}=\dfrac{m}{2}=\dfrac{n}{3}$

Hence the direction ratios of the required chord are 1, 2, 3 and sicne it passes through (1, 2, 3) so its equations are

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}. \qquad \text{Hence proved.}$$

Ex. 2. Show that the diametral plane $2x+3y=4$ and $3x-4y=7$ are conjugate for the paraboloid $x^2+2y^2=4z$.

Solution. Here the paraboloid is $x^2+2y^2=4z$. ...(i)

The diametral plane $2x+3y-4=0$ bisects the chords, of the above paraboloid, with direction ratios $(2/1)$, $(3/2)$, $(4/2)$ *i.e.* 2, $(3/2)$, 2.

...See § 10.16 Page 21 of this chapter

Now we know that two diametral planes are said to be conjugate if each bisects chords parallel to the other.

$\therefore$ If the given diametral planes are conjugate then the chords with direction ratios 2, $(3/2)$, 2 are parallel to the second diametal plane $3x-4y=7$ *i.e.* are perpendicular to the normal of the plane $3x-4y=7$ whose direction ratios are 3, -4, 0.

Now $2.3+(3/2)(-4)+2.0=6-6+0=0$

Hence the above is true *i.e.* the given diametral planes are conjugate.

Exercises on § 10.15—§10.16.

Ex. 1. Find the equations to the chord through $(3, 4, 5)$ which is bisected by the diametral plane $3x+4y=1$ of the paraboloid $5x^2+6y^2=2z$.

Ex. 2. Show that the diametral planes $x+3y=3$ and $2x-y=1$ are conjugate diametral planes for the paraboloid $2x^2+3y^2=4z$.

§ 10.17. Enveloping cone of the paraboloid $ax^2+by^2=2cz$ with vertex at the point (α, β, γ).

The equations of a line through (α, β, γ) are $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ ...(i)

Any point on this line is $(\alpha+lr, \beta+mr, \gamma+nr)$ ...(ii)

If the line (i) meets the given paraboloid at a distance r from the point (α, β, γ), then the point given by (ii) must lie on the given paraboloid and so we have

$$a(\alpha+lr)^2+b(\beta+mr)^2=2c(\gamma+nr)$$

or

$$r^2(al^2+bm^2)+2r(al\alpha+bm\beta-cn)+(a\alpha^2+b\beta^2-2c\gamma)=0 \quad ...(iii)$$

If the line (i) is a tangent of the given paraboloid, then the line (i) should meet the paraboloid in two coincident points, the condition for the same is that the roots of (iii) are equal *i.e.* "$B^2=4AC$"

or

$$4(al\alpha+bm\beta-cn)^2=4(al^2+bm^2)(a\alpha^2+b\beta^2-2c\gamma) \quad ...(iv)$$

The locus of the line (i) which is tangent to the given paraboloid is obtained by eliminating l, m, n between (i) and (iv) and is

$$[a\alpha(x-\alpha)+b\beta(y-\beta)-c(z-\gamma)]^2$$
$$=[a(x-\alpha)^2+b(y-\beta)^2](a\alpha^2+b\beta^2-2c\gamma) \quad ...(v)$$

If $S\equiv ax^2+by^2-2cz$, $S_1=a\alpha^2+b\beta^2-2c\gamma$ and

$T\equiv a\alpha x+b\beta y-c(z+\gamma)$, then (v) can be written as

$$(T-S_1)^2=(S+S_1-2T)\,S_1 \qquad \textbf{(Note)}$$

or $$T^2+S_1^2-2TS_1=SS_1+S_1^2-2TS_1 \quad \text{or} \quad \boxed{\mathbf{SS_1=T^2}}$$

or $$(ax^2+by^2-2cz)\,(a\alpha^2+b\beta^2-2c\gamma)=[a\alpha x+b\beta y-c\,(z+\gamma)]^2 \qquad \text{...(v)}$$

is the required equation of the enveloping cone of the given paraboloid.

Cor. *To find the locus of the points from which three mutually perpendicular tangents can be drawn to the paraboloid $ax^2+by^2=2cz$.*

(*Rohilkhand* 95)

Here we are to apply the condition that the enveloping cone, of the given paraboloid, with vertex at (α, β, γ) may have three mutually perpendicular generators and we know that the condition for the same is that the sum of the coefficients of x^2, y^2 and z^2 in the equation of the cone is zero.

(See Chapter on Cone).

$\therefore$ From (vi) above we get

$$[a\,(a\alpha^2+b\beta^2-2c\gamma)-a^2\alpha^2]+[b\,(a\alpha^2+b\beta^2-2c\gamma)-b^2\beta^2]-c^2=0$$

or $$ab\beta^2-2ca\gamma+ba\alpha^2-2cb\gamma-c^2=0$$

or $$ab\,(\alpha^2+\beta^2)-2c\,(a+b)\,\gamma-c^2=0$$

Hence the required locus of the point (α, β, γ) is

$$ab\,(x^2+y^2)-2c\,(a+b)\,z-c^2=0 \qquad \text{...(vi)}$$

****S. Example. Find the locus of the points from which three mutually perpendicular tangents can be drawn to the paraboloid**

$$\mathbf{(x^2/a^2)-(y^2/b^2)=2z} \qquad (\textit{Gorakhpur } 95)$$

Sol. As in Cor of § 10.17 above, we are to apply the condition that the enveloping cone, of the given paraboloid, with vertex at (α, β, γ) may have three mutually perpendicular generators.

Now the equation of the envelopoing cone of the given paraboloid with vertex at the point (α, β, γ) is $\qquad SS_1=T^2 \qquad$ ...(i)

where $$S=(x^2/a^2)-(y^2/b^2)-2z,$$
$$S_1=(\alpha^2/a^2)-(\beta^2/b^2)-2\gamma$$
and $$T=(\alpha x/a^2)-(\beta y/b^2)-(z+\gamma)$$

$\therefore$ From (i), the equation of the enveloping cone of the given paraboloid with vertex at (α, β, γ) is

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-2z\right)\left(\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}-2\gamma\right)=\left(\frac{\alpha x}{a^2}-\frac{\beta y}{b^2}-z-\gamma\right)^2$$

Also we know that if this cone has three mutually perpendicular generators, then sum of coefficients of x^2, y^2 and z^2 in it must be zero.

(See chapter on cone)

i.e. $$\left[\frac{1}{a^2}\left(\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}-2\gamma\right)-\frac{\alpha^2}{a^4}\right]+\left[-\frac{1}{b^2}\left(\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}-2\gamma\right)-\frac{\beta^2}{b^4}\right]+\left[\;-1\;\right]=0$$

or $$-\frac{1}{a^2}\left(\frac{\beta^2}{b^2}+2\gamma\right)-\frac{1}{b^2}\left(\frac{\alpha^2}{a^2}-2\gamma\right)-1=0$$

or $$\alpha^2+\beta^2-2\gamma(a^2-b^2)+a^2b^2=0$$

$\therefore$ Required locus of the point (α, β, γ) is

$$x^2+y^2-2(a^2-b^2)z+a^2b^2=0 \qquad \textbf{Ans.}$$

§ **10.18. Enveloping cylinder** of the paraboloid $ax^2+by^2=2cz$, having its generators parallel to the line $x/l=y/m=z/n$.

Let $P(\alpha, \beta, \gamma)$ be any point on the cylinder. Then the equations of the generator of the cylinder through P drawn parallel to the given line are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r \text{ (say)} \qquad ...(i)$$

Any point on this generator is $(\alpha+lr, \beta+mr, \gamma+nr)$. If this point lies on the given paraboloid we have $a(\alpha+lr)^2+b(\beta+mr)^2=2c(\gamma+nr)$

or $$r^2(al^2+bm^2)+2r(al\alpha+bm\beta-cn)+(a\alpha^2+b\beta^2-2c\gamma)=0 \qquad ...(ii)$$

Since the generator (i) is a tangent line to the given paraboloid, so the two values of r given by (ii) must be equal and the condition for the same is

$$\text{“}B^2=4AC\text{”}$$

i.e. $$4(al\alpha+bm\beta-cn)^2=4(al^2+bm^2)(a\alpha^2+b\beta^2-2c\gamma)$$

or $$(al\alpha+bm\beta-cn)^2=(al^2+bm^2)(a\alpha^2+b\beta^2-2c\gamma)$$

$\therefore$ The equation of the required enveloping cylinder or the locus of $P(\alpha, \beta, \gamma)$ is $$(alx+bmy-cn)^2=(al^2+bm^2)(ax^2+by^2-2cz) \qquad ...(ii)$$

Exercises on Chapter X

Ex. 1. What is the locus represented by the equation

$x^2+y^2=2cz$? **Ans.** Paraboloid, whose axis is z-axis and its section by planes parallel to xy-axis are circles.

Ex. 2. Two perpendicular tangent planes to the paraboloid $x^2/a+y^2/b=2z$ intersect in a line lying on the plane $x=0$. Prove that the line touches the parabola $y^2=(a+b)(2z+a), x=0$.

Ex. 3. Show that the direction ratios of the chords of the paraboloid $2x^2+4y^2=9z$ bisected by the plane $x+y=1$ are 18, 9, 8.

***Ex. 4.** Show that the locus of points from which three mutually perpendicular tangents can be drawn to the paraboloid $ax^2+by^2=2z$ is given by $$ab(x^2+y^2)-2(a+b)z-1=0 \qquad \textit{(Rohilkhand 95)}$$

[**Hint** : See Cor of § 10.17 Page 23 Here $c=1$].

CHAPTER XI

Plane Sections of a Conicoid

§ **11.01. Introduction.** We have already read in chapters IX & X that every plane section of a conicoid is a conic and parallel plane sections are similar and similarly situated conics.

In the present chapter we shall find out the nature of these plane sections (*i.e.* whether these sections are parabola, ellipse or hypebola) and the lengths and direction ratios of its axes.

Here a parabola, an ellipse or a hyperbola is orthogonally projected into a parabola or an ellipse or a hyperbola respectively and thus we can find whether these curves of intersection of a conicoid and a plane is a parabola, an ellipse or a hyperbola by finding the equation of the projection of the section in any one of the coordinate planes. The plane section of the conicoid will be a conic similar to the above projection.

If, however, the plane of the section passes therough the centre of the conicoid, then the section is called **central plane section** otherwise merely a plane section.

§ **11.02. The nature of a plane section of a central conicoid.**

Let the equation of the central conicoid and the plane section be respectively

$$ax^2+by^2+cz^2=1 \qquad \text{...(i)}$$

and

$$lx+my+nz=p \qquad \text{...(ii)}$$

Now we know that the equation of the cylinder passing through the section of (i) and (ii) and having its generators parallel to the x-axis is obtained by eliminating x between (i) and (ii) and so is given by

$$a\,[(p-my-nz)/l\,]^2+by^2+cz^2=1$$

or

$$y^2\,(am^2+bl^2)+2\,amnyz+z^2\,(an^2+cl^2)-2pamy$$

$$-2panz+(ap^2-l^2)=0, \text{ on simplifying}$$

The plane $x=0$, being perpendicular to the generators of this cylinder which are parallel to the x-axis, cuts it in the conic whose equations are $x=0$ and $(am^2+bl^2)\,y^2+2amnyz+(an^2+cl^2)\,z^2-2apmy-2apnz+(ap^2-l^2)=0$

...(iii)

The above conic represents the projection of the given section in the plane $x=0$ and if this conic is parabola, an ellipse or a hyperbola then the given section is also a similar conic.

Now we know that the condition for a conic

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$$

to be a hyperbola or parabola or an ellipse is $H^2-AB>0$ or $=0$ or <0

(Remember)

$\therefore$ Here from (iii) we have

$$\text{"}H^2 - AB\text{"} = (amn)^2 - (am^2 + bl^2)(an^2 + cl^2) \qquad \textbf{(Note)}$$
$$= -[acl^2m^2 + abl^2n^2 + bcl^4]$$
$$= -l^2(bcl^2 + cam^2 + abn^2)$$

$\therefore$ The conic (*i.e.* projection of the given section on the plane $x = 0$) given by (iii) is a hyperbola, a parabola or an ellipse according as

$$-l^2(bcl^2 + cam^2 + abn^2) > \quad \text{or} \quad = \quad \text{or} < 0$$

i.e. $\qquad l^2(bcl^2 + cam^2 + abn^2) < \quad \text{or} \quad = \quad \text{or} > 0$

i.e. $\qquad (bcl^2 + cam^2 + abn^2) < \quad \text{or} \quad = \quad \text{or} > 0$

Hence the plane section of conicoid (i) by the plane (ii) will be

an ellipse if $\mathbf{bcl^2 + cam^2 + abn^2 > 0}$

a parabola if $\mathbf{bcl^2 + cam^2 + abn^2 = 0}$

and **a hyperbola if $\mathbf{bcl^2 + cam^2 + abn^2 < 0}$**

§ 11.03. The nature of a plane section of paraboloid.

Let the equation of the paraboloid be $\quad ax^2 + by^2 = 2cz \qquad$ ...(i)

and the equation of the plane be $\quad lx + my + nz = p \qquad$ ...(ii)

Now the equation of the cylinder passing through the section of (i) and (ii) and having its generators parallel to the x-axis is obtained by eliminating x between (i) and (ii) and so is given by $a[(p - my - nz)/l]^2 + by^2 - 2cz = 0$

or $\quad y^2(am^2 + bl^2) + 2amnyz + ax^2z^2 + 2apmy$

$$-2(apn + cl^2)z + ap^2 = 0, \text{ on simplifying.}$$

The plane $x = 0$ being perpendicular to the generators of the cylinder which are parallel to the x-axis, cuts it in the conic whose equations are $x = 0$ and $(am^2 + bl^2)y^2 - 2amnyz + an^2z^2 - 2apmy - 2(apn + cl^2)z + ap^2 = 0 \quad$ ...(iii)

The above conic represents the projection of the given section on the plane $x = 0$ and if this conic is a parabola, an ellipse or a hyperbola then the given section is also a similar conic and (as in § 11.02 above) the condition for this conic to be a hyperbola or a parabola or an ellipse is

$$\text{'}H^2 - AB\text{'} > 0 \quad \text{or} \quad = 0 \quad \text{or} \quad < 0$$

$\therefore$ From (iii) we have "$H^2 - AB$" $= (-amn)^2 - (am^2 + bl^2)(an^2) = -abl^2n^2$

$\therefore$ The conic (*i.e.* projection of the given section on the plane $x = 0$) given by (iii) is a hyperbola, a parabola or an ellipse according as

$-abl^2n^2 >$ or $=$ or < 0

i.e. $\quad abl^2n^2 <$ or $=$ or > 0 multiplying both sides by -1 and changing the sign of inequalities.

Now $l \neq 0$ since if $l = 0$ the plane is perpendicular to $x = 0$ and the question of projection on the plane $x = 0$ does not arise.

Hence for parabola n = 0 and if n ≠ 0, then the section will be an ellipse if ab > 0 and a hyperbola if ab < 0.

****§ 11.04. Lengths and Direction Ratios of the axes of the central section of a central conicoid.** (*Gorakhpur* 96)

Let the equations of the central conicoid and the plane section be

$$ax^2 + by^2 + cz^2 = 1 \qquad ...(i)$$

and
$$lx + my + nz = 0 \qquad ...(ii)$$

Then the centre of conicoid is also the centre of the section.

Lengths of axes of central section.

Now as the semi-diameters of the length r of the conicoid are also the semi-diameters of the sphere $x^2 + y^2 + z^2 = r^2$, the equation of the cone whose vertex is the origin (*i.e.* centre of the conicoid) and which passes through the above semi-diameters is obtained by making the equation of the conicoid homogeneous with the help of equation of the sphere. **(Note)**

∴ The equation of this cone is $r^2 (ax^2 + by^2 + cz^2) = r^2 = x^2 + y^2 + z^2$

or
$$x^2 (ar^2 - 1) + y^2 (br^2 - 1) + z^2 (cr^2 - 1) = 0 \qquad ...(iii)$$

[Alternatively if L, M, N be the direction cosines of a semi-diameter of length r of the conicoid (i), then the extremity of the diameter is the point (Lr, Mr, Nr).

As this point lies on the conicoid (i), so we have

$$a (Lr)^2 + b (Mr)^2 + c (Nr)^2 = 1 = L^2 + M^2 + N^2 \qquad \textbf{(Note)}$$

or
$$(ar^2 - 1) L^2 + (br^2 - 1) M^2 + (Cr^2 - 1) N^2 = 0$$

∴ The semi-diameters of the conicoid of length r lie on the cone

$$(ar^2 - 1) x^2 + (br^2 - 1) y^2 + (cr^2 - 1) z^2 = 0]$$

Now the semi-diameters of length r of the section are the lines of intersection of the plane (ii) and the cone (iii). These diameters are equally inclined to the axes of the section and coincide when r is equal to either semi-axis of the section. In this case the plane (ii) intersects the cone (iii) in two coincident generators *i.e.* the plane (ii) touches the cone (iii) and the generator of contact is one of the axes.

Hence the condition for the plane (ii) to be a tangent plane to the cone (iii) is
$$\frac{l^2}{(ar^2 - 1)} + \frac{m^2}{(br^2 - 1)} + \frac{n^2}{(cr^2 - 1)} = 0$$

or
$$l^2 (br^2 - 1)(cr^2 - 1) + m^2 (ar^2 - 1)(cr^2 - 1) + n^2 (ar^2 - 1)(br^2 - 1) = 0$$

or
$$\mathbf{r^4 (bcl^2 + cam^2 + abn^2) - r^2 [(b + c) l^2 + (c + a) m^2 + (a + b) n^2] + (l^2 + m^2 + n^2) = 0} \qquad ...(iv)$$

This being a quadratic equation in r^2 gives us two values of r^2 which are the squares of the semi-axes of the section.

Also from (iv) we have
$$r_1^2 \cdot r_2^2 = \frac{l^2 + m^2 + n^2}{bcl^2 + cam^2 + abn^2} \qquad ...(v)$$

Note. If the conicoid (i) be the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$, then replacing a, b, c in (iv) by $1/a^2, 1/b^2, 1/c^2$ respectively, we get the quadratic in r^2 as $\mathbf{r^4\,[a^2l^2+b^2m^2+c^2n^2]-r^2\,[a^2\,(b^2+c^2)\,l^2+b^2\,(c^2+a^2)\,m^2}$
$$\mathbf{+\,c^2\,(a^2+b^2)\,n^2]+a^2b^2c^2\,(l^2+m^2+n^2)=0} \quad \text{...(vi)}$$

Direction ratios of the axes.

If the direction cosines of semi-axes of length r be L, M, N, the plane (ii) touches the cone (iii) along the line $x/L=y/M=z/N$ and the coordinates of an extremity of this diameter are (Lr, Mr, Nr). The plane (ii) is a tangent plane to the cone (iii) at the point (Lr, Mr, Nr) and so the equation of the plane (ii) should be the same as the tangent plane to the cone (iii) at this point (Lr, Mr, Nr)

i.e.
$$Lx\,(ar^2-1)+My\,(br^2-1)+Nz\,(cr^2-1)=0 \quad \text{...(vii)}$$

Comparing (ii) and (vii), we get
$$\frac{L\,(ar^2-1)}{l}=\frac{M\,(br^2-1)}{m}=\frac{N\,(cr^2-1)}{n} \quad \text{...(viii)}$$

Putting the values of r^2 from (iv), we get the direction ratios of the axes of the conic *i.e.* the central plane section of (i).

In a similar manner we can discuss for the central plane section of an ellipsoid and in that case the result (viii) above reduces to
$$\frac{L\,(r^2-a^2)}{a^2l}=\frac{M\,(r^2-b^2)}{b^2m}=\frac{N\,(r^2-c^2)}{c^2n},$$
where the values of r^2 are given by (vi).

****Cor. 1.** *Axes of the section of the conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=0$ lies on the cone*

$(b-c)\,(l/x)+(c-a)\,(m/y)+(a-b)\,(n/z)=0$ (*Avadh* 93; *Rohilkhand* 94)

Proof. If L, M, N be the direction-ratios of the axes of the section, then from (viii) above we have
$$\frac{L\,(ar^2-1)}{l}=\frac{M\,(br^2-1)}{m}=\frac{N\,(cr^2-1)}{n}=k \text{ (say)}$$
$$\therefore \quad \frac{lk}{L}=ar^2-1,\ \frac{mk}{M}=br^2-1,\ \frac{nk}{N}=cr^2-1$$

Multiplying these by $b-c$, $c-a$ and $a-b$ respectively and adding, we get
$$k\left[\frac{l}{L}(b-c)+\frac{m}{M}(c-a)+\frac{n}{N}(a-b)\right]$$
$$=(ar^2-1)\,(b-c)+(br^2-1)\,(c-a)+(cr^2-1)\,(a-b)=0$$

$\therefore$ This axis $x/L=y/M=z/N$ lies on the cone
$$\frac{l}{x}(b-c)+\frac{m}{y}(c-a)+\frac{n}{z}(a-b)=0. \quad \text{...(ix)}$$

****Cor. 2. Area of the central plane section.**

If the central plane section is an ellipse and length of its semi-axes be r_1 and r_2, then its area $= \pi r_1 r_2$

$$= \pi \cdot \frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{(bcl^2 + cam^2 + abn^2)}},$$

by (v) of § 11.04 Page 3 of this chapter ...(x)

Another form of the area of central plane section.

The equation of any plane parallel to the plane $lx + my + nz = 0$ is

$$lx + my + nz = \lambda.$$

If it is a tangent plane to the conicoid $ax^2 + by^2 + cz^2 = 1$, then

$$\lambda^2 = (l^2/a) + (m^2/b) + (n^2/c) \qquad \text{...See chapter IX.}$$

Also if p be the length of perpendicular from origin on the above plane $lx + my + nz = \lambda$, then

$$\therefore p = \frac{\lambda}{\sqrt{(l^2 + m^2 + n^2)}} = \frac{\sqrt{[(l^2/a) + (m^2/b) + (n^2/c)]}}{\sqrt{(l^2 + m^2 + n^2)}}$$

or $$p\sqrt{(abc)} = \frac{\sqrt{(bcl^2 + cam^2 + abn^2)}}{\sqrt{(l^2 + m^2 + n^2)}}$$

$\therefore$ From (x), the area of the section $= \dfrac{\pi}{p\sqrt{(abc)}}$, ...(xi)

where p is the length of the perpendicular from origin on the tangent plane to the conicoid, which is parallel to the given plane $lx + my + nz = 0$.

Note. In case the conicoid be an ellipsoid, then replacing a, b, c in (x) by $1/a^2, 1/b^2, 1/c^2$ respectively, the area of central plane section of the ellipsoid

$$= \pi\, abc \frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{(a^2l^2 + b^2m^2 + c^2n^2)}} \qquad \text{...(xii)}$$

****Cor. 3. Condition for the section tto be a rectangular hyperbola.**

We know that in the case of a rectangular hyperbola, the sum of the squares of the semi-axes is zero *i.e.* $r_1^2 + r_2^2 = 0$ **(Remember)**

Now from (iv) of § 11.04 Page 3 of this chapter, we find that $r_1^2 + r_2^2 = 0$

gives $$(b + c)\, l^2 + (c + a)\, m^2 + (a + b)\, n^2 = 0$$

This condition shows that the normal to the plane $lx + my + nz = 0$ is a generator of the cone $(b + c)\, x^2 + (c + a)\, y^2 + (a + b)\, z^2 = 0$.

Hence the plane touches the reciprocal cone

$$\frac{x^2}{(b + c)} + \frac{y^2}{(c + a)} + \frac{z^2}{(a + b)} = 0$$

Thus we conclude that the section of the conicoid $ax^2+by^2+cz^2=1$ by a tangent plane to the cone $\dfrac{x^2}{b+c}+\dfrac{y^2}{c+a}+\dfrac{z^2}{a+b}=0$ is a rectangular hyperbola.

(*Avadh* 95; *Gorakhpur* 97, 95)

****§ 11.05. Lengths and Direction Ratios of the axes of the section of the paraboloid $ax^2+by^2=2cz$ by the plane $lx+my+nz=p$.**

Here the paraboloid is $\quad ax^2+by^2=2cz \quad$ ...(i)

and the plane is $\quad lx+my+nz=p \quad$ ...(ii)

Let (α, β, γ) be the centre of the section, so that the equation of the plane is given by

$$T=S_1$$

i.e.
$$a\alpha x+b\beta y-c\,(z+\gamma)=a\alpha^2+b\beta^2-2c\,\gamma$$

or
$$a\alpha x+b\beta y-cz=a\alpha^2+b\beta^2-c\,\gamma \quad \text{...(iii)}$$

Comparing (ii) and (iii), we get $\dfrac{a\alpha}{l}=\dfrac{b\beta}{m}=\dfrac{-c}{n}=\dfrac{a\alpha^2+b\beta^2-c\gamma}{p}$ ...(iv)

$\Rightarrow$
$$\alpha=\frac{-cl}{an},\quad \beta=\frac{-cm}{bn} \quad \text{...(v)}$$

and
$$c\gamma=a\alpha^2+b\beta^2+\left(\frac{cp}{n}\right)=a\left(\frac{c^2l^2}{a^2n^2}\right)+b\left(\frac{c^2m^2}{b^2n^2}\right)+\frac{cp}{n}$$

or
$$c\gamma=\frac{c^2}{n^2}\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{np}{c}\right)=\frac{c^2k}{n^2} \quad \text{or} \quad \gamma=\frac{kc}{n^2},$$

where $\quad k=(l^2/a)+(m^2/b)+(np/c)$

$\therefore$ The co-ordinates of the centre are $\left(\dfrac{-cl}{an},\dfrac{-cm}{bn},\dfrac{kc}{n^2}\right)$ ...(vi)

Now if the origin is shifted to this centre, then the equation of the paraboloid becomes $a\left(x-\dfrac{cl}{an}\right)^2+b\left(y-\dfrac{cm}{bn}\right)^2=2c\left(z+\dfrac{kc}{n^2}\right)$ **(Note)**

or
$$ax^2+by^2-\frac{2c}{n}(lx+my+nz)+c^2\left(\frac{l^2}{an^2}+\frac{m^2}{bn^2}-\frac{2k}{n^2}\right)=0$$

or
$$ax^2+by^2-\frac{2c}{n}(lx+my+nz)+\left(a\alpha^2+b\beta^2-\frac{2kc^2}{n^2}\right)=0, \quad \text{...(vii)}$$

substituting values from (v)

Also from (iv) we have $a\alpha^2+b\beta^2-c\gamma=\dfrac{-cp}{n}$

or
$$a\alpha^2+b\beta^2=c\left(\gamma-\frac{p}{n}\right)=\frac{c^2k}{n^2}-\frac{pc}{n};\quad \because \quad c\,\gamma=\frac{c^2k}{n^2}$$

$\therefore$ From (vii), we get

$$ax^2+by^2-\frac{2c}{n}(lx+my+nz)+\left[\frac{c^2k}{n^2}-\frac{pc}{n}-2\frac{kc^2}{n^2}\right]=0$$

or
$$ax^2+by^2-\frac{2c}{n}(lx+my+nz)-\frac{c}{n^2}(kc+np)=0 \qquad \text{...(viii)}$$

And the equation of the plane (ii) reduces to

$$l\left(x-\frac{cl}{an}\right)+m\left(y-\frac{cm}{bn}\right)+n\left(z+\frac{kc}{n^2}\right)=p$$

or
$$lx+my+nz=p+\frac{cl^2}{an}+\frac{cm^2}{bn}-\frac{kc}{n}=p+\frac{c}{n}\left(\frac{l^2}{a}+\frac{m^2}{b}-k\right)$$

$$=p+\frac{c}{n}\left[\frac{l^2}{a}+\frac{m^2}{b}-\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{np}{c}\right)\right],$$

substituting the value of k

$$=p-p=0$$

or
$$lx+my+nz=0 \qquad \text{...(ix)}$$

Now the conic given by (viii) and (ix) is the same as

$$ax^2+by^2=\frac{c}{n^2}(kc+np),\quad lx+my+nz=0 \qquad \text{...(x)}$$

or
$$ax^2+by^2=\frac{p_0^2}{n^2},\ lx+my+nz=0,$$

where
$$p_0^2=c\,(kc+np)=c^2\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{np}{c}\right)+cnp$$

or
$$p_0^2=c\left(\frac{l^2c}{a}+\frac{m^2c}{b}+2np\right) \qquad \text{...(xi)}$$

The equation (x) gives the equation of the conic of intersection of (i) and (ii).

Lengths of the axes.

Let r be the length of a semi-diameter of the paraboloid and if L, M, N be the direction cosines of this semi-diameter, then its equations are

$$\frac{x}{L}=\frac{y}{M}=\frac{z}{N}$$

$\therefore$ The co-ordinates of the extremity of the semi-diameter are (Lr, Mr, Nr) and this point lies on $ax^2+by^2=p_0^2/n^2$

$\therefore$
$$a\,(Lr)^2+b\,(Mr)^2=\frac{p_0^2}{n^2}=\frac{p_0^2\,(L^2+M^2+N^2)}{n^2} \qquad \textbf{(Note)}$$

or
$$L^2\,(ar^2n^2-p_0^2)+M^2\,(br^2n^2-p_0^2)-N^2p_0^2=0 \qquad \text{...(xii)}$$

This shows that the semi-diameters of length r with direction cosines, L, M, N are the generators of the cone

$$x^2(a^2r^2n^2-p_0^2)+y^2(br^2n^2-p_0^2)-p_0^2z^2=0 \qquad \text{...(xiii)}$$

The lines of section of this cone (xiii) by the plane $lx+my+nz=0$ are semi-diameters of length r of the conic.

$\therefore$ If r is the length of either semi-axes of the conic in which the plane cuts the paraboloid (i), then the plane should touch the above cone at the extremity of the semi-diameter.

On applying the condition that the plane $lx+my+nz=0$ may touch (iii), we get

$$\frac{l^2}{ar^2n^2-p_0^2}+\frac{m^2}{br^2n^2-p_0^2}-\frac{n^2}{p_0^2}=0$$

or $\quad l^2(br^2n^2-p_0^2)p_0^2+m^2(ar^2n^2-p_0^2)p_0^2-n^2(ar^2n^2-p_0^2)(br^2n^2-p_0^2)=0$

or $\quad \mathbf{abn^6r^4-n^2p_0^2[(a+b)n^2+am^2+bl^2]r^2+p_0^4(l^2+m^2+n^2)=0} \qquad \text{...(xiv)}$

This is a quadratic in r^2 and gives us two values of r^2 which correspond to the squares of the semi-axes of the conic.

Also if r_1^2 and r_2^2 be the roots of (xiv), then

$$r_1^2r_2^2=p_0^4(l^2+m^2+n^2)/(abn^6) \qquad \text{...(xv)}$$

Direction Ratios of the Axes.

If the direction cosines of semi-diameter of length r be L, M, N, then the plane $lx+my+nz=0$ is a tangent plane to the cone given by (xiii) at the extremity of a semi-diameter whose coordinates are (Lr, Mr, Nr) and it should be the same as the equation of the tangent plane to (xiii) at this point

i.e. $\quad Lx(ar^2n^2-p_0^2)+My(br^2n^2-p_0^2)-Np_0^2z=0 \qquad \text{...(xvi)}$

Comparing $lx+my+nz=0$ and the plane (xvi), we get

$$\frac{L(ar^2n^2-p_0^2)}{l}=\frac{M(br^2n^2-p_0^2)}{m}=\frac{-Np_0^2}{n} \qquad \text{...(xvii)}$$

Substituting the two values of r^2 from (xiv) in it, we can obtain the direction-ratios of the axes of the conic in which the plane (ii) intersects the paraboloid (i).

Cor. 1. Area of the section.

If the plane section is an ellipse and the length of its semi-axes be r_1 and r_2, then its area

$$=\pi r_1r_2=\frac{\pi p_0^2\sqrt{(l^2+m^2+n^2)}}{n^3\sqrt{(ab)}}, \text{ from (xv)}$$

$$=\frac{\pi c}{n^3}\left(\frac{l^2c}{a}+\frac{m^2c}{b}+2np\right)\sqrt{\left(\frac{l^2+m^2+n^2}{ab}\right)}, \qquad \text{...(xviii)}$$

putting the value of p^2 from (xi) on Page 7 of this chapter.

Cor. 2. Condition for the section to be a rectangular hyperbola.

We know in the case of a rectangular hyperbola, the sum of the squares of the semi-axes is zero *i.e.* $r_1^2 + r_2^2 = 0$

i.e. $$n^2 p_0^2 [(a+b) n^2 + am^2 + bl^2] = 0, \text{ from (xiv) above.}$$

i.e. $$(a+b) n^2 + am^2 + bl^2 = 0 \qquad \text{...(xix)}$$

Cor. 3. Angle between the asymptotes.

If θ be the angle between the asymptotes, then we have

$$\tan^2 \theta = \frac{-4r_1^2 r_2^2}{(r_1^2 + r_2^2)^2} \qquad \textbf{(Remember)}$$

$$= \frac{-4p_0^4 (l^2 + m^2 + n^2)/(abn^6)}{\{n^2 p_0^2 [(a+b) n^2 + am^2 + bl^2]/(abn^6)\}^2}$$

$$= -4abn^2 (l^2 + m^2 + n^2)/[(a+b) n^2 + am^2 + bl^2] \qquad \text{...(xx)}$$

This result being free from p shows that angle between the asymptotes of parallel plane sections of the paraboloid are the same. **(Note)**

Solved Examples on § 11.01—§11.05.

Plane Section of a central conicoid.

***Ex. 1. Find the lengths and equations of the axes of the conic**

$$\mathbf{x^2 + y^2 + (z^2/4) = 1,\ 2x + 2y + z = 0} \qquad \textit{(Rohilkhand 92)}$$

Sol. Comparing $x^2 + y^2 + (z^2/4) = 1$ with $ax^2 + by^2 + cz^2 = 1$ we get $a = 1, b = 1$ and $c = 1/4$ and $2x + 2y + z = 0$ with $lx + my + nz = 0$ we get $l = 2$, $m = 2$ and $n = 1$.

Also we know [See § 11.04 Page 3 of this chapter] that the axes of the section are given by $$\frac{l^2}{(ar^2 - 1)} + \frac{m^2}{(br^2 - 1)} + \frac{n^2}{(cr^2 - 1)} = 0$$

which here reduces to $$\frac{4}{(r^2 - 1)} + \frac{4}{(r^2 - 1)} + \frac{1}{(\frac{1}{4}r^2 - 1)} = 0,$$

on substituting values of l, m, n, a, b and c.

or $$\frac{1}{r^2 - 1} + \frac{1}{r^2 - 1} + \frac{1}{r^2 - 4} = 0$$

or $$(r^2 - 1)(r^2 - 4) + (r^2 - 1)(r^2 - 4) + (r^2 - 1)(r^2 - 1) = 0 \qquad \textbf{(Note)}$$

or $$(r^2 - 1)[2(r^2 - 4) + (r^2 - 1)] = 0 \quad \text{or} \quad 3(r^2 - 1)(r^2 - 3) = 0$$

or $$r^2 = 1, 3 \text{ which gives } r_1 = 1,\ r_2 = \sqrt{3}.$$

$\therefore$ Required lengths of the axes are $2r_1, 2r_2$ *i.e.* $2, 2\sqrt{3}$.

Also the direction ratios L, M, N of the axes are given by the equations

$$\frac{L(ar^2 - 1)}{l} = \frac{M(br^2 - 1)}{m} = \frac{N(cr^2 - 1)}{n}$$

...See § 11.04 (viii) Page 4 of this chapter

or $$\frac{L(r^2-1)}{2}=\frac{M(r^2-1)}{2}=\frac{N(\frac{1}{4}r^2-1)}{1}$$

or $$\frac{L}{r^2-4}=\frac{M}{r^2-4}=\frac{N}{2(r^2-1)}$$

$\therefore$ For $r^2=1$, we have $\dfrac{L}{1-4}=\dfrac{M}{1-4}=\dfrac{N}{2(1-1)}$ *i.e.* $\dfrac{L}{1}=\dfrac{M}{1}=\dfrac{N}{0}$

And for $r^2=3$, we have $\dfrac{L}{3-4}=\dfrac{M}{3-4}=\dfrac{N}{2(3-1)}$ or $\dfrac{L}{1}=\dfrac{M}{1}=\dfrac{N}{-4}$

$\therefore$ The equations of the axes [which evidently pass through the origin (0, 0, 0)] are $\dfrac{x}{1}=\dfrac{y}{1}=\dfrac{z}{0}$ and $\dfrac{x}{1}=\dfrac{y}{1}=\dfrac{z}{-4}$ **Ans.**

Ex. 2. Prove that the section of the conicoid $ax^2+by^2+cz^2=1$ by the plane whose normal lies on the cone $(x^2/a)+(y^2/b)+(z^2/c)=0$ is a parabola.

Sol. Let the plane be $lx+my+nz=0$.

As its normal lies on the given cone, so

$(l^2/a)+(m^2/b)+(n^2/c)=0$ or $bcl^2+cam^2+abn^2=0$

But it is the condition that the section is a parabola, as shown in § 11.02 Page 2 of this chapter.

***Ex. 3. Prove that the central section of an ellipsoid whose area is constant touches a cone of second degree.**

Sol. Let the area of the central section of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ be the constant $\pi abc/\lambda$.

Then $$\frac{\pi abc\sqrt{(l^2+m^2+n^2)}}{\sqrt{(a^2l^2+b^2m^2+c^2n^2)}}=\frac{\pi abc}{\lambda}$$ ...See § 11.04 (xii) Page 5

or $$(a^2l^2+b^2m^2+c^2n^2)=\lambda^2(l^2+m^2+n^2)$$

or $$l^2(a^2-\lambda^2)+m^2(b^2-\lambda^2)+n^2(c^2-\lambda^2)=0$$

This relation shows that the normal to the plane $lx+my+nz=0$ is a generator of the cone $(a^2-\lambda^2)x^2+(b^2-\lambda^2)y^2+(c^2-\lambda^2)z^2=0$

$\therefore$ The plane $lx+my+nz=0$ touches the reciprocal cone

$$\frac{x^2}{a^2-\lambda^2}+\frac{y^2}{b^2-\lambda^2}+\frac{z^2}{c^2-\lambda^2}=0.$$ Hence proved.

****Ex. 4. Prove that the axes of the section of the conicoid $ax^2+by^2+cz^2=1$ which pass through the line $x/l=y/m=z/n$ lie on the cone $\dfrac{b-c}{x}(mz-ny)+\dfrac{c-a}{y}(nz-lz)+\dfrac{a-b}{z}(ly-mx)=0.$** (*Garhwal* 92)

Sol. Let the equations to the axes of the section through the line

$$x/l=y/m=z/n \quad \text{...(i)}$$

be $x/l_1 = y/m_1 = z/n_1$ ...(ii) and $x/l_2 = y/m_2 = z/n_2$ ...(iii)
then as these axes are mutually perpendicular so we have

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \quad \text{...(iv)}$$

Also if the two lines (ii) and (iii) be the axes of the section, then each of them will bisect chords which are parallel to the other. So by the definition of diametral plane, each of the two lines will belong to the diametral plane which is conjugate to the other.

Now the equation of the diametral plane conjugate to (ii) is

$$a\, l_1 x + b\, m_1 y + c\, n_1 z = 0$$

and if it contains the line (iii), then we have $a l_1 l_2 + b m_1 m_2 + c n_1 n_2 = 0$ ...(v)

Now as the lines (i), (ii) and (iii) are coplanar, so we have

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} l_1 & m_1 & n_1 \\ l & m & n \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or $$l_1 (m n_2 - m_2 n) + m_1 (n l_2 - l n_2) + n_1 (l m_2 - l_2 m) = 0 \quad \text{...(vi)}$$

Now for obtaining the required locus, we are to eliminate l_1, m_1, n_1 and l_2, m_2, n_2.

Eliminating l_1, m_1, n_1 from (iv), (v) and (vi), we get

$$\begin{vmatrix} l_2 & m_2 & n_2 \\ a l_2 & b m_2 & c n_2 \\ m n_2 - m_2 n & n l_2 - l n_2 & l m_2 - l_2 m \end{vmatrix} = 0 \quad \text{...(viii)}$$

Again eliminating l_2, m_2, n_2 between (vii) and (ii), we have

$$\begin{vmatrix} x & y & z \\ ax & by & cz \\ mz - ny & nx - lz & ly - mx \end{vmatrix} = 0 \quad \textbf{(Note)}$$

or $$x\,[by\,(ly - mx) - cz\,(nx - lz)] + y\,[cz\,(mz - ny) - ax\,(ly - mx)] + z\,[ax\,(nx - lz) - by\,(mz - ny)] = 0$$

or $$yz\,(mz - ny)\,(c - b) + zx\,(nx - lz)\,(a - c) + xy\,(ly - mx)\,(b - a) = 0$$

Dividing each term by $-xyz$, we get the required locus as

$$\frac{(b - c)}{x}(mz - ny) + \frac{(c - a)}{y}(nx - lz) + \frac{a - b}{z}(ly - mx) = 0 \qquad \text{Hence proved.}$$

Ex. 5. Prove that if l_1, m_1, n_1 and l_2, m_2, n_2 are the direction ratios of the principal axes of any plane section of the quadric $ax^2 + by^2 + cz^2 = 1$, then $\dfrac{l_1 l_2}{b - c} = \dfrac{m_1 m_2}{c - a} = \dfrac{n_1 n_2}{a - b}$.

Sol. As l_1, m_1, n_1 and l_2, m_2, n_2 are the direction ratios of the principal axes of any plane section of the conicoid $ax^2 + by^2 + cz^2 = 1$, then as in Ex. 4. above we can prove that $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

and $a l_1 l_2 + b m_1 m_2 + c n_1 n_2 = 0$ (To be proved in Exam.)

Solving these simutaneously for l_1l_2, m_1m_2 and n_1n_2, we have

$$\frac{l_1l_2}{c-b}=\frac{m_1m_2}{a-c}=\frac{n_1n_2}{b-a} \quad \text{or} \quad \frac{l_1l_2}{b-c}=\frac{m_1m_2}{c-a}=\frac{n_1n_2}{a-b}$$ Hence proved.

***Ex. 6. One axis of a central section of the conicoid $ax^2+by^2+cz^2=1$ lies in the plane $px+qy+rz=0$. Show that the other lies on the cone $(b-c)\,pyz+(c-a)\,qzx+(a-b)\,rxy=0$.**

Sol. If we take $x/l_1=y/m_1=z/n_1$ and $x/l_2=y/m_2=z/n_2$ as the equations of the two axes of the central section of the given conicoid, then as in Ex. 4 above we can have $l_1l_2+m_1m_2+n_1n_2=0$ ...(i)

and $al_1l_2+bm_1m_2+cn_1n_2=0$ (To be proved in the exam.)

Also if the axis given by $x/l_1=y/m_1=z/n_1$ lies on the given plane $px+qy+rz=0$ then $pl_1+qm_1+rn_1=0$ ...(iii)

Eliminating l_1, m_1, n_1 between (i), (ii) and (iii) we get

$$\begin{vmatrix} l_2 & m_2 & n_2 \\ al_2 & bm_2 & cn_2 \\ p & q & r \end{vmatrix}=0$$

or $$p\,(c-b)\,m_2n_2+q\,(a-c)\,n_2l_2+r\,(b-a)\,l_2m_2=0$$

$\therefore$ The locus of the other axis $x/l_2=y/m_2=z/n_2$ is

$$p\,(c-b)\,yz+q\,(a-c)\,zx+r\,(b-a)\,xy=0$$

or $$pyz\,(b-c)+qzx\,(c-a)+rxy\,(a-b)=0.$$ Hence proved.

***Ex. 7. Find the equation of the central plane section of the conicoid $ax^2+by^2+cz^2=1$, which has one of its axes along the line**

$$x/l_1=y/m_1=z/n_1.$$

Sol. Let the other axis of the section by the plane $lx+my+nz=0$ ...(i) be given by $x/l_2=y/m_2=z/n_2$, then as in Ex. 4 Page 10 of this chapter we can have (To be proved in the exam.)

$$l_1l_2+m_1m_2+n_1n_2=0 \quad \text{and} \quad al_1l_2+bm_1m_2+cn_1n_2=0$$

Solving these simutaneously for l_2, m_2, n_2, we get

$$\frac{l_2}{m_1n_1\,(c-b)}=\frac{m_2}{n_1l_1\,(a-c)}=\frac{n_2}{l_1m_1\,(b-a)}=\lambda_1 \text{ (say)} \quad \text{...(ii)}$$

Again if l, m, n be the direction-ratios of the normal to the plane through the two axes, then $ll_1+mm_1+nn_1=0$ and $ll_2+mm_2+nn_2=0$

Solving these simutaneously for l, m, n we get

$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}=\lambda_2 \text{ (say)}$$

From these, we get $l=\lambda_2\,(m_1n_2-m_2n_1)$

or $$l=\lambda_2\,[m_1\,\lambda_1\,l_1\,m_1\,(b-a)-\lambda_1\,n_1l_1\,(a-c)\,.\,n_1] \text{ from (ii)}$$

$$=-\lambda_1\lambda_2l_1\,[m_1^2\,(a-b)+n_1^2\,(a-c)],$$

$$m=\lambda_2\,(n_1l_2-n_2l_1)$$

$$= -\lambda_1\lambda_2 m_1\,[n_1^2\,(b-c) + l_1^2\,(b-a)], \text{ as before}$$

and $$n = -\lambda_1\lambda_2 n_1\,[l_1^2\,(c-a) + m_1^2\,(c-b)]$$

Substituting these values in (i), the required equation of the plane is

$$\Sigma\,[-\lambda_1\lambda_2\,l_1\,\{m_1^2\,(a-b) + n_1^2\,(a-c)\}]\;x = 0$$

or $$\Sigma\,\{m_1^2\,(a-b) + n_1^2\,(a-c)\}\;l_1 x = 0.$$ **Ans.**

***Ex. 8. If A_1, A_2, A_3 are the areas of three mutually perpendicular central sections of an ellipsoid, show that $A_1^{-2} + A_2^{-2} + A_3^{-2}$ is constant.**

Sol. Let l_1, m_1, n_1 ; l_2, m_2, n_2 and l_3, m_3, n_3 be the direction cosines of the normals of the three mutually perpendicular central sections of the ellipsoid

$$(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1.$$

Then if A_1 be the area of the section of the above ellipsoid by the plane $l_1 x + m_1 y + n_1 z = 0$, then

$$A_1 = \frac{\pi\, abc\,\sqrt{(l_1^2 + m_1^2 + n_1^2)}}{\sqrt{(a^2 l_1^2 + b^2 m_1^2 + c^2 n_1^2)}}$$ ...See result (xii) of § 11.04 Page 5

or $$A_1 = \pi\, abc\,(1)/\sqrt{(a^2 l_1^2 + b^2 m_1^2 + c^2 n_1^2)}$$

or $$A_1^{-2} = 1/A_1^2 = (a^2 l_1^2 + b^2 m_1^2 + c^2 n_1^2)/(\pi^2\, a^2 b^2 c^2)$$

Similarly $$A_2^{-2} = (a^2 l_2^2 + b^2 m_2^2 + c^2 n_2^2)/(\pi^2\, a^2 b^2 c^2)$$

and $$A_3^{-2} = (a^2 l_3^2 + b^2 m_3^2 + c^2 n_3^2)/(\pi^2\, a^2 b^2 c^2)$$

$$\therefore\; A_1^{-2} + A_2^{-2} + A_3^{-2} = \frac{1}{\pi^2\, a^2 b^2 c^2}\,[a^2\,\Sigma\, l_1^2 + b^2\,\Sigma\, m_1^2 + c^2\,\Sigma\, n_1^2]$$

$$= \frac{a^2 + b^2 + c^2}{\pi^2\, a^2 b^2 c^2}, \quad \because\quad \Sigma\, l_1^2 = 1 = \Sigma\, m_1^2 = \Sigma\, n_1^2,$$

as the planes are mutually perpendicular.

i.e. $$A_1^{-2} + A_2^{-2} + A_3^{-2} = \text{constant.}$$ Hence proved.

Ex. 9. If a length PQ be taken on the normal at any point P of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$ equal in length to $k^2 A/(\pi\, abc)$, where k is a constant and A is the area of the section of the ellipsoid by the diametral plane of OP, show that the locus of Q is

$$\frac{a^2 x^2}{(a^2 + k^2)^2} + \frac{b^2 y^2}{(b^2 + k^2)^2} + \frac{c^2 z^2}{(c^2 + k^2)^2} = 1.$$

Sol. Let P be the point (α, β, γ), then as it lies on the given ellipsoid, so

$$(\alpha^2/a^2) + (\beta^2/b^2) + (\gamma^2/c^2) = 1 \qquad \text{...(i)}$$

Also the equation of the diametral plane OP is

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = 0 \quad \text{or} \quad Lx + My + Nz = 0, \text{ say} \qquad \text{...(ii)}$$

where $$L = \alpha/a^2,\; M = \beta/b^2,\; N = \gamma/c^2.$$

If A be the area of the section of the given ellipsoid by the plane (ii),

then $A = \dfrac{\pi\, abc \sqrt{(L^2 + M^2 + N^2)}}{\sqrt{(a^2L^2 + b^2M^2 + c^2N^2)}}$, See § 11.04 (xii) Page 5 Ch. XI

$= \pi\, abc/p$ ...(iii)

If $PQ = r$, then we are given that $r = \dfrac{k^2 A}{\pi\, abc} = \dfrac{k^2}{p}$, from (iii)

Also the equations to the normals at (α, β, γ) to the ellipsoid are

$$\frac{x-\alpha}{(p\alpha/a^2)} = \frac{y-\beta}{(p\beta/b^2)} = \frac{z-\gamma}{(p\gamma/c^2)} = r = \frac{k^2}{p} \text{ say, for } Q$$

If (x_1, y_1, z_1) be the co-ordinates of the point Q, which is at a distance $r = k^2/p$ form the point $P(\alpha, \beta, \gamma)$, then

$$x_1 = \alpha + \frac{k^2}{p} \cdot \frac{p\alpha}{a^2},\ y_1 = \beta + \frac{k^2}{p} \cdot \frac{p\beta}{b^2},\ z^2 = \gamma + \frac{k^2}{p} \cdot \frac{p\gamma}{c^2}$$

or $$x_1 = \alpha\left(1 + \frac{k^2}{a^2}\right),\ y_1 = \beta\left(1 + \frac{k^2}{b^2}\right),\ z_1 = \gamma\left(1 + \frac{k^2}{\lambda^2}\right)$$

or $$\frac{ax_1}{a^2 + k^2} = \frac{\alpha}{a},\ \frac{by_1}{b^2 + k^2} = \frac{\beta}{b},\ \frac{cz_1}{c^2 + k^2} = \frac{\gamma}{c}$$

Squaring and adding these, we get

$$\frac{a^2x_1^2}{(a^2 + k^2)^2} + \frac{b^2y_1^2}{(b^2 + k^2)^2} + \frac{c^2z_1^2}{(c^2 + k^2)^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = 1, \text{ from (i)}$$

$\therefore$ The required locus of $Q(x_1, y_1, z_1)$ is $\Sigma \dfrac{a^2x^2}{(a^2 + k^2)^2} = 1$. Hence proved.

***Ex. 10. If A_1, A_2, A_3 are the areas of the sections of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$ by the diametral planes of three mutually perpendicular semi-diameters of lengths r_1, r_2 and r_3, then show that**

$$\frac{A_1^2}{r_1^2} + \frac{A_2^2}{r_2^2} + \frac{A_3^2}{r_3^2} = \pi^2\left(\frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2} + \frac{a^2b^2}{c^2}\right)$$

Sol. Let us consider a semi-diameter of lengths r_1, whose equations are

$$x/l_1 = y/m_1 = z/n_1 = r_1 \quad ...(i)$$

Then the co-ordinates of its extremity are (l_1r_1, m_1r_1, n_1r_1) and it will lie on the given conicoid, provided $r_1^2\left(\dfrac{l_1^2}{a^2} + \dfrac{m_1^2}{b^2} + \dfrac{n_1^2}{c^2}\right) = 1$ ...(ii)

Also the equation of the diametral plane of (i) is

$$\frac{l_1x}{a^2} + \frac{m_1y}{b^2} + \frac{n_1z}{c^2} = 0 \quad ...(iii)$$

Now if A_1 be the area of the section of the given ellipsoid by the plane (iii),

then $$A_1 = \pi\, abc \frac{\sqrt{\left(\dfrac{l_1^2}{a^4}+\dfrac{m_1^2}{b^4}+\dfrac{n_1^2}{c^4}\right)}}{\sqrt{\left(\dfrac{l_1^2}{a^2}+\dfrac{m_1^2}{b^2}+\dfrac{n_1^2}{c^2}\right)}}$$ ...See § 11.04 (xii) Page 5

or $$\frac{A_1^2}{r_1^2} = \pi^2 a^2b^2c^2 \left(\frac{l_1^2}{a^4}+\frac{m_1^2}{b^4}+\frac{n_1^2}{c^4}\right), \text{ using (ii)}$$

Similarly we can evaluate A_2^2/r_2^2 and A_3^2/r_3^2.
Adding these we get

$$\Sigma \frac{A_1^2}{r_1^2} = \pi^2 a^2b^2c^2 \left[\frac{1}{a^4}\Sigma l_1^2 + \frac{1}{b^4}\Sigma m_1^2 + \frac{1}{c^4}\Sigma n_1^2\right]$$

$$= \pi^2 a^2b^2c^2 \left[(1/a^4)(1) + (1/b^4)(1) + (1/c^4)(1)\right]$$

$\because$ $\Sigma l_1^2 = 1 = \Sigma m_1^2 = \Sigma n_1^2$ as l_1, m_1, n_1 ; l_2, m_2, n_2 and l_3, m_3, n_3 are the d.cosines of three mutually perpendicular lines.

$$\therefore \quad \Sigma \frac{A_1^2}{r_1^2} = \pi^2 \left[\frac{b^2c^2}{a^2} + \frac{c^2a^2}{b^2} + \frac{a^2b^2}{c^2}\right]$$ Hence proved.

***Ex. 11. If A_1, A_2, A_3 ; B_1, B_2, B_3 are the areas of the sections of the ellipsoids $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$, $(x^2/\alpha^2)+(y^2/\beta^2)+(z^2/\gamma^2)=1$ by three conjugate diametral planes of the former, then**

$$\frac{A_1^2}{B_1^2}+\frac{A_2^2}{B_2^2}+\frac{A_3^2}{B_3^2} = \frac{a^2b^2c^2}{\alpha^2\beta^2\gamma^2}\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)$$

Sol. The equations of conjugate diametral planes of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ are

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0, \text{ etc.} \quad \text{or} \quad L_1x+M_1y+N_1z=0, \text{ etc.}$$

where $$L_1 = x_1/a^2,\ M_1 = y_1/b^2,\ N_1 = z_1/c^2$$

$$\therefore \frac{A_1^2}{B_1^2} = \frac{\pi^2 a^2b^2c^2 (L_1^2+M_1^2+N_1^2)}{(a^2L_1^2+b^2M_1^2+c^2N_1^2)} \times \frac{(\alpha^2 L_1^2+\beta^2 M_1^2+\gamma^2 N_1^2)}{\pi^2 \alpha^2\beta^2\gamma^2 (L_1^2+M_1^2+N_1^2)}$$

...See § 11.04 (xii) Page 5 of this chapter.

$$= \frac{a^2b^2c^2}{\alpha^2\beta^2\gamma^2}\left[\frac{\alpha^2(x_1/a^2)^2+\beta^2(y_1/b^2)^2+\gamma^2(z_1/c^2)^2}{a^2(x_1/a^2)^2+b^2(y_1/b^2)^2+c^2(z_1/c^2)^2}\right], \text{ from (i)}$$

$$=\frac{a^2b^2c^2}{\alpha^2\beta^2\gamma^2}\left[\frac{(x_1^2\alpha^2/a^4)+(y_1^2\beta^2/b^4)+(z_1^2\gamma^2/c^4)}{(x_1^2/a^2)+(y_1^2/b^2)+(z_1^2/c^2)}\right]$$

$$=\frac{a^2b^2c^2}{\alpha^2\beta^2\gamma^2}\left[\frac{\alpha^2}{a^4}x_1^2+\frac{\beta^2}{b^4}y_1^2+\frac{\gamma^2}{c^4}z_1^2\right], \quad \because \quad \Sigma\frac{x_1^2}{a^2}=1$$

as (x_1, y_1, z_1) lies on $\Sigma(x^2/a^2)=1$

or
$$\frac{A_1^2}{B_1^2}=\frac{a^2b^2c^2}{\alpha^2\beta^2\gamma^2}\left[\frac{\alpha^2}{a^4}x_1^2+\frac{\beta^2}{b^4}y_1^2+\frac{\gamma^2}{c^4}z_1^2\right]$$

Similarly we can calculate A_2^2/B_2^2 and A_3^2/B_3^2.

Adding we get $\Sigma\frac{A_1^2}{B_1^2}=\frac{a^2b^2c^2}{\alpha^2\beta^2\gamma^2}\left[\frac{\alpha^2}{a^4}\Sigma x_1^2+\frac{\beta^2}{b^4}\Sigma y_1^2+\frac{\gamma^2}{c^4}\Sigma z_1^2\right]$ ...(ii)

Also from relations between the co-ordinates of the extremities of set of three semi-conjugate diameters of an ellipsoid (See Ch. IX) we have

$$\Sigma x_1^2=x_1^2+x_2^2+x_3^2=a^2,\ \Sigma y_1^2=y_1^2+y_2^2+y_3^2=b^2,\ \Sigma z_1^2=z_1^2+z_2^2+z_3^2=c^2.$$

$\therefore$ From (ii) we have $\Sigma\frac{A_1^2}{B_1^2}=\frac{a^2b^2c^2}{\alpha^2\beta^2\gamma^2}\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right].$

****Ex. 12. If through the centre of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ a perpendicular is drawn to any central section and lengths equal to the axes of section are marked off along the perpendicular, the locus of their extremities is given by $\frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}+\frac{c^2z^2}{r^2-c^2}=0$,**

where $\quad r^2=x^2+y^2+z^2.$

Sol. Let the central section be $\quad lx+my+nz=0,$ ...(i)

where l, m, n are the actual direction cosines of the normal to (i) and the lengths of the axes of the central section are given by

$$r^4(a^2l^2+b^2m^2+c^2n^2)-r^2[a^2(b^2+c^2)l^2+b^2(c^2+a^2)m^2+c^2(a^2+b^2)n^2]+a^2b^2c^2=0, \quad ...(ii)$$

as $l^2+m^2+n^2=1$ and See § 11.04 (vi) Page 3 of this ch.

Equations of perpendicular from origin to (i) are

$$x/l=y/m=z/n=r_1 \quad \text{or} \quad r_2 \quad ...(iii)\ \textbf{(Note)}$$

$\therefore$ Co-ordinates of the extremities are $(lr_1, mr_1, nr_1), (lr_2, mr_2, nr_2)$

$\therefore$ If the co-ordinates of the extremity be (x, y, z), then

$$x^2+y^2+z^2=(l^2+m^2+n^2)r_1^2 \quad \text{or} \quad (l^2+m^2+n^2)r_2^2$$

or
$$x^2+y^2+z^2=r^2, \quad ...(iv)$$

where r stands for r_1 or r_2 and $l^2+m^2+n^2=1$

Multiply (ii) by r^2 and rewrite it as

$$r^4 (a^2l^2r^2 + b^2m^2r^2 + c^2n^2r^2) - r^2 [a^2 (b^2 + c^2) l^2r^2 + b^2 (c^2 + a^2) m^2r^2 + c^2 (a^2 + b^2) n^2r^2] + a^2b^2c^2r^2 = 0$$

or $$r^4 (a^2x^2 + b^2y^2 + c^2z^2) - r^2 [a^2 (b^2 + c^2) x^2 + b^2 (c^2 + a^2) y^2 + c^2 (a^2 + b^2) z^2] + a^2b^2c^2 (x^2 + y^2 + z^2) = 0,$$

with the help of (iii) and (iv).

or $$a^2x^2 [r^4 - (b^2 + c^2) r^2 + b^2c^2] + b^2y^2 [\quad] + c^2z^2 [\quad] = 0$$

or $$a^2x^2 (r^2 - b^2)(r^2 - c^2) + b^2y^2 (r^2 - c^2)(r^2 - a^2) + c^2z^2 (r^2 - a^2)(r^2 - b^2) = 0$$

or $$\frac{a^2x^2}{r^2 - a^2} + \frac{b^2y^2}{r^2 - b^2} + \frac{c^2z^2}{r^2 - c^2} = 0, \text{ where } r^2 = x^2 + y^2 + z^2.$$ Proved.

****Ex. 13. Prove that the normals to the central sections of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$ which are of given eccentricity e lie on the cone $a^2b^2c^2 (e^2 - 2)^2 (x^2 + y^2 + z^2)(a^2x^2 + b^2y^2 + c^2z^2)$**

$$= (1 - e^2) [a^2 (b^2 + c^2) x^2 + b^2 (c^2 + a^2) y^2 + c^2 (a^2 + b^2) z^2]^2$$

Sol. If r_1, r_2 be the lengths of the semi-axes of the central section of the given ellipsoid by the plane $lx + my + nz = 0,$ ...(i)

then r_1^2 and r_2^2 are the roots of

$$r^4 (a^2l^2 + b^2m^2 + c^2n^2) - r^2 [a^2 (b^2 + c^2) l^2 + b^2 (c^2 + a^2) m^2 + c^2 (a^2 + b^2) n^2] + a^2b^2c^2 (l^2 + m^2 + n^2) = 0,$$

...See § 11.04 (vi) Page 4 of this chapter

$$\therefore r_1^2 + r_2^2 = \frac{a^2 (b^2 + c^2) l^2 + b^2 (c^2 + a^2) m^2 + c^2 (a^2 + b^2) n^2}{a^2l^2 + b^2m^2 + c^2n^2} \quad ...(ii)$$

and $$r_1^2 r_2^2 = \frac{a^2b^2c^2 (l^2 + m^2 + n^2)}{a^2l^2 + b^2m^2 + c^2n^2} \quad ...(iii)$$

Also from "$b^2 = a^2 (1 - e^2)$" we have $r_2^2 = r_1^2 (1 - e^2)$

or $$\frac{r_2^2}{r_1^2} = \frac{1 - e^2}{1} \quad \text{or} \quad \frac{r_2^2 + r_1^2}{r_2^2 - r_1^2} = \frac{(1 - e^2) + 1}{(1 - e^2) - 1} = \frac{2 - e^2}{-e^2}$$

or $$e^4 (r_2^2 + r_1^2)^2 = (2 - e^2)^2 (r_2^2 - r_1^2)^2 = (2 - e^2)^2 [(r_2^2 + r_1^2)^2 - 4r_1^2r_2^2]$$

or $$(r_2^2 + r_1^2)^2 [(2 - e^2)^2 - e^4] = 4r_1^2r_2^2 (2 - e^2)^2$$

or $$(r_2^2 + r_1^2)^2 (4 - 4e^2) = 4r_1^2r_2^2 (2 - e^2)^2$$

or $$\left[\frac{\Sigma a^2 (b^2 + c^2) l^2}{\Sigma a^2l^2}\right]^2 (1 - e^2) = \frac{a^2b^2c^2 (l^2 + m^2 + n^2)}{\Sigma a^2l^2} (2 - e^2)^2,$$

from (ii) and (iii)

$\therefore$ The locus of the normal *i.e.* $x/l = y/m = z/n$ is

$$\left[\frac{\Sigma a^2 (b^2+c^2) x^2}{\Sigma a^2 x^2}\right]^2 (1-e^2) = \frac{a^2b^2c^2 (x^2+y^2+z^2)}{\Sigma a^2x^2} (2-e^2)^2$$

or $\quad (1-e^2) [a^2 (b^2+c^2) x^2 + b^2 (c^2+a^2) y^2 + c^2 (a^2+b^2) z^2]^2$

$$= a^2b^2c^2 (e^2-2)^2 (x^2+y^2+z^2) (a^2x^2+b^2y^2+c^2z^2) \quad \text{Proved.}$$

Ex. 14. Prove that all plane sections of the conicoid $ax^2+by^2+cz^2=1$ which are rectangular hyperbolas and which pass through the point (α, β, γ) touch the cone $\frac{(x-\alpha)^2}{b+c}+\frac{(y-\beta)^2}{c+a}+\frac{(z-\gamma)^2}{a+b}=0$.

Sol. The condition for the section to be a rectangular hyperbola is

$$(b+c)\, l^2 + (c+a)\, m^2 + (a+b)\, n^2 = 0 \qquad \text{...(i)}$$

The plane through (α, β, γ) is $l(x-\alpha)+m(y-\beta)+n(z-\gamma)=0$

Shifting the origin to the point (α, β, γ), the above equation of the plane reduces to

$$lx + my + nz = 0 \qquad \text{...(ii)}$$

Relations (i) and (ii) show that normal to the plane (ii) is a generator of the cone $\quad (b+c)x^2+(c+a)y^2+(a+b)z^2=0$

or $\quad$ the plane (ii) touches the reciprocal cone $\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0$

Shifting the origin back, we get

$$\frac{(x-\alpha)^2}{b+c}+\frac{(y-\beta)^2}{c+a}+\frac{(z-\gamma)^2}{a+b}=0. \qquad \text{Hence proved.}$$

****Ex. 15. Find the condition that the section of the conicoid $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$ by the plane $lx+my+nz=0$ may be rectangular hyperbola.**

Sol. If L, M, N be the direction-cosines of a semi-diameter of length r of the given conicoid, then the extremity of the diameter is the point (Lr, Mr, Nr).

As this point lies on the given conicoid, so we have

$$r^2 (aL^2+bM^2+cN^2+2fMN+2gNL+2hLM)=1$$

or

$$1/r^2 = aL^2+bM^2+cN^2+2fMN+2gNL+2hLM \qquad \text{...(i)}$$

Now choose any three perpendicular semi-diameters r_1, r_2, r_3 and their correspoinding direction cosines L_1, M_1, N_1; L_2, M_2, N_2 and L_3, M_3, N_3.

Then as these semi-diameters are mutually perpendicular, so

$$L_1^2+L_2^2+L_3^2=1, \text{ etc. and } L_1M_1+L_2M_2+L_3M_3=0, \text{ etc.} \qquad \text{...(ii)}$$

Also from (i) we have

$$1/r_1^2 = aL_1^2+bM_1^2+cN_1^2+2fM_1N_1+2gN_1L_1+2hL_1M_1, \text{ etc}$$

$$\therefore \frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2} = a\Sigma L_1^2+b\Sigma M_1^2+c\Sigma N_1^2+2f\Sigma M_1N_1+2g\,\Sigma N_1L_1+2h\Sigma L_1M_1$$

$= a(1) + b(1) + c(1) + 2f(0) + 2g(0) + 2h(0)$, from (ii)

or $$(1/r_1^2) + (1/r_2^2) + (1/r_3^2) = a + b + c \quad \text{...(iii)}$$

Also if the section of the given conicoid by the plane $lx + my + nz = 0$ be a rectangular hyperbola, then $\quad r_1^2 + r_2^2 = 0 \quad$ ...(iv)

Now from (ii), $\dfrac{r_1^2 + r_2^2}{r_1^2 r_2^2} + \dfrac{1}{r_3^2} = a + b + c \quad$ or $\quad \dfrac{1}{r_3^2} = a + b + c$, from (iv)

$\therefore$ In general $1/r^2 = a + b + c$, ...(v)

where r is the length of the semi-diameter which is perpendicular to the plane of a section which is a rectangular hyperbola.

$\therefore$ The direction-cosines of the axis *i.e.* L, M, N are the same as that of the normal to the plane $lx + my + nz = 0$ *i.e.* l, m, n.

$\therefore$ From (i) and (v), we have the required condition as

$$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = a + b + c$$

or $$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = (a + b + c)(l^2 + m^2 + n^2)$$

$$\because l^2 + m^2 + n^2 = 1$$

or $$(b + c)l^2 + (c + a)m^2 + (a + b)n^2 - 2fmn - 2gnl - 2hlm = 0. \quad \textbf{Ans.}$$

Ex. 16. Find the angle between the asymptotes of any plane section of a conicoid and hence prove that if θ be the angle between the asymptotes of the section of $ax^2 + by^2 + cz^2 = 1$ by the plane $lx + my + nz = 0$, then

$$\tan^2\theta = -4(bcl^2 + cam^2 + abn^2)/[l^2(b + c) + m^2(c + a) + n^2(a + b)]^2$$

Sol. We know that the section of a conicoid by any plane is a conic and if the lengths of the semi-axes be α and β, then the equation of the conic can be written in the form $(x^2/\alpha^2) + (y^2/\beta^2) = 1$ after suitable transformations.

Also from our knowledge of co-ordinate geometry of two dimensions we know that the equations of the asymptotes differs from the equation of the conic in constant term only, so the equation of the asymptoes is

$$(x^2/\alpha^2) + (y^2/\beta^2) = \lambda$$

Also if θ be the angle between the asymptotes, then

$$\tan\theta = \frac{\text{“}2\sqrt{(h^2 - ab)}\text{”}}{a + b} = \frac{2\sqrt{[0^2 - (1/\alpha^2)(1/\beta^2)]}}{(1/\alpha^2) + (1/\beta^2)}$$

or $$\tan\theta = 2i\alpha\beta/(\alpha^2 + \beta^2)$$

or $$\frac{2\tan(\theta/2)}{1 - \tan^2(\theta/2)} = \frac{2i\alpha\beta}{\alpha^2 + \beta^2} = \frac{2(i\beta/\alpha)}{1 - (i\beta/\alpha)^2} \quad \text{(Note)}$$

or $$\tan(\theta/2) = i\beta/\alpha$$

Also if r_1 and r_2 be the lengths of axes of the section of the conicoid $ax^2 + by^2 + cz^2 = 1$ by the plane $lx + my + nz = 0$, then r_1^2 and r_2^2 are the roots of the equation $r^4(bcl^2 + cam^2 + abn^2) - r^2[(b + c)l^2 + (c + a)m^2 + (a + b)n^2] + (l^2 + m^2 + n^2) = 0$ ...See § 11.04 (iv) Page 3

$$\therefore r_1^2 + r_2^2 = \frac{[(b+c)\,l^2 + (c+a)\,m^2 + (a+b)\,n^2]}{bcl^2 + cam^2 + abn^2}$$

and $$r_1^2 r_2^2 = (l^2 + m^2 + n^2)/(bcl^2 + cam^2 + abn^2)$$

$$\therefore \text{From (i), } \tan^2\theta = \frac{-4\alpha^2\beta^2}{(\alpha^2+\beta^2)} = \frac{-4r_1^2 r_2^2}{(r_1^2 + r_2^2)^2}$$

or $$\tan^2\theta = \frac{-4\,(l^2+m^2+n^2)\,(bcl^2 + cam^2 + abn^2)}{[(b+c)\,l^2 + (c+a)\,m^2 + (a+b)\,n^2]^2},$$

substituting values of $r_1^2 + r_2^2$ and $r_1^2 r_2^2$.

Plane Sections of a Paraboloid.

***Ex. 17 (a). Find the lengths of the semi-axes of the section of the paraboloid $2x^2 + y^2 = z$ by the plane $x + 2y + z = 4$.** *(Rohilkhand 97)*

Sol. Comparing $ax^2 + by^2 = 2cz$ with $2x^2 + y^2 = z$ we get 'a' = 2, 'b' = 1, 'c' = 1/2 and comparing $lx + my + nz = p$ with $x + 2y + z = 4$ we get 'l' = 1, 'm' = 2, 'n' = 1 and 'p' = 4.

Also we know [See § 11.05 Page 6 Ch. XI] that the lengths of the axes of the section are given by $$\frac{l^2}{ar^2n^2 - p_0^2} + \frac{m^2}{br^2n^2 - p_0^2} - \frac{n^2}{p_0^2} = 0$$

which here reduces to

$$\frac{1}{2r^2 \cdot 1^2 - p_0^2} + \frac{4}{1 . r^2 . 1^2 - p_0^2} - \frac{1}{p_0^2} = 0, \text{ substituting values of } l, m, n, a \text{ and } b.$$

or $$\frac{1}{2r^2 - p_0^2} + \frac{4}{r^2 - p_0^2} - \frac{1}{p_0^2} = 0, \quad \text{...(i)}$$

where $$p_0^2 = \text{“}c\left(\frac{l^2c}{a} + \frac{m^2c}{b} + 2np\right)\text{”}$$ ...See § 11.05 (xi) P. 7 Ch. XI

$$= \frac{1}{2}\left[\frac{1^2 . (1/2)}{2} + \frac{2^2 . (1/2)}{1} + 2.1.4\right],$$

substituting values of l, m, n, a, b, c and p

$$= \frac{1}{2}\left[\frac{1}{4} + 2 + 8\right] = \frac{41}{8} \quad \text{...(ii)}$$

$$\therefore \text{(i) reduces to } \frac{1}{2r^2 - (41/8)} + \frac{4}{r^2 - (41/8)} - \frac{1}{41/8} = 0$$

or $$\frac{1}{16r^2 - 41} + \frac{4}{8r^2 - 41} - \frac{1}{41} = 0, \text{ on simplifying}$$

or $$41\,(8r^2 - 41) + 4\,(41)\,(16r^2 - 41) - (16r^2 - 41)\,(8r^2 - 41) = 0$$

or $$64r^4 - 1968r^2 + 5043 = 0 \quad \text{...(iii)}$$

The required lengths of the semi-axes of the section are given by (iii).

***Ex. 17 (b). Find the lengths and direction cosines of the axes of the section of a paraboloid $x^2 + y^2 = 2az$ by a plane $ux + vy + wz = 1$.**

(Gorakhpur 97)

Sol. Comparing $ax^2 + by^2 = 2cz$ with $x^2 + y^2 = 2az$ we get 'a' = 1, 'b' = 1, 'c' = a and comparing $lx + my + nz = p$ with $ux + vy + wz = 1$, we get 'l' = u, 'm' = v, 'n' = w, 'p' = 1

Also we know [See § 11.05 Page 6 ch. XI] that the lengths of the axes of the section are given by
$$\frac{l^2}{ar^2n^2 - p_0^2} + \frac{m^2}{br^2n^2 - p_0^2} - \frac{n^2}{p_0^2} = 0$$
which here reduces to
$$\frac{u^2}{r^2w^2 - p_0^2} + \frac{v^2}{r^2w^2 - p_0^2} - \frac{w^2}{p_0^2} = 0, \text{ substuting values of } l, m, n, a \text{ and } b$$
or
$$p_0^2 (u^2 + v^2) - w^2 (r^2w^2 - p_0^2) = 0$$
or
$$r^2 = (u^2 + v^2 + w^2)\, p_0^2/w^4 \quad \text{...(i)}$$

Required lengths of axes are $2r_1, 2r_2$, where r_1, r_2 are the values of r given by (i) *i.e.* $2p_0 \sqrt{(u^2 + v^2 + w^2)}/w^2$ and $-2\, p_0 \sqrt{(u^2 + v^2 + w^2)}/w^2$,

where
$$p_0^2 = \text{“}c\left(\frac{l^2c}{a} + \frac{m^2c}{b} + 2np\right)\text{”}$$
$$= a\,(u^2a + v^2a + 2w), \text{ substituting values of } a, b, c, l, m, n \text{ and } p$$
i.e.
$$p_0^2 = a^2\,(u^2 + v^2) + 2wa$$

$\therefore$ From (i), we get $r^2 = (u^2 + v^2 + w^2)\,(a^2u^2 + a^2v^2 + 2wa)/w^4$

Also the direction ratios, L, M, N of the axes are given by the equations
$$\frac{L\,(ar^2n^2 - p_0^2)}{l} = \frac{M\,(br^2n^2 - p_0^2)}{m} = \frac{-N p_0^2}{n},$$
...See § 11.05 (xvii) Page 8 ch. XI

which here reduces to
$$\frac{L\,(r^2w^2 - p_0^2)}{u} = \frac{M\,(r^2w^2 - p_0^2)}{v} = \frac{-N p_0^2}{w}, \quad \text{...(iv)}$$
where r^2 and p_0^2 are given by (iii) and (ii).

From (ii), (iii) and (iv) we can find the proportional values of L, M, N and whence required d.c.'s can be obtained.

****Ex. 18. Find the locus of the centres of sections of the paraboloid $(x^2/a^2) + (y^2/b^2) = 2z$ which are of constant area πk^2.**

(Gorakhpur 95; Rohilkhand 95)

Sol. We know that the area of the section of the paraboloid $ax^2 + by^2 = 2cz$ by the plane $lx + my + nz = p$ is given by
$$A = \frac{\pi c}{n^3}\left[\left(\frac{l^2c}{a} + \frac{m^2c}{b} + 2np\right)\sqrt{\left(\frac{l^2 + m^2 + n^2}{ab}\right)}\right]$$

...See § 11.05 (xviii) Page 8

Here the given paraboloid is $(x^2/a^2)+(y^2/b^2)=2z$, so replacing a, b and c by $1/a^2$, $1/b^2$ and 1 respectively the above result reduces to

$$A=\frac{\pi}{n^3}\left[(a^2l^2+b^2m^2+2np)\sqrt{\{(l^2+m^2+n^2)\}a^2b^2\}}\right] \qquad \text{...(i)}$$

Also if (α, β, γ) be the centre, then the plane $lx+my+nz=p$ is the same as

$$T=S_1 \qquad \textbf{(Note)}$$

i.e. $$\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}-(z+\gamma)=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-2\gamma$$

or $$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}-z=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-\gamma \qquad \text{...(ii)}$$

Comparing (ii) with $lx+my+nz=p$ we get

$$\frac{\alpha/a^2}{l}=\frac{\beta/b^2}{m}=\frac{-1}{n}=\frac{(\alpha^2/a^2)+(\beta^2/b^2)-\gamma}{p} \qquad \text{...(iii)}$$

$\therefore$ From (i), $A=\pi ab\left[\left(\frac{a^2l^2}{n^2}+\frac{b^2m^2}{n^2}+\frac{2p}{n}\right)\sqrt{\left(\frac{l^2}{n^2}+\frac{m^2}{n^2}+1\right)}\right]$

or $$\pi k^2=\pi ab\left[\left\{a^2\frac{\alpha^2}{a^4}+b^2\frac{\beta^2}{b^4}-2\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}-\gamma\right)\right\}\times\sqrt{\left(\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+1\right)}\right], \text{ by (iii)}$$

or $$k^4=a^2b^2\left(-\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}+2\gamma\right)^2\cdot\left(\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+1\right), \text{ squaring both sides}$$

$\therefore$ The required locus of the centre (α, β, γ) of the section is

$$a^2b^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-2z\right)^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+1\right)=k^4 \qquad \textbf{Ans.}$$

***Ex. 19. If the area of the section of $ax^2+by^2=2cz$ be constant and equal to πk^2, then prove that the locus of the centre is**

$$(a^2x^2+b^2y^2+c^2)(ax^2+by^2-2cz)^2=abc^2k^4.$$

Sol. We know that the area of the section of the given paraboloid by the plane

$$lx+my+nz=p \qquad \text{...(i)}$$

is $$\frac{\pi c}{n^3}\left[\left(\frac{l^2c}{a}+\frac{m^2c}{b}+2np\right)\sqrt{\left(\frac{l^2+m^2+n^2}{ab}\right)}\right]$$ ...See § 11.05 (xviii) Page 8.

$\therefore$ $$\pi k^2=\frac{\pi c}{n^3}\left[\left(\frac{l^2c}{a}+\frac{m^2c}{b}+2np\right)\sqrt{\left(\frac{l^2+m^2+n^2}{ab}\right)}\right] \qquad \text{...(ii)}$$

Also if (α, β, γ) be the centre, then the plane $lx + my + nz = p$ is the same as

$$T = S_1 \qquad \textbf{(Note)}$$

i.e.
$$a\alpha x + b\beta y - c(z + \gamma) = a\alpha^2 + b\beta^2 - 2c\gamma$$

or
$$a\alpha x + b\beta y - cz = a\alpha^2 + b\beta^2 - c\gamma \qquad \text{...(iii)}$$

Comparing (i) and (iii) we get $\dfrac{a\alpha}{l} = \dfrac{b\beta}{m} = \dfrac{-c}{n} = \dfrac{a\alpha^2 + b\beta^2 - c\gamma}{p}$ ...(iv)

$\therefore$ Now (ii) can be rewritten as

$\therefore$
$$k^2 = \frac{c}{\sqrt{(ab)}}\left[\left\{\frac{c}{a}\left(\frac{l^2}{n^2}\right) + \frac{c}{b}\left(\frac{m^2}{n^2}\right) + 2\left(\frac{p}{n}\right)\right\}\sqrt{\left(\frac{l^2}{n^2} + \frac{m^2}{n^2} + 1\right)}\right]$$

or
$$k^2\sqrt{(ab)} = c\left[\left\{\frac{c}{a}\left(\frac{a^2\alpha^2}{c^2}\right) + \frac{c}{b}\left(\frac{b^2\beta^2}{c^2}\right) + 2\left(\frac{a\alpha^2 + b\beta^2 - c\gamma}{-c}\right)\right\}\right]$$
$$\times\sqrt{\left(\frac{a^2\alpha^2}{c^2} + \frac{b^2\beta^2}{c^2} + 1\right)}, \text{ from (iv)}$$

or
$$k^4\, abc^2 = \{a\alpha^2 + b\beta^2 - 2(a\alpha^2 + b\beta^2 - c\gamma)\}^2\,(a^2\alpha^2 + b^2\beta^2 + c^2)$$

or
$$k^4\, abc^2 = (a\alpha^2 + b\beta^2 - 2c\gamma)^2\,(a^2\alpha^2 + b^2\beta^2 + c^2)$$

$\therefore$ The required locus of the centre (α, β, γ) is

$$(ax^2 + by^2 - 2cz)^2\,(a^2x^2 + b^2y^2 + c^2) = abc^2k^4 \qquad \textbf{Proved.}$$

Ex. 20. Prove that the axes of the section of the conicoid $ax^2 + by^2 = 2z$ by the plane $lx + my + nz = 0$ lie on the cone

$$\frac{bl}{x} - \frac{am}{y} + \frac{(a-b)\,n}{z} = 0$$

Sol. If L, M, N be the direction ratios of the axes of the section of the given paraboloid by the given plane, then as in § 11.05 result (xvii) Page 8 we can obtain
$$\frac{L\,(ar^2n^2 - p\delta^2)}{l} = \frac{M\,(br^2n^2 - p\delta^2)}{m} = \frac{-N\,p\delta^2}{n} = \lambda \text{ (say)}$$

(To be obtained in the exam.)

or
$$\frac{l}{L} = \frac{ar^2n^2 - p\delta^2}{\lambda},\ \frac{m}{M} = \frac{br^2n^2 - p\delta^2}{\lambda},\ \frac{n}{N} = -\frac{p\delta^2}{\lambda}$$

$$\therefore \frac{bl}{L} - \frac{am}{M} + \frac{(a-b)\,n}{N}$$
$$= \frac{b\,(ar^2n^2 - p\delta^2)}{\lambda} - \frac{a\,(br^2n^2 - p\delta^2)}{\lambda} + \frac{(a-b)\,(-p\delta^2)}{\lambda}$$
$$= \frac{1}{\lambda}\,[bar^2n^2 - bp\delta^2 - abr^2n^2 + ap\delta^2 - ap\delta^2 + bp\delta^2] = 0$$

$\therefore$ The axis $\dfrac{x}{L} = \dfrac{y}{M} = \dfrac{z}{N}$ lies on the cone

$$\frac{bl}{x}-\frac{am}{y}+\frac{(a-b)\,n}{z}=0.$$ Hence proved.

***Ex. 21. Prove that the section of the paraboloid $ax^2+by^2=2cz$ by a tangent plane to the cone $(x^2/b)+(y^2/a)+[z^2/(a+b)]=0$ is a rectangular hyperbola.**

Sol. If $lx+my+nz=0$ be a tangent plane to the given cone, then its normal $x/l=y/m=z/n$ is a generator of the reciprocal cone

$$bx^2+ay^2+(a+b)\,z^2=0$$

$$\therefore \quad bl^2+am^2+(a+b)\,n^2=0 \qquad \text{...(i)}$$

Now if the section be a rectangular hyperbola, then the sum of the squares of the semi-axes is zero *i.e.* $r_1^2+r_2^2=0$, where r_1^2 and r_2^2 are the roots of

$$abn^6r^4-n^2p_0^2\,[(a+b)\,n^2+am^2+bl^2]\,r^2+p_0^4\,(l^2+m^2+n^2)=0$$

...See § 11.05 (xiv) Page 8

$\therefore$ $r_1^2+r_2^2=0$ gives $(a+b)\,n^2+am^2+bl^2=0$, which is true by virtue of (i).

Hence proved.

****Ex. 22. Planes are drawn through a fixed point (α, β, γ) so that their sections of the paraboloid $ax^2+by^2=2z$ are rectangular hyperbolas. Prove that they touch the cone**

$$\frac{(x-\alpha)^2}{b}+\frac{(y-\beta)^2}{a}+\frac{(z-\gamma)^2}{a+b}=0$$ *(Rohilkhand 96)*

Sol. The equation of any plane through (α, β, γ) is

$$l\,(x-\alpha)+m\,(y-\beta)+n\,(z-\gamma)=0 \qquad \text{...(i)}$$

or $$lx+my+nz=l\alpha+m\beta+n\gamma=p,\ \text{say} \qquad \text{...(ii)}$$

Also we know (from § 11.05 (xiv) Page 8 of this chapter) that the lengths of the axes of the section of the given paraboloid by the plane (i) are given by

$$abn^6r^4-n^2p_0^2\,[(a+b)\,n^2+am^2+bl^2]\,r^2+p_0^2\,(l^2+m^2+n^2)=0, \qquad \text{...(iii)}$$

where $$p_0^2=\frac{l^2}{a}+\frac{m^2}{b}+2np=\frac{l^2}{a}+\frac{m^2}{b}+2n\,(l\alpha+m\beta+n\gamma),\ \text{from (ii)}$$

Now if the section be a rectangular hypebola, then $r_1^2+r_2^2=0$, where r_1 and r_2 are the lengths of the semi-axes.

i.e. $$(a+b)\,n^2+am^2+bl^2=0,\ \text{from (iii)} \qquad \text{...(iv)}$$

Again the equations of the normal to the plane (i) through (α, β, γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{...(v)}$$

Relations (iv) and (v) show that the normal generates the cone

$$(a+b)\,(z-\gamma)^2+a\,(y-\beta)^2+b\,(x-\alpha)^2=0 \qquad \text{...(vi)}$$

Now by definition of the reciprocal cone, the plane (i) touches the reciprocal cone of (vi), whose equation is

$$\frac{(x-\alpha)^2}{b}+\frac{(y-\beta)^2}{a}+\frac{(z-\gamma)^2}{a+b}=0$$ Hence proved.

Exercises on § 11.01—§11.05

Plane central sections of central conicoid.

Ex. 1. Find the condition that any two lines

$$x/l_1 = y/m_1 = z/n_1, \quad x/l_2 = y/m_2 = z/n_2$$

be the axes of the section of the conicoid $ax^2 + by^2 + cz^2 = 1$ by a plane through them.

***Ex. 2.** The director circle of a plane central section of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$ has a radius of constant length r. Prove that its plane section touches the cone

$$\frac{x^2}{a^2(b^2+c^2-r^2)} + \frac{y^2}{b^2(c^2+a^2-r^2)} + \frac{z^2}{c^2(a^2+b^2-r^2)} = 0.$$

Ex. 3. Planes are drawn through the origin so as to cut the conicoid $ax^2 + by^2 + cz^2 = 1$ in rectangular hyperbolas. Prove that the normals to the planes through the origin lie on the cone $(b+c)x^2 + (c+a)y^2 + (a+b)z^2 = 0$.

Plane sections of a paraboloid.

***Ex. 4.** If the area of the section of $(y^2/b^2) + (z^2/c^2) = 2x$ be constant and equal to a^2, then show that the locus of the centre is

$$a^4\left(1 + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^{-1} = \pi^2 bc\left(2x - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right),$$

***§ 11.06. Axes and direction cosines of non-central plane sections of a conicoid.** (*Rohilkhand* 97)

Let the equations of the conicoid and the plane section be

$$ax^2 + by^2 + cz^2 = 1 \quad \text{...(i)}$$

and

$$lx + my + nz = p \quad \text{...(ii)}$$

Here the centre of the section is not the centre (0, 0, 0) of the conicoid, but some other point (α, β, γ), say.

If (α, β, γ) be the centre of the plane section, then the plane (ii) is identical with the equation of the plane given by $T = S_1$

i.e.

$$a\alpha x + b\beta y + c\gamma z = a\alpha^2 + b\beta^2 + c\gamma^2 \quad \textbf{(Note)}$$

Comparing this with (ii), we have

$$\frac{a\alpha}{l} = \frac{b\beta}{m} = \frac{c\gamma}{n} = \frac{a\alpha^2 + b\beta^2 + c\gamma^2}{p} = \lambda \text{ (say)}$$

$$\therefore \alpha = l\lambda/a, \quad \beta = m\lambda/b, \quad \gamma = n\lambda/c \quad \text{...(iii)}$$

and

$$a\alpha^2 + b\beta^2 + c\lambda^2 = p\lambda$$

or

$$a(l\lambda/a)^2 + b(m\lambda/b)^2 + c(n\lambda/c)^2 = p\lambda, \text{ from (iii)}$$

or

$$\lambda^2\left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}\right) = p\lambda \quad \text{or} \quad \lambda = p/\left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}\right) = p/p_0^2, \quad \text{...(iv)}$$

where $$p_0^2 = (l^2/a) + (m^2/b) + (n^2/c) \qquad \text{...(v)}$$

This is the condition that the plane $lx + my + nz = p_0$ be a tangent plane to the conicoid (i).

Now from (iii) and (iv), we have $\alpha = lp/ap_0^2$, $\beta = mp/bp_0^2$, $\gamma = np/cp_0^2$

$\therefore$ The co-ordinates of the centre of the plane section are

$$\left(\frac{lp}{ap_0^2}, \frac{mp}{bp_0^2}, \frac{np}{cp_0^2}\right) \qquad \text{...(vi)}$$

where p_0^2 is given by (v). *(Rohilkhand 97)*

Now shifting the origin to the centre whose co-ordinates are given by (vi), the equation of the conicoid (i) reduces to

$$a\left(x + \frac{lp}{ap_0^2}\right)^2 + b\left(y + \frac{mp}{bp_0^2}\right)^2 + c\left(z + \frac{np}{cp_0^2}\right)^2 = 1$$

or $$ax^2 + by^2 + cz^2 + \frac{2p}{p_0^2}(lx + my + nz) + \frac{p^2}{p_0^4}\left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}\right) = 1$$

or $$ax^2 + by^2 + cz^2 + \frac{2p}{p_0^2}(lx + my + nz) + \frac{p^2}{p_0^2} = 1, \text{ from (v)} \qquad \text{...(vi)}$$

And the equation of the plane (ii) reduces to

$$l\left(x + \frac{lp}{ap_0^2}\right) + m\left(y + \frac{mp}{bp_0^2}\right) + n\left(z + \frac{np}{cp_0^2}\right) = p$$

or $$lx + my + nz + \frac{p}{p_0^2}\left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}\right) - p = 0$$

or $$lx + my + nz + \frac{p}{p_0^2}(p_0^2) - p = 0, \text{ from (v)}$$

or $$lx + my + nz = 0 \qquad \text{...(vii)}$$

Now the conic given by equations (vi) and (vii) is the same as the conic given by $ax^2 + by^2 + cz^2 = 1 - \dfrac{p^2}{p_0^2}$ and $lx + my + nz = 0$

or $$Ax^2 + By^2 + Cz^2 = 1,\ lx + my + nz = 0, \qquad \text{...(viii)}$$

where $A = a/k^2$, $B = b/k^2$, $C = c/k^2$ and $k^2 = 1 - (p^2/p_0^2)$

Here the plane $lx + my + nz = 0$ evidently passes through the centre $(0, 0, 0)$ of the conicoid $Ax^2 + By^2 + Cz^2 = 1$.

$\therefore$ Proceeding as in § 11.04 Pages 3-6 of this chapter, the lengths of the axes are given by

$$\frac{l^2}{Ar^2-1}+\frac{m^2}{Br^2-1}+\frac{n^2}{Cr^2-1}=0, \text{ where } A=\frac{a}{k^2} \text{ etc.}$$

or
$$\frac{l^2}{(a/k^2)\,r^2-1}+\frac{m^2}{(b/k^2)\,r^2-1}+\frac{n^2}{(c/k^2)\,r^2-1}=0 \qquad \text{...(ix)}$$

And the direction-cosines L, M, N of the axes are given by

$$\frac{L\,(Ar^2-1)}{l}=\frac{M\,(Br^2-1)}{m}=\frac{N\,(Cr^2-1)}{n} \qquad \text{...(x)}$$

or
$$\frac{L\,\{(a/k^2)\,r^2-1\}}{l}=\frac{M\,\{(b/k^2)\,r^2-1\}}{m}=\frac{N\,\{(c/k^2)\,r^2-1\}}{n} \qquad \text{...(xi)}$$

Thus we find that in this case the discussion is the same as in § 11.04 Pages 3-6 and here we take A, B, C in place of a, b, c or $a/k^2, b/k^2, c/k^2$ in place of a, b, c

and $\qquad k^2=1-(p^2/p_0^2),\; p_0^2=(l^2/a)+(m^2/b)+(n^2/c)$

and $\quad p$ is the length of the perpendicular from the origin to the given plane.

Cor. 1. Area of the plane section.

From § 11.04 Cor. 2 Page 5 of this chapter we know that the area of the section of the conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=0$ is

$$\frac{\pi\sqrt{(l^2+m^2+n^2)}}{\sqrt{(bc\,l^2+ca\,m^2+ab\,n^2)}}=A_0 \text{ (say)} \qquad \text{...(xii)}$$

$\therefore$ The area of the section of the conicoid $ax^2+by^2+cz^2=1$ by the parallel plane $lx+my+nz=p$ is obtained by replacing a, b, c by $a/k^2, b/k^2, c/k^2$ respectively and so is given by

$$\mathbf{A}=\frac{\pi \mathbf{k}^2\sqrt{(\mathbf{l}^2+\mathbf{m}^2+\mathbf{n}^2)}}{\sqrt{(\mathbf{bcl}^2+\mathbf{cam}^2+\mathbf{abn}^2)}}, \text{ where } \mathbf{k}^2=1-\frac{\mathbf{p}^2}{\mathbf{p}_0^2} \qquad \text{...(xiii)}$$

$$=\mathbf{k}^2\mathbf{A}_0=\left(1-\frac{\mathbf{p}^2}{\mathbf{p}_0^2}\right)\mathbf{A}_0, \qquad \text{...(xiv)}$$

where A_0 is given by (xii) above.

Comparision of Lengths of the axes.

We know [§ 11.04 Pages 3-6 of this chapter] that lengths of the axes of the conic section of the conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=0$ are given by $\dfrac{l^2}{ar^2-1}+\dfrac{m^2}{br^2-1}+\dfrac{n^2}{cr^2-1}=0$ ...(xv)

Also the lengths of the axes of the section of $ax^2+by^2+cz^2=1$, by the plane $lx+my+nz=p$ are given by [See § 11.06 (ix) above]

$$\frac{l^2}{a(r^2/k^2)-1}+\frac{m^2}{b(r^2/k^2)-1}+\frac{n^2}{c(r^2/k^2)-1}=0 \qquad \text{...(xvi)}$$

Thus if r_1^2, r_2^2 be the roots of (xv) and R_1^2, R_2^2 be the roots of (xvi), then comparing (xv) and (xvi), we find that

$$R_1^2=k^2r_1^2 \text{ and } R_2^2=k^2r_2^2 \quad \textit{i.e.} \quad \mathbf{R_1=kr_1} \text{ and } \mathbf{R_2=kr_2}$$

Also the area A_0 in the former case $=\pi\, r_1 r_2$

and the area A in the later case $=\pi\, R_1\, R_2=\pi\, kr_1\,.\, kr_2$

i.e. $\quad A=k^2 A_0=\left(1-\frac{p^2}{p_0^2}\right)A_0$

which has already been proved above.

Thus we conclude that **parallel sections of a central conicoid are similarly situated conics.**

Solved Examples on § 11.06.

***Ex. 1. Find the locus of the centres of the sections of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ whose area is constant and equal to $\pi\, k^2$.**

(*Avadh* 95)

Sol. If (α, β, γ) be the centre of the section, then the equation of the plane of section is $\qquad T=S_1.$

i.e. $$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}-1=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1$$

or $\quad (\alpha/a^2)\,x+(\beta/b^2)\,y+(\gamma/c^2)\,z=(\alpha^2/a^2)+(\beta^2/b^2)+(\gamma^2/c^2)$

or $\qquad lx+my+nz=p$ (say), $\qquad$ ...(i)

where $\quad l=\alpha/a^2,\ m=\beta/b^2,\ n=\gamma/c^2,\ p=(\alpha^2/a^2)+(\beta^2/b^2)+(\gamma^2/c^2) \qquad$ ...(ii)

$\therefore$ If A_0 be the area of the parallel central section, then

$$A=[1-(p^2/p_0^2)]\,A_0 \qquad \text{...(iii)}$$

where $\quad p_0^2=a^2l^2+b^2m^2+c^2n^2=a^2(\alpha^2/a^4)+b^2(\beta^2/b^4)+c^2(\gamma^2/c^4)$

$\qquad =(\alpha^2/a^2)+(\beta^2/b^2)+(\gamma^2/c^2)$ and p is given by (ii)

$$\therefore\ 1-\frac{p^2}{p_0^2}=1-\frac{\{(\alpha^2/a^2)+(\beta^2/b^2)+(\gamma^2/c^2)\}^2}{(\alpha^2/a^2)+(\beta^2/b^2)+(\gamma^2/c^2)}=1-\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)$$

And $A_0=\pi\, abc\,\dfrac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(a^2l^2+b^2m^2+c^2n^2)}}$

or $$A_0=\frac{\pi\, abc\,\sqrt{\{(\alpha^2/a^4)+(\beta^2/b^4)+(\gamma^2/c^4)\}}}{\sqrt{\{(\alpha^2/a^2)+(\beta^2/b^2)+(\gamma^2/c^2)\}}}\text{, from (ii)}$$

Also $A=\pi\, k^2$

Substituting these values in (iii) and squaring, we get

$$\pi^2 k^4 = \left[1 - \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}\right)\right]^2 \times \frac{\pi^2 a^2 b^2 c^2 \left[(\alpha^2/a^4) + (\beta^2/b^4) + (\gamma^2/c^4)\right]}{(\alpha^2/a^2) + (\beta^2/b^2) + (\gamma^2/c^2)}$$

or $$k^4 \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}\right) = a^2 b^2 c^2 \left(\frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4} + \frac{\gamma^2}{c^4}\right)\left[1 - \frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} - \frac{\gamma^2}{c^2}\right]^2$$

$\therefore$ The requird locus of the centre (α, β, γ) of the section is

$$k^4 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = a^2 b^2 c^2 \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right]^2.$$

****Ex. 2. (a). Find the coordinates of the centre and the lengths of the axes of the section of the ellipsoid $5x^2 + 5y^2 + 9z^2 = 20$ by the plane $2x + y + z = 9$.** (*Avadh* 92)

Sol. If (α, β, γ) be the centre of the section, then the equation of the plane of section is
$$T = S_1$$
i.e. $$5\alpha x + 5\beta y + 9\gamma z - 20 = 5\alpha^2 + 5\beta^2 + 9\gamma^2 - 20$$
or $$(5\alpha)\, x + (5\beta)\, y + (9\gamma)\, z = 5\alpha^2 + 5\beta^2 + 9\gamma^2 \quad \text{...(i)}$$

Also the equation of the plane section is given as $2x + y + z = 9$ ...(ii)

Comparing (i) and (ii) we have
$$\frac{5\alpha}{2} = \frac{5\beta}{1} = \frac{9\gamma}{1} = \frac{5\alpha^2 + 5\beta^2 + 9\gamma^2}{9} = \lambda \text{ (say)} \quad \text{...(iii)}$$

$\therefore \alpha = 2\lambda/5,\ \beta = \lambda/5,\ \gamma = \lambda/9$ ...(iv)

and so $5\alpha^2 + 5\beta^2 + 9\gamma^2 = 9\lambda$ gives $5\,(2\lambda/5)^2 + 5\,(\lambda/5)^2 + 9\,(\lambda/9)^2 = 9\lambda$

or $$(4\lambda/5) + (\lambda/5) + (\lambda/9) = 9 \quad \text{or} \quad \lambda = 81/10$$

$\therefore$ From (iv), $\alpha = \dfrac{162}{50},\ \beta = \dfrac{81}{50},\ \gamma = \dfrac{9}{10}$

$\therefore$ The required coordinates of the centre are $\left(\dfrac{162}{50}, \dfrac{81}{50}, \dfrac{9}{10}\right)$. **Ans.**

Now shifting the origin to (162/50, 81/50, 9/10), the equatiton of the ellipsoid $5x^2 + 5y^2 + 9z^2 = 20$ reduces to
$$5\left(x + \frac{162}{50}\right)^2 + 5\left(y + \frac{81}{50}\right)^2 + 9\left(z + \frac{9}{10}\right)^2 = 20$$
or $$5x^2 + 5y^2 + 9z^2 + \frac{162}{5}x + \frac{81}{5}y + \frac{81}{5}z + \frac{529}{10} = 0$$
or $$5x^2 + 5y^2 + 9z^2 + \frac{81}{5}(2x + y + z) + \frac{529}{10} = 0 \quad \text{...(v)}$$

And the equation of the given plane $2x + y + z = 9$ reduces to

$$2\left(x+\frac{162}{50}\right)+\left(y+\frac{81}{50}\right)+\left(z+\frac{9}{10}\right)=9 \quad \text{or} \quad 2x+y+z=0 \qquad \text{...(vi)}$$

$\therefore$ With the help of (vi), (v) reduces to $5x^2+5y^2+9z^2+\frac{529}{200}=0$

$\therefore$ The equation of the conic is

$$5x^2+5y^2+9z^2+(529/200)=0, \quad 2x+y+z=0$$

or
$$-\left(\frac{1000}{529}\right)x^2-\left(\frac{1000}{529}\right)y^2-\left(\frac{1800}{529}\right)z^2=1, \; 2x+y+z=0$$

or
$$Ax^2+By^2+Cz^2=1, \; 2x+y+z=0$$

We know that the length of the axes are given by

$$\text{“}\frac{l^2}{Ar^2-1}+\frac{m^2}{Br^2-1}+\frac{n^2}{Cr^2-1}=0\text{”}$$

or
$$\frac{(2)^2}{-\frac{1000}{529}r^2-1}+\frac{(1)^2}{-\frac{1000}{529}r^2-1}+\frac{(1)^2}{-\frac{1800}{529}r^2-1}=0,$$

putting the values of A, B, C and $l=2$, $m=1=n$

or
$$\frac{2116}{1000r^2+529}+\frac{529}{1000r^2+529}+\frac{529}{1800r^2+529}=0$$

or
$$\frac{5}{1000r^2+529}+\frac{1}{1800r^2+529}=0$$

or
$$5\,(1800\,r^2+529)+(1000r^2+529)=0$$

or
$$10{,}000\,r^2+(6\times 529)=0 \quad \text{or} \quad r^2=\frac{-6\times 529}{10000}=\frac{-1587}{5000}$$

which gives the required lengths of the axes.

Ex. 2. (b). Find the coordinates of the centre and the lengths of the semi-axes of the section of the ellipsoid $3x^2+3y^2+6z^2=10$ by the plane $x+y+z=1$. *(Avadh 91)*

Sol. If (α, β, γ) be the required centre of the section, then the equation of the plane is
$$T=S_1$$

or
$$3\alpha x+3\beta y+6\gamma z-10=3\alpha^2+3\beta^2+6\gamma^2-10$$

or
$$(3\alpha)\,x+(3\beta)\,y+(6\gamma)\,z=3\alpha^2+3\beta^2+6\gamma^2 \qquad \text{...(i)}$$

Also the equation of the plane section is given as $x+y+z=1$ ...(ii)

Comparing (i) and (ii) we have

$$\frac{3\alpha}{1}=\frac{3\beta}{1}=\frac{6\gamma}{1}=\frac{3\alpha^2+3\beta^2+6\gamma^2}{1}=\lambda \text{ (say)} \qquad \text{...(iii)}$$

or
$$\alpha=\lambda/3, \; \beta=\lambda/3, \; \gamma=\lambda/6 \qquad \text{...(iv)}$$

and so $3\alpha^2+3\beta^2+6\gamma^2=\lambda$ gives $3\,(\lambda^2/9)+3\,(\lambda^2/9)+6\,(\lambda^2/36)=\lambda$

or $$\lambda\left(\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{6}\right)=1 \quad \text{or} \quad \lambda=6/5$$

$\therefore$ From (iv), $\alpha=2/5$, $\beta=2/5$, $\gamma=1/5$

$\therefore$ The required coordinates of the centre are $(2/5,\ 2/5,\ 1/5)$. **Ans.**

Now shifting the origin to $(2/5,\ 2/5,\ 1/5)$, the equation of the given ellipsoid $3x^2+3y^2+6z^2=10$ reduces to $3\left(x+\frac{2}{5}\right)^2+3\left(y+\frac{2}{5}\right)^2+6\left(z+\frac{1}{5}\right)^2=10$

or $$3x^2+3y^2+6z^2+\frac{12}{5}(x+y+z)=\frac{44}{5} \qquad \text{...(v)}$$

And the equation of the given plane $x+y+z=1$ reduces to

$$\left(x+\tfrac{2}{3}\right)+\left(y+\tfrac{2}{3}\right)+\left(z+\tfrac{1}{5}\right)=1 \quad \text{or} \quad x+y+z=0 \qquad \text{...(vi)}$$

$\therefore$ With the help of (vi), (v) reduces to $3x^2+3y^2+6z^2=\dfrac{44}{5}$

$\therefore$ The equation of the conic is $3x^2+3y^2+6z^2=44/5,\ x+y+z=0$

or $$\frac{15}{44}x^2+\frac{15}{44}y^2+\frac{15}{22}z^2=1,\ x+y+z=0$$

or $$Ax^2+By^2+Cz^2=1,\ x+y+z=0 \qquad \text{...(vii)}$$

where $$A=15/44,\ \ B=15/44,\ \ C=15/22$$

Also we know that the lengths of the axes are given by

$$\text{“}\frac{l^2}{Ar^2-1}+\frac{m^2}{Br^2-1}+\frac{n^2}{Cr^2-1}=0\text{”}$$

or $$\frac{1}{(15/44)\,r^2-1}+\frac{1}{(15/44)\,r^2-1}+\frac{1}{(15/22)\,r^2-1}=0,$$

putting values of A, B, C and $l=1=m=n$

or $$\frac{44}{15r^2-44}+\frac{44}{15r^2-44}+\frac{22}{15r^2-22}=0$$

or $$\frac{4}{15r^2-44}+\frac{1}{15r^2-22}=0 \quad \text{or} \quad 4(15r^2-22)+(15r^2-44)=0$$

or $$75r^2=132 \quad \text{or} \quad 25r^2=44 \quad \text{or} \quad r^2=44/25,$$

which gives the required lengths of the axes.

****Ex. 3. Prove that the axes of the section of the cone $ax^2+by^2+cz^2=0$ by the plane $lx+my+nz=p$ are given by**

$$\frac{l^2}{ap_0^2r^2+p^2}+\frac{m^2}{bp_0^2r^2+p^2}+\frac{n^2}{cp_0^2r^2+p^2}=0,$$

where $$p^2=(l^2/a)+(m^2/b)+(n^2/c) \qquad (Avadh\ 92, 91)$$

Sol. If (α, β, γ) be the required centre of the secton, then the equation of the plane is $$T=S_1$$

or $$a\alpha x+b\beta y+c\gamma z=a\alpha^2+b\beta^2+c\gamma^2 \qquad \text{...(i)}$$

Also the equation of the plane section is given by $lx+my+nz=p$ ...(ii)

Comparing (i) and (ii), we have

$$\frac{a\alpha}{l}=\frac{b\beta}{m}=\frac{c\gamma}{n}=\frac{a\alpha^2+b\beta^2+c\gamma^2}{p}=\lambda, \text{ (say)} \quad ...(iii)$$

or $$\alpha=l\lambda/a,\ \beta=m\lambda/b,\ \gamma=n\lambda/c \quad ...(iv)$$

and so $a\alpha^2+b\beta^2+c\gamma^2=p\lambda$ gives

$$\lambda^2\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)=p\lambda \quad \text{or} \quad \lambda=p\Big/\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)$$

or $$\lambda=p/p_0^2, \text{ where } p_0^2=(l^2/a)+(m^2/b)+(n^2/c) \quad ...(v)$$

∴ From (iv) the centre of the plane section (α, β, γ) is given by

$$(lp/ap_0^2,\ mp/bp_0^2,\ np/cp_0^2)$$

Now shifting the origin to the above centre, the equation of the cone

reduces to $$a\left(x+\frac{lp}{ap_0^2}\right)^2+b\left(y+\frac{mp}{bp_0^2}\right)^2+c\left(z+\frac{np}{cp_0^2}\right)^2=0$$

or $$ax^2+by^2+cz^2+\frac{2p}{p_0^2}(lx+my+nz)+\frac{p^2}{p_0^4}\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)=0$$

or $$ax^2+by^2+cz^2+\frac{2p}{p_0^2}(lx+my+nz)+\frac{p^2}{p_0^2}=0, \text{ from (v)} \quad ...(vi)$$

And the equation of the given plane reduces to $lx+my+nz=0$ ...(vii)

Now the conic given by the equations (vi) and (vii) is the same as the

conic given by $$ax^2+by^2+cz^2+\frac{p^2}{p_0^2}=0,\ lx+my+nz=0$$

or $$(a/k^2)x^2+(b/k^2)y^2+(c/k^2)z^2=1,\ lx+my+nz=0, \text{ where } k^2=-p^2/p_0^2$$

or $$Ax^2+By^2+Cz^2=1,\ lx+my+nz=0$$

∴ As in § 11.06 Page 25 of this chapter, the lengths of the axes are given by

$$\frac{l^2}{Ar^2-1}+\frac{m^2}{Br^2-1}+\frac{n^2}{Cr^2-1}=0, \text{ where } A=\frac{a}{k^2}, \text{ etc.}$$

or $$\frac{l^2}{(a/k^2)r^2-1}+\frac{m^2}{(b/k^2)r^2-1}+\frac{n^2}{(c/k^2)r^2-1}=0$$

or $$\frac{l^2}{ar^2-k^2}+\frac{m^2}{br^2-k^2}+\frac{n^2}{cr^2-k^2}=0, \text{ where } k^2=-\frac{p^2}{p_0^2}$$

or $$\frac{l^2}{ar^2p_0^2+p^2}+\frac{m^2}{br^2p_0^2+p^2}+\frac{n^2}{cr^2p_0^2+p^2}=0, \text{ where } p^2 \text{ is given by (v).}$$

Hence proved.

***Ex. 4. The locus of the centres of the sections of the cone $ax^2+by^2+cz^2=0$ such that the sum of the squares of the axes is constant and equal to k^2 is the conicoid**

$$a\left(\frac{1}{b}+\frac{1}{c}\right)x^2+b\left(\frac{1}{c}+\frac{1}{a}\right)y^2+c\left(\frac{1}{a}+\frac{1}{b}\right)z^2+k^2=0.$$

Sol. If (α, β, γ) be the centre, then the plane is

$$T=S_1 \quad \text{or} \quad a\alpha x+b\beta y+c\gamma z=a\alpha^2+b\beta^2+c\gamma^2$$

which is
$$lx+my+nz=p$$

Comparing $l=a\alpha,\ m=b\beta,\ n=c\gamma,\ p=a\alpha^2+b\beta^2+c\gamma^2$

Also as in Ex. 3. above $p_0^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=a\alpha^2+b\beta^2+c\gamma^2$

Also as in Ex. 3. above we can prove that the axes of the section are given by

$$\frac{l^2}{ap_0^2r^2+p^2}+\frac{m^2}{bp_0^2r^2+p^2}+\frac{n^2}{cp_0^2r^2+p^2}=0$$

or
$$\Sigma\, l^2\,(bp_0^2r^2+p^2)\,(cp_0^2r^2+p^2)=0$$

or
$$\Sigma\,[l^2\,\{bcp_0^4r^4+p_0^2p^2r^2\,(b+c)+p^4\}]=0$$

or
$$p_0^4r^4\,(bcl^2+cam^2+abn^2)+p_0^2p^2r^2\,[l^2\,(b+c)+m^2\,(c+a)+n^2\,(a+b)]+p^4\,(l^2+m^2+n^2)=0$$

If r_1^2 and r_2^2 be its roots, then

$$r_1^2+r_2^2=\frac{-p_0^2p^2\,[l^2\,(b+c)+m^2\,(c+a)+n^2\,(a+b)]}{p_0^4\,(bcl^2+cam^2+abn^2)}$$

$$=k^2 \text{ (say)}$$

or
$$k^2=-\frac{p^2}{p_0^2}\,\frac{[a^2\alpha^2\,(b+c)+b^2\beta^2\,(c+a)+c^2\gamma^2\,(a+b)]}{bca^2\alpha^2+cab^2\beta^2+abc^2\gamma^2}, \quad \because\ l=a\alpha, \text{ etc.}$$

$$=-\frac{(a\alpha^2+b\beta^2+c\gamma^2)^2\,[a^2\,(b+c)\,\alpha^2+b^2\,(c+a)\,\beta^2+c^2\,(a+b)\,\gamma^2]}{(a\alpha^2+b\beta^2+c\gamma^2)\,[abc\,(a\alpha^2+b\beta^2+c\gamma^2)]},$$

putting values of p^2 and p_0^2

or
$$\frac{(b+c)\,a\alpha^2}{bc}+\frac{(c+a)\,b\beta^2}{ca}+\frac{(a+b)\,c\gamma^2}{ab}+k^2=0$$

or
$$a\left(\frac{1}{b}+\frac{1}{c}\right)\alpha^2+b\left(\frac{1}{c}+\frac{1}{a}\right)\beta^2+c\left(\frac{1}{a}+\frac{1}{b}\right)\gamma^2+k^2=0$$

$\therefore$ The required locus of (α, β, γ) is

$$a\left(\frac{1}{b}+\frac{1}{c}\right)x^2+b\left(\frac{1}{c}+\frac{1}{a}\right)y^2+c\left(\frac{1}{a}+\frac{1}{b}\right)z^2+k^2=0.$$

Ex. 5. **Prove that the area of the section of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ by the plane $(x/a)+(y/b)+(z/c)=1$ is**

$$(2\pi/3\sqrt{3})\,[b^2c^2+c^2a^2+a^2b^2]^{1/2}.$$ *(Gorakhpur 96)*

Sol. If A_0 be the area of the central section, then

$$A_0=\pi\,abc\left[\frac{\sqrt{(l^2+m^2+n^2)}}{\sqrt{(a^2l^2+b^2m^2+c^2n^2)}}\right]$$ ...See § 11.04 (xii) Page 5

$$=\pi\,abc\left[\frac{\sqrt{(1/a^2)+(1/b^2)+(1/c^2)}}{\sqrt{\{a^2(1/a^2)+b^2(1/b^2)+c^2(1/c^2)\}}}\right],$$

$\because$ Here $l=1/a^2$, $m=1/b^2$, $n=1/c^2$

$$=\pi\sqrt{[(b^2c^2+c^2a^2+a^2b^2)/3]}$$

Also here $p=1$ and $p_0^2=a^2l^2+b^2m^2+c^2n^2$

$$=a^2(1/a^2)+b^2(1/b^2)+c^2(1/c^2)=3$$

$\therefore$ Required area $A=\left(1-\dfrac{p^2}{p_0^2}\right)A_0=(1-\tfrac{1}{3})A_0$

$$=\frac{2\pi}{3\sqrt{3}}\sqrt{(b^2c^2+c^2a^2+a^2b^2)}$$ Proved.

****Ex. 6. Prove that the tangent planes to $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)+1=0$ which cut $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ in ellipses of constant area πk^2 have their points of contact on the surface**

$$(x^2/a^4)+(y^2/b^4)+(z^2/c^4)=k^4/(4a^2b^2c^2).$$ *(Garhwal 96)*

Sol. The equation of the tangent plane to $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)+1=0$ at any point (x_1, y_1, z_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}+1=0 \quad ...(i), \quad \text{where} \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}=-1 \quad ...(ii)$$

Again if A_0 be the area of the corresponding central section $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}-\dfrac{zz_1}{c^2}=0$ of the conicoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$

then $$A=\left(1-\frac{p^2}{p_0^2}\right)A_0 \quad ...(iii)$$

Now $A=\pi k^2$ (given), $p^2=(-1)^2=1$

and $$p_0^2=a^2l^2+b^2m^2+c^2n^2=a^2\left(\frac{x_1}{a^2}\right)^2+b^2\left(\frac{y_1}{b^2}\right)^2-c^2\left(\frac{z_1}{c^2}\right)^2$$ **(Note)**

$$=(x_1^2/a^2)+(y_1^2/b^2)-(z_1^2/c^2)=-1, \text{ from (ii)}$$

Also $A_0=\dfrac{\pi\, abc\, \sqrt{(l^2+m^2+n^2)}}{\sqrt{(a^2l^2+b^2m^2+c^2n^2)}}$ ...See § 11.04 (xii) Page 5

$$=\frac{\pi\, ab\, \sqrt{(-c^2)}\,.\,\sqrt{(l^2+m^2+n^2)}}{\sqrt{(a^2l^2+b^2m^2-c^2n^2)}}, \text{ replacing } c^2 \text{ by } -c^2$$

$$=\frac{\pi\, ab\, \sqrt{(-c^2)}\, \sqrt{[(x_1/a^2)^2+(y_1/b^2)^2+(z_1/c^2)^2]}}{\sqrt{[a^2\,(x_1/a^2)^2+b^2\,(y_1/b^2)^2-c^2\,(z_1/c^2)^2]}},$$

$$\because\ l=x_1/a^2,\ m=y_1/b^2,\ n=z_1/c^2$$

$$=\frac{\pi\, ab\, \sqrt{(-c^2)}\, \sqrt{[(x_1^2/a^4)+(y_1^2/b^4)+(z_1^2/c^4)]}}{\sqrt{[(x_1^2/a^2)+(y_1^2/b^2)-(z_1^2/c^2)]}}$$

$$=\pi\, ab\, \sqrt{(-c^2)}\, \sqrt{[(x_1^2/a^4)+(y_1^2/b^4)+(z_1^2/c^4)]}/\sqrt{(-1)}, \text{ from (ii)}$$

$\therefore$ From (iii) on squaring both sides we get

$$A^2=\left[1-\frac{1}{(-1)}\right]^2 A_0^2, \quad \because\ p^2=1,\ p_0^2=-1$$

or $$\pi^2 k^4=4A_0^2,\ \because A=\pi k^2$$

or $$\pi^2 k^2=4\,\pi^2 a^2 b^2\,(-c^2)\,[(x_1^2/a^4)+(y_1^2/b^4)+(z_1^2/c^4)]/(-1),$$

putting value of A_0

or $$\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}=\frac{k^4}{4a^2b^2c^2}$$

$\therefore$ The locus of (x_1, y_1, z_1) is $\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+\dfrac{z^2}{c^4}=\dfrac{k^4}{4a^2b^2c^2}$ Hence proved.

****Ex. 7. Prove that the areas of the sections of the greatest and least areas of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ which pass through the fixed line $x/l=y/m=z/n$ are $\pi\, abc/r_1$, $\pi\, abc/r_2$ where r_1, r_2 are the axes of the section by the plane $(lx/a)+(my/b)+(nz/c)=0$.**

Sol. Let the equation of the plane through the given line

$$x/l=y/m=z/n \text{ be } \alpha x+\beta y+\gamma z=0 \quad \text{...(i)}$$

where $l\alpha+m\beta+n\gamma=0$ ...(ii), and $\alpha^2+\beta^2+\gamma^2=1$ ...(iii)

The area of the section $=\dfrac{\pi\, abc\, \sqrt{(l^2+m^2+n^2)}}{\sqrt{(a^2l^2+b^2m^2+c^2n^2)}}$ ...See § 11.04 (xii) Page 5

$$=\frac{\pi\, abc\, \sqrt{(\alpha^2+\beta^2+\gamma^2)}}{\sqrt{(a^2\alpha^2+b^2\beta^2+c^2\gamma^2)}}, \text{ from (i)}$$

$$=\pi\, abc/\sqrt{(a^2\alpha^2+b^2\beta^2+c^2\gamma^2)}, \text{ from (iii)} \quad \text{...(iv)}$$

This area will be maximum or minimum according as $a^2\alpha^2+b^2\beta^2+c^2\gamma^2$ is minimum or maximum

Let $$u = a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 \qquad \text{...(v)}$$

where $$l\alpha + m\beta + n\gamma = 0 \quad \text{and} \quad \alpha^2 + \beta^2 + \gamma^2 = 1 \qquad \text{...(vi)}$$

Let $V = (a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2) + \lambda_1 (l\alpha + m\beta + n\gamma) + \lambda_2 (\alpha^2 + \beta^2 + \gamma^2 - 1)$

$\therefore\ dV = (a^2 \,.\, 2\alpha + \lambda_1 l + \lambda_2 \,.\, 2\alpha)\, d\alpha + \ldots + \ldots = 0$

Equating to zero the coefficients of $d\alpha$, $d\beta$ and $d\gamma$, we get

$2a^2\alpha + \lambda_1 l + 2\lambda_2\alpha = 0,\ 2b^2\beta + \lambda_1 m + 2\lambda_2\beta = 0$

$$\text{and} \quad 2c^2\gamma + \lambda_1 n + 2\lambda_2\gamma = 0 \qquad \text{...(vi)}$$

Multiplying the above relations by α, β, γ and adding, we get

$2(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2) + \lambda_1 (l\alpha + m\beta + n\gamma) + 2\lambda_2 (\alpha^2 + \beta^2 + \gamma^2) = 0$

or $\quad 2(u) + \lambda_1 (0) + 2\lambda_2 (1)$, from (v) and (vi)

or $\quad \lambda_2 = -u$

Hence from (vi), we get $2\alpha (a^2 - u) + \lambda_1 l = 0$,

$$2\beta (b^2 - u) + \lambda_1 m = 0,\ 2\gamma (c^2 - u) + \lambda_1 n = 0$$

$$\Rightarrow \alpha = \frac{-\lambda_1 l}{2(a^2 - u)};\ \beta = \frac{-\lambda_1 m}{2(b^2 - u)},\ \gamma = \frac{-\lambda_1 n}{2(c^2 - u)}$$

Putting these values of α, β, γ in (ii), we have

$$\frac{l^2}{(a^2 - u)} + \frac{m^2}{(b^2 - u)} + \frac{n^2}{(c^2 - u)} = 0 \qquad \text{...(viii)}$$

Also we know that the axes of the section of the conicoid $ax^2 + by^2 + cz^2 = 1$ by the plane $lx + my + nz = 0$ is given by

$$\frac{l^2}{ar^2 - 1} + \frac{m^2}{br^2 - 1} + \frac{n^2}{cr^2 - 1} = 0$$

Hence the axes of the section of the given ellipsoid by the given plane $(l/a)\,x + (m/b)\,y + (n/c)\,z = 0$ is given by

$$\frac{(l/a)^2}{(1/a^2)\,r^2 - 1} + \frac{(m/b)^2}{(1/b^2)\,r^2 - 1} + \frac{(n/c)^2}{(1/c^2)\,r^2 - 1} = 0 \qquad \textbf{(Note)}$$

or $$\frac{l^2}{a^2 - r^2} + \frac{m^2}{b^2 - r^2} + \frac{n^2}{c^2 - r^2} = 0 \qquad \text{...(ix)}$$

If its roots be r_1^2 and r_2^2, then comparing (viii) and (ix) we find that

$$u = r_1^2 \quad \text{or} \quad r_2^2.$$

Hence from (iv) the area is $\dfrac{\pi abc}{\sqrt{u}}$ *i.e.* $\dfrac{\pi abc}{r_1}$ or $\dfrac{\pi abc}{r_2}$.

***Ex. 8. If OP, OQ, OR are conjugate semi-diameters of an ellipsoid, then prove that the area of the section of the ellipsoid by the plane PQR is two-thirds the area of the parallel central section.**

Sol. From Chapter IX on Conicoids we know that the equation of the plane PQR through the extremities of the three conjugate semi-diameters is

$$\left(\frac{x_1+x_2+x_3}{a^2}\right)x+\left(\frac{y_1+y_2+y_3}{b^2}\right)y+\left(\frac{z_1+z_2+z_3}{c^2}\right)z=1.$$

or $lx+my+nz=1$, say, where $l=(x_1+x_2+x_3)/a^2$, etc.

The equation of the plane parallel to it and through the centre is

$$lx+my+nz=0$$

Again $\dfrac{A}{A_0}=1-\dfrac{p^2}{p_0^2}=1-\dfrac{1}{p_0^2}$, as here $p=1$ ...(i)

and $p_0^2=a^2l^2+b^2m^2+c^2n^2$, for the given ellipsoid

$$=a^2\left(\frac{x_1+x_2+x_3}{a^2}\right)^2+b^2\left(\frac{y_1+y_2+y_3}{b^2}\right)^2+c^2\left(\frac{z_1+z_2+z_3}{c^2}\right)^2$$

$$=\frac{(x_1+x_2+x_3)^2}{a^2}+\frac{(y_1+y_2+y_3)^2}{b^2}+\frac{(z_1+z_2+z_3)^2}{c^2}$$

$$=\frac{x_1^2+x_2^2+x_3^2}{a^2}+\frac{y_1^2+y_2^2+y_3^2}{b^2}+\frac{z_1^2+z_2^2+z_3^2}{c^2}$$

$$+2\left(\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}\right)+2\,\Sigma\left(\frac{x_2x_3}{a^2}\right)+2\,\Sigma\left(\frac{x_3x_1}{a^2}\right)$$

$$=1+1+1+2\,(0)+2\,(0)+2\,(0),$$

$$\because x_1^2+x_2^2+x_3^2=a^2 \text{ etc. and } \frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0, \text{ etc.}$$

(See Chapter IX on Conicoids)

or $p_0^2=3$.

$\therefore$ From (i), $A/A_0=1-(1/3)=2/3$ or $A=(2/3)\,A_0$ Hence proved.

***Ex. 9. Through a given point (α,β,γ) planes are drawn parallel to three conjugate diametral planes of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$. Show that sum of the ratios of the areas of the sections by these planes to the areas of the parallel diametral planes is**

$$3-(\alpha^2/a^2)-(\beta^2/b^2)-(\gamma^2/c^2).$$

Sol. Equations of the conjugate diametral planes are

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0, \text{ etc.} \quad \text{...(i)}$$

and that of a parallel plane through (α,β,γ) is

$$(x-\alpha)\frac{x_1}{a^2}+(y-\beta)\frac{y_1}{b^2}+(z-\gamma)\frac{z_1}{c^2}=0$$

or
$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=\frac{\alpha x_1}{a^2}+\frac{\beta y_1}{b^2}+\frac{\gamma z_1}{c^2}=p_1, \text{ say} \quad \text{...(ii)}$$

If A_1 be the area of the section of the given ellipsoid by the plane (i) and A_1' be the corresponding area of the section by (ii), then we have

$$A_1' = \left(1 - \frac{p^2}{p_0^2}\right) A_1 \quad \text{or} \quad \frac{A_1'}{A_1} = 1 - \frac{p^2}{p_0^2} \qquad \text{...(iii)}$$

where $\quad p_0^2 = \dfrac{l^2}{a} + \dfrac{m^2}{b} + \dfrac{n^2}{c}$, when the conicoid is $ax^2 + by^2 + cz^2 = 1$

$$= \frac{(x_1/a^2)^2}{(1/a^2)} + \frac{(y_1/b^2)^2}{(1/b^2)} + \frac{(z_1/c^2)^2}{(1/c^2)} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1$$

$\therefore$ From (iii), $\dfrac{A_1'}{A_1} = 1 - \dfrac{p_1^2}{1} = 1 - p_1^2 = 1 - \left(\dfrac{\alpha x_1}{a^2} + \dfrac{\beta y_1}{b^2} + \dfrac{\gamma z_1}{c^2}\right)^2$, from (ii)

In a similar manner the values of A'_2/A_2 and A'_3/A_3 can be written and adding these results we have the required sum of the ratios of the areas

$$= \left[1 - \left(\frac{\alpha x_1}{a^2} + \frac{\beta y_1}{b^2} + \frac{\gamma z_1}{c^2}\right)\right]^2 + \left[1 - \left(\frac{\alpha x_2}{a^2} + \frac{\beta y_2}{b^2} + \frac{\gamma z_2}{c^2}\right)^2\right]$$

$$+ \left[1 - \left(\frac{\alpha x_3}{a^2} + \frac{\beta y_3}{b^2} + \frac{\gamma z_3}{c^2}\right)^2\right]$$

$$= 3 - \left[\left\{\frac{\alpha^2}{a^4}(x_1^2 + x_2^2 + x_3^2) + \frac{\beta^2}{b^4}(y_1^2 + y_2^2 + y_3^2) + \frac{\gamma^2}{c^4}(z_1^2 + z_2^2 + z_3^2)\right\}\right.$$

$$\left. + \frac{2\alpha\beta}{a^2 b^2}(x_1 y_1 + x_2 y_2 + x_3 y_3) + \ldots + \ldots\right]$$

$$= 3 - \left[\left\{\frac{\alpha^2}{a^4}(a^2) + \frac{\beta^2}{b^4}(b^2) + \frac{\gamma^2}{c^4}(c^2)\right\} + \frac{2\alpha\beta}{a^2 b^2}(0) + 0 + 0\right],$$

$\because x_1^2 + x_2^2 + x_3^2 = a^2$, etc. and $x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$, etc.

...See Ch. IX on Conicoids

$= 3 - [(\alpha^2/a^2) + (\beta^2/b^2) + (\gamma^2/c^2)]$ Hence proved.

***Ex. 10. Prove that if l_1, m_1, n_1 ; l_2, m_2, n_2 are the direction cosines of the axes of any plane section of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$, then $\dfrac{l_1 l_2}{a^2(b^2 - c^2)} = \dfrac{m_1 m_2}{b^2(c^2 - a^2)} = \dfrac{n_1 n_2}{c^2(a^2 - b^2)}$.**

Sol. We know that the direction ratios L, M, N of the axes $x/L = y/M = z/N$ of the section of the conicoid $ax^2 + by^2 + cz^2 = 1$ by any plane $lx + my + nz = p$ are given by

$$\frac{L\{(a/k^2)r^2-1\}}{l}=\frac{M\{(b/k^2)r^2-1\}}{m}=\frac{N\{(c/k^2)r^2-1\}}{n}$$

...See § 11.06 (xi) Page 27 Ch. XI

$\therefore$ The direction ratios of the axes $x/L=y/M=z/N$ of the section of the given ellipsoid by any plane $lx+my+nz=p$ are given by

$$\frac{L\{(r^2/a^2k^2)-1\}}{l}=\frac{M\{(r^2/b^2k^2)-1\}}{m}=\frac{N\{(r^2/c^2k^2)-1\}}{n}=\lambda \text{ (say)},$$

putting $1/a^2$, $1/b^2$, $1/c^2$, for a, b, c respectively.

This gives $(r^2-a^2k^2)/(la^2k^2)=\lambda/L$ or $(r^2-a^2k^2)/(\lambda k^2)=la^2/L$

Similarly $(r^2-b^2k^2)/(\lambda k^2)=mb^2/M$, $(r^2-c^2k^2)/(\lambda k^2)=nc^2/N$

Multiplying these by (b^2-c^2), (c^2-a^2) and (a^2-b^2) respectively and adding we get $\dfrac{la^2(b^2-c^2)}{L}+\dfrac{mb^2(c^2-a^2)}{M}+\dfrac{nc^2(a^2-b^2)}{N}$

$$=\frac{1}{\lambda k^2}[(r^2-a^2k^2)(b^2-c^2)+(r^2-b^2k^2)(c^2-a^2)+(r^2-c^2k^2)(a^2-b^2)]$$

$$=(1/\lambda k^2)[0]=0$$

i.e. $$\frac{la^2(b^2-c^2)}{L}+\frac{mb^2(c^2-a^2)}{M}+\frac{nc^2(a^2-b^2)}{N}=0 \quad \text{...(i)}$$

Also as $x/L=y/M=z/N$ lies on the plane $lx+my+nz=p$, so we have

$$lL+mM+nN=0 \quad \text{...(ii)}$$

Eliminating N between (i) and (ii), we get

$$\frac{la^2(b^2-c^2)}{L}+\frac{mb^2(c^2-a^2)}{M}+\frac{nc^2(a^2-b^2)}{[-(lL+mM)/n]}=0$$

or $$la^2(b^2-c^2)M(lL+mM)+mb^2(c^2-a^2)L(lL+mM)-n^2c^2(a^2-b^2)LM=0$$

or $$lmb^2(c^2-a^2)L^2+[l^2a^2(b^2-c^2)+m^2b^2(c^2-a^2)-n^2c^2(a^2-b^2)]LM+[lma^2(b^2-c^2)]M^2=0$$

or $$lmb^2(c^2-a^2)(L/M)^2+[l^2a^2(b^2-c^2)+m^2b^2(c^2-a^2)-n^2c^2(a^2-b^2)](L/M)+lma^2(b^2-c^2)=0 \quad \text{...(iii)}$$

$\therefore$ If L_1, M_1, N_1 and L_2, M_2, N_2 be the direction-cosines of the axes, then L_1/M_1 and L_2/M_2 are the roots of (iii) and so

$$\frac{L_1}{M_1}\cdot\frac{L_2}{M_2}=\frac{lma^2(b^2-c^2)}{lmb^2(c^2-a^2)}=\frac{a^2(b^2-c^2)}{b^2(c^2-a^2)}$$

$$\therefore \frac{L_1L_2}{a^2(b^2-c^2)}=\frac{M_1M_2}{b^2(c^2-a^2)}=\frac{N_1N_2}{c^2(a^2-b^2)}, \text{ by symmetry}$$

Now replacing L_1, M_1, N_1 and L_2, M_2, N_2 by the given quantities l_1, m_1, n_1 and l_2, m_2, n_3 respectively, we get the required result.

***Ex. 11. The normal section of an envelopoing cylinder of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ has a given area πk^2. Prove that the planes of contact of the cylinder and ellipsoid touch the cone**

$$\frac{x^2}{a^4(b^2c^2-k^4)}+\frac{y^2}{b^4(c^2a^2-k^2)}+\frac{z^2}{c^4(a^2b^2-k^4)}=0.$$

Sol. If the generators of the enveloping cylinder be parallel to the line $x/l=y/m=z/n$, then the equation of the envelopoing cylinder is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)=\left(\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}\right)^2 \quad ...(i)$$

...See Section on Enveloping cylinder.

Now the plane of contact is the diametral plane of $x/l=y/m=z/n$ and so the equation of the plane of contact is

$$(lx/a^2)+(my/b^2)+(nz/c^2)=0 \quad ...(ii)$$

Also we know that the area of the section of the given ellipsoid by the plane $lx+my+nz=0$ is $\dfrac{\pi abc\sqrt{(l^2+m^2+n^2)}}{\sqrt{(a^2l^2+b^2m^2+c^2n^2)}}$

$\therefore$ If A be the area of the section of the given ellipsoid by the plane (ii), then

$$A=\frac{\pi abc\sqrt{[(l/a^2)^2+(m/b^2)^2+(n/c^2)^2]}}{\sqrt{[a^2(l/a^2)^2+b^2(m/b^2)^2+c^2(n/c^2)^2]}}$$

or

$$A=\frac{\pi abc\sqrt{[(l^2/a^4)+(m^2/b^4)+(n^2/c^4)]}}{\sqrt{[(l^2/a^2)+(m^2/b^2)+(n^2/c^2)]}} \quad ...(iii)$$

Now the equation of the central normal section of this cylinder is

$$lx+my+nz=0 \quad ...(iv)$$

Here the central normal section has been taken, since all the normal sections of a cylinder are equal.

If θ be the angle between the planes (ii) and (iv), then

$$\cos\theta=\frac{l.(l/a^2)+m.(m/b^2)+n.(n/c^2)}{\sqrt{[(l^2+m^2+n^2).\{(l/a^2)^2+(m/b^2)^2+(n/c^2)^2\}]}} \quad \text{(Note)}$$

or

$$\cos\theta=\frac{[(l^2/a^2)+(m^2/b^2)+(n^2/c^2)]}{\sqrt{[(l^2+m^2+n^2)\{(l^2/a^4)+(m^2/b^4)+(n^2/c^4)\}]}} \quad ...(v)$$

Now if A' be the area of the normal section, then evidently

$$A'=A\cos\theta=\pi k^2 \text{ (given)}$$

or $\pi k^2=A.\cos\theta=\pi abc\dfrac{\sqrt{[(l^2/a^2)+(m^2/b^2)+(n^2/c^2)]}}{\sqrt{(l^2+m^2+n^2)}}$, from (iii) and (v) or $\quad k^4(l^2+m^2+n^2)=a^2b^2c^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)$, on squaring

or $\quad (b^2c^2-k^4)\,l^2+(c^2a^2-k^4)\,m^2+(a^2b^2-k^4)\,n^2=0$.

or $\quad a^4(b^2c^2-k^4)\cdot\frac{l^2}{a^4}+b^4(c^2a^2-k^4)\cdot\frac{m^2}{b^4}+c^4(a^2b^2-k^4)\cdot\frac{n^2}{c^4}=0 \quad$ (Note)

This shows that the normal to the plane (ii) generates the cone

$$a^4(b^2c^2-k^4)\,x^2+b^4(c^2a^2-k^4)\,y^2+c^2(a^2b^2-k^4)\,z^2=0 \qquad \text{...(vi)}$$

Hence the plane of contact given by (ii) touches the cone reciprocal to (vi) *i.e.* the cone $\dfrac{x^2}{a^4(b^2c^2-k^4)}+\dfrac{y^2}{b^4(c^2a^2-k^4)}+\dfrac{z^2}{c^4(a^2b^2-k^4)}=0$

Ex. 12. Prove that the area of the section of the cone $\mathbf{bcx^2+cay^2+abz^2=0}$ by the plane $\mathbf{lx+my+nz=p}$ is

$$\pi \mathbf{p^2\sqrt{(abc)}/(al^2+bm^2+cn^2)^{3/2}}.$$

Sol. Let the equation of the given cone be $Ax^2+By^2+Cz^2=0$, where

$$A=bc,\ B=ca,\ C=ab.$$

Also from Ex. 4. Page 33 of this chapter, we know that r_1^2 and r_2^2 are the roots of the equation $p_0^4r^4(bcl^2+cam^2+abn^2)+p_0^2p^2r^2\,[\Sigma\, l^2(b+c)]$

$$+\,p^4(l^2+m^2+n^2)=0$$

$\therefore\ r_1^2r_2^2=p^4(l^2+m^2+n^2)/[p_0^4(bcl^2+cam^2+abn^2)]$

$$=\frac{p^4(l^2+m^2+n^2)}{p_0^4(BCl^2+CAm^2+ABn^2)}=\frac{p^4(l^2+m^2+n^2)}{p_0^4\,abc\,(al^2+bm^2+cn^2)} \qquad \text{...(i)}$$

where $\quad p_0^2=\dfrac{l^2}{a}+\dfrac{m^2}{b}+\dfrac{n^2}{c}=\dfrac{l^2}{A}+\dfrac{m^2}{B}+\dfrac{n^2}{C}$

$$=\frac{l^2}{bc}+\frac{m^2}{ca}+\frac{n^2}{ab}=\frac{1}{abc}(al^2+bm^2+cn^2) \qquad \text{...(ii)}$$

$\therefore$ Required area $=\pi r_1r_2$

$$=\frac{\pi p^2\sqrt{(l^2+m^2+n^2)}}{p_0^2\sqrt{(abc)}\sqrt{(al^2+bm^2+cn^2)}}$$

$$=\frac{\pi p^2\cdot 1}{(1/abc)(al^2+bm^2+cn^2)\cdot\sqrt{[abc\,(al^2+bm^2+cn^2)]}}, \text{ from (ii)}$$

$=\pi p^2\sqrt{(abc)}/(al^2+bm^2+cn^2)^{3/2}$. Hence proved.

****§ 11.07. Circular Sections of an ellipsoid.**

We know that the parallel sections of an ellipsoid are similar and so we consider only sections through the centre.

Let the equation of the ellipsoid be $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ ...(i)

which can be rewritten as

$$\frac{1}{a^2}(x^2+y^2+z^2-a^2)+\left(\frac{1}{b^2}-\frac{1}{a^2}\right)y^2+\left(\frac{1}{c^2}-\frac{1}{a^2}\right)z^2=0 \quad \text{...(ii)}$$

Now if the equation of a conicoid be $C=0$, that of a sphere be $S=0$ and $u=0$, $v=0$ be the equations of two planes then $C=S+\lambda uv$ **(Note)**

From here we find that the common points of the conicoid and the two planes lie on the sphere and so we conclude that the sections of the conicoid by the planes are circles as every plane meets a sphere in a circle.

$\therefore$ From (ii) we find that the two planes given by

$$\left(\frac{1}{b^2}-\frac{1}{a^2}\right)y^2+\left(\frac{1}{c^2}-\frac{1}{a^2}\right)z^2=0 \quad \text{...(iii)}$$

meet the ellipsoid (i) where they meet the sphere $x^2+y^2+z^2-a^2=0$ ...(iv)

$\therefore$ The sections of the ellipsoid by the plane (iii) are circles of radius a.

Again the equation (i) of the ellipsoid can be written in following two ways also :—

$$\frac{1}{b^2}(x^2+y^2+z^2-b^2)+\left(\frac{1}{a^2}-\frac{1}{b^2}\right)x^2+\left(\frac{1}{c^2}-\frac{1}{b^2}\right)z^2=0 \quad \text{...(v)}$$

and

$$\frac{1}{c^2}(x^2+y^2+z^2-c^2)+\left(\frac{1}{a^2}-\frac{1}{c^2}\right)x^2+\left(\frac{1}{b^2}-\frac{1}{c^2}\right)y^2=0 \quad \text{...(vi)}$$

Again arguing as above, we find the section of the ellipsoid (i) by the pair of planes

$$\left(\frac{1}{a^2}-\frac{1}{b^2}\right)x^2+\left(\frac{1}{c^2}-\frac{1}{b^2}\right)z^2=0 \quad \text{...(vii)}$$

and

$$\left(\frac{1}{a^2}-\frac{1}{c^2}\right)x^2+\left(\frac{1}{b^2}-\frac{1}{c^2}\right)y^2=0 \quad \text{...(viii)}$$

are also circles of radii b and c respectively.

[**Note.** Here as the plane has been taken to pass through the centre, so the section is a great circle whose radius is the same as that of the sphere.]

If $a^2>b>c^2$ *i.e.* $\frac{1}{a^2}<\frac{1}{b^2}<\frac{1}{c^2}$, then $\frac{1}{a^2}-\frac{1}{b^2}$ is negative and $\frac{1}{c^2}-\frac{1}{b^2}$ is positive and so the planes given by (vii) are real. Thus real circular sections are obtained by planes (vii) *i.e.* $\dfrac{(a^2-b^2)}{a^2b^2}x^2 = \dfrac{(b^2-c^2)}{b^2c^2}z^2$ or $\dfrac{x}{a}\sqrt{(a^2-b^2)} \pm \dfrac{z}{c}\sqrt{(b^2-c^2)} = 0$ ...(ix)

Again as the parallel plane sections of a conicoid are similar, so the planes

$$\frac{\mathbf{x}}{\mathbf{a}}\sqrt{(\mathbf{a}^2-\mathbf{b}^2)} + \frac{\mathbf{z}}{\mathbf{c}}\sqrt{(\mathbf{b}^2-\mathbf{c}^2)} = \lambda, \quad \frac{\mathbf{x}}{\mathbf{a}}\sqrt{(\mathbf{a}^2-\mathbf{b}^2)} - \frac{\mathbf{z}}{\mathbf{c}}\sqrt{(\mathbf{b}^2-\mathbf{c}^2)} = \mu$$

which are parallel to the planes given by (ix) will cut the ellipsoid (i) in circles for all values of λ and μ.

***Condition for the section of ellipsoid (i) by the plane $lx+my+nz=p$ to be a circle.**

If the section by the given plane $lx+my+nz=p$ of the ellipsoid (i) is a circle, then the section by a parallel plane $lx+my+nz=0$ is also a circle.

Also we know that the real central circular sections of the ellipsoid (i) are given by (ix)$'$ above.

$\therefore$ On comparing we get the required condition is

$$\frac{l}{\sqrt{(a^2-b^2)}/a} = \frac{m}{0} = \frac{n}{\pm\sqrt{(b^2-c^2)}/c}$$

or

$$\frac{al}{\sqrt{(a^2-b^2)}} = \frac{m}{0} = \frac{cn}{\pm\sqrt{(b^2-c^2)}} \quad \text{...(x)}$$

Cor. *Any two circular sections of an ellipsoid which are not parallel lie on a sphere.*

Let the equations of the planes of any two circulr sections of ellipsoid which are not parallel be $\dfrac{x}{a}\sqrt{(a^2-b^2)} + \dfrac{z}{c}\sqrt{(b^2-c^2)} = \lambda$ ...(xi)

and $\dfrac{x}{a}\sqrt{(a^2-b^2)} - \dfrac{z}{c}\sqrt{(b^2-c^2)} = \mu$ ...(xii)

The conicoid given by

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right) + k\left\{\frac{x}{a}\sqrt{(a^2-b^2)} + \frac{z}{c}\sqrt{(b^2-c^2)} - \lambda\right\}$$

$$\times\left\{\frac{x}{a}\sqrt{(a^2-b^2)} - \frac{z}{c}\sqrt{(b^2-c^2)} - \mu\right\} = 0 \quad \text{...(xiii)}$$

for all values of k passing through the two circular sections given by (xi) and (xii) will represent a sphere if k can be so chosen that the coefficients of x^2, y^2 and z^2 are equal and there are no terms of xy, yz or zx.

Thus we must choose k, such that the coefficients of x^2, y^2 and z^2 in (xiii) are equal

i.e.

$$\frac{1}{a^2} + \frac{k(a^2-b^2)}{a} = \frac{1}{b^2} = \frac{1}{c^2} - \frac{k(b^2-c^2)}{c^2} \quad \text{...(xiv)}$$

If we choose $k = 1/b^2$, then the above equations (xiv) are satisfied and each coefficients is equal to $1/b^2$ and the equations of the sphere from (xiii) is

$$x^2 + y^2 + z^2 - \frac{(\lambda + \mu)\sqrt{(a^2 - b^2)}}{a} x + \frac{(\lambda - \mu)\sqrt{(b^2 - c^2)}}{c} z + \lambda\mu - b^2 = 0 \quad \text{...(xv)}$$

Solved Examples on § 11.07.

***Ex. 1. Find the radius of the circle in which the plane $\sqrt{(a^2 - b^2)}\,\frac{x}{a} + \sqrt{(b^2 - c^2)}\,\frac{z}{c} = \lambda$ cuts the ellipsoid**

$$(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1. \quad \textit{(Avadh 93; Gorakhpur 96)}$$

Sol. From § 11.07 (ix) Page 43 Ch. XI we know that the pair of planes

$$\frac{x}{a}\sqrt{(a^2 - b^2)} \pm \frac{z}{c}\sqrt{(b^2 - c^2)} = 0$$

cuts the given ellipsoid in a circle of radius b.

The given plane is parallel to one of these planes and we know that parallel plane sections are similar.

$\therefore$ The radius of the circle of parallel section will be

$$b\sqrt{[1 - (p^2/p_0^2)]}, \quad \text{...(i)}$$

where $\quad p = \lambda \quad$ and $\quad p_0^2 = a^2 l^2 + b^2 m^2 + c^2 n^2$

i.e. $\quad p_0^2 = a^2 \frac{(a^2 - b^2)}{a^2} + b^2 (0) + c^2 \frac{(b^2 - c^2)}{c^2} = a^2 - c^2 \quad$ **(Note)**

$\therefore$ From (i), the required radius $= b\sqrt{\left[1 - \frac{\lambda^2}{a^2 - c^2}\right]}$ **Ans.**

Ex. 2. Prove that the radius of a circular section of the ellipsoid $\Sigma (x^2/a^2) = 1$ at a distance p from the centre is $b\sqrt{[1 - (p^2 b^2/a^2 c^2)]}$.

Sol. From Ex. 1. above we know (to be proved in exam.) that the radius r of the circular section $\quad (x/a)\sqrt{(a^2 - b^2)} + (z/c)\sqrt{(b^2 - c^2)} = \lambda \quad$ ...(i)

is given by $\quad r = b\sqrt{\left[1 - \frac{\lambda^2}{a^2 - c^2}\right]} \quad$ ...(ii)

Here p is the distance of the plane (i) from centre (0, 0, 0) of the ellipsoid and so $\quad p = \frac{\lambda}{\sqrt{[\{(a^2 - b^2)/a^2\} + \{(b^2 - c^2)/c^2\}]}}$

or $\quad p^2 = \frac{\lambda^2}{\left(\frac{a^2 - b^2}{a^2}\right) + \left(\frac{b^2 - c^2}{c^2}\right)} = \frac{\lambda^2}{b^2\left(\frac{1}{c^2} - \frac{1}{a^2}\right)} = \frac{\lambda^2 a^2 c^2}{b^2 (a^2 - c^2)}$

or $$\frac{b^2p^2}{a^2c^2}=\frac{\lambda^2}{a^2-c^2}$$

$\therefore$ From (ii), the required radius $r=b\sqrt{\left[1-\frac{b^2p^2}{a^2c^2}\right]}$ Hence proved.

****Ex. 3. Show that the circular sections of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ passing through one extremity of x-axis are both of radius r, where $\frac{r^2}{b^2}=\frac{b^2-c^2}{a^2-c^2}$.** *(Garhwal 94)*

Sol. We know [See § 11.07 (xi) and (xii) Page 43] that the real circular sections of the ellipsoid are $$\frac{x}{a}\sqrt{(a^2-b^2)}+\frac{z}{c}\sqrt{(b^2-c^2)}=\lambda \quad \text{...(i)}$$

and $$\frac{x}{a}\sqrt{(a^2-b^2)}-\frac{z}{c}\sqrt{(b^2-c^2)}=\mu \quad \text{...(ii)}$$

Also as in Ex. 1 Page 44 of this chapter we can prove that the radius r of the circle in which the plane (i) cuts the given ellipsoid is given by $r=b\sqrt{[1-\{\lambda^2/(a^2-c^2)\}]}$ and of that in which the plane (ii) cuts the given ellipsoid is given by $r=b\sqrt{\left[1-\frac{\mu^2}{a^2-c^2}\right]}$, on replacing c by $-c$ and λ by μ.

Again as the plane (i) passes through an extremity $(a, 0, 0)$ of x-axis, so $$\sqrt{(a^2-b^2)}=\lambda$$

$$\therefore r=b\sqrt{\left[1-\frac{a^2-b^2}{a^2-c^2}\right]}=b\sqrt{\left[\frac{b^2-c^2}{a^2-c^2}\right]} \quad \text{or} \quad \frac{r^2}{b^2}=\frac{b^2-c^2}{a^2-c^2}$$

Similarly if the plane (ii) passes through the extremity $(a, 0, 0)$ of x-axis, then $$\sqrt{(a^2-b^2)}=\mu$$

$\therefore$ and so $$r=b\sqrt{\left[1-\frac{\mu^2}{a^2-c^2}\right]}=b\sqrt{\left[1-\frac{a^2-b^2}{a^2-c^2}\right]}=b\sqrt{\left(\frac{b^2-c^2}{a^2-c^2}\right)}$$

or $$r^2/b^2=(b^2-c^2)/(a^2-c^2)$$ Hence proved.

****Ex. 4. Prove that the real central circular sections of the ellipsoid $x^2+2y^2+6z^2=8$ are given by $x\pm 2z=0$.** *(Gorakhpur 95)*

Sol. The given equation of the ellipsoid can be written as $$[x^2+2y^2+6z^2-\lambda(x^2+y^2+z^2)]+\lambda(x^2+y^2+z^2)-8=0$$

The equation $x^2+2y^2+6z^2-\lambda(x^2+y^2+z^2)=0$

or $\quad (1-\lambda)\,x^2+(2-\lambda)\,y^2+(6-\lambda)\,z^2=0$ represents a pair of planes if

$$(1-\lambda)\,(2-\lambda)\,(6-\lambda)=0 \qquad \textbf{(Note)}$$

or $$\lambda=1,2,6$$

Now as only the mean value of λ gives real planes, so taking $\lambda=2$, we have $\quad x^2+2y^2+6z^2-2\,(x^2+y^2+z^2)=-x^2+4z^2=-(x+2z)\,(x-2z)$

Hence the required real central circular sections are given by

$$x+2z=0 \quad \text{and} \quad x-2z=0 \quad \text{or} \quad x\pm 2z=0 \qquad \textbf{Ans.}$$

***Ex. 5. Find the locus of the centres of a sphere of constant radius k which cut the ellipsoid $\Sigma\,(x^2/a^2)=1$ in a pair of circles.**

Sol. We know that any two circular sections of the given ellipsoid by the planes $\quad \dfrac{x}{a}\sqrt{(a^2-b^2)}+\dfrac{z}{c}\sqrt{(b^2-c^2)}=\lambda,\ \dfrac{x}{a}\sqrt{(a^2-b^2)}-\dfrac{z}{c}\sqrt{(b^2-c^2)}=\mu$

lie on a sphere whose equation [See § 11.07 (xv) Page 44 Ch. XI] is

$$x^2+y^2+z^2-(\lambda+\mu)\frac{\sqrt{(a^2-b^2)}}{a}\,x+(\lambda-\mu)\frac{\sqrt{(b^2-c^2)}}{c}\,z+\lambda\,\mu-b^2=0 \quad \text{...(i)}$$

If (f,g,h) be its centre, then

$$f=\frac{(\lambda+\mu)\,\sqrt{(a^2-b^2)}}{2a},\ g=0,\ h=\frac{-(\lambda-\mu)\,\sqrt{(b^2-c^2)}}{2c} \quad \text{...(ii)}$$

$$\therefore \left[\frac{2af}{\sqrt{(a^2-b^2)}}\right]^2-\left[\frac{2ch}{\sqrt{(b^2-c^2)}}\right]^2=(\lambda+\mu)^2-(\lambda-\mu)^2=4\lambda\mu \quad \text{...(iii)}$$

Again according to the problem radius of the sphere $=k$, constant.

$$\therefore\ k^2=\text{“}g^2+f^2-c\text{”}=\frac{(\lambda+\mu)^2\,(a^2-b^2)}{a^2}+\frac{(\lambda-\mu)^2\,(b^2-c^2)}{c^2}$$
$$-(\lambda\,\mu-b^2),\ \text{from (i)}$$

or $\quad k^2=f^2+h^2-(\lambda\mu-b^2)$, from (ii)

or $\quad f^2+h^2+b^2-k^2=\lambda\,\mu=\dfrac{a^2f^2}{a^2-b^2}-\dfrac{c^2h^2}{b^2-c^2}$, from (iii)

or $$f^2\left(\frac{a^2}{a^2-b^2}-1\right)-h^2\left(\frac{c^2}{b^2-c^2}+1\right)=b^2-k^2$$

or $$\frac{b^2f^2}{a^2-b^2}-\frac{h^2b^2}{b^2-c^2}=b^2\left(1-\frac{k^2}{b^2}\right) \quad \text{or} \quad \frac{f^2}{a^2-b^2}-\frac{h^2}{b^2-c^2}=1-\frac{k^2}{b^2}$$

$\therefore$ The locus of the centre (f,g,h) is

$$\frac{x^2}{a^2-b^2}-\frac{z^2}{b^2-c^2}=1-\frac{k^2}{b^2},\ y=0$$ **Ans.**

Ex. 6. Find the locus of the centres of the spheres which pass through the origin and cut the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ in a pair of real circles.

Sol. As in Ex. 5. above, the equation of the sphere is

$$x^2+y^2+z^2-\frac{(\lambda+\mu)\sqrt{(a^2-b^2)}}{a}x+\frac{(\lambda-\mu)\sqrt{(b^2-c^2)}}{c}z+\lambda\mu-b^2=0$$ ...(i)

If it passes through the origin, then $\lambda\mu=b^2$ ...(ii)

Again if (f, g, h) be the centre of the sphere (i), then as in Ex. 5 above we can prove (To be proved in the exam.) that

$$\left[\frac{2af}{\sqrt{(a^2-b^2)}}\right]^2-\left[\frac{2ch}{\sqrt{(b^2-c^2)}}\right]^2=4\lambda\mu \text{ and } g=0$$ ...See (iii) of Ex. 5 above

or $$\frac{a^2f^2}{a^2-b^2}-\frac{c^2h^2}{b^2-c^2}=b^2 \text{ and } g=0, \text{ by (ii)}$$

$\therefore$ The locus of the centre (f, g, h) is

$$\frac{a^2x^2}{a^2-b^2}-\frac{c^2z^2}{b^2-c^2}=b^2 \quad \text{and} \quad y=0.$$ **Ans.**

[**Note.** This locus will represent a hyperbola if $a^2-b^2>0$ and $b^2-c^2>0$ *i.e.* if $a^2>b^2>c^2$.]

***Ex. 7. If p_1, p_2, p_3 and π_1, π_2, π_3 be the perpendiculars from the extremities P_1, P_2, P_3 of conjugate semi-diameters on the two central circular sections of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$, then evaluate $p_1\pi_1+p_2\pi_2+p_3\pi_3$.**

Sol. We know that the two real circular sections of the given ellipsoid are

$$\frac{x}{a}\sqrt{(a^2-b^2)}\pm\frac{z}{c}\sqrt{(b^2-c^2)}=0$$

According to the problem, p_1 and π_1 are their perpendicular distances from the point $P_1(x_1, y_1, z_1)$.

$$\therefore p_1\pi_1=\frac{\left[\frac{x_1}{a}\sqrt{(a^2-b^2)}+\frac{z_1}{c}\sqrt{(b^2-c^2)}\right]}{\sqrt{\left(\frac{a^2-b^2}{a^2}+\frac{b^2-c^2}{c^2}\right)}}\times\frac{\left[\frac{x_1}{a}\sqrt{(a^2-b^2)}-\frac{z_1}{c}\sqrt{(b^2-c^2)}\right]}{\sqrt{\left[\frac{a^2-b^2}{a^2}+\frac{b^2-c^2}{c^2}\right]}}$$

$$= \frac{\frac{x_1^2}{a^2}(a^2-b^2) - \frac{z_1^2}{c^2}(b^2-c^2)}{\left(\frac{b^2}{c^2} - \frac{b^2}{a^2}\right)}$$

$$= \frac{a^2c^2}{b^2(a^2-c^2)}\left[\frac{a^2-b^2}{a^2}x_1^2 - \frac{b^2-c^2}{c^2}z_1^2\right]$$

$\therefore p_1\pi_1 + p_2\pi_2 + p_3\pi_3$

$$= \frac{a^2c^2}{b^2(a^2-c^2)}\left[\frac{a^2-b^2}{a^2}(x_1^2+x_2^2+x_3^2) - \frac{b^2-c^2}{c^2}(z_1^2+z_2^2+z_3^2)\right],$$

where P_2 and P_3 are (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively.

$$= \frac{a^2c^2}{b^2(a^2-c^2)}\left[\frac{a^2-b^2}{a^2}(a^2) - \frac{b^2-c^2}{c^2}(c^2)\right], \quad \because \quad x_1^2+x_2^2+x_3^2 = a^2,$$

$z_1^2+z_2^2+z_3^2 = c^2$, from ch. IX

$$= \frac{a^2c^2}{b^2(a^2-c^2)}[(a^2-b^2)-(b^2-c^2)] = \frac{a^2c^2}{b^2(a^2-c^2)}[a^2+c^2-2b^2] \qquad \textbf{Ans.}$$

***Ex. 8. If p_1, p_2, p_3 be the lengths of the perpendiculars from P_1, P_2, P_3, the extremities of conjugate semi-diameters on one of the planes of central circular section of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$, then evaluate $p_1^2+p_2^2+p_3^2$.**

Sol. Let (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be the coordinates of P_1, P_2 and P_3 respectively, then from chapter on Conicoids, we have

$$x_1^2+x_2^2+x_3^2 = a^2, \; y_1^2+y_2^2+y_3^2 = b^2, \; z_1^2+z_2^2+z_3^2 = c^2 \qquad \text{...(A)}$$

and $\quad x_1y_1+x_2y_2+x_3y_3 = 0, \; y_1z_1+y_2z_2+y_3z_3 = 0, \; z_1x_1+z_2x_2+z_3x_3 = 0 \quad$ ...(B)

Now from § 11.07 Page 42 Ch. XI we know that one of the real central circular sections of the given ellipsoid is

$$\frac{x}{a}\sqrt{(a^2-b^2)} + \frac{z}{c}\sqrt{(b^2-c^2)} = 0 \qquad \text{...(i)}$$

Its distance from $P_1(x_1, y_1, z_1)$ is given as p_1.

$$\therefore p_1 = \frac{\frac{x_1}{a}\sqrt{(a^2-b^2)} + \frac{z_1}{c}\sqrt{(b^2-c^2)}}{\sqrt{\left[\frac{a^2-b^2}{a^2} + \frac{b^2-c^2}{c^2}\right]}}$$

or $$p_1^2=\frac{(x_1^2/a^2)(a^2-b^2)+(z_1^2/c^2)(b^2-c^2)+2(x_1z_1/ac)\sqrt{[(a^2-b^2)(b^2-c^2)]}}{b^2[(1/c^2)-(1/a^2)]}$$

or $$p_1^2=\frac{a^2c^2}{b^2(a^2-c^2)}\left[\frac{a^2-b^2}{a^2}x_1^2+\frac{b^2-c^2}{c^2}z_1^2+\frac{2x_1z_1}{ac}\sqrt{\{(a^2-b^2)(b^2-c^2)\}}\right]$$

$$\therefore p_1^2+p_2^2+p_3^2=\Sigma p_1^2$$

$$=\frac{a^2c^2}{b^2(a^2-c^2)}\left[\frac{a^2-b^2}{a^2}\Sigma x_1^2+\frac{b^2-c^2}{c^2}\Sigma z_1^2+\frac{2\sqrt{\{(a^2-b^2)(b^2-c^2)\}}}{ac}\Sigma x_1z_1\right]$$

$$=\frac{a^2c^2}{b^2(a^2-c^2)}\left[\frac{a^2-b^2}{a^2}(a^2)+\frac{b^2-c^2}{c^2}(c^2)+0\right]\text{, from (A) and (B)}$$

$$=\frac{a^2c^2}{b^2(a^2-c^2)}[\{(a^2-b^2)+(b^2-c^2)\}]=\frac{a^2c^2}{b^2}\qquad\textbf{Ans.}$$

***Ex. 9. A cone is drawn with its vertex at the centre of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ and its base is a circular section of the ellipsoid. If the cone contains three mutually perpendicular generators, prove that the distance of the section from the centre of the ellipsoid is**

$$\mathbf{abc/\sqrt{(b^2c^2+c^2a^2+a^2b^2)}}.$$

Sol. The equation of a circular section of the given ellipsoid

$$(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1 \qquad ...(i)$$

is $$\frac{x}{a}\sqrt{(a^2-b^2)}+\frac{z}{c}\sqrt{(b^2-c^2)}=\lambda \qquad ...(ii)$$

We are to find the equation of the cone with its vertex at the centre (0, 0, 0) of the ellipsoid (i) and whose base is the circular section given by (ii).

$\therefore$ The equation of the cone is obtained by making the equation (i) homogeneous with the help of (ii) and so is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{\lambda^2}\left\{\frac{x}{a}\sqrt{(a^2-b^2)}+\frac{z}{c}\sqrt{(b^2-c^2)}\right\}^2 \qquad \textbf{(Note)}$$

If this cone has three mutually perpendicular generators then the sum of the coefficients of x^2, y^2 and z^2 in its equation must be zero

i.e. $$\left[\frac{1}{a^2}-\frac{(a^2-b^2)}{a^2\lambda^2}\right]+\frac{1}{b^2}+\left[\frac{1}{c^2}-\frac{b^2-c^2}{c^2\lambda^2}\right]=0$$

or $$\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=\frac{1}{\lambda^2}\left(\frac{a^2-b^2}{a^2}+\frac{b^2-c^2}{c^2}\right) \qquad ...(iii)$$

Again the distance of the section (ii) from the centre (0, 0, 0) of (i) is

$$\frac{\lambda}{\sqrt{\left[\frac{a^2-b^2}{a^2}+\frac{b^2-c^2}{c^2}\right]}}=-\frac{\lambda}{\lambda\sqrt{\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}}, \text{ from (iii)}$$

$$= abc/\sqrt{(b^2c^2+c^2a^2+a^2b^2)} \qquad \text{Proved.}$$

****Ex. 10. Prove that the planes $2x+3z-5=0$, $2x-3z+7=0$ meet the conicoid $3y^2+12z^2-x^2=75$ in circles which lie on the sphere**

$$3x^2+3y^2+3z^2+4x+36z-110=0.$$

Sol. Any surface through the sections of the given conicoid by the given planes is $$C+\lambda\, u\, v=0 \qquad \textbf{(Note)}$$

or $$(3y^2+12z^2-x^2-75)+\lambda\,(2x+3z-5)\,(2x-3z+7)=0 \qquad ...(i)$$

If λ is chosen equal to 1, then (i) reprents a sphere as in that case the coefficients of x^2, y^2, z^2 are equal and (i) is a second degree equation in x, y and z.

Hence from (i), the sections which are circles lie on the sphere

$$(3y^2+12z^2-x^2-75)+(2x+3z-5)\,(2x-3z+7)=0$$

or $$3y^2+12z^2-x^2-75+(4x^2+9z^2+14x+21z-10x+15z-35)=0$$

or $$3x^2+3y^2+3z^2+4x+36z-110=0 \qquad \text{Hence proved.}$$

****Ex. 11. If the section of the cone whose vertex is $P(\alpha, \beta, \gamma)$ and base $z=0$, $ax^2+by^2=1$, by the plane $x=0$ is a circle, then P lies on $y=0$, $ax^2-bz^2=1$ and the section of the cone by the plane $(a-b)\gamma x-2a\alpha z=0$, $x=0$ is also a circle.**

Sol. We can find that the equation of the cone is

$$a\,(\gamma x-\alpha z)^2+b\,(\gamma y-\beta z)^2=(z-\gamma)^2 \qquad ...(i)$$

...See chapter on cone (ch. VIII)

Its section by the plane $x=0$ is $a\alpha^2z^2+b\,(\gamma y-\beta z)^2=(z-\gamma)^2$, $x=0$

or $$(a\alpha^2+b\beta^2-1)\,z^2+b^2\gamma^2y^2-2b\beta\gamma yz+2\gamma z-\gamma^2=0,\; x=0$$

It will be a circle if the coefficents of y^2 and z^2 are equal and that of yz is zero *i.e.* $a\alpha^2+b\beta^2-1=b\gamma^2$ and $b\beta\gamma=0$

Now $b\neq 0$ and $\gamma\neq 0$, so the vertex of the cone will lie in its plane. Hence $\beta=0$ and then $$a\alpha^2-1=b\gamma^2 \qquad ...(ii)$$

$\therefore$ The vertex $P(\alpha, \beta, \gamma)$ lies on $ax^2-bz^2=1$, $y=0$

Also when $\beta=0$, the equation (i) of the cone reduces to

$$a(\gamma x-\alpha z)^2+b\gamma^2y^2=(z-\gamma)^2$$

or $$a(\gamma^2x^2+\alpha^2z^2-2\alpha\gamma xz)+b\gamma^2y^2-(z^2+\gamma^2-2\gamma z)=0$$

or $$a\gamma^2x^2+b\gamma^2y^2+(a\alpha^2-1)z^2-2a\alpha\gamma zx+2\gamma z-\gamma^2=0,$$

or $$a\gamma^2x^2+b\gamma^2y^2+b\gamma^2z^2-2a\alpha\gamma zx+2\gamma z-\gamma^2=0, \text{ from (ii)}$$

or $$b\gamma^2(x^2+y^2+z^2)-b\gamma^2x^2+a\gamma^2x^2-2a\alpha\gamma zx+2\gamma z-\gamma^2=0$$

or $$[b\gamma^2(x^2+y^2+z^2)+2\gamma z-\gamma^2]+\gamma x[(a-b)\gamma x-2a\alpha z]=0$$

$\therefore$ The circular sections are given by $(a-b)\gamma x-2a\alpha z=0$ and $x=0$.

Ex. 12. Prove that the real central circular sections of the hypeboloids $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ and $(x^2/a^2)-(y^2/b^2)-(z^2/c^2)=1$ are given by the planes $\frac{y}{b}\sqrt{(a^2-b^2)}\pm\frac{z}{c}\sqrt{(a^2+c^2)}=0$

and $$\frac{x}{a}\sqrt{(a^2+b^2)}\pm\frac{z}{c}\sqrt{(b^2-c^2)}=0$$ *(Garhwal 95)*

Sol. The equation $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ can be written as

$$\frac{x^2+y^2+z^2-a^2}{a^2}+y^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)-z^2\left(\frac{1}{c^2}+\frac{1}{a^2}\right)=0 \quad \text{...(i)}$$

or $$\frac{x^2+y^2+z^2-b^2}{b^2}+x^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)-z^2\left(\frac{1}{c^2}+\frac{1}{b^2}\right)=0 \quad \text{...(ii)}$$

or $$\frac{x^2+y^2+z^2-c^2}{c^2}+x^2\left(\frac{1}{a^2}-\frac{1}{c^2}\right)+y^2\left(\frac{1}{b^2}-\frac{1}{c^2}\right)-\frac{2z^2}{c^2}=0 \quad \text{...(iii)}$$

If $a>b>c$ *i.e.* $\frac{1}{a}<\frac{1}{b}<\frac{1}{c}$

then $$\frac{1}{a^2}-\frac{1}{b^2}<0,\ \frac{1}{b^2}-\frac{1}{c^2}>0$$

and so only **real circular sections** will be given by planes

$$y^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)-z^2\left(\frac{1}{c^2}+\frac{1}{a^2}\right)=0, \text{ from (i)}$$

and from (ii) and (iii) we get imaginary sections.

$$\therefore \frac{y^2}{b^2}(a^2-b^2)=\frac{z^2}{c^2}(a^2+c^2) \quad \text{or} \quad \frac{y}{b}\sqrt{(a^2-b^2)}\pm\frac{z}{c}\sqrt{(a^2+c^2)}=0$$

Hence proved.

In a similar way writing $(x^2/a^2)-(y^2/b^2)-(z^2/c^2)=1$ as

$$\left(\frac{x^2+y^2+z^2+b^2}{b^2}\right)+x^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-z^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)=0,$$

the **real** central circular section is given by

$$x^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=z^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right) \quad \text{or} \quad \frac{x^2(a^2+b^2)}{a^2}=\frac{z^2(b^2-c^2)}{c^2}$$

or $$\frac{x}{a}\sqrt{(a^2+b^2)}\pm\frac{z}{c}\sqrt{(b^2-c^2)}=0$$ Hence proved.

***Ex. 13. The normals to the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ at all points of a central circular section are parallel to the plane that makes an angle $\cos^{-1}[ac/\{b\sqrt{(a^2-b^2+c^2)}\}]$.**

Sol. We know that if $a>b>c$, then the real central circular section of the given ellipsoid is $$\frac{x}{a}\sqrt{(a^2-b^2)}+\frac{z}{c}\sqrt{(b^2-c^2)}=0 \qquad ...(i)$$

Let $x/l=y/m=z/n$ be a generator of the enveloping cylinder which touches the given ellipsoid along the plane (i).

$\therefore$ The diametral plane which bisects chords parallel to the above generator is $$(lx/a^2)+(my/b^2)+(nz/c^2)=0 \qquad ...(ii)$$

As (i) and (ii) are identical, so comparing them we get

$$\frac{(l/a^2)}{(1/a)\sqrt{(a^2-b^2)}}=\frac{m/b^2}{0}=\frac{n/c^2}{(1/c)\sqrt{(b^2-c^2)}}$$

or $$\frac{l}{a\sqrt{(a^2-b^2)}}=\frac{m}{0}=\frac{n}{c\sqrt{(b^2-c^2)}} \qquad ...(iii)$$

Also the normals to the given ellipsoid will be parallel to the normal section of the cylinder $lx+my+nxz=0$

or $$a\sqrt{(a^2-b^2)}\,x+c\sqrt{(b^2-c^2)}\,z=0, \text{ from (iii)} \qquad ...(iv)$$

$\therefore$ If θ be the angle between planes (i) and (iv), then

$$\cos\theta=\frac{a\sqrt{(a^2-b^2)}\,.\,(1/a)\sqrt{(a^2-b^2)}+c\sqrt{(b^2-c^2)}\,.\,(1/c)\sqrt{(b^2-c^2)}}{\sqrt{\left(\frac{a^2-b^2}{a^2}+\frac{b^2-c^2}{c^2}\right)}\sqrt{[a^2(a^2-b^2)+c^2(b^2-c^2)]}}$$

$$=\frac{(a^2-b^2)+(b^2-c^2)}{\sqrt{\left(\frac{b^2}{c^2}-\frac{b^2}{a^2}\right)}\,.\,\sqrt{[(a^4-c^4)-b^2(a^2-c^2)]}}$$

$$= \frac{ac\,(a^2-c^2)}{b\sqrt{(a^2-c^2)}\sqrt{[(a^2-c^2)(a^2+c^2-b^2)]}}$$

or $\cos\theta = \dfrac{ac}{b\sqrt{(a^2+c^2-b^2)}}$ or $\theta = \cos^{-1}\left[\dfrac{ac}{b\sqrt{(a^2+c^2-b^2)}}\right]$ Proved.

§ 11.08. Circular sections of any central conicoid.

The equation of any central conicoid is of the form

$$f(x, y, z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy-1=0. \quad ...(i)$$

which can be put in the form

$$(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)-\lambda(x^2+y^2+z^2)$$
$$+\lambda\left(x^2+y^2+z^2+\frac{1}{\lambda}\right)=0 \quad ...(ii)$$

Now if $ax^2+by^2+cz^2+2fyz+2gzx+2hxy-\lambda(x^2+y^2+z^2)=0$ ...(ii)
represents a pair of planes, then

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad \text{(Note)}$$

This will give a cubic in λ and so will give three values of λ which will always be real. It can be found that only one of the values (viz. mean value) of λ will give real planes.

Solved Examples on § 11.08.

***Ex. 1. Find the equations to the circular sections of the conicoid**

$$\mathbf{yz}\left(\frac{\mathbf{b}}{\mathbf{c}}+\frac{\mathbf{c}}{\mathbf{b}}\right)+\mathbf{zx}\left(\frac{\mathbf{c}}{\mathbf{a}}+\frac{\mathbf{a}}{\mathbf{c}}\right)+\mathbf{xy}\left(\frac{\mathbf{a}}{\mathbf{b}}+\frac{\mathbf{b}}{\mathbf{a}}\right)+\mathbf{1}=\mathbf{0}.$$

Sol. The given equation of the conicoid can be rewritten as

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)-\lambda(x^2+y^2+z^2)$$
$$+\lambda\left(x^2+y^2+z^2+\frac{1}{\lambda}\right)=0 \quad ...(i)$$

Now choose λ in such a way that

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)-\lambda(x^2+y^2+z^2)=0 \quad ...(ii)$$

represents a pair of planes and the condition for the same is

$$\begin{vmatrix} -\lambda & \dfrac{a^2+b^2}{2ab} & \dfrac{c^2+a^2}{2ac} \\ \dfrac{a^2+b^2}{2ab} & -\lambda & \dfrac{b^2+c^2}{2bc} \\ \dfrac{c^2+a^2}{2ac} & \dfrac{b^2+c^2}{2bc} & -\lambda \end{vmatrix} = 0$$

or $$-\lambda^3+\lambda\left[\frac{(b^2+c^2)^2}{4b^2c^2}+\frac{(a^2+b^2)^2}{4a^2b^2}+\frac{(c^2+a^2)^2}{4a^2c^2}\right]$$

$$+2\left(\frac{a^2+b^2}{2ab}\right)\left(\frac{b^2+c^2}{2bc}\right)\left(\frac{c^2+a^2}{2ac}\right)=0$$

which is satisfied by $\lambda=-1$ and so from (ii), the equations of the planes reduces to $$x^2+y^2+z^2+yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

or $$\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)(ax+by+cz)=0,$$

which gives the planes corresponding to central circular sections.

$\therefore$ The required equations of circular sections are

$$(x/a)+(y/b)+(z/c)=k_1 \text{ and } ax+by+cz=k_2$$

which are parallel to the central circular sections.

****Ex. 2. Find the real circular sections of the conicoid**

$$\mathbf{3x^2+5y^2+3z^2+2zx=4} \qquad \textit{(Garhwal 93)}$$

Sol. The given equation of the conicoid can be written in the form

$$\{3x^2+5y^2+3z^2+2zx-\lambda(x^2+y^2+z^2)\}+\lambda(x^2+y^2+z^2)-4=0.$$

The equation $3x^2+5y^2+3z^2+2zx-\lambda(x^2+y^2+z^2)=0$

or $(3-\lambda)x^2+(5-\lambda)y^2+(3-\lambda)z^2+2zx=0$ represents a pair of planes if

$$(3-\lambda)(5-\lambda)(3-\lambda)-(5-\lambda)(1)^2=0 \qquad \text{(Note)}$$

or $(5-\lambda)[(3-\lambda)^2-1]=0$ or $(5-\lambda)(\lambda^2-6\lambda+8)=0$

or $(5-\lambda)(\lambda-4)(\lambda-2)=0$ or $\lambda=2, 4, 5$

Now as only the mean value of λ gives real planes, **(Note)**

so taking $\lambda=4$, we have

$$3x^2+5y^2+3z^2+2zx-4(x^2+y^2+z^2)=-x^2+y^2-z^2+2zx$$

$$=y^2-(x^2+z^2-2zx)=y^2-(x-z)^2=(y+x-z)(y-x+z)$$

Hence the real circular sections of the surface are given by planes parallel to $y+x-z=0$ and $y-x+z=0$ *i.e.* parallel to the planes $x+y-z=0$ and $x-y-z=0$. Ans.

Ex. 3. Show that the plane $x+y-z=0$ cuts the conicoid $4x^2+2y^2+z^2+3yz+zx-1=0$ in a circle. Find also the radius of this circle.

Sol. The given equation of the conicoid can be rewritten as

$$[(4x^2+2y^2+z^2+3yz+zx)-\lambda(x^2+y^2+z^2)]$$
$$+\lambda[x^2+y^2+z^2-(1/\lambda)]=0 \quad ...(i)$$

Choose λ such that $4x^2+2y^2+z^2+3yz+zx-\lambda(x^2+y^2+z^2)=0$

or $\quad (4-\lambda)x^2+(2-\lambda)y^2+(1-\lambda)z^2+3yz+zx=0 \quad ...(ii)$

represents a pair of planes, the condition for the same is

$$\begin{vmatrix} 4-\lambda & 0 & 1/2 \\ 0 & 2-\lambda & 3/2 \\ 1/2 & 3/2 & 1-\lambda \end{vmatrix}=0 \qquad \textbf{(Note)}$$

or $\quad (4-\lambda)[(2-\lambda)(1-\lambda)-(3/2)(3/2)]+(1/2)[0-(1/2)(2-\lambda)]=0$

or $\quad 4(4-\lambda)(2-\lambda)(1-\lambda)-9(4-\lambda)-(2-\lambda)=0$

or $\quad -4\lambda^3+28\lambda^2-46\lambda-6=0$, on simplifying

or $\quad 2\lambda^3-14\lambda^2+23\lambda+3=0 \quad$ or $\quad (\lambda-3)(2\lambda^2-3\lambda-1)=0$

or $\quad \lambda=3$ (real)

$\therefore$ Substituting 3 for λ in (ii), we get

$$[4x^2+2y^2+z^2+3yz+zx-3(x^2+y^2+z^2)]=0$$

or $\quad x^2-y^2-2z^2+3yz+zx=0 \quad$ or $\quad (x+y-z)(x-y+2z)=0.$

$\therefore$ The plane $x+y-z=0$ cuts the given conicoid in a circle.

Also as the plane $x+y-z=0$ passes through the centre of the sphere $x^2+y^2+z^2-(1/\lambda)=0$, where $\lambda=3$

i.e. the sphere $x^2+y^2+z^2=1/3$.

$\therefore$ The circle is a great circle whose radius is the same as that of the sphere *i.e.* $1/\sqrt{3}$. **Ans.**

Ex. 4. Find the conditions that the equation $f(x,y,z)=1$, $lx+my+nz=0$ should determine a circle, where

$$f(x,y,z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy.$$

Sol. Proceeding as in § 11.08 Page 53 of this chapter we find that the equation $\quad f(x,y,z)-\lambda(x^2+y^2+z^2)=0 \quad ...(i)$

should represent a pair of planes and one of these should be the given plane

$$lx+my+nz=0.$$

$\therefore$ We have $f(x,y,z)-\lambda(x^2+y^2+z^2)$ as

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy-\lambda(x^2+y^2+z^2)$

$$= (a-\lambda)\,x^2 + (b-\lambda)\,y^2 + (c-\lambda)\,z^2 + 2fyz + 2gzx + 2hxy$$
$$\equiv (lx + my + nz) + \left[\frac{a-\lambda}{l}x + \frac{b-\lambda}{m}y + \frac{c-\lambda}{n}z\right]$$

In this identity the coefficients of x^2, y^2 and z^2 are equal and so the required conditions are obtained by equating the coefficients of yz, zx and xy

$$\therefore\ 2f = \frac{m\,(c-\lambda)}{n} + \frac{n\,(b-\lambda)}{m}, \qquad \text{...(i)}$$

$$2g = \frac{l\,(c-\lambda)}{n} + \frac{n\,(a-\lambda)}{l},\quad 2h = \frac{l\,(b-\lambda)}{m} + \frac{m\,(a-\lambda)}{l} \qquad \text{...(ii)}$$

From (i),
$$\lambda\left(\frac{m}{n} + \frac{n}{m}\right) = \frac{cm}{n} + \frac{bn}{m} - 2f$$

or
$$\lambda\left(\frac{m^2+n^2}{mn}\right) = \frac{cm^2 + bn^2 - 2fmn}{mn}$$

or
$$\lambda = \frac{bn^2 + cm^2 - 2fmn}{m^2+n^2} = \frac{cl^2 + an^2 - 2gnl}{n^2+l^2} = \frac{am^2 + bl^2 - 2hlm}{l^2+m^2}, \text{ from (ii)}$$

Ex. 5. Prove that the real central circular sections of the conicoid $5y^2 - 8z^2 + 18yz - 14zx - 10xy + 27 = 0$ are given by $x - 2y - 5z = 0$ and $3x - 4y + z = 0$.

Sol. The given equation of the conicoid can be rewritten as

$$[5y^2 - 8z^2 + 18yz - 14zx - 10xy - \lambda\,(x^2+y^2+z^2)]$$
$$+ \lambda\,(x^2+y^2+z^2) + 27 = 0$$

The equation $5y^2 - 8z^2 + 18yz - 14zx - 10xy - \lambda\,(x^2+y^2+z^2) = 0$

or $-\lambda x^2 + (5-\lambda)\,y^2 - (8+\lambda)\,z^2 + 18yz - 14zx - 10xy = 0$ represents a pair of planes if

$$\begin{vmatrix} -\lambda & -5 & -7 \\ -5 & 5-\lambda & 9 \\ -7 & 9 & -8-\lambda \end{vmatrix} = 0$$

or $\quad -\lambda\,[-(5-\lambda)\,(8+\lambda) - 81] + 5\,[5\,(8+\lambda) + 63] - 7\,[-45 + 7\,(5-\lambda)] = 0$

or $\quad \lambda^3 + 3\lambda^2 - 195\,\lambda - 585 = 0$, on simplifying

or $\quad (\lambda+3)\,(\lambda^2 - 195) = 0 \quad$ or $\quad \lambda = -3,\ \pm\sqrt{(195)}$

Now as only the mean value of λ gives real planes, so taking $\lambda = -3$, we have

$$5y^2 - 8z^2 + 18yz - 14zx - 10xy + 3\,(x^2+y^2+z^2)$$
$$= 3x^2 + 8y^2 - 5z^2 + 18yz - 14zx - 10xy$$
$$= (3x - 4y + z)\,(x - 2y - 5z)$$

Hence the required real central circular sections of the given conicoid are $3x - 4y + z = 0$ and $x - 2y - 5z = 0$ Hence proved.

Exercises on § 11.08

Ex. 1. Find the real circular sections of the surface

$4x^2+2y^2+z^2+3yz+zx=1$ **Ans.** $x+y-z=0,\ x-y+2z=0$

Ex. 2. Find the real central circular sections of the conicoid

$2x^2+5y^2+2z^2-yz-4zx-xy+4=0$ **Ans.** $x+y+z=0,\ 2x-y+2z=0$

Ex. 3. Find the equation of the sphere which contains the two circular sections of the conicoid $x^2-3y^2+2z^2=4$ through the point (1, 2, 3).

(*Gorakhpur 97*)

§ 11.09. Circular sections of the paraboloid $ax^2+by^2=2cz$.

The above equation of the paraboloid can be written in the form

$$a\left(x^2+y^2+z^2-\frac{2cz}{a}\right)+(b-a)\,y^2-az^2=0$$

and
$$b\left(x^2+y^2+z^2-\frac{2cz}{b}\right)+(a-b)\,x^2-bz^2=0$$

∴ The two pairs of planes given by the equations

$$(b-a)\,y^2-az^2=0 \quad \text{and} \quad (a-b)\,x^2-bz^2=0$$

give the circular sections of the paraboloid through the origin.

We can easily verify that of the two pairs of the planes found above, one will be real if a and b are of the same sign but both pairs of planes will be imaginary if a and b are of different signs.

∴ It follows that there are no real circular sections of a hyperbolic paraboloid.

If $a>b>0$, the real central circular sections are given by

$$x\sqrt{(a-b)}\pm z\sqrt{b}=0$$

so that the two systems of the circular sections are given by

$$x\sqrt{(a-b)}+z\sqrt{(b)}=\lambda \quad \text{and} \quad x\sqrt{(a-b)}-z\sqrt{(b)}=\mu$$

Again the equation of the paraboloid $ax^2+by^2=2cz$ can also be written as

$$ax^2+by^2-(0\,.\,x^2+0\,.\,y^2+0\,.\,z^2+2cz)=0$$

where $0\,.\,x^2+0\,.\,y^2+0\,.\,z^2+2cz=0$ represents a sphere of infinite radius, being the limiting case of the equation

$$\lambda x^2+\lambda y^2+\lambda\,[z+(c/\lambda)]^2=c^2/\lambda \text{ as } \lambda\to 0$$

∴ If a and b are of different signs, the only real circular sections of the hyperbolic paraboloid are given by the planes $ax^2+by^2=0$ and are of infinite radius.

∴ The circular sections in this case reduce to straight lines which are the lines of section of the paraboloid by the plane $z=0$.

Note. : We can prove as in § 11.07 Cor. Page 43 of this chapter that any two circularr sections of a paraboloid of opposite system lie on a sphere.

Solved Examples on § 11.09.

Ex. 1. Find the real circular sections of the paraboloid $x^2 + 10z^2 = 2y$.

Sol. The given equation of the paraboloid can be written in the form

$$(x^2 + y^2 + z^2 - 2y) - y^2 + 9z^2 = 0$$

$\therefore$ The planes given by the equation $-y^2 + 9z^2 = 0$ *i.e.* $y \pm 3z = 0$ *i.e.* $y + 3z = 0$ and $y - 3z = 0$ give the circular sections of the given paraboloid through the origin.

$\therefore$ The required equations of the circular sections are

$$y + 3z = \lambda \quad \text{and} \quad y - 3z = \mu \qquad \textbf{Ans.}$$

which are parallel to the circular sections through the origin.

Ex. 2. Show that the plane $7x + 2z = 5$ cuts the paraboloid $53x^2 + 4y^2 = 8z$ in a circle.

Sol. The given equation of the parabolid can be written in the form

$$53\left(x^2 + y^2 + z^2 - \frac{8z}{53}\right) - 49\,y^2 - 53z^2 = 0 \qquad \text{...(i)}$$

or

$$4\,(x^2 + y^2 + z^2 - 2z) + 49\,x^2 - 4z^2 = 0 \qquad \text{...(ii)}$$

The form (ii) only gives real circular sections and the planes given by $49x^2 - 4z^2 = 0$ *i.e.* $7x + 2z = 0$ and $7x - 2z = 0$ give the circular sections of the given paraboloid through the origin.

Also the plane $7x + 2z = 5$ which is parallel to $7x + 2z = 0$ gives circular section of the given paraboloid. Hence proved.

§ 11.10. Umbilics.

If a series of planes parallel to either of the ciruclar sections of an ellipsoid be drawn, then the circles in which these planes cut the ellipsoid go on becoming smaller and smaller as the planes move farther and farther away from the centre of the ellipsoid and ultimately the circular section reduces to a point circle and the plane of this point circle becomes the tangent plane to the ellipsoid at that point. This point is known as an **umbilic.**

Definition. *An umbilic is a circular section of zero radius on the surface of the ellipsoid.*

Again we know (See chapter on Conicoids) that the locus of the centres of parallel sections of an ellipsoid is a diameter and the tangent plane at an extremity of the diameter is parallel to these sections. Thus the centres of one system of circular sections lie on a diameter and the extremities of the diameter are the umbilics, the tangent planes at these points being the limiting positions of the plane sections whose radii are zero.

From above we conclude that an ellipsoid has four umbilics and also an hyperboloid of two sheets has four umbilics. Whereas an hyperboloid of one sheet has no real umbilics as the smallest circular section of this hypeboloid is the central section of radius a.

§ 11.11. Umbilics of an ellipsoid.

Let (α, β, γ) be the coordinates of an umbilic of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$, so that the equation of the tangent plane to the ellipsoid at (α, β, γ) is

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=1 \qquad \text{...(i)}$$

By definition this plane should be parallel to either of the real circular sections of the ellipsoid viz.

$$\frac{x}{a}\sqrt{(a^2-b^2)} \pm \frac{z}{c}\sqrt{(b^2-c^2)}=0 \qquad \text{...(ii)}$$

Comparing (i) and (ii), we get.

$$\beta=0 \text{ and } \frac{\alpha}{a\sqrt{(a^2-b^2)}}=\pm\frac{\gamma}{c\sqrt{(b^2-c^2)}}=\lambda, \text{ (say)} \qquad \text{...(iii)}$$

Again as (α, β, γ) lies on the ellipsoid $\Sigma(x^2/a^2)=1$,

so $$(\alpha^2/a^2)+(\beta^2/b^2)+(\gamma^2/c^2)=1$$

or $$\lambda^2[(a^2-b^2)+0+(b^2-c^2)]=1, \text{ from (iii)}$$

or $$\lambda^2(a^2-c^2)=1 \text{ or } \lambda=\pm 1/\sqrt{(a^2-c^2)}$$

$$\therefore \text{ From (iii), } \alpha=\frac{\pm a\sqrt{(a^2-b^2)}}{\sqrt{(a^2-c^2)}},\ \beta=0,\ \gamma=\pm\frac{c\sqrt{(b^2-c^2)}}{\sqrt{(a^2-c^2)}} \qquad \text{...(iv)}$$

(iv) gives the coordinates of the four real umbilics of the ellipsoid

$$(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1.$$

$$\text{Also } \alpha^2+\beta^2+\gamma^2=\frac{a^2(a^2-b^2)}{(a^2-c^2)}+0+\frac{c^2(b^2-c^2)}{(a^2-c^2)}, \text{ from (iv)}$$

$$=\frac{a^2[(a^2-c^2)+(c^2-b^2)]+c^2[(a^2-c^2)+(b^2-a^2)]}{(a^2-c^2)} \qquad \textbf{(Note)}$$

$$=\frac{a^2(a^2-c^2)}{(a^2-c^2)}+\frac{c^2(a^2-c^2)}{(a^2-c^2)}+\frac{(a^2c^2-a^2b^2+c^2b^2-c^2a^2)}{(a^2-c^2)}$$

or $$\alpha^2+\beta^2+\gamma^2=a^2+c^2-b^2$$

This shows tht the **umbilics** of the ellipsoid $\Sigma(x^2/a^2)=1$ **lie on the sphere** $$\mathbf{x^2+y^2+z^2=a^2-b^2+c^2}$$ *(Gorakhpur 96)*

§ 11.12. Umbilics of a paraboloid.

Let $P(\alpha, \beta, \gamma)$ be an umbilic of the paraboloid

$$ax^2 + by^2 = 2cz, \; a > b > 0.$$

$\therefore$ The equaiton of the tangent plane to the paraboloid at $P(\alpha, \beta, \gamma)$ is

$$a\alpha x + b\beta y - c(z + \gamma) = 0 \qquad \text{...(i)}$$

By definition this plane should be parallel to either of the real circular sections through the origin of this paraboloid viz.

$$x\sqrt{(a-b)} \pm \sqrt{(b)}\, z = 0 \qquad \text{...(ii)}$$

Comparing (i) and (ii) we find that

$$\beta = 0, \; \frac{a\alpha}{\sqrt{(a-b)}} = \pm \frac{c}{\sqrt{(b)}} \quad \text{or} \quad \alpha = \pm \frac{c}{a} \frac{\sqrt{(a-b)}}{\sqrt{(b)}} \qquad \text{...(iii)}$$

Also as $P(\alpha, \beta, \gamma)$ lies on the paraboloid $ax^2 + by^2 = 2cz$.

so $$a\alpha^2 + b\beta^2 = 2c\gamma$$

$$\therefore a\left[\frac{c^2(a-b)}{a^2 b}\right] + b(0) = 2c\gamma, \text{ putting values of } \alpha, \beta \text{ from (iii)}$$

or $$\gamma = (a-b)\, c/(2ab)$$

$\therefore$ The coordinates of two real umbilics of the paraboloid $ax^2 + by^2 = 2cz, \; a > b > 0$ are $\left[\pm \frac{c\sqrt{(a-b)}}{a\sqrt{(b)}}, 0, \frac{(a-b)\, c}{2ab}\right]$

Note. If $b > a$, then the real circular sections will be given by

$$y\sqrt{(b-a)} \pm z\sqrt{a} = 0.$$

Solved Examples on Umbilics.

****Ex. 1. Prove that the central circular sections of the conicoid $(a - b)x^2 + ay^2 + (a + b)z^2 = 1$ are at right angles and that the umbilics are given by** $$x = \pm\sqrt{\left[\frac{a+b}{2a(a-b)}\right]}, y = 0, z = \pm\sqrt{\left[\frac{a-b}{2a(a+b)}\right]}$$

(*Garhwal* 92; *Gorakhpur* 95)

Sol. The given equation of the conicoid can be written in the form

$$\{a(x^2 + y^2 + z^2) - 1\} + b(z^2 - x^2) = 0.$$

$\therefore$ The circular sections are evidently given by $z^2 - x^2 = 0$

or $z = \pm x$, which are at right angles.

Again if (α, β, γ) be an umbilic, then tangent plane at (α, β, γ) of the given conicoid is $(a-b)\alpha x + a\beta y + (a+b)\gamma z = 1$,

which by definition should be parallel to $x = \pm z$

$\therefore$ The direction ratios of the normals are proportional and so,

$$\frac{(a-b)\,\alpha}{1}=\frac{a\beta}{0}=\frac{(a+b)\,\gamma}{\pm 1}=\lambda, \text{ (say)}$$

$\therefore\ \alpha=\lambda/(a-b),\ \beta=0,\ \gamma=\pm\lambda/(a+b)$ ...(i)

Also as (α, β, γ) lies on the given conicoid, so we have

$$(a-b)\,\alpha^2+a\beta^2+(a+b)\,\gamma^2=1$$

or $$\lambda^2\left[(a-b)\cdot\frac{1}{(a-b)^2}+a\,(0)^2+(a+b)\,\frac{1}{(a+b)^2}\right]=1,$$

substituting values of α, β, γ from (i)

or $$\lambda^2\left[\frac{1}{a-b}+\frac{1}{a+b}\right]=1 \quad\text{or}\quad \lambda=\pm\sqrt{\left(\frac{a^2-b^2}{2a}\right)} \qquad \text{...(ii)}$$

$\therefore$ From (i) with the help of (ii), the umbilics are

$$\alpha=\pm\frac{1}{a-b}\sqrt{\left(\frac{a^2-b^2}{2a}\right)}=\pm\sqrt{\left[\frac{a+b}{2a\,(a-b)}\right]},\ \beta=0$$

and $$\gamma=\pm\frac{1}{a+b}\sqrt{\left(\frac{a^2-b^2}{2a}\right)}=\pm\sqrt{\left[\frac{a-b}{2a\,(a+b)}\right]}$$ Hence proved.

***Ex. 2. Find the real umbilics of the hyperboloid of two sheets**

$$(\mathbf{x^2/a^2})-(\mathbf{y^2/b^2})-(\mathbf{z^2/c^2})=\mathbf{1} \qquad (\textit{Garhwal } 93)$$

Sol. Let (α, β, γ) be the coordinates of an umbilic of the given hyperboloid, so that the equation of the tangent plane to this hyperboloid at (α, β, γ) is

$$\frac{\alpha x}{a^2}-\frac{\beta y}{b^2}-\frac{\gamma z}{c^2}=1. \qquad \text{...(i)}$$

By definition this plane should be parallel to either of the real circular sections of the hyperboloid viz.

$$\frac{x}{a}\sqrt{(a^2+b^2)}\pm\frac{z}{c}\sqrt{(b^2-c^2)}=0$$ ...See Ex. 12 Page 51 Ch. XI.

Comparing (i) and (ii) we get

$$\beta=0 \quad\text{and}\quad \frac{\alpha}{a\sqrt{(a^2+b^2)}}=\frac{\mp\gamma}{c\sqrt{(b^2-c^2)}}=\lambda \text{ (say)} \qquad \text{...(iii)}$$

Also as (α, β, γ) lies on the given hyperboloid, so

$$(\alpha^2/a^2)-(\beta^2/b^2)-(\gamma^2/c^2)=1$$

or $$\lambda^2\,[(a^2+b^2)-0-(b^2-c^2)]=1 \quad\text{or}\quad \lambda=\pm 1/\sqrt{(a^2+c^2)}.$$

$$\therefore \text{ From (iii), } \alpha=\pm\frac{a\sqrt{(a^2+b^2)}}{\sqrt{(a^2+c^2)}},\ \beta=0,\ \gamma=\frac{\pm c\sqrt{(b^2-c^2)}}{\sqrt{(a^2+c^2)}}$$ Ans.

Ex. 3. Prove that the umbilics of the conicoid $\dfrac{x^2}{a+b}+\dfrac{y^2}{a}+\dfrac{z^2}{a-b}=1$ are the extremities of the equal conjugate diameters of the ellipse

$$y=0,\ [x^2/(a+b)]+[z^2/(a-b)]=1. \qquad (\textit{Garhwal } 94)$$

Sol. Let (α, β, γ) be the umbilic of the given conicoid, so that the equation of the tangent to the given conicoid at the point (α, β, γ) is

$$\frac{\alpha x}{a+b}+\frac{\beta y}{a}+\frac{\gamma z}{a-b}=1 \qquad \text{...(i)}$$

We can write the given equation of the conicoid as

$$a(a-b)x^2+(a^2-b^2)y^2+a(a+b)z^2=a(a^2-b^2)$$

which can be written in the form.

$$[(a^2-b^2)(x^2+y^2+z^2)-a(a^2-b^2)]+b(b-a)x^2+(ab+b^2)z^2=0$$

or $\quad (a^2-b^2)(x^2+y^2-z^2-a)+b(a+b)z^2-b(a-b)x^2=0$

∴ Circular sections are evidently given by

or $\quad (a+b)z^2-(a-b)x^2=0 \quad$ or $\quad \sqrt{(a-b)}\,x=\pm\sqrt{(a+b)}\,z \qquad$ ...(ii)

By definition the plane (i) should be parallel to either of the real circular sections of the conicoid.

∴ Comparing (i) and (ii), we get

$$\beta=0,\ \frac{\alpha}{(a+b)\sqrt{(a-b)}}=\frac{\gamma}{\pm(a-b)\sqrt{(a+b)}}$$

or $$\beta=0,\ \frac{\alpha}{\sqrt{(a+b)}}=\frac{\gamma}{\pm\sqrt{(a-b)}}=\lambda, \text{ say} \qquad \text{...(iii)}$$

Also as (α, β, γ) lies on the given conicoid, so we have

$$\frac{\alpha^2}{(a+b)}+\frac{\beta^2}{a}+\frac{\gamma^2}{(a-b)}=1$$

or $\quad \lambda^2[1+0+1]=1$, substituting value of α, β, γ from (iii)

or $\quad \lambda^2=1/2 \quad$ or $\quad \lambda=1/\sqrt{2}$

∴ From (iii) we have $\alpha=\pm\sqrt{\left(\dfrac{a+b}{2}\right)},\ \beta=0,\ \gamma=\pm\sqrt{\left(\dfrac{a-b}{2}\right)} \qquad$ ...(iv)

Also from coordinate geometry we know that the extremities of equi-conjugate diameters of the ellipsoid $(x^2/A^2)+(y^2/B^2)=1$ are $\pm A\cos\phi$, $\pm B\sin\phi$, where $\phi=\pi/4$.

∴ The extremities of the equi-conjugate diameters of the ellipse

$$\frac{x^2}{a+b}+\frac{z^2}{a-b}=1,\ y=0 \text{ are } \left[\pm\sqrt{(a+b)}\cos\frac{\pi}{4},\ 0,\ \pm\sqrt{(a-b)}\sin\frac{\pi}{4}\right]$$

i.e. $\left[\pm\sqrt{\left(\frac{a+b}{2}\right)}, 0, \pm\sqrt{\left(\frac{a-b}{2}\right)}\right]$

which are the umbilics of the given conicoid as proved in (iv).

***Ex. 4. Prove that the perpendicular distance from centre to the tangent plane at an umbilic of the ellipsoid $\Sigma (x^2/a^2) = 1$ is ac/b.**

(*Garhwal* 96, 95)

Sol. Let $P(\alpha, \beta, \gamma)$ be an umbilic so that tangent plane at $P(\alpha, \beta, \gamma)$ to the given ellipsoid is $\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = 1$

Its distance p from the centre (0, 0, 0) is given by

$$p = 1/\sqrt{[(\alpha^2/a^4) + (\beta^2/b^4) + (\gamma^2/c^4)]} \quad \text{...(i)}$$

Also as in § 11.11 Pages ~~58~~-59 Ch. XI we can find (To be found in exam.) that

$$\alpha = \frac{\pm a\sqrt{(a^2-b^2)}}{\sqrt{(a^2-c^2)}}, \ \beta = 0, \ \gamma = \frac{\pm c\sqrt{(b^2-c^2)}}{\sqrt{(a^2-c^2)}} \quad \text{...(ii)}$$

$$\therefore \frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4} + \frac{\gamma^2}{c^4} = \frac{1}{(a^2-c^2)}\left[\frac{a^2-b^2}{a^2} + 0 + \frac{b^2-c^2}{c^2}\right], \text{ from (ii)}$$

$$= \frac{1}{(a^2-c^2)}\left[\frac{c^2a^2 - c^2b^2 + b^2a^2 - a^2c^2}{a^2c^2}\right] = \frac{b^2(a^2-c^2)}{(a^2-c^2)a^2c^2} = \frac{b^2}{a^2c^2}$$

$\therefore$ From (i), we have $p = 1/\sqrt{(b^2/a^2c^2)} = ac/b$ Hence proved.

***Ex. 5. Find the umbilics of the paraboloid**

$$(x^2/a^2) + (y^2/b^2) = 2z, \ a > b. \quad (\textit{Gorakhpur } 97)$$

Sol. Let $P(\alpha, \beta, \gamma)$ be an umbilic of the given paraboloid.

$\therefore$ The equation of the tangent plane to the given paraboloid at $P(\alpha, \beta, \gamma)$ is

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} - (z + \gamma) = 0 \quad \text{...(i)}$$

By definition this plane should be parallel to either of the real circular sections through the origin of the given paraboloid viz.

$$y\sqrt{\left(\frac{1}{b^2} - \frac{1}{a^2}\right)} \pm z\sqrt{\left(\frac{1}{a^2}\right)} = 0, \quad \because \ a > b \quad \text{...(ii)}$$

...See § 11.09 Page 57 Ch. XI

Comparing (i) and (ii) we find that

$$\alpha = 0, \quad \frac{\beta/b^2}{\sqrt{\left(\frac{1}{b^2}-\frac{1}{a^2}\right)}} = \frac{\pm 1}{\sqrt{\left(\frac{1}{a^2}\right)}}$$

or $$\alpha = 0, \quad \beta = \pm b\sqrt{(a^2-b^2)} \quad \text{...(iii)}$$

Also as $P(\alpha, \beta, \gamma)$ lies on the given paraboloid.

$\therefore$ $$(\alpha^2/a^2) + (\beta^2/b^2) = 2\gamma$$

or $0 + (a^2 - b^2) = 2\gamma$, putting values of α, β from (iii)

or $$\gamma = (a^2 - b^2)/2.$$

Hence the required umbilics are given by

$$[0, \pm b\sqrt{(a^2-b^2)}, \tfrac{1}{2}(a^2-b^2)]$$ **Ans.**

Excercises on Chapter XI

*1. Show that the plane $x + y - z = 0$ cuts the conicoid $4x^2 + 2y^2 + z^2 + 3yz + zx - 1 = 0$ in a circle. What is the radius of the circle? **Ans.** $1/\sqrt{3}$

2. Find the umbilics of the ellipsoid $2x^2 + 3y^2 + 6z^2 = 6$

Ans. $(\pm\frac{1}{2}\sqrt{6}, 0, \pm\frac{1}{2}\sqrt{2})$.

CHAPTER XII

Reduction of General Equation of Second Degree

§ 12.01. General Equation of the Second Degree.

The most general equation of second degree is written as

$$F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \quad \text{...(i)}$$

or $\quad F(x, y, z) \equiv f(x, y, z) + 2ux + 2vy + 2wz + d = 0, \quad$ ...(ii)

where $f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \quad$ ...(iii)

The equation (i) contains in fact **nine** disposable constants, since the whole equation can be divided by any one constant and then nine ratios of ten constants a, b, c etc. will be left.

$\therefore$ A surface of second degree can be made to satisfy **nine** conditions and no more and each of these nine conditions gives one relation involving these constants.

Note 1. In all discussions in this chapter we shall take $F(x, y, z)$ and $f(x, y, z)$ as defined in (i) and (iii) above *i.e.* $f(x, y, z)$ will be taken as the homogeneous part of $F(x, y, z)$.

Note 2. Here $\dfrac{\partial f}{\partial x} = 2(ax + by + gz)$, $\dfrac{\partial f}{\partial y} = 2(hx + by + fz)$;

$$\frac{\partial f}{\partial z} = 2(gx + fy + cz)\ ; \quad \frac{\partial F}{\partial x} = 2(ax + hy + gz + u);$$

$$\frac{\partial F}{\partial y} = 2(hx + by + fz + v); \quad \frac{\partial F}{\partial z} = 2(gx + fy + cz + w).$$

*§ 12.02. Determination of the centre of surface F (x, y, z) = 0.

Let (x_1, y_1, z_1) be the centre of the surface $F(x, y, z) = 0$. Shifting the origin to the centre (x_1, y_1, z_1), the transformed equation of the surface is

$$F(x + x_1,\ y + y_1,\ z + z_1) = 0$$

i.e. $\quad a(x + x_1)^2 + b(y + y_1)^2 + c(z + z_1)^2 + 2f(y + y_1)(z + z_1) + 2g(z + z_1)(x + x_1) + 2h(x + x_1)(y + y_1) + 2u(x + x_1) + 2v(y + y_1) + 2w(z + z_1) + d = 0$

or $\quad f(x, y, z) + 2x(ax_1 + hy_1 + gz_1 + u) + 2y(hx_1 + by_1 + fz_1 + v) + 2z(gx_1 + fy_1 + cz_1 + w) + (ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 + 2ux_1 + 2vy_1 + 2wz_1 + d) = 0 \quad$ ...(I)

Now as the centre of (I) is origin, so it should be homogeneous in (x, y, z) [since if (x', y', z') is a point on it, $(-x', -y', -z')$ must also lie on it as $(0, 0, 0)$, the mid-point of the chord joining (x', y', z') and $(-x', -y', -z')$, is the centre of the surface $F(x, y, z) = 0$ and therefore only second degree terms must exist in (I).]

$\therefore$ From (I) we have $ax_1 + hy_1 + gz_1 + u = 0$ ...(II)

$hx_1 + by_1 + fz_1 + v = 0$ ...(III)

$gx_1 + fy_1 + cz_1 + w = 0$ ...(IV)

Also constant term in (V) can be rewritten as

$x_1 (ax_1 + hy_1 + gz_1 + u) + y_1 (hx_1 + by_1 + fz_1 + v)$
$+ z_1 (gx_1 + fy_1 + cz_1 + w) + (ux_1 + vy_1 + wz_1 + d)$

$= ux_1 + vy_1 + wz_1 + d$, with the help of (II), (III), (IV).

$= d'$, (say)

Then (I) reduces to $f(x, y, z) + d' = 0$ ...(V)

where $d' = ux_1 + vy_1 + wz_1 + d.$ ...(VI)

And x_1, y_1, z_1 is obtained from (II), (III) and (IV), which can be obtained from

$\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$ replacing x, y, z by x_1, y_1, z_1.

Hence **centre of the surface** $F(x, y, z) = 0$, is given by solving

$$\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0 \text{ for } x, y, z$$

and the **equation of the surface referred to centre as origin** is

$$f(x, y, z) + (ux_1 + vy_1 + wz_1 + d) = 0, \quad \text{...(VII)}$$

where (x_1, y_1, z_1) is the centre of the surface. **(Remember)**

Note. The equations (II), (III), (IV) may or may not give a unique centre. There may be more than one centre, a line of centres or a plane of centres depending upon the nature of solutions of the above three equations.

From (II), (III), (IV) we find that the centre (x_1, y_1, z_1) lies on the planes

$$\left.\begin{aligned} ax + hy + gz + u &= 0 \\ hx + by + fz + v &= 0 \\ \text{and} \quad gx + fy + cz + w &= 0 \end{aligned}\right\} \quad \text{...(VIII)}$$

These planes are known as **central planes** and any point common to these planes is a **centre.**

§ 12.03. Transformation of f (x, y, z).

To show that by the rotation of axes the expression $f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ *transforms to* $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0$, *where* $\lambda_1, \lambda_2, \lambda_3$ *are the roots of the cubic*

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

or $\lambda^3 - \lambda^2 (a + b + c) + \lambda (ab + bc + ca - f^2 - g^2 - h^2)$
$- (abc + 2fgh - af^2 - bg^2 - ch^2) = 0$

or $\lambda^3 - \lambda^2 (a + b + c) + \lambda (A + B + C) - D = 0,$

where A, B, C are the cofactors of a, b, c respectively in the determinant

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

We know that the expression $x^2 + y^2 + z^2$ is an invariant when the rectangular axes are rotated through the same origin.

(See chapter on Change of axes)

$\therefore$ If we put $x = l_1 x + m_1 y + n_1 z,\ y = l_2 x + m_2 y + n_2 z$

and $z = l_3 x + m_3 y + n_3 z$ in $x^2 + y^2 + z^2$, then by the relations

$l_1^2 + m_1^2 + n_1^2 = 1,\ l_2^2 + m_2^2 + n_2^2 = 1,\ l_3^2 + m_3^2 + n_3^2 = 1;$

$l_1^2 + l_2^2 + l_3^2 = 1,\ m_1^2 + m_2^2 + m_3^2 = 1,\ n_1^2 + n_2^2 + n_3^2 = 1,$

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0,\ l_2 l_3 + m_2 m_3 + n_2 n_3 = 0,\ l_3 l_1 + m_3 m_1 + n_3 n_1 = 0$

$l_1 m_1 + l_2 m_2 + l_3 m_3 = 0,\ m_1 n_1 + m_2 n_2 + m_3 n_3 = 0,\ n_1 l_1 + n_2 l_2 + n_3 l_3 = 0,$

it remains unchanged.

Now if the axes are rotated in such a manner that $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ becomes $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2$, then the expression

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy - \lambda (x^2 + y^2 + z^2) \qquad ...(i)$$

should reduce to $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 - \lambda (x^2 + y^2 + z^2)$ ...(ii)

i.e. both the expressions (i) and (ii) will be the product of linear factors for the same value of λ.

Now if (i) *i.e.* if $(a - \lambda) x^2 + (b - \lambda) y^2 + (c - \lambda) z^2 + 2fyz + 2gzx + 2hxy$ is the product of two linear factors then we must have

$$\begin{vmatrix} a - \lambda & h & g \\ h & b - \lambda & f \\ g & f & c - \lambda \end{vmatrix} = 0 \qquad ...(iii)$$

or $\lambda^3 - \lambda^2 (a + b + c) + \lambda (ab + bc + ca - f^2 - g^2 - h^2)$
$- (abc + 2fgh - af^2 - bg^2 - ch^2) = 0$

or $\lambda^3 - \lambda^2 (a + b + c) + \lambda (A + B + C) - D = 0$,

where A, B, C are the cofactors of corresponding small letters in the determidant $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

And the expression $(\lambda_1 - \lambda) x^2 + (\lambda_2 - \lambda) y^2 + (\lambda_3 - \lambda) z^2$ in (ii) will be the product of two linear factors, if

$$\begin{vmatrix} \lambda_1 - \lambda & 0 & 0 \\ 0 & \lambda_2 - \lambda & 0 \\ 0 & 0 & \lambda_3 - \lambda \end{vmatrix} = 0 \ \textit{i.e.}\ (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) = 0$$

i.e. when $\lambda = \lambda_1$ or λ_2 or λ_3.

The same values of λ should be obtained from (iii) also.

Hence $\lambda_1, \lambda_2, \lambda_3$ are the roots of the cubic (iii) in λ, which is called **discriminating cubic.**

Let λ be any root (real or imaginary) of the discriminating cubic (iii) and that l, m, n be the principal direction cosines (which may also be real or imaginary) corresponding to this value of λ, then

$$al + hm + gn = \lambda\, l$$
$$hl + bm + fn = \lambda\, m$$
$$gl + fm + cn = \lambda\, n \quad \textbf{[See chapter on Change of axes]}$$

or
$$(a - \lambda)\, l + hm + gn = 0$$
$$hl + (b - \lambda)\, m + fn = 0$$
$$gl + fm + (c - \lambda)\, n = 0$$

where λ is to be replaced by $\lambda_1, \lambda_2, \lambda_3$, to get the corresponding direction-cosines of the axes.

§ 12.04. Various Forms of General Equation of Second Degree.

The general equation of second degree viz $F(x, y, z) = 0$, as given in § 12.01 Page 1 of this chaper, can be reduced to any one of the following forms :—

S. No.	Equation	Name of the surface
1.	$Ax^2 + By^2 + Cz^2 = 1$	Ellipsoid
2.	$Ax^2 + By^2 - Cz^2 = 1$	Hyperboloid of one sheet
3.	$Ax^2 - By^2 - Cz^2 = 1$	Hyperboloid of two sheets
4.	$Ax^2 + By^2 + Cz^2 = 0$	Cone
5.	$Ax^2 + By^2 + 2kz = 0$	Elliptic paraboloid
6.	$Ax^2 - By^2 + 2kz = 0$	Hyperbolic paraboloid
7.	$Ax^2 + By^2 + d = 0$	Elliptic cylinder
8.	$Ax^2 - By^2 + d = 0$	Hyperbolic cylinder
9.	$Ax^2 - By^2 = 0$	Pair of planes
If second degree terms form a perfect square, then		
10.	$Ax^2 + Bx + C = 0$	Pair of parallel planes
11.	$y^2 = Ax$	Parabolic cylinder
12.	$A(x^2 + y^2) + Bz = 0$	Paraboloid of revolution
13.	$A(x^2 + y^2) + Cz^2 = 1$	Ellipsoid of revolution
14.	$A(x^2 - y^2) + Cz^2 = 1$	Hyperboloid of revolution

Note. In discussion to follow we shall take

$$f(l, m, n) \equiv al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm,$$

whence
$$\frac{\partial f}{\partial l} = 2(al + bm + gn),$$

$$\frac{\partial f}{\partial m}=2\,(hl+bm+fn),\ \frac{\partial f}{\partial n}=2\,(gl+fm+cn)$$

§ 12.05. Equation of surface referred to centre as origin.

From § 12.02. result (VIII) on Page 2 of this chapter we know that the centre (x_1, y_1, z_1) of $F(x, y, z)=0$, lies on the planes given by

$$ax+hy+gz+u=0 \qquad \text{...(i)}$$
$$hx+by+fz+v=0 \qquad \text{...(ii)}$$
$$gx+fy+cz+w=0 \qquad \text{...(iii)}$$

Multiplying (i), (ii) and (iii) by A, H, G respectively and adding we get

$$Dx+(Au+Hv+Gw)=0,$$

where A, B, C, F, G, H are the cofactors of the corresponding small letters viz. a, b, c, f, g, h in the determinant $D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

Similarly multiplying (i), (ii), (iii) by H, B, F and G, F, C respectively and adding separately, we get $\quad Dy+(Hu+Bv+Fw)=0 \qquad$ ...(v)

$$Dz+(Gu+Fv+Cw)=0 \qquad \text{...(vi)}$$

$\therefore$ From (iv), (v) and (vi) the coordinates of the centre are given by

$$\frac{x}{Au+Hv+Gw}=\frac{y}{Hu+Bv+Fw}=\frac{z}{Gu+Fv+Cw}=-\frac{1}{D} \qquad \text{...(A)}$$

Cor. 1. The equation of a diametral plane of the surface (conicoid) $F(x, y, z)=0$ is $\quad \boxed{l\frac{\partial F}{\partial x}+m\frac{\partial F}{\partial y}+n\frac{\partial F}{\partial z}=0} \qquad$ ...(B)

and so any diametral plane passes through the centre or centres.

Cor. 2. From (II), (III), (IV) and (VI) of § 12.02 Page 2 of this chapter we have

$$ax_1+hy_1+gz_1+u=0$$
$$hx_1+by_1+fz_1+v=0$$
$$gx_1+fy_1+cz_1+w=0$$

and $\quad ux_1+vy_1+wz_1+(d-d')=0$

which on eliminating x_1, y_1, z_1 gives $\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d-d' \end{vmatrix}=0$

or $\quad \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}-d'\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$

or $\quad P-d'D=0 \quad$ or $\quad d'=P/D, \qquad$ ...(C)

where $\quad P\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}$ and $D=\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Hence referred to centre as origin [See result (VII) of § 12.02. Page 2 of this chapter] the equation of the surface $F(x, y, z) = 0$ is

$$\boxed{f(x, y, z) + (P/D) = 0} \quad \text{...(E)}$$

***§ 12.06. Some properties of determinant D.**

We know $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ and A, B, C, F, G, H denote the cofactors of the corresponding small letters in this determinant D.

$$\therefore \quad A = bc - f^2,\ B = ca - g^2,\ C = ab - h^2,$$
$$F = gh - af,\ G = hf - bg,\ H = fg - ch.$$

Also $BC - F^2 = (ca - g^2)(ab - h^2) - (gh - af)^2$
$$= a^2bc - abg^2 - ach^2 + g^2h^2 - g^2h^2 - a^2f^2 + 2afgh$$
$$= a(abc + 2fgh - af^2 - bg^2 - ch^2) = aD$$

Similarly $CA - G^2 = bD,\ AB - H^2 = cD,$
$$GH - AF = fD,\ HF - BG = gD,\ FG - CH = hD.$$

And from the properties of determinants **(See Author's Algebra or Matrices)** we know that

$$Aa + Hh + Gg = D,\ Ha + Bh + Fg = 0,\ Ga + Fh + Cg = 0$$

and similar other results.

(i) If $D = 0$, from above we have
$$BC = F^2,\ CA = G^2,\ AB = H^2,\ GH = AF,\ HF = BG,\ FG = CH$$

(ii) If $D = 0$ and $A = 0$, then we have $G = 0,\ H = 0$.

(iii) If $D = 0$ and $A = 0$, $B = 0$, then $F = 0$, $G = 0$, $H = 0$ but C may or may not be zero.

(iv) If $D = 0$ and $H = 0$, then either $A = 0, G = 0$ or $B = 0,\ F = 0$.

(v) If $D = 0$ and $A + B + C = 0$, then
$$A = B = C = F = G = H = 0. \quad \textbf{(Note)}$$

since, A, B, C have the same sign when $D = 0$ and so $A + B + C = 0$ gives $A = B = C = 0$, whence $F = 0 = G = H$.

§ 12.07. Some facts about planes (to be remembered).

Let there be two equatons

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \end{aligned}\right\} \quad \text{...(I)}$$

and each representing a plane.

These two equatons will **represent the same plane**, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2} \quad \text{...(II)}$$

i.e.
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} = 0$$

The planes given by (I) **will be parallel but not the same** provided

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}\neq\frac{d_1}{d_2} \qquad \text{...(III)}$$

i.e. $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}=0,\ \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}\neq 0$$

and will intersect in a line provided

$$\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\neq\frac{c_1}{c_2} \qquad \text{...(IV)}$$

i.e. $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\neq 0,\ \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\neq 0$$

§ 12.08. Various Cases.

Here we shall consider the various cases which depend on the solution of the equations (i), (ii), (iii) of § 12.05 on Page 5 of this chapter.

Case I. D ≠ 0.

In this case the coordinates of the centre as obtained from (A) of § 12.05 Page 5 of this chapter are finite and unique.

∴ The surface (conicoid) $F(x, y, z)=0$ has a unique centre at a finite distance.

Case II. D = 0 and Au + Hv + Gw ≠ 0.

In this case the coordinates of the centre as obtained from (A) of § 12.05 Page 5 of this chapter are infnite, provided

$Au+Hv+Gw$, $Hu+Bv+Fw$ and $Gu+Fv+Cw$ are not zero.

Thus the surface $F(x, y, z)=0$ has a single centre at infinity.

Case III. D = 0, Au + Hv + Gw = 0.

If we denote the equations (i), (ii) and (iii) of § 12.05 on Page 5 of this chapter by $S_1=0$, $S_2=0$ and $S_3=0$ respectively then we can see that

$$AS_1+HS_2+GS_3=0.$$

∴ The central planes (see definition on § 12.02 Page 2 of this chapter) have a common line of intersection.

Also if $A=bc-f^2\neq 0$, then the planes $S_2=0$ and $S_3=0$ are neither identical nor parallel. So there is a definite line of intersection and the surface $F(x, y, z)=0$ in this case possesses a line of centres at a finite distance.

We can easily see that when $D=0$ and $Au+Hv+Gw=0$ but $A\neq 0$, then $Hu+Bv+Fw=0$ and $Gu+Fv+Cw$ are also zero, since in this case from § 12.06 (i) Page 6 we have $F=\sqrt{(BC)},\ G=\sqrt{(CA)},\ H=\sqrt{(AB)}$ ...(α)

$$\therefore\ Au+Hv+Gw=0 \Rightarrow \sqrt{A}\,(\sqrt{A}u+\sqrt{B}v+\sqrt{C}w)=0$$

$$\Rightarrow \sqrt{A}u+\sqrt{B}v+\sqrt{C}w=0,\ \because\ A\neq 0 \qquad \text{...(β)}$$

Now $Hu+Bv+Fw=\sqrt{(AB)}\,u+Bv+\sqrt{(BC)}\,w$

$$=\sqrt{B}\,[\sqrt{A}u+\sqrt{B}v+\sqrt{C}w]=0, \text{ from (β)}$$

Similarly we can prove that $Gu + Fv + Cw = 0$

[Also we can see from above that if $D = 0$, $A = 0$, then from (α) we get

$$G = 0, \ H = 0.$$

Hence in this case $Au + Hv + Gw = 0$ but $Hu + Bv + Fw$ and $Gu + Fv + Cw$ may or may not be zero].

Case IV. A, B, C, F, G, H are all zero.

As in case III above, the central planes have a common line of intersection. But these planes are parallel as is evident from § 12.07 Page 6 of this chapter.

Also we assume that $fu - gv \neq 0$, because otherwise the two planes given by (i) and (ii) of § 12.05 Page 5 of this chapter would be identical and similarly $fu - hw \neq 0$, as otherwise the planes given by (i) and (iii) of § 12.05 Page 5 of this chapter would be identical.

Hence in this case central planes [given by (i), (ii) and (iii) of § 12.05 Page 5 of this chapter] are parallel but not coincident and so the surface $F(x, y, z) = 0$ has a line of centres at an infinite distance.

Case V. A, B, C, F, G, H are all zero and fu = gv = hw.

In this case if f, g, h are not zero, the central planes (as discussed above in Case IV) are identical and so the surface $F(x, y, z) = 0$ has a plane of centres.

In case all or two of f, g, h are zero, we can deal the case directly.

§ 12.09. Reduction of general equation.

In § 12.04 Page 4 of this chapter, we have seen the various forms of the surfaces represented by the general equation of second degree. Now we shall discuss in articles to follow the reduction to the standard forms depending upon the various cases as given in § 12.08 on Pages 7–8 Ch. XII.

§ 12.10. Case I. D ≠ 0.

In this case there is a unique centre at a finite distance. Also none of the roots of the discriminating cubic (or λ –cubic) vanishes and so $D \neq 0$.

Here the forms to any one of which the given equation can reduce are :—

(i) $\mathbf{Ax^2 + By^2 + Cz^2 = 1}$ **(Ellipsoid)**

(ii) $\mathbf{Ax^2 + By^2 - Cz^2 = 1}$ **(Hyperboloid of one sheet)**

(iii) $\mathbf{Ax^2 - By^2 - Cz^2 = 1}$ **(Hyperboloid of two sheets)**

(iv) $\mathbf{Ax^2 + By^2 + Cz^2 = 0}$ **(Cone)**

Method of Procedure.

(i) Find the coordinates (x_1, y_1, z_1) of the centre of the given surface $F(x, y, z) = 0$ by solving the equations

$$\frac{\partial F}{\partial x} = 0, \ \frac{\partial F}{\partial y} = 0, \ \frac{\partial F}{\partial z} = 0.$$

(ii) Shift the origin to the centre (x_1, y_1, z_1) and then the equation of the surface referred to centre as origin is

$f(x, y, z) + d' = 0$, where $d' = ux_1 + vy_1 + wz_1 + d$.

(iii) By rotation of axes, transform the given equation to the form $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$, where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the discriminating cubic, which can be reduced to any one of the forms given above.

(iv) The **direction ratios of axes** can be obtained by solving any two of the following three equations :—

$$(a - \lambda)\, l + hm + gn = 0$$
$$hl + (b - \lambda)\, m + fn = 0$$
$$gl + fm + (c - \lambda)\, n = 0.$$

Putting the three values of λ, the direction ratios of the three axes can be obtained and so their equations can be obtained *i.e.* we can find the equations of three lines through the centre and having above direction-ratios.

(v) The **principal planes** are given by

$$\lambda\, (lx + my + nz) + (ul + vm + wn) = 0.$$

(vi) If $d' = 0$, then the surface is a cone. **(Remember)**

Note : For the solution of a cubic equation, students should go through the section on solution of cubic equations from **Author's Theory of Equations.** It is not always possible to solve a cubic equation when all its roots are real, but with the help of Descarte's Rule of signs, we can find the number of its positive and negative roots.

Solved Examples on § 12.10.

***Ex. 1. Reduce the equation $3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy - 4x - 8z + 5 = 0$ to the standard form. Find the nature of the conicoid, its centre and equations of its axes.**

Sol. Let $F(x, y, z) \equiv 3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy - 4x - 8z + 5 = 0$.

Then the coordinates of the centre are given by

$$\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0,$$

i.e.

$$\left.\begin{array}{lcl} 6x + 2z + 2y - 4 = 0 & \text{or} & 3x + y + z - 2 = 0 \\ 10y + 2z + 2x = 0 & \text{or} & x + 5y + z = 0 \\ 6z + 2y + 2x - 8 = 0 & \text{or} & x + y + 3z - 4 = 0 \end{array}\right\} \quad \ldots(\text{I})$$

Solving the equations of (I) we get the centre (x_1, y_1, z_1) as

$$(1/3, -1/3, 4/3) \quad i.e. \quad x_1 = 1/3,\ y_1 = -1/3,\ z_1 = 4/3.$$

Shifting the origin to the centre $(1/3, -1/3, 4/3)$, the equation of the surface reduces to

$$f(x, y, z) + d' = 0, \quad \ldots(\text{II})$$

where $d' = ux_1 + vy_1 + wz_1 + d$

$$= (-2)(1/3) + (0)(-1/3) + (-4)(4/3) + 5 = -1$$

$\therefore$ From (II), the reduced equation of the surface is

$$(3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy) + (-1) = 0, \quad \ldots(\text{III})$$

as $f(x, y, z)$ is the homogeneous part of $F(x, y, z)$.

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 5-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

or $(3-\lambda)[(5-\lambda)(3-\lambda)-1]-[(3-\lambda)-1]+[1-(5-\lambda)]=0$

or $$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0. \qquad \text{...(IV)}$$

By trial we find that $\lambda = 2$ satisfies (IV), so we have $(\lambda - 2)$ as a factor of L.H.S. of (IV) and so we can rewrite (IV) as

$$(\lambda-2)(\lambda^2-9\lambda+18)=0 \quad \text{or} \quad (\lambda-2)(\lambda-3)(\lambda-6)=0.$$

$\therefore$ The roots of the discriminating cubic (IV) are 2, 3, 6.

$\therefore$ By rotation of axes, the given equation transforms to

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0 \quad i.e. \quad 2x^2 + 3y^2 + 6z^2 - 1 = 0, \qquad \text{...(V)}$$

substituting values of $\lambda_1, \lambda_2, \lambda_3$ and d'.

Then equation (V) can be rewritten as $2x^2 + 3y^2 + 6z^2 = 1$, which represents an ellipsoid.

The direction-ratios of axes can be obtained by solving two of the following three equations

$$(a-\lambda)l + hm + gn = 0, \; hl + (b-\lambda)m + fn = 0, \; gl + fm + (c-\lambda)n = 0$$

or $(3-\lambda)l + m + n = 0, \; l + (5-\lambda)m + n = 0, \; l + m + (3-\lambda)n = 0 \qquad \text{...(VI)}$

Taking $\lambda = 2$, we have $l + m + n = 0, \; l + 3m + n = 0, \; l + m + n = 0.$

Solving $l + m + n = 0, \; l + 3m + n = 0$, we get

$$\frac{l}{1-3} = \frac{m}{1-1} = \frac{n}{3-1} \quad \text{or} \quad \frac{l}{1} = \frac{m}{0} = \frac{n}{-1}$$

$\therefore$ The equations of the axis, corresponding to $\lambda = 2$, are

$$\frac{x-(1/3)}{1} = \frac{y-(1/3)}{0} = \frac{z-(4/3)}{-1} \qquad \textbf{Ans.}$$

Similarly corresponding to $\lambda = 3$ and $\lambda = 6$ the direction ratios of the axes (*i.e.* the principal directions) are

$$\frac{l}{1} = \frac{m}{-1} = \frac{n}{1} \quad \text{and} \quad \frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$$

As the equation of the corresponding axes are

$$\frac{x-(1/3)}{1} = \frac{y-(1/3)}{-1} = \frac{z-(4/3)}{1};$$

$$\frac{x-(1/3)}{1} = \frac{y-(1/3)}{-2} = \frac{z-(4/3)}{1};$$

Ex. 2. Reduce the equation $3x^2 - y^2 - z^2 + 6yz - 6x + 6y - 2z - 2 = 0$ to the standard form. Also find its centre and the equation referred to centre as origin. (*Avadh* 95)

Solution. Given $F(x, y, z) \equiv 3x^2 - y^2 - z^2 + 6yz - 6x + 6y - 2z - 2 = 0.$

$\therefore$ The coordinates of the centre are given by

$$\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial z}=0$$

i.e. $6x-6=0$ or $x-1=0$ or $x=1$ ...(i)

$-2y+6z+6=0$ or $y-3z-3=0$ ...(ii)

and $-2z-2=0$ or $z+1=0$ or $z=-1$ ...(iii)

$\therefore$ Solving (i), (ii) and (iii) we get the centre (x_1, y_1, z_1) as $(1, 0, -1)$. **Ans.**

Shifting the origin to the centre $(1, 0, -1)$ the equation of the surface reduces to $f(x, y, z)+d'=0$,

where $d'=ux_1+vy_1+wz_1+d$

$=(-3)(1)+(3)(0)+(-1)(-1)-2=-4$

$\therefore$ From (iv), the equation of the surface referred to centre as origin is

$(3x^2-y^2-z^2+6yz)+(-4)=0$ or $3x^2-y^2-z^2+6yz-4=0$. **Ans.**

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & -1-\lambda & 3 \\ 0 & 3 & -1-\lambda \end{vmatrix}=0,$$

putting values of a, b, c, f, g, h

or $(3-\lambda)[(1+\lambda)^2-9]=0$ or $(\lambda-3)[\lambda^2+2\lambda-8]=0$

or $(\lambda-3)(\lambda-2)(\lambda+4)=0$, or $\lambda=2, 3, -4$.

$\therefore$ By rotation of axes, the given equation transforms to

"$\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2+d'=0$" *i.e.* $2x^2+3y^2-4z^2-4=0$

or $2x^2+3y^2-4z^2=4$, which represents a hyperboloid of one sheet

[$\because$ it is of the form $Ax^2+By^2-Cz^2=1$]. **Ans.**

Ex. 3. Show that the equation $x^2+y^2+z^2-6yz-2zx-2xy-6x-2y-2z+2=0$ represents a hyperboloid of two sheets.

Solution. Comparing the given equation $F(x, y, z)=0$ with the equation $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$, we have $a=1$, $b=1, c=1, f=-3, g=-1, h=-1, u=-3, v=-1, w=-1, d=2$. ...(i)

Now coordinates of the centre (x_1, y_1, z_1) of the given surface are given by

$$\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial z}=0.$$

i.e. $2x_1-2y_1-2z_1-6=0$ or $x_1-y_1-z_1=3$ ...(ii)

$2y_1-6z_1-2x_1-2=0$ or $x_1-y_1+3z_1=-2$ ...(iii)

$2z_1-6y_1-2x_1-2=0$ or $x_1+3y_1-z_1=-2$ ...(iv)

Solving (ii), (iii) and (iv) we get $x_1=1/2$, $y_1=-5/4$, $z_1=-5/4$

$\therefore$ centre of the given surface is $(1/2, -5/4, -5/4)$.

Also $d'=ux_1+vy_1+wz_1+d$

$=(-3)(1/2)+(-1)(-5/4)+(-1)(-5/4)+2=3$ ...(v)

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -3 \\ -1 & -3 & 1-\lambda \end{vmatrix}=0$$

or $$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & \lambda-4 \\ -1 & -3 & 4-\lambda \end{vmatrix}=0, \text{ applying, } C_3-C_2.$$

or $\lambda^3-3\lambda^2-8\lambda+16=0$ or $(\lambda-4)(\lambda^2+\lambda-4)=0$

$\therefore$ Either $\lambda=4$ or $\lambda^2+\lambda-4=0$.

Now $\lambda^2+\lambda-4=0$ gives $\lambda=[-1\pm\sqrt{(1+16)}]/2$.

Thus we find that two values of λ are +ve and one −ve.

$\therefore$ By rotation of axes, the given equation transforms to

$\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2+d'=0$ *i.e.* $-\dfrac{\lambda_1}{3}x^2-\dfrac{\lambda_2}{3}y^2-\dfrac{\lambda_3}{3}z^2=1, \because d'=3.$

Now two values of λ being positive and one negative, from above the equation of the surface transforms to the form $Ax^2+By^2+Cz^2=1$, where two of A, B, C are negative and third positive, so that the given surface is a hyperboloid of two sheets.

Ex. 4. Reduce the equation $2x^2-7y^2+2z^2-10yz-8zx-10xy+6x+12y-6z+5=0$ to the standard form. What does it represent?

Sol. Comparing the given equation $F(x, y, z)=0$ with the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0,$$

we have $a=2, b=-7, c=2, f=-5, g=-4, h=-5, u=3, v=6, w=-3, d=5$.

Now coordinates of the centre (x_1, y_1, z_1) of the given surface are given by

$$\frac{\partial F}{\partial x}=0,\ \frac{\partial F}{\partial y}=0, \text{ and } \frac{\partial F}{\partial z}=0.$$

i.e. $4x_1-8z_1-10y_1+6=0$ or $2x_1-5y_1-4z_1+3=0$; ...(ii)

$-14y_1-10z_1-10x_1+12=0$ or $5x_1+7y_1+5z_1-6=0$; ...(iii)

$4z_1-10y_1-8x_1-6=0$ or $4x_1+5y_1-2z_1+3=0$; ...(iv)

Solving (ii), (iii) and (iv) we get $x_1=1/3, y_1=-1/3, z_1=4/3$.

$\therefore$ Centre of the given surface is $(1/3, -1/3, 4/3)$.

Also $d'=$ '$ux_1+vy_1+wz_1+d$'

$=3(1/3)+6(-1/3)+(-3)(4/3)+5=1-2-4+5=0$ ...(v)

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} 2-\lambda & -5 & -4 \\ -5 & -7-\lambda & -5 \\ -4 & -5 & 2-\lambda \end{vmatrix}=0$$

or $(2-\lambda)[-(7+\lambda)(2-\lambda)-25]+5[-5(2-\lambda)-20]-4[25-4(7+\lambda)]=0$

or $\lambda^3+3\lambda^2-90\lambda+216=0$, on simplifying or $(\lambda-3)(\lambda^2+6\lambda-72)=0$ or $(\lambda-3)(\lambda-6)(\lambda+12)=0$

or $\lambda=3, 6, -12$

$\therefore$ Let $\lambda_1=3, \lambda_2=6, \lambda_3=-12$

$\therefore$ By rotation of axes, the given equation transforms to

$$\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2+d'=0$$

or $3x^2+6y^2-12z^2+0=0$, substituting values of $\lambda_1, \lambda_2, \lambda_3, d'$

or $x^2+2y^2-4z^2=0$, which is the required standard form and represents a cone. [See § 12.04 (4) Page 4 Ch. XII]

Also the vertex of the cone (and not centre) is $(1/3, -1/3, 4/3)$ as calculated above.

Exercises on § 12.10 (Case I).

Ex. 1. Reduce the equation $11x^2+10y^2+6z^2-8yz+4zx-12xy+72x-72y+36z+150=0$ to the standard form and show that it represents an ellipsoid and find the equations of the axes. (*Avadh* 91; *Garhwal* 94, 92)

Ans. Centre $(-2, 2, -1)$; $3x^2+6y^2+18z^2=12$ (ellipsoid)

d.r's of the axes are $1, 1, 2; 2, 1, -2; -2, 2, -1$.

Ex. 2. Reduce $3x^2+6yz-y^2-z^2-6x+6y-2z-2=0$ to the standard form. What surface does it represent?

Ans. $2x^2+3y^2-4z^2=4$; Hyperboloid of one sheet.

Ex. 3. Reduce $2x^2-y^2-10z^2+20yz-8zx-28xy+16x+26y+16z-34=0$ to the standard form. What does it represent ?

Ans. $2x^2-y^2-2z^2=1$; Hyperboloid of two sheets.

Ex. 4. For the conicoid $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$, show that

(i) all the roots of the discriminating cubic are real.

and (ii) principal directions are perpendicular to each other. (***Rohilkhand*** **91**)

Ex. 5. Reduce $x^2+3y^2+3z^2-2yz-2x-2y+6z+3=0$ to the standard form and show that the surface represented by it is an ellipsoid.

Ans. $x^2+2y^2+4z^2=1$.

§ 12.11. Case II. D = 0 and Au + Hv + Gw ≠ 0.

$D=0 \Rightarrow$ one root of the discriminating cubic is zero. **(Note)**

Here the forms to any one of which the given equation can reduce are

$$\mathbf{Ax^2+By^2+Cz=0} \quad \text{(Elliptic Paraboloid)}$$

and

$$\mathbf{Ax^2-By^2+Cz=0} \quad \text{(Hyperbolic Paraboloid)}$$

Method of Procedure.

(i) Find the discriminating cubic viz $\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0$

One root of this cubic will be zero in this case. **(Note)**

(ii) Put $\lambda = 0$ in the above determinant and associate each row with l_3, m_3, n_3.

i.e. $al_3 + hm_3 + gn_3 = 0,\ hl_3 + bm_3 + fn_3 = 0,\ gl_3 + fm_3 + cn_3 = 0.$

Solve any two of these, which will give the direction ratios of the axis corresponding to $\lambda = 0$.

(iii) Evaluate $k = ul_3 + vm_3 + wn_3$, **(Remember)**

where l_3, m_3, n_3 are actual direction-cosines.

If $k \neq 0$, then reduced equation is

$\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$, where λ_1, λ_2 are non-zero roots of the discriminating cubic.

This equation represents an elliptic or hyperbolic paraboloid according as λ_1 and λ_2 have the same or opposite signs.

(iv) **Vertex.** The coordinates of the vertex of the paraboloid in this case are obtained by solving any two of the three equations

$$\frac{\left(\dfrac{\partial F}{\partial x}\right)}{l_3} = \frac{\left(\dfrac{\partial F}{\partial y}\right)}{m_3} = \frac{\left(\dfrac{\partial F}{\partial z}\right)}{n_3} = 2k,$$

with the equation $k(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$ **(Remember)**

Solved Examples on § 12.11.

****Ex. 1. Determine completely what is represented by the equation**

$$\mathbf{2x^2 + 2y^2 + z^2 + 2yz - 2zx - 4xy + x + y = 0.}$$

Find the coordinates of its vertex and the equations to its axis

(*Garhwal* 93)

Solution. Here 'a' $= 2$, 'b' $= 2$, 'c' $= 1$, 'f' $= 1$, 'g' $= -1$.

'h' $= -2$, 'u' $= 1/2$, 'v' $= 1/2$, 'w' $= 0$ and 'd' $= 0$

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 2-\lambda & -2 & -1 \\ -2 & 2-\lambda & 1 \\ -1 & 1 & 1-\lambda \end{vmatrix} = 0 \qquad \text{...(i)}$$

or $\quad (2-\lambda)[(2-\lambda)(1-\lambda) - 1] + 2[-2(1-\lambda) + 1] - [-2 + (2-\lambda)] = 0.$

or $\quad \lambda^3 - 5\lambda^2 + 2\lambda = 0$ or $\lambda(\lambda^2 - 5\lambda + 2) = 0$ or $\lambda = 0, \frac{1}{2}[5 \pm \sqrt{(17)}]$

$\therefore$ Let $\lambda_1 = \frac{1}{2}[5 + \sqrt{(17)}],\ \lambda_2 = \frac{1}{2}[5 - \sqrt{(17)}],\ \lambda_3 = 0$

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$2l_3 - 2m_3 - n_3 = 0,\ -2l_3 - 2m_3 + n_3 = 0,\ -l_3 + m_3 + n_3 = 0$

Solving last two equations simultaneously for l_3, m_3, n_3; we get

$$\frac{l_3}{2-1} = \frac{m_3}{-1+2} = \frac{n_3}{-2+2}$$

i.e. $$\frac{l_3}{1} = \frac{m_3}{1} = \frac{n_3}{0} = \frac{\sqrt{(l_3^2 + m_3^2 + n_3^2)}}{\sqrt{(1^2 + 1^2 + 0^2)}} = \frac{1}{\sqrt{2}}$$

$\therefore\ l_3 = 1/\sqrt{2},\ m_3 = 1/\sqrt{2},\ n_3 = 0.$

These gives d.c.'s of the axis corresponding to $\lambda = 0$.

Now $k =$ '$ul_3 + vm_3 + wn_3$' $= (\frac{1}{2})(1/\sqrt{2}) + (\frac{1}{2})(1/\sqrt{2}) + 0 = 1/\sqrt{2}$

$\therefore$ The required reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$

or $\quad \frac{1}{2}[5 + \sqrt{(17)}]\, x^2 + \frac{1}{2}[5 - \sqrt{(17)}]\, y^2 + 2(1/\sqrt{2})\, z = 0,$

which represents an elliptic paraboloid ~~as both λ_1, λ_2 are positive~~. **(Note)**

Also if $F(x, y, z) = 0$ be the given surface then the coordinates of its vertex are given by solving any two of the three equations

$$\frac{\left(\dfrac{\partial F}{\partial x}\right)}{l_3} = \frac{\left(\dfrac{\partial F}{\partial y}\right)}{m_3} = \frac{\left(\dfrac{\partial F}{\partial z}\right)}{n_3} = 2k.$$

along with $\quad k(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$

...See § 12.11 (iv) Page 14 Ch. XII

i.e. any two of the equations $4x - 2z - 4y + 1 = 2(1/\sqrt{2})(1/\sqrt{2})$

or $\quad 2x - 2y - z = 0;$

$4y + 2z - 4x + 1 = 2(1/\sqrt{2})(1/\sqrt{2})$ or $2x - 2y - z = 0;$

$2z + 2y + 2x = 2(0)(1/\sqrt{2})$ or $x - y - z = 0$

with $$\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y + 0z\right] + \frac{1}{2}x + \frac{1}{2}y + 0 + 0 = 0$$

i.e. $\quad 2x - 2y - z = 0,\ x - y - z = 0,\ x + y = 0$

Solving these we get $x = 0, y = 0, z = 0$ *i.e.* the coordinates of the vertex are $(0, 0, 0)$. **Ans.**

$\therefore$ The equations to its axis are

$$\frac{x-0}{l_3} = \frac{y-0}{m_3} = \frac{z-0}{n_3} \quad i.e. \quad \frac{x-0}{(1/\sqrt{2})} = \frac{y-0}{(1/\sqrt{2})} = \frac{z-0}{0}$$

i.e. $\quad x = y, z = 0.$ **Ans.**

***Ex. 2. Reduce the equation $3z^2 - 6yz - 6zx - 7x - 5y + 6z + 3 = 0$ to standard form and find its nature.** (*Avadh* 94)

Sol. Here 'a' $= 0$, 'b' $= 0$, 'c' $= 3$, 'f' $= -3$, 'g' $= -3$, 'h' $= 0$, 'u' $= -7/2$, 'v' $= -5/2$, 'w' $= 3$ and 'd' $= 3$

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 0-\lambda & 0 & -3 \\ 0 & 0-\lambda & -3 \\ -3 & -3 & 3-\lambda \end{vmatrix} = 0 \qquad \text{...(i)}$$

or $\quad -\lambda[-\lambda(3-\lambda) - 9] + 0 - 3[-3\lambda] = 0 \quad$ or $\quad \lambda^3 - 3\lambda^2 - 18\lambda = 0$

or $\quad \lambda(\lambda^2 - 3\lambda - 18) = 0 \quad$ or $\quad \lambda(\lambda - 6)(\lambda + 3) = 0 \quad$ or $\quad \lambda = 0, 6, -3$

Let $\lambda_1 = 6, \lambda_2 = -3, \lambda_3 = 0.$

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$0 . l_3 + 0 . m_3 - 3n_3 = 0,\ 0 . l_3 + 0 . m_3 - 3n_3 = 0,\ -3l_3 - 3m_3 + 3n_3 = 0$

These gives $n_3 = 0,\ l_3 + m_3 = 0$

i.e. $\dfrac{l_3}{1} = \dfrac{m_3}{-1} = \dfrac{n_3}{0} = \dfrac{\sqrt{(l_3^2 + m_3^2 + n_3^2)}}{\sqrt{[1^2 + (-1)^2 + 0^2]}} = \dfrac{1}{\sqrt{2}}$

$\therefore l_3 = 1/\sqrt{2},\ m_3 = -1/\sqrt{2},\ n_3 = 0$

These gives d.c's of the axis corresponding to $\lambda = 0$.

Now $k = \text{'}ul_3 + vm_3 + wn_3\text{'} = -\dfrac{7}{2}\left(\dfrac{1}{\sqrt{2}}\right) - \dfrac{5}{2}\left(\dfrac{1}{\sqrt{2}}\right) + 3\,(0) = -\dfrac{1}{\sqrt{2}}$

$\therefore$ Required reduced equation is

$$\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0 \quad \text{or} \quad 6x^2 - 3y^2 - \sqrt{2}z = 0,$$

which represents a hyperbolic paraboloid as λ_1 and λ_2 are of opposite signs.

Ans.

****Ex. 3. Find the coordinates of the vertex and equation to the axis of the hyperbolic paraboloid $4x^2 - y^2 - z^2 + 2yz - 8x - 4y + 8z - 2 = 0$.**

(Rohilkhand 95)

Solution. Here 'a' $= 4$, 'b' $= -1$, 'c' $= -1$, 'f' $= 1$, 'g' $= 0$, 'h' $= 0$, 'u' $= -4$, 'v' $= -2$, 'w' $= 4$ and 'd' $= -2$

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

or $(4-\lambda)\,[(1+\lambda)^2 - 1] = 0$ or $\lambda\,(\lambda + 2)\,(\lambda - 4) = 0$ or $\lambda = 0, -2, 4$.

$\therefore$ Let $\lambda_1 = -2,\ \lambda_2 = 4,\ \lambda_3 = 0$

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$$4l_3 = 0,\ -m_3 + n_3 = 0,\ m_3 - n_3 = 0 \Rightarrow l_3 = 0,\ m_3 = n_3.$$

But $l_3^2 + m_3^2 + n_3^2 = 1$, so $0 + m_3^2 + m_3^2 = 1$ or $m_3 = 1/\sqrt{2} = n_3$

$\therefore$ We have $l_3 = 0, m_3 = 1/\sqrt{2},\ n_3 = 1/\sqrt{2}$

Now $k = ul_3 + vm_3 + wn_3 = -4\,(0) - 2\,(1/\sqrt{2}) + 4\,(1/\sqrt{2}) = \sqrt{2}$

$\therefore$ Required reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$

or $\quad -2x^2 + 4y^2 + 2\sqrt{2}z = 0 \quad$ or $\quad x^2 - 2y^2 - z\sqrt{2} = 0$,

which represents a hyperbolic paraboloid as λ_1 is $-$ ve and λ_2 is $+$ ve.

Also if $F(x, y, z) = 0$ be the given surface then the coordinates of its vertex are given by solving any two of these equations.

$$\frac{(\partial F/\partial x)}{l_3} = \frac{(\partial F/\partial y)}{m_3} = \frac{(\partial F/\partial z)}{n_3} = 2k$$

and $\quad k\,(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$ ...See § 12.11 (iv) Page 14.

i.e. any two of the equations

$$8x-8=2(\sqrt{2})(0) \quad i.e. \quad x=1;$$
$$-2y+2z-4=2(\sqrt{2})(1/\sqrt{2}) \quad i.e. \quad y-z+3=0$$
$$-2z+2y+8=2(\sqrt{2})(1/\sqrt{2}) \quad i.e. \quad y-z+3=0$$

with $$\sqrt{2}\left[0.x+\frac{1}{\sqrt{2}}y+\frac{1}{\sqrt{2}}z\right]-4x-2y+4z-2=0$$

i.e. $x=1,\ y-z+3=0,\ 4x+y-5z+2=0$

Solving these we get $x=1,\ y=-9/4,\ z=3/4.$

$\therefore$ Coordinates of the vertex are $(1,-9/4,3/4)$ **Ans.**

And the equations of its axis are $\dfrac{x-1}{l_3}=\dfrac{y-(-9/4)}{m_3}=\dfrac{z-(3/4)}{n_3}$

or $$\frac{x-1}{0}=\frac{y+(9/4)}{1/\sqrt{2}}=\frac{z-(3/4)}{1/\sqrt{2}} \quad i.e. \quad \frac{x-1}{0}=\frac{4y+9}{1}=\frac{4z-3}{1}$$ **Ans.**

R ***Ex. 4. Show that the following equation represents a paraboloid. Find its vertex and equations to the axis.**

$$\mathbf{4y^2+4z^2+4yz-2x-14y-22z+33=0.}$$ (*Rohilkhand* 92, 90)

Solution. Here 'a' $=0$, 'b' $=4$, 'c' $=4$, 'f' $=2$, 'g' $=0$, 'h' $=0$, 'u' $=-1$, 'v' $=-7$, 'w' $=-11$ and 'd' $=33$

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & 4-\lambda & 2 \\ 0 & 2 & 4-\lambda \end{vmatrix}=0 \quad \text{...(i)}$$

or $-\lambda[(4-\lambda)^2-4]=0$ or $\lambda[\lambda^2-8\lambda+12]=0$

or $\lambda(\lambda-2)(\lambda-6)=0$ or $\lambda=0,2,6$

$\therefore$ Let $\lambda_1=2,\ \lambda_2=6$ and $\lambda_3=0$

Now putting $\lambda=0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3 we have $4m_3+2n_3=0,\ 2m_3+4n_3=0 \Rightarrow m_3=0=n_3$

But $l_3^2+m_3^2+n_3^2=1$, so $l_3^2+0+0=1 \Rightarrow l_3=1$

Now $k=ul_3+vm_3+wn_3=-1(1)-7(0)-11(0)=-1$

$\therefore$ Required reduced equation is $\lambda_1x^2+\lambda_2y^2+2kz=0$

or $$2x^2+6y^2+2(-1)z=0 \quad \text{or} \quad x^2+3y^2-z=0,$$

which represents an elliptic paraboloid as both λ_1 and λ_2 are positive.

Also if $F(x,y,z)=0$ be the given surface then the coordinates of its vertex are given by solving any two of these equations

$$\frac{\partial F/\partial x}{l_3}=\frac{\partial F/\partial y}{m_3}=\frac{\partial F/\partial z}{n_3}=2k$$

and $k(l_3x+m_3y+n_3z)+ux+vy+wz+d=0$ ...See § 12.11 (iv) P. 14.

i.e. any two of the equations

$-2=2(-1)(0)$ which is absurd.

$8y + 4z - 14 = 2(-1).0 \Rightarrow 4y + 2z = 7$;

$8x + 4y - 22 = 2(-1)(0) \Rightarrow 2y + 4z = 11$;

with $(-1)[1.x + 0.y + 0.z] - x - 7y - 11z + 33 = 0$ or $2x + 7y + 11z = 33$

i.e. $4y + 2z = 7,\ 2y + 4z = 11,\ 2x + 7y + 11z = 33$

Solving these we get $x = 1, y = 1/2, z = 5/2$

$\therefore$ Coordinates of the vertex are $(1, 1/2, 5/2)$ **Ans.**

And the equations of its axis are $\dfrac{x-1}{l_3} = \dfrac{y-(1/2)}{m_3} = \dfrac{z-(5/2)}{n_3}$

or $\dfrac{x-1}{1} = \dfrac{y-(1/2)}{0} = \dfrac{z-(5/2)}{0}$ or $\dfrac{x-1}{1} = \dfrac{2y-1}{0} = \dfrac{2z-5}{0}$.

or $2y - 1 = 0,\ 2z - 5 = 0$ or $y = 1/2, z = 5/2$.

****Ex. 5. Prove that $z(ax + by + cz) + \alpha x + \beta y = 0$ represents a paraboloid and the equations to the axis are**

$$ax + by + 2cz = 0,\ (a^2 + b^2)z + a\alpha + b\beta = 0. \quad (\textit{Rohilkhand } 93)$$

Sol. Given equation is $cz^2 + byz + azx + \alpha x + \beta y = 0$

$\therefore$ Here 'a' $= 0$, 'b' $= 0$, 'c' $= c$, 'b' $= b/2$, 'g' $= a/2$, 'h' $= 0$, 'u' $= \alpha/2$, 'v' $= \beta/2$ 'w' $= 0$ and 'd' $= 0$

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 0-\lambda & 0 & a/2 \\ 0 & 0-\lambda & b/2 \\ a/2 & b/2 & c-\lambda \end{vmatrix} = 0 \quad \text{...(i)}$$

or $-\lambda[-\lambda(c-\lambda) - (b^2/4)] + (a/2)[a\lambda/2] = 0$

or $4\lambda^3 - 4a\lambda^2 - (a^2 + b^2)\lambda = 0$ or $\lambda[4\lambda^2 - 4a\lambda - a^2 - b^2] = 0$

or $\lambda = 0$ and $\lambda = \dfrac{4a \pm \sqrt{(16a^2 + 16a^2 + 16b^2)}}{8} = \dfrac{a \pm \sqrt{(2a^2 + b^2)}}{2}$

Let $\lambda_1 = \frac{1}{2}\left[a + \sqrt{(2a^2 + b^2)}\right],\ \lambda_2 = \frac{1}{2}\left[a - \sqrt{(2a^2 + b^2)}\right],\ \lambda_3 = 0$

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$$0.l_3 + 0.m_3 + (a/2)n_3 = 0,\ 0.l_3 + 0.m_3 + (b/2)n_3 = 0$$

and $$(a/2)l_3 + (b/2)m_3 + cn_3 = 0$$

These gives $n_3 = 0$ and $al_3 + bm_3 = 0$

i.e. $\dfrac{l_3}{b} = \dfrac{m_3}{-a} = \dfrac{n_3}{0} = \dfrac{\sqrt{(l_3^2 + m_3^2 + n_3^2)}}{\sqrt{(b^2 + a^2 + 0)}} = \dfrac{1}{\sqrt{(a^2 + b^2)}}$

$\therefore l_3 = b/\sqrt{(a^2 + b^2)},\ m_3 = -a/\sqrt{(a^2 + b^2)},\ n_3 = 0$

Now $k =$ '$ul_3 + vm_3 + wn_3$'

$$= \frac{\alpha}{2} \cdot \frac{b}{\sqrt{(a^2 + b^2)}} + \frac{\beta}{2}\left[\frac{-a}{\sqrt{(a^2 + b^2)}}\right] + 0 = \frac{b\alpha - a\beta}{2\sqrt{(a^2 + b^2)}} \neq 0$$

$\therefore$ Reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$

or $\frac{1}{2}\left[a+\sqrt{(2a^2+b^2)}\right]x^2+\frac{1}{2}\left[a-\sqrt{(2a^2+b^2)}\right]y^2+\frac{(b\alpha-a\beta)}{\sqrt{(a^2+b^2)}}z=0$ ...(ii)

Now as $a+\sqrt{(2a^2+b^2)}>0$ and $a-\sqrt{(2a^2+b^2)}<0$, so (ii) represents a hypebolic paraboloid.

Also if $F(x, y, z)=0$ be the given surface then the coordinates of its vertex are given by solving any two of the equations

$$\frac{\partial F/\partial x}{l_3}=\frac{\partial F/\partial y}{m_3}=\frac{\partial F/\partial z}{n_3}=2k$$

and $k(l_3x+m_3y+n_3z)+ux+vy+wz+d=0$ ...See § 12.11 (iv) P. 14 Ch. XII

i.e. any two of the equations

$$\frac{\alpha+az}{b/\sqrt{(a^2+b^2)}}=\frac{\beta+bz}{-a/\sqrt{(a^2+b^2)}}=\frac{ax+by+2cz}{0}=\frac{2(b\alpha-a\beta)}{2\sqrt{(a^2+b^2)}}$$

and $k\left[\frac{bx}{\sqrt{(a^2+b^2)}}-\frac{ay}{\sqrt{(a^2+b^2)}}+0\right]+\frac{\alpha}{2}x+\frac{\beta}{2}y=0,$ on substituting the values

Thus we have $\frac{\alpha+az}{b}=\frac{\beta+bz}{-a}$, $ax+by+2cz=0$

and $$\frac{b\alpha-a\beta}{2\sqrt{(a^2+b^2)}}\left[\frac{bx-ay}{\sqrt{(a^2+b^2)}}\right]+\frac{1}{2}(\alpha x+\beta y)=0$$

i.e. $(a^2+b^2)z+a\alpha+b\beta=0,\quad ax+by+2cz=0$

and $(b\alpha-a\beta)(bx-ay)+(\alpha x+\beta y)(a^2+b^2)=0$

Now if (x_1, y_1, z_1) be the vertex of the paraboloid then x, y, z satisfies above three equations

i.e. $$(a^2+b^2)z_1+a\alpha+b\beta=0, \quad ...(iii)$$

$$ax_1+by_1+2cz_1=0 \quad ...(iv)$$

and $$(b\alpha-a\beta)(bx-ay)+(\alpha x+\beta y)(a^2+b^2)=0 \quad ...(v)$$

And the equations of the axis are $\frac{x-x_1}{l_3}=\frac{y-y_1}{m_3}=\frac{z-z_1}{n_3}$

or $$\frac{x-x_1}{b/\sqrt{(a^2+b^2)}}=\frac{y-y_1}{-a/\sqrt{(a^2+b^2)}}=\frac{z-z_1}{0}, \quad ...(vi)$$

substituting values of l_3, m_3, n_3

These give $z-z_1=0$ or $z=z_1=-\frac{a\alpha+b\beta}{(a^2+b^2)}$, from (iii)

or $$(a^2+b^2)z+a\alpha+b\beta=0 \quad ...(vii)$$

Again from first two fractions of (vi), we get $a(x-x_1)+b(y-y_1)=0$

or $ax+by=ax_1+by_1=-2cz_1$, from (iv)

$=-2c\left[-\frac{a\alpha+b\beta}{a^2+b^2}\right]$, from (iii)

$=-2cz$, from (vii)

or $$ax + by + 2cz = 0 \qquad \ldots\text{(viii)}$$

Hence from (vii) and (viii) the equations of the axis of the paraboloid are

$$(a^2 + b^2)\, z + a\alpha + b\beta = 0 \mid ax + by + 2cz = 0$$

Exercises on § 12.11 (Case II)

Ex. 1. Prove that the surface represented by the equation $3x^2 + 4y^2 + 9z^2 + 12yz + 6zx + 4xy + 4x + 6y + 2z + 1 = 0$ is an elliptic paraboloid.

Ans. Reduced form is $[8 + \sqrt{(38)}]x^2 + [8 - \sqrt{(38)}]\, y^2 - [14/\sqrt{(13)}]\, z = 0$

***Ex. 2.** Find the coordinates of the vertex and equation to the axis of the elliptic paraboloid, $4x^2 + y^2 + z^2 - 2zx - 2xy + x + y - 4z - 6 = 0$.

Ans. $(-1, 2, -1)\,;\, x + 1 = -\frac{1}{2}(y - 2) = \frac{1}{2}(z + 1)$.

***Ex. 3.** Find the coordinates of the vertex and equation to the axis of the hyperbolic paraboloid

$$5x^2 - 16y^2 + 5z^2 + 8yz - 14zx + 8xy + 4x + 20y + 4z - 24 = 0.$$

Ans. $(1, 1, 1), \frac{1}{2}(x - 1) = y - 1 = \frac{1}{2}(z - 1)$.

§ 12.12. Case III. D = 0, Au + Hv + Gw = 0, A ≠ 0.

In this case the forms to any one of which the given equation can reduce are :—

(i) $Ax^2 + By^2 + C = 0$ **(Elliptic cylinder)**

(ii) $Ax^2 - By^2 + C = 0$ **(Hyperbolic cylinder)**

(iii) $Ax^2 - By^2 = 0$ **(Pair of Planes)**

In this case there is a line centres at a finite distance and the discriminating cubic has one root zero, say λ_3.

The line of centres is given by any two of $\dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 0,\ \dfrac{\partial F}{\partial z} = 0,$

where $F(x, y, z) = 0$ is the equation of the given surface.

Let (α, β, γ) be the coordinates of any point lying on this line. Then shifting the origin to (α, β, γ) and rotating the axes in such a manner that these coincide with a set of mutually perpendicular principal directions, the given equation reduces to the form $\lambda_1 x^2 + \lambda_2 y^2 + k = 0$, where $k = u\alpha + v\beta + w\gamma + d$.

Nature : If $k = 0$, this represents a pair of planes.

If $k \neq 0$, this represents an **elliptic** or **hyperbolic cylinder** according as the non-zero values of λ (*i.e.* the non-zero roots of the discriminating cubic) are both of the same or opposite signs. **(Remember)**

The line of intersection of the principal planes corresponding to non-zero values of λ is the axis of the cylinder. It is parallel to the principal direction corresponding to λ_3 which is zero and is also the line of the centres.

Solved Examples on § 12.12.

****Ex. 1. Show that the surface $26x^2 + 20y^2 + 10z^2 - 4yz - 16zx - 36xy + 52x - 36y - 16z + 25 = 0$ represents an elliptic cylinder. Also find the equations to its axis.**

Solution. Here the discriminating cubic is given by

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} 26-\lambda & -18 & -8 \\ -18 & 20-\lambda & -2 \\ -8 & -2 & 10-\lambda \end{vmatrix}=0$$

or $\quad (26-\lambda)\,[(20-\lambda)(10-\lambda)-4]+18\,[-18\,(10-\lambda)-16]$

$$-8\,[36+8\,(20-\lambda)]=0$$

or $\quad \lambda^3-56\lambda^2+588\lambda=0 \quad$ or $\quad \lambda\,(\lambda^2-56\lambda+588)=0$

or $\quad \lambda\,(\lambda-14)\,(\lambda-42)=0 \quad$ or $\quad \lambda=0,\ 14,\ 42.$

Let $\lambda_1=14, \lambda_2=42$ and $\lambda_3=0.$ ...(i)

Now putting $\lambda=0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have $26l_3-18m_3-8n_3=0,\ -18l_3+20m_3-2n_3=0,$

$$-8l_3-2m_3+10n_3=0.$$

Solving first and third of these simultaneously, we have

$$\frac{l_3}{1}=\frac{m_3}{1}=\frac{n_3}{1}=\frac{\sqrt{(l_3^2+m_3^2+n_3^2)}}{\sqrt{(1^2+1^2+1^2)}}=\frac{1}{\sqrt{3}}$$

i.e. $\quad l_3=1/\sqrt{3}=m_3=n_3.$

The line of centres is given by any two of $\dfrac{\partial F}{\partial x}=0,\ \dfrac{\partial F}{\partial y}=0,\ \dfrac{\partial F}{\partial z}=0$

$$\frac{\partial F}{\partial x}=0 \Rightarrow 52x-36y-16z+52=0 \ \ i.e.\ \ 13x-9y-4z+13=0$$

$$\frac{\partial F}{\partial y}=0 \Rightarrow 40y-4z-36x-36=0 \ \ i.e.\ \ 9x-10y+z+9=0$$

$$\frac{\partial F}{\partial z}=0 \Rightarrow 20z-4y-16x-16=0 \ \ i.e.\ \ 4x+y-5z+4=0.$$

Let (α, β, γ) be any point on the line of centres.

Choosing $\alpha=-1,\ \beta=0,\ \gamma=0$ we find $(-1, 0, 0)$ is a point on the line of centres.

[**Note.** The method of choosing α, β, γ is not unique]

Now $k=u\alpha+v\beta+w\gamma+d=26\,(-1)+(-18)\,(0)+(-8)\,(0)+25=-1\neq 0.$

Hence the given surface reduces to $\lambda_1x^2+\lambda_2y^2+k=0$

i.e. $\quad 14x^2+42y^2-1=0$, which represents an elliptic cylinder as λ_1, λ_2 are both of the same sign. [See § 12.12 Page 20 Ch. XII]

Also the equations of the axis of cylinder are

$$\frac{x-(-1)}{l_3}=\frac{y-0}{m_3}=\frac{z-0}{n_3} \quad \text{or} \quad \frac{x+1}{1}=\frac{y}{1}=\frac{z}{1}, \text{ from (iii)}$$

***Ex. 2. Prove that the surface represented by the equation**

$$\mathbf{5x^2+5y^2+8z^2+8yz+8zx-2xy+12x-12y+6=0}$$

represents a cylinder whose cross-section is an ellipse of eccentricity $1/\sqrt{2}$ and find the equations to its axis. (*Garhwal* 95)

Solution. Here 'a' $=5$, 'b' $=5$, 'c' $=8$, 'f' $=4$, 'g' $=4$, 'h' $=-1$, 'u' $=6$, 'v' $=-6$, 'w' $=0$ and 'd' $=6$.

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 5-\lambda & -1 & 4 \\ -1 & 5-\lambda & 4 \\ 4 & 4 & 8-\lambda \end{vmatrix} = 0$$

or $(5-\lambda)[(5-\lambda)(8-\lambda)-16]+[-(8-\lambda)-16]+4[-4-4(5-\lambda)]=0$

or $\lambda^3 - 18\lambda^2 + 72\lambda = 0$ or $\lambda(\lambda^2 - 18\lambda + 72) = 0$

or $\lambda(\lambda-6)(\lambda-12)=0$ or $\lambda = 0, 6, 12.$

$\therefore$ Let $\lambda_1 = 6,\ \lambda_2 = 12,\ \lambda_3 = 0.$ ...(ii)

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$5l_3 - m_3 + 4n_3 = 0,\ -l_3 + 5m_3 + 4n_3 = 0,\ 4l_3 + 4m_3 + 8n_3 = 0.$

Solving last two equations simultaneously, we get

$$\frac{l_3}{4-10} = \frac{m_3}{-2-4} = \frac{n_3}{5+1} \quad \text{or} \quad \frac{l_3}{-1} = \frac{m_3}{-1} = \frac{n_3}{1} = \frac{1}{\sqrt{3}} \qquad \text{...(iii)}$$

Also the line of centres is given by any two of

$$\partial F/\partial x = 0,\ \partial F/\partial y = 0,\ \partial F/\partial z = 0$$

$\dfrac{\partial F}{\partial x} = 0 \Rightarrow 10x + 8z - 2y + 12 = 0$ or $5x - y + 4z + 6 = 0$

$\dfrac{\partial F}{\partial y} = 0 \Rightarrow 10y + 8z - 2x - 12 = 0$ or $x - 5y - 4z + 6 = 0$

$\dfrac{\partial F}{\partial z} = 0 \Rightarrow 16z + 8y + 8x = 0$ or $x + y + 2z = 0$

Let (α, β, γ) be any point on the line of centres. Choosing $\gamma = 0, \beta = 1, \alpha = -1$ we find $(-1, 1, 0)$ is a point on the line of centres.

Now $k = u\alpha + v\beta + w\gamma + d = (6)(-1) + (-6)(1) + 0 + 6 = -6 \neq 0$

Hence the given surface reduces to $\lambda_1 x^2 + \lambda_2 y^2 + k = 0$

i.e. $6x^2 + 12y^2 - 6 = 0$, *i.e.* $x^2 + 2y^2 - 1 = 0$ ...(iv)

which represents an elliptic cylinder as λ_1, λ_2 are both of the same sign.

Also (iv) can be rewritten as $\dfrac{x^2}{1} + \dfrac{y^2}{(1/2)} = 1.$

And so if e be the required eccentricity, then

$$b^2 = a^2(1-e^2) \Rightarrow 1/2 = (1-e^2) \text{ or } e = 1/\sqrt{2}. \qquad \textbf{Ans.}$$

Also the equations of the axis of cylinder are

$$\frac{x-(-1)}{l_3} = \frac{y-1}{m_3} = \frac{z-0}{n_3} \quad \text{or} \quad \frac{x+1}{-1} = \frac{y-1}{-1} = \frac{z}{1} \qquad \textbf{Ans.}$$

****Ex. 3. Determine completely the surface represented by**

$$\mathbf{2y^2 - 2yz + 2zx - 2xy - x - 2y + 3z - 2 = 0.}$$

Sol. Here 'a' = 0, 'b' = 2, 'c' = 0, 'f' = − 1, 'g' = 1, 'h' = − 1, 'u' = − 1/2, 'v' = − 1, 'w' = 3/2 and 'd' = − 2.

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} -\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} = 0 \qquad \ldots\text{(i)}$$

or $\quad -\lambda\,[-(2-\lambda)\,\lambda - 1] + [\lambda + 1] + [1 - (2-\lambda)] = 0$

or $\quad -\lambda^3 + 2\lambda^2 + \lambda + \lambda + 1 + 1 - 2 + \lambda = 0 \quad$ or $\quad \lambda^3 - 2\lambda^2 - 3\lambda = 0$

or $\quad \lambda\,(\lambda^2 - 2\lambda - 3) = 0 \quad$ or $\quad \lambda\,(\lambda + 1)\,(\lambda - 3) = 0 \quad$ or $\quad \lambda = 0, -1, 3$

or $\quad \lambda_1 = 3,\ \lambda_2 = -1,\ \lambda_3 = 0.$

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have $-m_3 + n_3 = 0,\ -l_3 + 2m_3 - n_3 = 0,\ l_3 - m_3 = 0$

From these on solving we get $l_3 = m_3 = n_3 = 1/\sqrt{3}$ **(Note)**

Further the line of centres is given by any two of

$$\partial F/\partial x = 0,\ \partial F/\partial y = 0,\ \partial F/\partial z = 0$$

Now $\dfrac{\partial F}{\partial x} = 0 \Rightarrow 2z - 2y - 1 = 0;$

$\dfrac{\partial F}{\partial y} = 0 \Rightarrow 4y - 2z - 2x - 2 = 0$ or $x - 2y + z + 1 = 0;$

$\dfrac{\partial F}{\partial z} = 0 \Rightarrow -2y + 2x + 3 = 0$ or $2x - 2y + 3 = 0$

Let (α, β, γ) be any point on the line of centres. Choosing $\gamma = 0,\ \beta = -1/2,\ \alpha = -2$ we find that $(-2, -1/2, 0)$ is a point on the line of centres.

Now $k = u\alpha + v\beta + w\gamma + d = (-\frac{1}{2})\,(-2) + (-1)\,(-\frac{1}{2}) + (\frac{3}{2})\,(0) - 2 = -\frac{1}{2} \neq 0$

Hence the given surface reduces to $\lambda_1 x^2 + \lambda_2 y^2 + k = 0$

i.e. $$3x^2 - y^2 - (1/2) = 0$$

which represents a hyperbolic cylinder as λ_1, λ_2 are of different signs.

Also the equations of the axis of the cylinder are

$$\frac{x-(-2)}{l_3} = \frac{y-(-1/2)}{m_3} = \frac{z-0}{n_3} \quad \text{or} \quad \frac{x+2}{1} = \frac{y+(1/2)}{1} = \frac{z}{1} \qquad \textbf{Ans.}$$

***Ex. 4 (a). Prove that the equation $5x^2 - 4y^2 + 5z^2 + 4yz - 14zx + 4xy + 16x + 16y - 32z + 8 = 0$ represents a pair of planes which pass through the line $x + 2 = y - 1 = z$ and are inclined at an angle $2\tan^{-1}(1/\sqrt{2})$.**

Solution. Here 'a' = 5, 'b' = −4, 'c' = 5, 'f' = 2, 'g' = −7, 'h' = 2, 'u' = 8, 'v' = 8, 'w' = −16 and 'd' = 8.

∴ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 5-\lambda & 2 & -7 \\ 2 & -4-\lambda & 2 \\ -7 & 2 & 5-\lambda \end{vmatrix} = 0 \qquad \ldots\text{(i)}$$

or $\quad (5-\lambda)\,[-(4+\lambda)\,(5-\lambda) - 4] - 2\,[2\,(5-\lambda) + 14] - 7\,[4 - 7\,(4+\lambda)] = 0$

or $$\lambda^3 - 6\lambda^2 - 72\lambda = 0$$

or $\lambda(\lambda^2 - 6\lambda - 72) = 0$ or $\lambda(\lambda+6)(\lambda-12) = 0$ or $\lambda = 0, -6, 12.$

Let $$\lambda_1 = 12,\ \lambda_2 = -6,\ \lambda_3 = 0.$$

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$$5l_3 + 2m_3 - 7n_3 = 0,\ 2l_3 - 4m_3 + 2n_3 = 0,\ -7l_3 + 2m_3 + 5n_3 = 0.$$

Solving first two equations simultaneously, we get

$$\frac{l_3}{4-28} = \frac{m_3}{-14-10} = \frac{n_3}{-20-4} \text{ or } \frac{l_3}{1} = \frac{m_3}{1} = \frac{n_3}{1} = \frac{1}{\sqrt{3}} \quad \textbf{(Note)} \text{ ...(ii)}$$

Further the line of centres is given by any two of

$$\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0.$$

Now $\frac{\partial F}{\partial x} = 0 \Rightarrow 10x - 14z + 4y + 16 = 0$ or $5x + 2y - 7z + 8 = 0$

$\frac{\partial F}{\partial y} = 0 \Rightarrow -8y + 4z + 4x + 16 = 0$ or $x - 2y + z + 4 = 0$

$\frac{\partial F}{\partial z} = 0 \Rightarrow 10z + 4y - 14x - 32 = 0$ or $7x - 2y - 5z + 16 = 0$

Let (α, β, γ) be any point on the line of centres.

Choosing $\gamma = 0,\ \beta = 1,\ \alpha = -2$ we find that $(-2, 1, 0)$ is a point on the line of the centres.

Now $k = u\alpha + v\beta + w\gamma + d = 8(-2) + 8(1) - 16(0) + 8 = 0.$

Hence the reduced equation of the given surface is

$$\lambda_1 x^2 + \lambda_2 y^2 + k = 0 \text{ or } 12x^2 - 6y^2 + 0 = 0 \text{ or } 2x^2 - y^2 = 0 \quad \text{...(iii)}$$

which represents a pair of planes whose line of section is the line through $(-2, 1, 0)$ and direction ratios from (ii) are 1, 1, 1.

$\therefore$ The equations of this line through which the two planes given by (ii) pass are $$\frac{x-(-2)}{1} = \frac{y-1}{1} = \frac{z-0}{1} \text{ or } x+2 = y-1 = z.$$

Again the planes represented by (iii) are $y^2 = 2x^2$.

i.e. $y = x\sqrt{2}$ and $y = -x\sqrt{2}$ *i.e.* $x\sqrt{2} - y = 0$ and $x\sqrt{2} + y = 0$

$\therefore$ The direction ratios of their normals are $\sqrt{2}, -1, 0$ and $\sqrt{2}, 1, 0$.

$\therefore$ If θ be the angle between these planes, then

$$\cos\theta = \frac{\sqrt{2}.\sqrt{2} + (-1).(1) + 0.0.}{\sqrt{[(\sqrt{2})^2 + (-1)^2 + 0^2].[(\sqrt{2})^2 + (1)^2 + 0^2]}} = \frac{1}{3}$$

$$\therefore \frac{1 - \tan^2(\theta/2)}{1 + \tan^2(\theta/2)} = \frac{1}{3} \text{ or } 3 - 3\tan^2\frac{\theta}{2} = 1 + \tan^2\frac{\theta}{2} \text{ or } 2\tan^2\frac{\theta}{2} = 1$$

or $$\tan(\theta/2) = 1/\sqrt{2} \text{ or } \theta = 2\tan^{-1}(1/\sqrt{2}).$$ Hence proved.

Ex. 4 (b). In Ex. 4 (a) above prove that the two planes pass through the line x + 3 = y = z + 1 and the angle between them is $\tan^{-1}(2\sqrt{2})$

Hint. Proceed exactly as in Ex. 4 (a) above.

Here prove that if (α, β, γ) be any point on the line of centres then choosing $\beta = 0, \gamma = -1, \alpha = -3$ we find that $(-3, 0, -1)$ is a point on the line of centres.

$\therefore$ The equations of the line, through which the planes given by (iii) of Ex. 4 (a) above pass, are $\dfrac{x-(-3)}{1} = \dfrac{y-0}{1} = \dfrac{z-(-1)}{1}$ or $x + 3 = y = z + 1$

Also in Ex. 4 (a) above $\cos\theta = 1/3$ *i.e.* $\sec\theta = 3$

$\Rightarrow \tan^2\theta + 1 = \sec^2\theta = 9 \Rightarrow \tan^2\theta = 8 \Rightarrow \tan\theta = 2\sqrt{2}$

$\Rightarrow \theta = \tan^{-1}(2\sqrt{2})$. Hence proved.

Exercises on § 12.12. (Case III)

Ex. 1. Reduce $2x^2 + 5y^2 + 2z^2 - 2yz + 4zx - 2xy + 14x - 16y + 14z + 26 = 0$ to the standard form. What does is represent ?

Ans. $2x^2 + y^2 = 1$, elliptic cylinder whose axis is $\dfrac{x+3}{-1} = \dfrac{y-1}{0} = \dfrac{z}{1}$

Ex. 2. Reduce $x^2 - y^2 + 4yz + 4zx - 6x - 2y - 8z + 5 = 0$ to the standard form. What does it represent ?

Ans. Hyperbolic cylinder $x^2 - y^2 = 1$, axis is $\dfrac{x-1}{-2} = \dfrac{y-1}{2} = \dfrac{z-1}{1}$

Ex. 3. Find the condition that the homogeneous equation of second degree in x, y, z represent a pair of planes. (*Kanpur* 92)

§ 12.13. Case IV. A, B, C, F, G, H are all zero, f u ≠ g v.

In this case there is a line of centres at infinity and the two roots of discriminating cubic are zero, say λ_2 and λ_3. Also third root $\lambda_1 \neq 0$.

If the axes through the same origin is so rotated that they are parallel to a set of three mutually perpendicular principal directions then the transformed equation is $\lambda_1 x^2 + 2x(ul_1 + vm_1 + wn_1) + 2y(ul_2 + vm_2 + wn_2)$

$$+ 2z(ul_3 + vm_3 + wn_3) + d = 0 \quad \text{...(i)}$$

The direction cosines l_2, m_2, n_2 and l_3, m_3, n_3 corresponding to zero roots λ_2 and λ_3 satisfy the equation $al + hm + gn = 0$ ...(ii)

Choose l_2, m_2, n_2 such that $ul_2 + vm_2 + wn_2 = 0$ ...(iii)

Then from (i), (iii) we have $\lambda_1 x^2 + 2px + 2rz + d = 0$, ...(iv)

where $p = ul_1 + vm_1 + wn_1$, $r = ul_3 + vm_3 + wn_3$ ...(v)

From (iv), $\lambda_1\left(x^2 + \dfrac{2p}{\lambda_1}x + \dfrac{p^2}{\lambda_1^2}\right) + 2rz + \left(d - \dfrac{p^2}{\lambda_1}\right) = 0$ **(Note)**

or $$\lambda_1\left(x + \frac{p}{\lambda_1}\right)^2 + 2r\left[z + \frac{1}{2r}\left(d - \frac{p^2}{\lambda_1}\right)\right] = 0 \quad \text{...(vi)}$$

Shifting the origin to the point $\left[-\frac{p}{\lambda_1},\ 0,\ -\frac{1}{2r}\left(d-\frac{p^2}{\lambda_1}\right)\right]$

the equation (vi) transforms to $\lambda_1 x^2 + 2rz = 0$ or $x^2 + (2r/\lambda_1)\, z = 0$, ...(vii)
which is the required reduced form and represents a **parabolic cylinder.**

The latus rectum of a normal section is

$$2r/\lambda_1 \text{ i.e. } (2/\lambda_1)(ul_3 + vm_3 + wn_3), \text{ from (v).}$$

Alternative method.

$\because A = 0 = B = C$, so we have $bc - f^2 = 0$, $ca - g^2 = 0$, $ab - h^2 = 0$.

These imply that a, b, c have the same sign, say positive

Also $F = 0$, $G = 0$, $H = 0$ give $gh - af = 0$, $hf - bg = 0$, $fg - ch = 0$
and so either f, g, h are all positive or two negative and one positive. **(Note)**

$$\therefore f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$
$$= [\sqrt{ax} \pm \sqrt{by} \pm \sqrt{cz}]^2$$

i.e. the terms of the second degree in the general equation $F(x, y, z) = 0$ form a perfect square.

Now if $fu \neq gv$, then we proceed as follows :—

General equation $F(x, y, z) = 0$ can be rewritten as

$$[\sqrt{ax} + \sqrt{by} + \sqrt{cz} + \lambda]^2 = 2x\,[\lambda \sqrt{a} - u] + 2y\,[\lambda \sqrt{b} - v]$$
$$+ 2z\,[\lambda \sqrt{c} - w] + (\lambda^2 - d) \quad \text{...(I)}$$

Now choose λ in such a way that the planes $\sqrt{ax} + \sqrt{by} + \sqrt{cz} + \lambda = 0$ and $2x(\lambda \sqrt{a} - u) + 2y(\lambda \sqrt{b} - v) + 2z(\lambda \sqrt{c} - w) + (\lambda^2 - d) = 0$ are at right angles

so $\quad \sqrt{a}\,(\lambda \sqrt{a} - u) + \sqrt{b}\,(\lambda \sqrt{b} - v) + \sqrt{c}\,(\lambda \sqrt{c} - w) = 0$

or $\quad \lambda\,(a + b + c) = u\sqrt{a} + v\sqrt{b} + w\sqrt{c}$

or $\quad \lambda = (u\sqrt{a} + v\sqrt{b} + w\sqrt{c})/(a + b + c)$ ...(II)

The equation (I) with the help of (II) can be rewritten as

$$\left[\frac{\sqrt{ax} + \sqrt{by} + \sqrt{cz} + \lambda}{\sqrt{(a + b + c)}}\right]^2$$
$$= k\left[\frac{2x(\lambda \sqrt{a} - u) + 2y(\lambda \sqrt{b} - v) + 2z(\lambda \sqrt{c} - w) + (\lambda^2 - d)}{2\sqrt{\{(\lambda \sqrt{a} - u)^2 + (\lambda \sqrt{b} - v)^2 + (\lambda \sqrt{c} - w)^2\}}}\right],$$

where $k = 2\,[\sqrt{\{(\lambda\sqrt{a} - u)^2 + (\lambda \sqrt{b} - v)^2 + (\lambda \sqrt{c} - w)^2\}}]/(a + b + c)$.

i.e. The above equation takes the from $X^2 = kY$,

where $\quad X = (\sqrt{ax} + \sqrt{by} + \sqrt{cz} + \lambda)/\sqrt{(a + b + c)}$

and $\quad Y = \dfrac{2x(\lambda \sqrt{a} - u) + 2y(\lambda \sqrt{b} - v) + 2z(\lambda \sqrt{c} - w) + (\lambda^2 - d)}{2\sqrt{\{(\lambda \sqrt{a} - u)^2 + (\lambda \sqrt{b} - v)^2 + (\lambda \sqrt{c} - w)^2\}}}$

This represents a parabolic cylinder.

§ 12.14. Case V. A, B, C, F, G, H are all zero and fu = gv = hw.

In this case there is a plane of centres and two roots λ_2, λ_3 (say) of the discriminating cubic are zero.

If l_1, m_1, n_1 be the principal direction cosines corresponding to the non-zero root λ_1 of the discriminating cubic, then

$$\frac{al_1 + hm_1 + gn_1}{l_1} = \frac{hl_1 + bm_1 + fn_1}{m_1} = \frac{gl_1 + fm_1 + cn_1}{n_1} \qquad ...(i)$$

But $f^2 = bc$, $g^2 = ca$ and $h^2 = ab$, so

$al_1 + hm_1 + gn_1 = al_1 + \sqrt{(ab)}\, m_1 + \sqrt{(ca)}\, n_1 = \sqrt{a}\, [\sqrt{a}\, l_1 + \sqrt{b}\, m_1 + \sqrt{c}\, n_1]$

Similarly $hl_1 + bm_1 + fn_1 = \sqrt{b}\, [\sqrt{a}l_1 + \sqrt{b}m_1 + \sqrt{c}n_1]$,

and $\qquad gl_1 + fm_1 + cn_1 = \sqrt{c}\, [\sqrt{a}l_1 + \sqrt{b}m_1 + \sqrt{c}n_1]$

$\therefore$ From (i) we have $\qquad \dfrac{l_1}{\sqrt{a}} = \dfrac{m_1}{\sqrt{b}} = \dfrac{n_1}{\sqrt{c}} \qquad ...(ii)$

Also here $fu = gv = hw$

$\Rightarrow \sqrt{(bc)}\, u = \sqrt{(ca)}\, v = \sqrt{(ab)}\, w, \quad \because\ f^2 = bc$ etc.

$\Rightarrow u/\sqrt{(a)} = v/\sqrt{(b)} = w/\sqrt{(c)}$

From (ii), $\qquad u/l_1 = v/m_1 = w/n_1 \qquad ...(iii)$

Now if l_2, m_2, n_2 and l_3, m_3, n_3 be the principal direction cosines corresponding to zero roots λ_2 and λ_3, then

$$ul_2 + vm_2 + wn_2 = l_1l_2 + m_1m_2 + n_1n_2 = 0$$
$$ul_3 + vm_3 + wn_3 = l_1l_3 + m_1m_3 + n_1n_3 = 0.$$

Now as in § 12.13 Page 25 Ch. XII rotating the axes we find that the transformed equation is $\quad \lambda_1x^2 + 2x\,(ul_1 + vm_1 + wn_1) + d = 0$ **(Note)**

or $\quad \lambda_1x^2 + 2px + d = 0$, where $p = ul_1 + vm_1 + wn_1$

or $\qquad \lambda_1\left(x + \dfrac{p}{\lambda_1}\right)^2 + \left(d - \dfrac{p^2}{\lambda_1}\right) = 0 \qquad$ or $\qquad \lambda_1 x^2 + k = 0,$

changing the origin to $(-p/\lambda_1, 0, 0)$ and where $k = d - (p^2/\lambda_1)$

This equation represents a **pair of planes** which are identical or parallel according as $k = 0$ or $k \neq 0$

Alternative method.

As in the alternative method given in § 12.13 on Page 26, if A, B, C, F, G and H are zero, we can prove that $\quad f(x, y, z) = [\sqrt{a}\, x \pm \sqrt{b}y \pm \sqrt{c}z]^2$

i.e. the terms of the second degree in the general equation $F(x, y, z) = 0$ form a perfect square.

Now if $fu = gv = hw$, then as above we can get

$$u/\sqrt{a} = v/\sqrt{b} = w/\sqrt{c} = \mu \text{ (say)} \qquad ...(iv)$$

Also the general equation $F(x, y, z) = 0$ in this case can be written as

$$(\sqrt{a}x + \sqrt{b}y + \sqrt{c}z)^2 + 2\,(ux + vy + wz) + d = 0 \qquad \textbf{(Note)}$$

or $(\sqrt{ax}+\sqrt{by}+\sqrt{cz})^2+2\mu(\sqrt{ax}+\sqrt{by}+\sqrt{cz})+d=0$, from (iv)

or $\sqrt{ax}+\sqrt{by}+\sqrt{cz}=-\mu\pm\sqrt{(\mu^2-d)}$, solving as a quadratic equation in $\sqrt{ax}+\sqrt{by}+\sqrt{cz}$

This represents a **pair of parallel planes.**

Solved Examples on § 12.13 — § 12.14 (Case IV and V).

***Ex. 1. Reduce the equation $x^2+y^2+z^2-2yz+2zx-2xy+x-4y-z+1=0$ to the standard form and find the latus rectum of the principal section.** (*Avadh* 93)

Solution. As the terms of second degree form a perfect square, so the given equation can be rewritten as $(x-y+z)^2=-x+4y-z-1$

or $(x-y+z+\lambda)^2=(2\lambda-1)x-2(\lambda-2)y+(2\lambda-1)z+(\lambda^2-1)$ ...(i)

adding a constant λ within the brackets on L.H.S. and adding the corresponding terms on the R.H.S. **(Note)**

Now choose λ in such a way that the planes $x-y+z+\lambda=0$ and $(2\lambda-1)x-2(\lambda-2)y+(2\lambda-1)z+(\lambda^2-1)=0$ are at right angles.

Then $1.(2\lambda-1)+(-1)\{-2(\lambda-2)\}+1.(2\lambda-1)=0\Rightarrow\lambda=1$

$\therefore$ From (i), the given equation of the surface can be rewritten as

$$(x-y+z+1)^2=x+2y+z$$

or $$3\left[\frac{x-y+z+1}{\sqrt{3}}\right]^2=\sqrt{6}\left[\frac{x+2y+z}{\sqrt{6}}\right]$$ **(Note)**

or $3X^2=\sqrt{6}Y$ or $X^2=(1/3)\sqrt{6}\,Y$, which represents a paraboloic cylinder and the latus rectum of the principal paraboloic section is $\sqrt{6}/3$. **Ans.**

****Ex. 2. Show that the equation $x^2+4y^2+9z^2+12yz+6zx+4xy-54x-52y+62z+113=0$ represents a parabolic cylinder, and that the foci of the normal parabolic section lie on the line**

$$\mathbf{x+2y+3z+1=0=x+y-z-5.}$$

Solution. As the terms of second degree form a perfect square, so the given equation can be rewritten as $(x+2y+3z)^2=54x+52y-62z-113$

or $(x+2y+3z+\lambda)^2=2(\lambda+27)x+4(\lambda+13)y+2(3\lambda-31)z+(\lambda^2-113)$, ...(i)

adding a constant λ within the brackets on L.H.S. and adding the corresponding terms on R.H.S. **(Note)**

Now choose λ in such a way that the planes $x+2y+3z+\lambda=0$ and $2(\lambda+27)x+4(\lambda+13)y+2(3\lambda-31)z+\lambda^2-113=0$ are at right angles.

Then $1.\{2(\lambda+27)\}+2.\{4(\lambda+13)\}+3.\{2(3\lambda-31)\}=0$

or $2\lambda+54+8\lambda+104+18\lambda-186=0$ or $\lambda=1$.

$\therefore$ From (i), the given equation of the surface reduces to

$$(x+2y+3z+1)^2 = 56x+56y-56z-112$$

or $$(x+2y+3z+1)^2 = 56\,(x+y-z-2)$$

or $$14\left[\frac{x+2y+3z+1}{\sqrt{(1^2+2^2+3^2)}}\right]^2 = 56\sqrt{3}\left[\frac{x+y-z-2}{\sqrt{\{(1^2+1^2+(-1)^2)\}}}\right]$$

or $$Y^2 = 4\sqrt{3}X, \qquad \text{...(ii)}$$

which represents a parabolic cylinder and the latus rectum of the normal parabolic section is $4\sqrt{3}$.

[**Note :** The vertex of the parabolic cylinder lie on the line of intersection of the planes $x+2y+3z+1=0$, $x+y-z-2=0$, the latter being a tangent plane which touches the cylinder along the vertices.]

The foci evidently lie on the line of intersection of the plane $x+2y+3z+1=0$ *i.e.* the plane through the axis and a plane parallel to the tangent plane $x+y-z-2=0$ but at a distance (1/4)th of latus rectum (*i.e.* $\sqrt{3}$) from it. ...(iii)

Now any plane parallel to the tangent plane $x+y-z-2=0$, $x+y-z+k=0$ and it should be at a distance $\sqrt{3}$ from the tangent plane.

Now any point on the tangent plane is (2, 0, 0), putting $y=0$, $z=0$ in $x+y-z-2=0$.

$\therefore$ distance of the plane $x+y-z+k=0$ from (2, 0, 0) must be $\sqrt{3}$.

i.e. $$\frac{2+0-0+k}{\sqrt{[1^2+1^2+(-1)^2]}} = \sqrt{3} \quad \text{or} \quad 2+k=3 \quad \text{or} \quad k=1.$$

$\therefore$ Foci lie on the line of intersection of the planes

$x+2y+3z+1=0$ and $x+y-z+1=0$, from (iii) Hence proved.

****Ex. 4. Show that the equation $4x^2+9y^2+36z^2-36yz+24zx-12xy-10x+15y-30z+6=0$ represents a pair of parallel planes and find the reduced equation.**

Solution. As the second degree terms of the given equation form a perfect square, so it can be rewritten as

$$(2x-3y+6z)^2 = 10x-15y+30z-6 = 5\,(2x-3y+6z)-6 \qquad \text{...(i)}$$

or $$(2x-3y+6z)^2 - 5\,(2x-3y+6z)+6=0$$

or $$X^2-5X+6=0, \text{ where } X = 2x-3y+6z$$

or $$(X-2)(X-3)=0 \quad \text{or} \quad X=2, X=3$$

or $$2x-3y+6z=2,\ 2x-3y+6z=3 \qquad \text{...(ii)}$$

Hence the given equation represents a pair of parallel planes given by (ii).

Also from (i) we have

$$49\left[\frac{2x-3y+6z}{\sqrt{(2^2+3^2+6^2)}}\right]^2 = 7\left[\frac{5\,(2x-3y+6z)}{\sqrt{(2^2+3^2+6^2)}}\right]-6 \qquad \text{...(iii)}$$

Now choose $2x-3y+6z=0$ as $x=0$ *i.e.* if (x, y, z) be the coordinates of any point, then $x = \dfrac{2x-3y+6z}{\sqrt{(2^2+3^2+6^2)}}$

Then (iii) reduces to $49x^2 - 35x + 6 = 0$, which is the required reduced equation.

Ans.

Exercises on § 12.13 — § 12.14 (Cases IV—V)

****Ex. 1.** Reduce the equation $36x^2 + 4y^2 + z^2 - 4yz - 12zx + 24xy + 4x + 16y - 26z - 3 = 0$ to the standard form. Show that it represents a parabolic cylinder and find the latus rectum of a normal section. Also show that the foci of the normal parabolic sections lie on the line $6x + 2y - z + 1 = 0 = 2x - 3y + 6z + (90/41)$ (*Avadh* 91)

Ans. $41y^2 = 28x$, latus rectum $= 28/41$

Ex. 2. Reduce the equation $9x^2 + 4y^2 + 4z^2 + 8yz + 12zx + 12xy + 4x + y + 10z + 1 = 0$.

Ans. $17y^2 = 7x$, a parabolic cylinder

Ex. 3. What surface is represented by the equation

$$x^2 + 4y^2 + z^2 + 2zx - 4yz - 4xy - 2x + 4y - 2z - 3 = 0\ ?$$

Reduce it to the standard form.

Ans. A pair of parallel planes, $6x^2 - 2\sqrt{6}x - 3 = 0$.

Ex. 4. Show that $(3x - 4y + z)^2 + 9x - 12y + 3z - 10 = 0$ represents a pair of parallel planes. Also reduce it to the standard form $26x^2 - 3\sqrt{(26)}\,x - 10 = 0$.

§ 12.15. Conicoids of revolution.

Here two cases arise viz.

(i) Two roots of the discriminating cubic are equal and third root not equal to zero.

(ii) Two roots of the discriminating cubic are equal and third root equal to zero.

Under (i) the form to which the given surface can reduce are

$$\mathbf{A\,(x^2 + y^2) + Bz^2 = 1} \qquad \textbf{(Ellipsoid of revolution)}$$

and

$$\mathbf{A\,(x^2 - y^2) + Bz^2 = 1} \qquad \textbf{(Hyperboloid of revolution)}$$

Under (ii) the form to which the given surface can reduce are

$$\mathbf{A\,(x^2 + y^2) + Bz = 0} \qquad \textbf{(Paraboloid of revolution)}$$

and

$$\mathbf{A\,(x^2 + y^2) + D = 0} \qquad \textbf{(Right circular cylinder)}$$

∴ We conclude that if the **two roots of the discriminating cubic are equal,** then surface $F(x, y, z) = 0$ represents a **surface (or conicoid) of revolution.**

Here we proceed in the usual way and the direction ratios of the axis of rotation are obtained from the usual equations by taking that value of λ which is different from the equal values.

Solved Examples on § 12.15.

****Ex. 1. Show that the equation $x^2 + y^2 + z^2 + yz + zx + xy + 3x + y + 4z + 4 = 0$ represents a surface of revolution and determine the equations of its axis of rotation.**

Solution. Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & 1/2 & 1/2 \\ 1/2 & 1-\lambda & 1/2 \\ 1/2 & 1/2 & 1-\lambda \end{vmatrix} = 0 \qquad \text{...(i)}$$

or $(1-\lambda)\,[(1-\lambda)^2-(1/4)]-(1/2)\,[(1/2)(1-\lambda)-(1/4)]$
$+(1/2)\,[(1/4)-(1/2)(1-\lambda)]=0$

or $(1-\lambda)^3-(3/4)(1-\lambda)+(1/4)=0$

or $4(1-\lambda)^3-3(1-\lambda)+1=0$ or $4\lambda^3-12\lambda^2+9\lambda-2=0$

or $(\lambda-2)(2\lambda-1)^2=0$ or $\lambda=2, 1/2, 1/2.$

$\therefore$ We observe that two roots of discriminating cubic are equal and the third is different from zero, so the given equation represents a surface of revolution [either ellipsoid or hyperboloid of revolution]

...See § 12.15 (i) Page 30 Ch. XII.

The central planes are given by $\dfrac{\partial F}{\partial x}=0,\ \dfrac{\partial F}{\partial y}=0,\ \dfrac{\partial F}{\partial z}=0$

i.e. $2x+y+z+3=0,\ x+2y+z+1=0,\ x+y+2z+4=0.$

Solving these we get $x=-1, y=1, z=-2$

$\therefore$ Centre of the given surface is $(-1, 1, -2)$.

$\therefore d'=u\alpha+v\beta+w\gamma+d=(3/2)(-1)+(1/2)(1)+(2)(-2)+4=-1.$

$\therefore$ The reduced equation is $\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+d'=0$

or $(1/2)x^2+(1/2)y^2+2z^2-1=0$ or $\dfrac{x^2}{2}+\dfrac{y^2}{2}+\dfrac{z^2}{(1/2)}=1,$

which is an **ellipsoid** (of revolution), the squares of whose semiaxes are 2, 2, 1/2.

Now putting $\lambda=2$ in the determinant given by (i) and associating each row with l, m, n, the direction cosines of the principal axis (or axis of revolution), we have

$-l+(1/2)m+(1/2)n=0,\ (1/2)l-m+(1/2)n=0,$
$(1/2)l+(1/2)m-n=0$

i.e. $-2l+m+n=0,\ l-2m+n=0,\ l+m-2n=0$

and these gives $l=m=n=1/\sqrt{3},\ \because l^2+m^2+n^2=1.$

Now the required axis of rotation (or principal axis) is a line through the centre $(-1, 1, -2)$ of the surface of revolution and direction cosines $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ or direction ratios 1, 1, 1.

$\therefore$ The required equations of the axis of rotation are

$$\frac{x-(-1)}{1}=\frac{y-1}{1}=\frac{z-(-2)}{1} \quad \text{or} \quad x+1=y-1=z+2 \qquad \textbf{Ans.}$$

***Ex. 2. Reduce to standard form the equation**

$$7x^2+y^2+z^2+16yz+8zx-8xy+2x+4y-40z-14=0$$

and find the principal axis.

Solution. Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 7-\lambda & -4 & 4 \\ -4 & 1-\lambda & 8 \\ 4 & 8 & 1-\lambda \end{vmatrix} = 0 \qquad \text{...(i)}$$

or $(7-\lambda)\,[(1-\lambda)^2-64]+4\,[-4\,(1-\lambda)-32]+4\,[-32-4\,(1-\lambda)]=0$

or $\lambda^3-9\lambda^2-81\lambda+729=0$ or $(\lambda-9)\,(\lambda^2-81)=0$

or $(\lambda-9)\,(\lambda-9)\,(\lambda+9)=0$ or $\lambda=9, 9, -9.$

i.e. the two roots of discriminating cubic are equal and the third is different from zero, so the given equation represents either an ellipsoid of revolution or a hyperboloid of revolution.

The central planes are given by $\dfrac{\partial F}{\partial x}=0,\ \dfrac{\partial F}{\partial y}=0,\ \dfrac{\partial F}{\partial z}=0$

i.e. $14x-8y+8z+2=0,\ -8x+2y+16z+4=0,\ 8x+16y+2z-40=0$

i.e. $7x-4y+4z+1=0,\ -4x+y+8z+2=0,\ 4x+8y+z-20=0$

Solving these we get $x=1, y=2, z=0.$

$\therefore$ Centre of the given surface is $(1, 2, 0)$.

$\therefore\ d'=u\alpha+v\beta+w\gamma+d=(1)\,(1)+(2)\,(2)+(-20)\,(0)-14=-9$

$\therefore$ The reduced equation of the given surface is

$$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+d'=0 \quad \text{or} \quad 9x^2+9y^2-9z^2-9=0$$

or $x^2+y^2-z^2=1$, which represents a hyperboloid of revolution, the squares of whose semi-axes are 1, 1, 1.

Now putting $\lambda=-9$ in the determinant given by (i) and associating each row with l, m, n, the d c.'s of the principal axis, we have

$$16l-4m+4n=0,\ -4l+10m+8n=0,\ 4l+8m+10n=0$$

and these gives $\dfrac{l}{1}=\dfrac{m}{2}=\dfrac{n}{-2}=\dfrac{1}{3}$, $\because\ l^2+m^2+n^2=1.$

$\therefore$ The equations of the principal axis passing through the centre $(1, 2, 0)$ and d.r.'s $1, 2, -2$ are $\dfrac{x-1}{1}=\dfrac{y-2}{2}=\dfrac{z-0}{-2}.$ **Ans.**

Ex. 3. Show that the equation $x^2+2yz=1$ represents a surface of revolution and find the axis of revolution.

Solution. Given $F(x, y, z)\equiv x^2+2yz-1=0$

$\therefore$ Here 'a' $=1, b=c=0$, 'f' $=1, g=0=h=u=v=w$, 'd' $=-1$

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0 \qquad \text{...(i)}$$

or $(1-\lambda)\,[\lambda^2-1]=0$ or $\lambda=1, 1, -1.$

i.e. the two roots of discriminating cubic are equal and the third is different from zero, so the given equation represents either an ellipsoid of revolution or a hyperboloid of revolution.

The central planes are given by $\dfrac{\partial F}{\partial x}=0,\ \dfrac{\partial F}{\partial y}=0,\ \dfrac{\partial F}{\partial z}=0.$

i.e. $2x=0,\ 2z=0,\ 2y=0$ *i.e.* $x=0, y=0, z=0.$

$\therefore$ Centre of the given surface is (0, 0, 0).

$\therefore d'=u\alpha+v\beta+w\gamma+d=0+0+0-1=-1.$

$\therefore$ Reduced equation of the given surface is

$$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+d'=0 \quad \text{or} \quad x^2+y^2-z^2-1=0$$

or $x^2+y^2-z^2=1$, which represents a hyperboloid of revolution.

Now putting $\lambda=-1$ in the determinant given by (i) and associating each row with l, m, n, the d.c.'s of the axis of revolution (or principal axis) we have

$$2l=0,\ m+n=0,\ m+n=0 \Rightarrow \frac{l}{0}=\frac{m}{1}=\frac{n}{-1}$$

$\therefore$ The equations of required axis of revolution which passes through the centre (0, 0, 0) and whose d.r.'s are 0, 1, -1 are

$$\frac{x-0}{0}=\frac{y-0}{1}=\frac{z-0}{-1} \quad i.e. \quad x=0, y+z=0. \qquad \textbf{Ans.}$$

****Ex. 4. Show that the surface represented by the equation**

$$\mathbf{x^2+y^2+z^2-yz-zx-xy-3x-6y-9z+21=0}$$

is a paraboloid of revolution the coordinates of the focus being (1, 2, 3) and the equations to axis are x = y − 1 = z − 2. (*Avadh* 95; *Rohilkhand* 97, 96, 94)

Solution. Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & -(1/2) & -(1/2) \\ -(1/2) & 1-\lambda & -(1/2) \\ -(1/2) & -(1/2) & 1-\lambda \end{vmatrix}=0$$

or $(1-\lambda)\,[(1-\lambda)^2-(\tfrac{1}{4})]+(\tfrac{1}{2})\,[-(\tfrac{1}{2})(1-\lambda)-(\tfrac{1}{4})]-(\tfrac{1}{2})\,[(\tfrac{1}{4})+\tfrac{1}{2}(1-\lambda)]=0$

or $(1-\lambda)\,[8(1-\lambda)^2-2]-2\,[2(1-\lambda)+1]=0$ or $4\lambda^3-12\lambda^2+9\lambda=0$

or $\lambda\,[4\lambda^2-12\lambda+9]=0$ or $\lambda\,(2\lambda-3)^2=0$ or $\lambda=3/2, 3/2, 0$

As two roots of the discriminating cubic are equal and third root is zero, so it is either a paraboloid of revolution or a right circular cylinder.

[See § 12.15 (ii) Page 30 Ch. XII]

The direction ratios of the axis are given by

$$al+hm+gn=0,\ hl+bm+fn=0,\ gl+fm+cn=0$$

i.e. $l-\frac{1}{2}m-\frac{1}{2}n=0,\ -\frac{1}{2}l+m-\frac{1}{2}n=0,\ -\frac{1}{2}l-\frac{1}{2}m+n=0$

i.e. $2l-m-n=0,\ -l+2m-n=0,\ -l-m+2n=0.$

These give $l=m=n=1/\sqrt{3}$

Now $k=ul+vm+wn.$

or $k=(-3/2)(1/\sqrt{3})+(-3)(1/\sqrt{3})+(-9/2)(1/\sqrt{3})=-3\sqrt{3}\neq 0.$

$\therefore$ The reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$ **(Note)**

...See § 12.11 (iii) Page 14 Ch. XII

or $\quad (3/2) x^2 + (3/2) y^2 + 2(-3\sqrt{3}) z = 0$

or $\quad x^2 + y^2 = 4\sqrt{3}z$, which reprresents a paraboloid of revolution.

Also the coordinates of the vertex of the paraboloid are obtained by solving any two of the three equations

$$\frac{\left(\dfrac{\partial F}{\partial x}\right)}{l} = \frac{\left(\dfrac{\partial F}{\partial y}\right)}{m} = \frac{\left(\dfrac{\partial F}{\partial z}\right)}{n} = 2k$$ ...See § 12.11 (iv) Page 14 Ch. XII

or $\quad \dfrac{2x - y - z - 3}{1/\sqrt{3}} = \dfrac{2y - z - x - 6}{1/\sqrt{3}} = \dfrac{2z - y - x - 9}{1/\sqrt{3}} = -6\sqrt{3}.$

or $\quad 2x - y - z - 3 = 2y - z - x - 6 = 2z - y - x - 9 = -6$

or $\quad 2x - y - z + 3 = 0,\ x - 2y + z = 0,\ x + y - 2z + 3 = 0$ ...(I)

with the equation $k(lx + my + nz) + ux + vy + wz + d = 0$

i.e. $\quad -3\sqrt{3}\left(\dfrac{1}{\sqrt{3}}x + \dfrac{1}{\sqrt{3}}y + \dfrac{1}{\sqrt{3}}z\right) + \left(-\dfrac{3}{2}\right)x + (-3)y + \dfrac{-9}{2}z + 21 = 0$

i.e. $$3x + 4y + 5z - 14 = 0 \qquad \text{...(II)}$$

Solving $2x - y - z + 3 = 0,\ x - 2y + z = 0,\ 3x + 4y + 5z - 14 = 0$

we get $x = 0, y = 1, z = 2$. $\quad\therefore$ The required vertex is (0, 1, 2).

$\therefore$ Equations of the axis are $\dfrac{x-0}{1} = \dfrac{y-1}{1} = \dfrac{z-2}{1}$

or $$x = y - 1 = z - 2.$$ **Ans.**

Also the focus will be a point on the axis whose actual direction cosines are $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ and will be at a distance $(1/4)\,4\sqrt{3}$ *i.e.* $\sqrt{3}$ from the vertex (0, 1, 2)

$\therefore$ Coordinates of the focus are given by

$$\frac{x-0}{(1/\sqrt{3})} = \frac{y-1}{(1/\sqrt{3})} = \frac{z-2}{(1/\sqrt{3})} = \sqrt{3}$$ **(Note)**

or $$x = 1, y = 2, z = 3$$

$\therefore$ The required focus is (1, 2, 3). **Ans.**

****Ex. 5. Show that $13x^2 + 45y^2 + 40z^2 + 12yz + 36zx - 24xy - 49 = 0$ represents a right circular cylinder whose axis is $x/6 = y/2 = z/-3$ and radius 1.**

Solution. Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 13-\lambda & -12 & 18 \\ -12 & 45-\lambda & 6 \\ 18 & 6 & 40-\lambda \end{vmatrix} = 0$$

or $\quad (13-\lambda)[(45-\lambda)(40-\lambda) - 36] + 12[-12(40-\lambda) - 108]$
$\qquad + 18[-72 - 18(45-\lambda)] = 0$

or $\quad \lambda(\lambda - 49)^2 = 0 \quad$ or $\quad \lambda = 0, 49, 49.$

As two roots of the discriminating cubic are equal and third root is zero, so it is either a paraboloid of revolution or a right circular cylinder.

[See § 12.15 (ii) Page 30 Ch. XII]

The d. ratios of the axis are given by

$$al + hm + gn = 0,\ hl + bm + fn = 0,\ gl + fm + cn = 0$$

i.e. $13l - 12m + 18n = 0,\ -12l + 45m + 6n = 0;\ 18l + 6m + 40n = 0.$

Solving these we get $\dfrac{l}{6} = \dfrac{m}{2} = \dfrac{n}{-3} = \dfrac{1}{\sqrt{7}}$ **(Note)**

Now here $k = ul + vm + wn = 0,\ \because u = 0 = v = w.$

Also the line of centres is given by $\dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 0,\ \dfrac{\partial F}{\partial z} = 0$

or $26x - 24y + 36z = 0,\ -24x + 90y + 12z = 0,\ 36x + 12y + 80z = 0$

or $13x - 12y + 18z = 0,\ 8x - 30y - 4z = 0,\ 9x + 3y + 20z = 0,$

which gives $x = 0,\ y = 0,\ z = 0.$

$\therefore$ Any point on the line of centres is $(0, 0, 0)$

Also $d' = u\alpha + v\beta + w\gamma + d = 0 = 0 + 0 - 49 = -49$

$\therefore$ The reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + d' = 0$ **(Note)**

or $$49x^2 + 49y^2 - 49 = 0 \quad \text{or} \quad x^2 + y^2 = 1,$$

which is a right circular cylinder of radius 1, as any section of this surface by a plane $z = k$ is a circle $x^2 + y^2 = 1$, whose radius is 1.

And the equations of the axis are

$$\frac{x - 0}{l} = \frac{y - 0}{m} = \frac{z - 0}{n} \quad i.e. \quad \frac{x}{6} = \frac{y}{2} = \frac{z}{-3}$$

****Ex. 6. Prove that the equation $2y^2 + 4zx + 2x - 4y + 6z + 5 = 0$ represents a right circular cone. Show also that the semi-veritical angle of this cone is $\pi/4$ and its axis is given by $x + z + 2 = 0,\ y = 1.$** (*Garhwal* 96)

Solution. The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 0-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = 0 \quad \text{...(i)}$$

or $$-\lambda[-\lambda(2-\lambda)] + 2[-2(2-\lambda)] = 0.$$

or $$(2-\lambda)(\lambda^2 - 4) = 0 \quad \text{or} \quad \lambda = 2, 2, -2$$

As two roots of this cubic are equal and third is not zero, so the given surface is a surface of revolution. **(Note)**

Also the line of centres is given by any two of $\dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 0,\ \dfrac{\partial F}{\partial z} = 0$

i.e. $$4z + 2 = 0,\ 4y - 4 = 0,\ 4x + 6 = 0.$$

$\therefore$ If (α, β, γ) be any point on the line of centres, then

$4\gamma + 2 = 0,\ 4\beta - 4 = 0,\ 4\alpha + 6 = 0 \Rightarrow \alpha = -3/2,\ \beta = 1,\ \gamma = -1/2.$

$\therefore$ Any point on the line of centres is $(-3/2, 1, -1/2)$.

$\therefore\ d' = u\alpha + v\beta + w\gamma + d = 1\,(-3/2) - 2\,(1) + 3\,(-1/2) + 5 = 0.$

$\therefore$ The reduced form of the equation is

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0 \quad \text{or} \quad 2x^2 + 2y^2 - 2z^2 + 0 = 0$$

or $$x^2 + y^2 - z^2 = 0 \quad \text{or} \quad x^2 + y^2 = z^2 \tan^2 45°$$

which represents a right ciruclar cone of semi-vertical angle $\pi/4$.

[$\because x^2 + y^2 = z^2 \tan^2 \alpha$ represents a cone whose semi-vertical angle is α].

Now putting the unequal value of λ viz. -2 in the determinaant of (i) and associating each row with l, m, n we have $2l + 2n = 0,\ 4m = 0,\ 2l + 2n = 0$

These gives $$\frac{l}{1} = \frac{m}{0} = \frac{n}{-1} = \frac{1}{\sqrt{2}}$$

$\therefore$ The equations of its axis are

$$\frac{x - (-3/2)}{l} = \frac{y - 1}{m} = \frac{z - (-1/2)}{n} \quad \text{or} \quad \frac{x + (3/2)}{1} = \frac{y - 1}{0} = \frac{z + (1/2)}{-1}.$$

or $$-x - (3/2) = z + (1/2),\ y - 1 = 0 \quad \text{or} \quad x + z + 2 = 0,\ y = 1$$

Hence proved.

Exercises on § 12.15

Ex. 1. Prove that the equation $2x^2 + 5y^2 + z^2 - 4xy - 8x + 14y + 3 = 0$ is a surface of revolution. Also find the equations of its principal axis.

Ans. Reduced equation is $x^2 + y^2 + 6z^2 = 8$, axis $2x + y - 1 = 0 = z$.

Ex. 2. Find the reduced equation of the surface

$x^2 - y^2 + 2yz - 2zx - x - y + z = 0$. Also find its axis.

Ans. $3\,(x^2 - y^2) = z\,;\ x - (1/3) = y + (1/3) = z.$

***Ex. 3.** Discuss the nature of the surface $yz + zx + xy = a^2$.

Ans. A hyperboloid of revolution ; reduced equation is

$2x^2 - y^2 - z^2 = 2a^2$, axis is $x = y = z$.

Excercises on Chapter XII

Ex. 1. Reduce the surface $40x^2 + 50y^2 + 9z^2 - 8yz - 16zx + 26xy + 4x + 20y - 28z - 3 = 0$ into the standard form and find the latus rectum of a normal section. (*Avadh* 92)

Ex. 2. Reduce the equation $12x^2 + 10y^2 + 8z^2 - 9yz + zx - 13xy + 75x + 77y - 38z + 100 = 0$ into the standard form and also describe the nature of the surface and find the equations of its axes. (*Avadh* 92)

CHAPTER XIII

Generating Lines

§ 13.01. Ruled Surfaces.

We are already aware that cylinder and cone are the surfaces which are generated by the motion of a straight line. Similarly hyperboloid of one sheet and hyperbolic paraboloid are also generated by the motion of a straightt line. This type of surfaces which are generated by the motion of a straight line is known as a **ruled surface.** Thus through every point on a ruled surface a straight line can be drawn which lies wholly (entirely) on the ruled surface. These straight lines are called the **generating lines** or **generators.**

**§ 13.02. Generating Lines of a hyperboloid of one sheet.

(*Gorakhpur* 97)

We know equation of the hyperboloid of one sheet is

$$(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1 \qquad \text{...(i)}$$

Consider any straight line whose equations are

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right), \frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1+\frac{y}{b}\right). \qquad \text{...(ii)}$$

where λ is constant.

If we multiply above two equations given by (ii), we find that λ is eliminated and we get $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2}$ or $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$

which is the equation (i) of the hyperboloid of one sheet.

Thus we conclude that all those points which lie on the straight line (ii) *i.e.* whose coordinates satisfy the equation given by (ii), must also satisfy the equation (i) of the hyperboloid of one sheet.

Hence the straight line whose equations are given by (ii) lies wholly on the surface given by (i).

Assigning different values to the constant λ, we find that equations (ii) represents an infinite number of straight lines all of which lie wholly on the hyperboloid of one sheet given by the equation (i) so that these lines cover the whole surface. These lines are the generators or the generating lines of the surface given by (i).

In a manner similar to above we can show that the system of lines given by the equations

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right), \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right) \qquad \text{...(iii)}$$

lie wholly on the hyperboloid of one sheet given by (i) and as such are its generators or generating lines.

Hence as λ and μ vary, we cotain, two families of straight lines such that every member of each family lies wholly on the hyperboloid of one sheet given by (i). These two families of straight lines given by (ii) and (iii) are known as two systems of generating lines of (i).

§ 13.03. Properties of generating lines (or generators) of hyperboloid of one sheet.

Let the equation of hyperboloid of one sheet be

$$(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1 \quad \text{...(i)}$$

From § 13.02 above we know that the systems of generators of (i) are given by the equations $\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right), \frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1+\frac{y}{b}\right)$ ...(ii)

and $$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right), \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right) \quad \text{...(iii)}$$

Prop I. *One generator of each system passes through every point of the hyperboloid.*

Let $P(\alpha, \beta, \gamma)$ be any point on the hyperboloid (i), then

$$(\alpha^2/a^2)+(\beta^2/b^2)-(\gamma^2/c^2)=1 \quad \text{...(iv)}$$

Now the generators of first system of (i) given by (ii) will pass through the point $P(\alpha, \beta, \gamma)$ if and only if λ has a value equal to each of the fractions

$$\frac{(\alpha/a)-(\gamma/c)}{1-(\beta/b)}, \frac{1+(\beta/b)}{(\alpha/a)+(\gamma/c)} \text{ [obtained from (ii)]} \quad \text{...(v)}$$

or $[(\alpha/a)-(\gamma/c)][(\alpha/a)+(\gamma/c)]=[1+(\beta/b)][1-(\beta/b)]$,
equating the two values of γ from (v)

or $(\alpha^2/a^2)-(\gamma^2/c^2)=1-(\beta^2/b^2)$ or $(\alpha^2/a^2)+(\beta^2/b^2)-(\gamma^2/c^2)=1$,
which is true by virtue of (iv).

Thus if λ is chosen equal to the values given by either of the fractions (v), the corresponding line (generator) of the system of generators (ii) will pass through the point $P(\alpha, \beta, \gamma)$.

Similarly we can show that if μ is equal to either of the fraction $\frac{(\alpha/a)-(\gamma/c)}{1+(\beta/b)}$ or $\frac{1-(\beta/b)}{(\alpha/a)+(\gamma/c)}$ [obtained by evaluating μ from the equations given by (iii)], then a member of the system of generators (iii) corresponding to either of equal values of μ will pass through the point $P(\alpha, \beta, \gamma)$.

Prop II. *No two generators of the same system intersect.*

Consider two generators of the λ-system given by (ii) corresponding to two distinct values λ_1, λ_2 of λ.

$$\frac{x}{a}-\frac{z}{c}=\lambda_1\left(1-\frac{y}{b}\right), \quad \text{...(vi)}$$

$$\frac{x}{b}+\frac{z}{c}=\frac{1}{\lambda_1}\left(1+\frac{y}{b}\right), \quad \text{...(vii)}$$

and $$\frac{x}{c}-\frac{z}{c}=\lambda_2\left(1-\frac{y}{b}\right), \quad \text{...(viii)}$$

$$\frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda_2}\left(1+\frac{y}{b}\right), \quad \text{...(ix)}$$

Substracting (viii) from (vi) we have $(\lambda_1-\lambda_2)\left(1-\frac{y}{b}\right)=0$

or $1-(y/b)=0, \quad \because \quad \lambda_1 \neq \lambda_2$

or $y=b$

Similarly subtracting (ix) from (vii), we get $y=-b$

Thus we find that the four equations giving two generators of the same system are inconsistent and so we conclude that the two generators of the same system do not intersect.

Prop. III. *Any two generators of the different systems intersect.*

Let us consider two generator one of each system given by (ii) and (iii) *i.e.*

$$(x/a)-(z/c)=\lambda\,[1-(y/b)], \quad \text{...(x)}$$

$$(x/a)+(z/c)=(1/\lambda)\,[1+(y/b)], \quad \text{...(xi)}$$

and $$(x/a)-(z/c)=\mu\,[1+(y/b)], \quad \text{...(xii)}$$

$$(x/a)+(z/c)=(1/\mu)\,[1-(y/b)] \quad \text{...(xiii)}$$

Solving (x) and (xii), we get $\lambda\,[1-(y/b)]=\mu\,[1+(y/b)]$

or $$y/b=(\lambda-\mu)/(\lambda+\mu) \quad \text{...(xiv)}$$

$\Rightarrow$ $$1-(y/b)=1-[(\lambda-\mu)/(\lambda+\mu)]=2\mu/(\lambda+\mu) \quad \text{...(xv)}$$

$\therefore$ From (x) and (xi) with the help of (xv) we get

$(x/a)-(z/c)=2\lambda\,\mu/(\lambda+\mu)$ and $(x/a)+(z/c)=2/(\lambda+\mu)$

Solving these, we get $$\frac{x}{a}=\frac{1+\lambda\mu}{\lambda+\mu}, \frac{z}{c}=\frac{1-\lambda\mu}{\lambda+\mu} \quad \text{...(xvi)}$$

The values of x, y and z given by (xvi) and (xiv) also satisfy the equation (xiii) which show that the two generators of λ, μ systems intersect at the point

$$\left(\frac{a\,(1+\lambda\mu)}{\lambda+\mu}, \frac{b\,(\lambda-\mu)}{\lambda+\mu}, \frac{c\,(1-\lambda\mu)}{\lambda+\mu}\right) \quad \text{...(xvii)}$$

For all values of λ and μ the coordinates of this point satisfy the equation (i) of the hyperboloid of one sheet and hence the parametric equations of the hyperboloid of one sheet given by (i) can be taken as

$$x=\frac{a\,(1+\lambda\mu)}{\lambda+\mu},\ y=\frac{b\,(\lambda-\mu)}{\lambda+\mu}, z=\frac{c\,(1-\lambda\mu)}{\lambda+\mu} \quad \text{...(xviii)}$$

§ 13.04. An Imporatnt Theorem.

Statement. *If three points of any straight line* $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ *lie on the conicoid* $F(x, y, z)=0$, *then the line wholly lies on the conicoid.*

Proof. Any point on the given line is $(\alpha+lr,\ \beta+mr,\ \gamma+nr)$

If this point lies on the given conicoid $F(x, y, z) = 0$, which is a second degree equation in x, y and z, then we will get a second degree equation in r, say

$$L r^2 + M r + N = 0 \qquad \text{...(i)}$$

As three points of the line lie on the given conicoid, we will have three values of r which will satisfy (i) and as such it should be an identity which leads to $L = 0$, $M = 0$ and $N = 0$.

In this case, (i) is satisfied by all values of r which shows that every point on the line lies on the conicoid *i.e.* the line lies entirely (*i.e.* wholly) on the conicoid.

****§ 13.05. Condition for a given line to be a generator of a given conicoid.** (*Kumaun* 94, 93)

Let the equation of the conicoid be $ax^2 + by^2 + cz^2 = 1$ ...(i)

and those of the given straight line be $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n} = r$ (say) ...(ii)

Any point on the line (ii) is $(\alpha + lr, \beta + mr, \gamma + nr)$

If this point lies on the conicoid (i), then we have

$$a(\alpha + lr)^2 + b(\beta + mr)^2 + c(\gamma + nr)^2 = 1$$

or $$r^2(al^2 + bm^2 + cn^2) + 2r(al\alpha + bm\beta + cn\gamma) + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0 \qquad \text{...(iii)}$$

If the line (ii) is a generator of the conicoid (i), then it lies wholly on the conicoid and the conditions for which are $al^2 + bm^2 + cn^2 = 0$ ...(iv)

$$al\alpha + bm\beta + cn\gamma = 0 \qquad \text{...(v)}$$

and $$a\alpha^2 + b\beta^2 + c\gamma^2 - 1 = 0 \qquad \text{...(vi)}$$

[obtained with the help of § 13.04 wherein the conditions are given as $L = 0 = M = N$]

The avove three conditions are analysed as follows :

The condition (iv) shows that the lines parallel to the generating lines (ii) and passing through the centre (0, 0, 0) of the conicoid (i) *i.e.* the lines $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$ are generators of the cone $ax^2 + by^2 + cz^2 = 0$,

which is known as the **asymptotic cone.**

The condition (v) shows that the generating lines whose direction cosines are l, m, n should lie on the plane $a\alpha x + b\beta y + c\gamma z = 1$, which is the equation of the tangent plane to the conicoid (i) at the point (α, β, γ).

The condition (vi) shows that the point (α, β, γ) lies on the conicoid (i).

Also the equations (iv) and (v) give the **direction ratios** l, m, n of the generating lines.

Solved Examples on § 13.01—§ 13.05.

Ex. 1. Find the equations of the generting lines of the hyperboloid $yz + 2zx + 3xy + 6 = 0$ which pass through the point $(-1, 0, 3)$.

Sol. Any line through $(-1, 0, 3)$ is $\frac{x+1}{l}=\frac{y-0}{m}=\frac{z-3}{n}=r$ ...(i)

$\therefore$ any point on this line is $(lr-1, mr, nr+3)$ and it lies on the given hyperboloid if $mr(nr+3)+2(nr+3)(lr-1)+3(lr-1)(mr)+6=0$

or $$r^2(mn+2nl+3lm)+r(3m-2n+6l-3m)=0$$

or $$r^2(mn+2nl+3lm)+r(6l-2n)=0 \quad \text{...(ii)}$$

If the line (i) is a generator of the given hyperboloid, then (i) lies wholly on the hyperboloid and the conditions for which from (ii) are

$$mn+2nl+3lm=0 \quad \text{and} \quad 6l-2n=0 \qquad [L=0=M=N]$$

$\therefore$ $n=3l$ and hence $m(3l)+2(3l)l+3lm=0$, on eliminating n.

or $$6l(l+m)=0 \Rightarrow l=0 \quad \text{or} \quad l=-m$$

and when $l=-m$, $n=3l \Rightarrow \frac{l}{1}=\frac{m}{-1}=\frac{n}{3}$

Hence from (i), the generators are

$$\frac{x+1}{0}=\frac{y}{m}=\frac{z-3}{0} \quad \text{and} \quad \frac{x+1}{1}=\frac{y}{-1}=\frac{z-3}{3}$$

i.e. $$x+1=0,\ z-3=0 \quad \text{and} \quad \frac{x+1}{1}=\frac{y}{-1}=\frac{z-3}{3}$$ **Ans.**

***Ex. 2. Find the equations to the generating lines of the hyperbleid $(x^2/4)+(y^2/9)-(z^2/16)=1$ which pass through the points $(2, 3, -4)$ and $(2, -1, 4/3)$.** (*Garhwal* 95)

Sol. Any line through $(2, 3, -4)$ is $\frac{x-2}{l}=\frac{y-3}{m}=\frac{z+4}{n}=r$ (say) ...(i)

$\therefore$ Any point on this line is $(lr+2, mr+3, nr-4)$ and it lies on the given hyperboloid if $[(lr+2)^2/4]+[(mr+3)^2/9]-[(nr-4)^2/16]=1$

or $$r^2\left[\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}\right]+2r\left[\frac{2l}{4}+\frac{3m}{9}+\frac{4n}{16}\right]=0 \quad \text{...(ii)}$$

If the line (i) is a generator of the given hyperboloid, then (i) lies wholly on the hyperboloid and the conditions for which from (ii) are

$$\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}=0 \quad \text{and} \quad \frac{2l}{4}+\frac{3m}{9}+\frac{4n}{16}=0$$

i.e. $$(l^2/4)+(m^2/9)-(n^2/16)=0 \quad \text{and} \quad (l/2)+(m/3)=-(n/4) \quad \text{...(iii)}$$

Eliminating n, we get $\frac{l^2}{4}+\frac{m^2}{9}-\left(\frac{l}{2}+\frac{m}{3}\right)^2=0$

or $-(1/3)\,lm=0 \Rightarrow$ either $l=0$ or $m=0$

When $l=0$, from (iii) we get $m/3=-n/4$

When $m=0$, from (iii) we get $l/2=-n/4$ *i.e.* $l/1=-n/2$

Hence from (i), equations of the required generator through $(2, 3, -4)$ are

$$\frac{x-2}{0}=\frac{y-3}{3}=\frac{z+4}{-4} \quad \text{and} \quad \frac{x-2}{1}=\frac{y-3}{0}=\frac{z+4}{-2} \qquad \textbf{Ans.}$$

In a similar manner we can find that the generators through the point $(2, -1, 4/3)$ are

$$\frac{x-2}{0}=\frac{y+1}{3}=\frac{z-(4/3)}{-4} \quad \text{and} \quad \frac{x-2}{3}=\frac{y+1}{6}=\frac{z-(4/3)}{10} \qquad \textbf{Ans.}$$

****Ex. 3. Find the equations to the generators of the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ through any point of the principal elliptic section $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$, $z=0$ by the plane $z=0$.**

Or

Find the equations of the generators of the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ which pass through the point $(a\cos\theta, b\sin\theta, 0)$

(*Garhwal* 94, *Gorakhpur* 96)

Sol. Any point on the elliptic section of the hyperboloid is $(a\cos\theta, b\sin\theta, 0)$.

$\therefore$ Equations of any line through this point is

$$\frac{x-a\cos\theta}{l}=\frac{y-b\sin\theta}{m}=\frac{z-0}{n}=r, \text{ say} \qquad \text{...(i)}$$

Any point on this line is $(lr+a\cos\theta, mr+b\sin\theta, nr)$ and it lies on the given hyperboloid if

$$\frac{(lr+a\cos\theta)^2}{a^2}+\frac{(mr+b\sin\theta)^2}{b^2}-\frac{n^2r^2}{c^2}=1$$

or
$$\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}\right)r^2+2\left(\frac{l\cos\theta}{a}+\frac{m\sin\theta}{b}\right)r=0 \qquad \text{...(ii)}$$

If the line (i) is a generator of the given hyperboloid, then (i) lies wholly on the hyperboloid and the condition for which from (ii) are

$$(l^2/a^2)+(m^2/b^2)-(n^2/c^2)=0 \qquad \text{...(iii)}$$

and
$$\frac{l\cos\theta}{a}+\frac{m\sin\theta}{b}=0 \qquad \text{...(iv)}$$

From (iv) we get $\dfrac{l}{a\sin\theta}=\dfrac{m}{-b\cos\theta}$ or $\dfrac{(l/a)}{\sin\theta}=\dfrac{(m/-b)}{\cos\theta}$

$$\Rightarrow \frac{(l/a)}{\sin\theta}=\frac{(m/-b)}{\cos\theta}=\frac{\sqrt{[(l^2/a^2)+(m^2/b^2)]}}{\sqrt{(\sin^2\theta+\cos^2\theta)}}=\frac{\sqrt{(n^2/c^2)}}{1}, \text{ from (iii)}$$

$$\Rightarrow \frac{l}{a\sin\theta}=\frac{m}{-b\cos\theta}=\frac{n}{\pm c} \qquad \textbf{(Note)}$$

$\therefore$ The equation to the required generators from (i) are

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{\pm c} \qquad \textbf{Ans.}$$

***Ex. 4. A point 'm' on the parabola $y = 0$, $cx^2 = 2a^2z$ is $(2am, 0, 2cm^2)$ and the point 'n' on the parabola $x = 0$, $cy^2 = -2b^2z$ is $(0, 2bn, -2cn^2)$. Obtain the locus of the lines joining the points for which (i) $m = n$ and (ii) $m = -n$.**

Sol. The equations of the line joining "m" and 'n' points are

$$\frac{x-2am}{2am} = \frac{y-0}{-2bm} = \frac{z-2cm^2}{2cm^2+2cn^2} = r \text{ (say)}$$

If $m = \pm n$, then $\dfrac{x \mp 2an}{\pm 2an} = \dfrac{y}{-2bn} = \dfrac{z-2cn^2}{2c\,(2n^2)} = r$

$\Rightarrow x/(2a) = \pm n\,(r+1),\ y/(2b) = -nr,\ z = 2cn^2\,(2r+1)$

$$\therefore \frac{x^2}{4a^2} - \frac{y^2}{4b^2} = n^2\,(r+1)^2 - n^2r^2 = n^2\,(2r+1) = \frac{z}{2c}$$

$\therefore$ The required locus is $(x^2/a^2) - (y^2/b^2) = 2z/c$ **Ans.**

****Ex. 5. CP, CQ are any two conjugate semi-diametrs of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, $z = c$, CP′, CQ′ are the conjugate diameters of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, $z = -c$ drawn in the same directions as CP and CQ. Prove that the hyperboloid $(2x^2/a^2) + (2y^2/b^2) - (z^2/c^2) = 1$ is generated by either PQ′ or P′Q′.**

Sol. The coordinates of P, Q, P' and Q' are given by

$P\,(a\cos\theta, b\sin\theta, c)$, $Q\,(-a\sin\theta, b\cos\theta\ c)$,

$P'\,(a\cos\theta, b\sin\theta, -c)$ and $Q'\,(-a\sin\theta, b\cos\theta, -c)$

$\therefore$ Equations to PQ' are

$$\frac{x-a\cos\theta}{-a\sin\theta - a\cos\theta} = \frac{y-b\sin\theta}{b\cos\theta - b\sin\theta} = \frac{z-c}{-c-c} = r \text{ (say)}$$

$\therefore\ x - a\cos\theta = r\,[-a\,(\sin\theta + \cos\theta)]$,

$y - b\sin\theta = r\,[b\,(\cos\theta - \sin\theta)]$ and $z - c = -2cr$

$\Rightarrow x/a = \cos\theta - r\,(\sin\theta + \cos\theta)$, $y/b = \sin\theta + r\,(\cos\theta - \sin\theta)$

and $\quad z = c\,(1-2r)$ ...(i)

Eliminating r from these we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = [\cos\theta - r\,(\sin\theta + \cos\theta)]^2 + [\sin\theta + r\,(\cos\theta - \sin\theta)]^2$$

$$= 1 + r^2\,\{(\sin\theta + \cos\theta)^2 + (\cos\theta - \sin\theta)^2\} - 2r\,[\cos\theta\,(\sin\theta + \cos\theta) - \sin\theta\,(\cos\theta - \sin\theta)]$$

$$= 1 + r^2\,(2) - 2r\,(1) = 2r^2 - 2r + 1$$

or $\quad 2\left(\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}\right) = 4r^2 - 4r + 2 = (1-2r)^2 + 1 = \left(\dfrac{z}{c}\right)^2 + 1$, from (i)

or $\quad 2\,(x^2/a^2) + 2\,(y^2/b^2) - (z^2/c^2) = 1$, which is a hyperboloid. Proved.

Similarly we can show that the surface generated by $P'Q$ is also the above hyperboloid.

****Ex. 6. Show that if two generators of the surface $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ through the points P $(a \cos \alpha, b \sin \alpha, 0)$ and Q $(a \cos \beta, b \sin \beta, 0)$ intersect at right angles, their projection on the plane $z=0$ intersect at an angle θ, where $\tan \theta=[ab \sin (\alpha-\beta)]/c^2$**

Sol. As in Ex. 3. Page 6 we can prove that the generator of one system through $P(a \cos \alpha, b \sin \alpha, 0)$ is

$$\frac{x-a \cos \alpha}{a \sin \alpha}=\frac{y-b \sin \alpha}{-b \cos \alpha}=\frac{z-0}{c} \quad \text{...(i)}$$

Generator of other system through $Q(a \cos \beta, b \sin \beta, 0)$ is

$$\frac{x-a \cos \beta}{a \sin \beta}=\frac{y-b \sin \beta}{-b \cos \beta}=\frac{z-0}{-c} \quad \text{...(ii)}$$

These will be perpendicular if

$$(a \sin \alpha)(a \sin \beta)+(-b \cos \alpha)(-b \cos \beta)+(c)(-c)=0$$

or
$$a^2 \sin \alpha \sin \beta+b^2 \cos \alpha \cos \beta-c^2=0 \quad \text{...(iii)}$$

Now the projecton of above generators on the plane $z=0$ are tangents to the ellipse $(x^2/a^2)+(y^2/b^2)=1$ at $(a \cos \alpha, b \sin \alpha)$ and $(a \cos \beta, b \sin \beta)$ and their equations are

$$\frac{x \cos \alpha}{a}+\frac{y \sin \alpha}{b}=1 \quad \text{and} \quad \frac{x \cos \beta}{a}+\frac{y \sin \beta}{b}=1$$

[See Ex. 7. below]

Their slopes are $-(b \cos \alpha)/(a \sin \alpha)$ and $-(b \cos \beta)/(a \sin \beta)$

$\therefore$ If θ be the angle between them, then

$$\tan \theta=\frac{\left(\dfrac{-b \cos \alpha}{a \sin \alpha}\right)-\left(\dfrac{-b \cos \beta}{a \sin \beta}\right)}{1+\left(\dfrac{-b \cos \alpha}{a \sin \alpha}\right)\left(\dfrac{-b \cos \beta}{a \sin \beta}\right)}=\frac{-\dfrac{b}{a}\left[\dfrac{\cos \alpha}{\sin \alpha}-\dfrac{\cos \beta}{\sin \beta}\right]}{1+\dfrac{b^2 \cos \alpha \cos \beta}{a^2 \sin \alpha \sin \beta}}$$

$$=\frac{-ab[\sin \beta \cos \alpha-\sin \alpha \cos \beta]}{a^2 \sin \alpha \sin \beta+b^2 \cos \alpha \cos \beta}=\frac{ab \sin (\alpha-\beta)}{c^2}, \text{ from (iii)}$$

****Ex. 7. Prove that the projections of the generators of a hyperboloid on coordinate plane are tangents to the section of the hyperboloid by that plane.**

Sol. Let the equation of the hyperboloid be

$$(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1 \quad \text{...(i)}$$

From Ex. 3 Page 6 we know that a generator of the hyperboloid (i) is

$$\frac{x-a \cos \theta}{a \sin \theta}=\frac{y-b \sin \theta}{-b \cos \theta}=\frac{z}{c} \quad \text{...(ii)}$$

Now consider the coordinate plane $z=0$. The section of the hyperboloid (i) by this plane $z=0$ is given by $(x^2/a^2)+(y^2/b^2)=1, \ z=0$ ...(iii)

The projection of the generator (ii) on the plane $z=0$ is given by

$$\frac{x - a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{-b\cos\theta},\ z = 0$$

which is a plane through the generator perpendicular to the plane $z = 0$.

On simplifying it reduces to $\dfrac{x}{a\sin\theta} - \dfrac{\cos\theta}{\sin\theta} = \dfrac{y}{-b\cos\theta} + \dfrac{\sin\theta}{\cos\theta},\ z = 0$

i.e. $\quad \dfrac{x}{a\sin\theta} + \dfrac{y}{b\cos\theta} = \dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{\sin\theta\cos\theta},\ z = 0$

i.e. $\quad \dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1,\ z = 0$

which is evidently a tangent to the section (iii) of the hyperboloid (i) by the plane $z = 0$ at the point $(a\cos\theta, b\sin\theta, 0)$

Again consider the coordinate plane $x = 0$. The section of the hyperboloid (i) by this plane $x = 0$ is given by $(y^2/b^2) - (z^2/c^2) = 1,\ x = 0$ ...(iv)

The projection of the generator (ii) on the plane $x = 0$ is given by $\dfrac{y - b\sin\theta}{-b\cos\theta} = \dfrac{z}{c},\ x = 0$ which is a plane through the generator perpendicular to the plane $x = 0$.

On simplifying it reduces to $\dfrac{y}{-b\cos\theta} + \dfrac{\sin\theta}{\cos\theta} = \dfrac{z}{c},\ x = 0$

or $\quad \dfrac{y}{b} + \dfrac{z\cos\theta}{c} = \sin\theta, x = 0 \quad$ or $\quad \dfrac{y}{b}\operatorname{cosec}\theta + \dfrac{z}{c}\cot\theta = 1,\ x = 0$

which is evidently a tangent to the section (iv) of the hyperboloid (i) by the plane $x = 0$ at the point $(0, b\operatorname{cosec}\theta, -c\cot\theta)$.

Similarly we can prove the result by considering the plane $y = 0$.

****Ex. 8. Prove that in general two generators of the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ can be drawn to cut a given generator at right angles. Also show that if they meet the plane $z = 0$ in P and Q, PQ touches the ellipse $(x^2/a^6) + (y^2/b^6) = c^4/(a^4b^4)$.**

Sol. We know for the given hyperboloid, the generator belonging to λ-system is given by $\dfrac{x}{a} - \dfrac{z}{c} = \lambda\left(1 - \dfrac{y}{b}\right)$ and $\dfrac{x}{a} + \dfrac{z}{c} = \dfrac{1}{\lambda}\left(1 + \dfrac{y}{b}\right)$ ...(i)

or $\quad \dfrac{x}{a} + \dfrac{\lambda y}{b} - \dfrac{z}{c} = \lambda \quad$ and $\quad \lambda\dfrac{x}{a} - \dfrac{y}{b} + \lambda\dfrac{z}{c} = 1$

$\therefore$ If l_1, m_1, n_1 be the d.r.'s of the generator (i), then

$$\frac{l_1}{a} + \frac{\lambda m_1}{b} - \frac{n_1}{c} = 0 \quad \text{and} \quad \frac{\lambda l_1}{a} - \frac{m_1}{b} + \frac{\lambda n_1}{c} = 0$$

Solving these simultaneously, we get

$$\frac{l_1/a}{\lambda^2 - 1} = \frac{m_1/b}{-\lambda - \lambda} = \frac{n_1/c}{-1 - \lambda^2}$$

or $$\frac{l_1}{-a(\lambda^2 - 1)} = \frac{m_1}{2\lambda b} = \frac{n_1}{c(1 + \lambda^2)} \quad ...(ii)$$

Similarly the direction ratios l_2, m_2, n_2 of the generator belonging to μ-system viz. $\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right)$ and $\frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right)$ ...(iii)

are given by $$\frac{l_2}{a(\mu^2-1)}=\frac{m_2}{2b\mu}=\frac{n_2}{-c(\mu^2+1)} \quad ...(iv)$$

If these two generators given by (i) and (iii) are perpendicular then
$$-a^2(\lambda^2-1)(\mu^2-1)+4b^2\lambda\mu-c^2(1+\lambda^2)(\mu^2+1)=0 \quad ...(v)$$

Now if λ-generator is given, then λ is constant and (v) will be a quadratic equation in μ which gives two values of μ and this shows that there will be two generators of μ-system which will be perpendicular to a generator of λ-system.

Now let the generators of μ-system meet the plane $z=0$ in the points $P(a\cos\alpha, b\sin\alpha, 0)$ and $Q(a\cos\beta, b\sin\beta, 0)$

$\therefore$ The generator of the μ-system through these points are given by
$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{c} \quad ...(vi)$$
and
$$\frac{x-a\cos\beta}{a\sin\beta}=\frac{y-b\sin\beta}{-b\cos\beta}=\frac{z}{c} \quad ...(vii)$$

[See Ex. 3. Page 6 of this chapter]

These two generators intersect at right angles a generator of λ-system through any point $(a\cos\theta, b\sin\theta, 0)$ say whose equations are
$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{-c} \quad ...(viii)$$

As (vi) and (vii) are both perpendicular to (viii), so
$$a^2\sin\alpha\sin\theta+b^2\cos\alpha\cos\theta-c^2=0$$
and
$$a^2\sin\beta\sin\theta+b^2\cos\beta\cos\theta-c^2=0$$

Solving these simultaneouly for $a^2\sin\theta$, $b^2\cos\theta$ and $-c^2$, we get
$$\frac{a^2\sin\theta}{\cos\alpha-\cos\beta}=\frac{b^2\cos\theta}{\sin\beta-\sin\alpha}=\frac{-c^2}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$$

or
$$\frac{a\sin\theta}{2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}}=\frac{b^2\cos\theta}{2\cos\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}}$$
$$=\frac{-c^2}{\sin(\alpha-\beta)}=\frac{-c^2}{2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2}}$$

$$\Rightarrow \quad \frac{a^2\sin\theta}{c^2}=\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}, \qquad \frac{b^2\cos\theta}{c^2}=\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \quad ...(ix)$$

Also equation of the line joining P and Q is

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2},\ z=0$$

or $$\frac{x}{a}\left(\frac{b^2\cos\theta}{c^2}\right)+\frac{y}{b}\left(\frac{a^2\sin\theta}{c^2}\right)=1,\ z=0 \qquad \text{...(x)}$$

using the results of (ix).

Now in order to find its envelope, we should differentiate (x) with respect to θ and then eliminate θ.

Differentiating (x) w. r. to θ, we get

$$\frac{-xb^2}{ac^2}\sin\theta+\frac{ya^2}{bc^2}\cos\theta=0,\ z=0 \qquad \text{...(xi)}$$

Squaring and adding (x) and (xi), θ is eliminated and we get the required envelope of PQ as $\frac{x^2b^4}{a^2c^4}+\frac{y^2a^4}{b^2c^4}=1,\ z=0$ or $\frac{x^2}{a^6}+\frac{y^2}{b^6}=\frac{c^4}{a^4b^4},\ z=0$

which represents an ellipse on the plane $z=0$. Hence proved.

****Ex. 9. Find the locus of the point of intersection of perpendicular generators of a hyperboloid of one sheet.** *(Gorakhpur 97, 95)*

Sol. As in Ex. 8 above we can find that if one generator of λ-system and one generator of μ-system intersect at right angles, then

$$-a^2(\lambda^2-1)(\mu^2-1)+4b^2\lambda\mu-c^2(1+\lambda^2)(1+\mu^2)=0$$

...See result (v) Page 10 of this chapter

or $$a^2(\lambda^2\mu^2-\lambda^2-\mu^2+1)-4b^2\lambda\mu+c^2(\lambda^2\mu^2+\lambda^2+\mu^2+1)=0$$

or $$a^2[(1+\lambda\mu)^2-(\lambda+\mu)^2]+b^2[(\lambda-\mu)^2-(\lambda+\mu)^2]$$
$$+c^2[(1-\lambda\mu)^2+(\lambda+\mu)^2]=0 \quad \textbf{(Note)}$$

or $$a^2(1+\lambda\mu)^2+b^2(\lambda-\mu)^2+c^2(1-\lambda\mu)^2=(\lambda+\mu)^2(a^2+b^2-c^2)$$

This relation shows that the point of intersection of the above two generators *i.e.* $\left[\frac{a(1+\lambda\mu)}{(\lambda+\mu)},\frac{b(\lambda-\mu)}{\lambda+\mu},\frac{c(1-\lambda\mu)}{\lambda+\mu}\right]$

...See § 13.03 Prop. III Page 3 of this chapter

lies on the sphere $x^2+y^2+z^2=a^2+b^2-c^2$, which is known as the director sphere.

Hence the required locus is the curve of intersection of the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ and the director sphere

$$x^2+y^2+z^2=a^2+b^2-c^2.$$

[Alternative method is given in Ex. 20 Page 2_ of this chapter]

****Ex. 10. Prove that the tangent planes to the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ which are parallel to tangent planes to the cone $\frac{b^2c^2x^2}{c^2-b^2}+\frac{c^2a^2y^2}{c^2+a^2}-\frac{a^2b^2z^2}{a^2+b^2}=0$ cut the surface in perpendicular generators.**

Sol. We know that the equation of the cone reciprocal to the cone $ax^2+by^2+cz^2=0$ is $(x^2/a)+(y^2/b)+(z^2/c)=0$.

$\therefore$ The equation of the cone reciprocal to the given cone is

$$\frac{c^2-b^2}{b^2c^2}x^2+\frac{c^2-a^2}{c^2a^2}y^2+\frac{a^2+b^2}{a^2b^2}z^2=0. \quad \text{...(i)}$$

Let $lx+my+nz=0$ be a tangent plane to the given cone so that by definition its normal with d.ratios l, m, n is a generator of its reciprocal cone (i).

$\therefore$ We know $$\frac{c^2-b^2}{b^2c^2}l^2+\frac{c^2-a^2}{c^2a^2}m^2+\frac{a^2+b^2}{a^2b^2}n^2=0 \quad \text{...(ii)}$$

Let any plane parallel to the tangent plane to the given cone be

$$lx+my+nz=p \quad \text{...(iii)}$$

If it is a tangent plane to the given hyperboloid, then

$$p^2=a^2l^2+b^2m^2-c^2n^2 \quad \text{(See Ch. IX)} \quad \text{...(iv)}$$

Again if it is a tanegnt plane at the point (x_1, y_1, z_1) then its equation is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}=1 \quad \text{...(v)}$$

Comparing (iii) and (v), we get $\dfrac{x_1/a^2}{l}=\dfrac{y_1/b^2}{m}=\dfrac{z_1/c^2}{-n}=\dfrac{1}{p}$

or $$\frac{x_1}{la^2}=\frac{y_1}{mb^2}=\frac{z_1}{-nc^2}=\frac{1}{p} \quad \text{...(vi)}$$

Also the plane (iii) cuts the given hyperboloid in perpendicular generators if (x_1, y_1, z_1) lies on the director sphere

$$x^2+y^2+z^2=a^2+b^2-c^2 \quad \text{...See Ex. 9 Page 11}$$

$\therefore x_1^2+y_1^2+z_1^2=a^2+b^2-c^2$

or $$\left(\frac{a^2l}{p}\right)^2+\left(\frac{b^2m}{p}\right)^2+\left(\frac{-c^2n}{p}\right)^2=a^2+b^2-c^2, \text{ from (vi)}$$

or $$a^4l^2+b^4m^2+c^4n^2=(a^2+b^2-c^2)p^2$$
$$=(a^2+b^2-c^2)(a^2l^2+b^2m^2-c^2n^2), \text{ from (iv)}$$

or $$a^2l^2(b^2-c^2)+b^2m^2(a^2-c^2)-c^2n^2(a^2+b^2)=0$$

or $$\frac{l^2(c^2-b^2)}{b^2c^2}+\frac{m^2(c^2-a^2)}{c^2a^2}+\frac{n^2(a^2+b^2)}{a^2b^2}=0, \text{ dividing each term by } -a^2b^2c^2$$

which is true by virtue of (ii). Hence proved.

Ex. 11. Find the point of intersection P, Q of the generators of opposite system drawn through the points A $(a\cos\alpha, b\sin\alpha, 0)$ and B $(a\cos\beta, b\sin\beta, 0)$ of the principal elliptic section of the hyperboloid

$$(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1.$$

Hence show that if A and B are extremities of semi-conjugate diameters, the loci of the points P and Q are the ellipses

$$(x^2/a^2)+(y^2/b^2)=2, \quad z=\pm c.$$

Sol. Let the coordinates of the point P be (x_1, y_1, z_1).

The equation of the tangent plane to the given hyperboloid at P is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}=1$ and it meets the plane $z=0$ in the line

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,\ z=0 \qquad \text{...(i)}$$

which is the same as the line joining the points A and B

i.e. $$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right),\ z=0 \qquad \text{...(ii)}$$

Comparing (i) and (ii), we get

$$\frac{x_1/a^2}{(1/a)\cos\frac{\alpha+\beta}{2}}=\frac{y_1/b^2}{(1/b)\sin\frac{\alpha+\beta}{2}}=\frac{1}{\cos\frac{\alpha-\beta}{2}}$$

$$\Rightarrow \frac{x_1}{a}=\frac{\cos\{(\alpha+\beta)/2\}}{\cos\{(\alpha-\beta)/2\}},\ \frac{y_1}{b}=\frac{\sin\{(\alpha+\beta)/2\}}{\cos\{(\alpha-\beta)/2\}} \qquad \text{...(iii)}$$

Again $(x_1^2/a^2)+(y_1^2/b^2)-(z_1^2/c^2)=1$

$$\left[1/\cos^2\left(\frac{\alpha-\beta}{2}\right)\right]-\frac{z_1^2}{c^2}=1, \text{ substituting values from (iii)}$$

or $$\frac{z_1^2}{c^2}=\sec^2\left(\frac{\alpha-\beta}{2}\right)-1=\tan^2\left(\frac{\alpha-\beta}{2}\right)$$

or $$\frac{z_1}{c}=\pm\frac{\sin\{(\alpha-\beta)/2\}}{\cos\{(\alpha-\beta)/2\}} \qquad \text{...(iv)}$$

From (iii) and (iv) we get the coordinates of $P(x_1, y_1, z_1)$ as

$$\left(\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\ \frac{b\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\ \frac{\pm c\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right)$$

Again as A and B are extremities of two semi-conjugate diameters, we have

$$\alpha-\beta=\pi/2 \qquad \text{...(v)}$$

$\therefore$ From (iii) we get

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=\frac{1}{\cos^2\{(\alpha-\beta)/2\}}=\frac{1}{\cos^2(\pi/4)}, \text{ from (v)}$$

or $$(x_1^2/a^2)+(y_1^2/b^2)=2$$

And from (iv), $z_1=\pm c\tan\left(\frac{\alpha-\beta}{2}\right)=\pm c\tan\left(\frac{\pi}{4}\right)$, from (v)

$$z_1=\pm c$$

$\therefore$ The locus of P and Q are $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2,\ z=\pm c$ Proved.

****Ex. 12. The generators through points on the principal elliptic section of the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$, such that the eccentric angle of the one is double that of the other, intersect on the curve given by $x=\dfrac{a(1-3t^2)}{1+t^2},\ y=\dfrac{bt(3-t^2)}{1+t^2},\ z=\pm ct$.**

Sol. Let $A(a\cos\alpha, b\sin\alpha, 0)$ and $B(a\cos\beta, b\sin\beta, 0)$ be the two points on the pricncipal elliptic section of the given hyperbolid by the plane $z=0$.

Then the points of intersection P and Q of the generators of opposite system through them are given by

$$\frac{x}{a}=\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\ \frac{y}{b}=\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\ \frac{z}{c}=\pm\frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \quad ...(i)$$

...See Ex. 11 Pages 12-13 of this chapter.

Now here we are given $\alpha=2\beta$

Putting $\alpha=2\beta$ in (i), we get $\dfrac{z}{c}=\pm\tan\dfrac{\beta}{2}$ or $z=\pm c\tan\dfrac{\beta}{2}=\pm ct,$ ...(ii)

where $t=\tan(\beta/2)$.

$$\frac{x}{a}=\frac{\cos(3\beta/2)}{\cos(\beta/2)}=\frac{4\cos^3(\beta/2)-3\cos(\beta/2)}{\cos(\beta/2)}=4\cos^2\frac{\beta}{2}-3$$

$$=\frac{4-3\sec^2(\beta/2)}{\sec^2(\beta/2)}=\frac{4-3[1+\tan^2(\beta/2)]}{1+\tan^2(\beta/2)}$$

or $\dfrac{x}{a}=\dfrac{1-3t^2}{1+t^2}$, where $t=\tan\dfrac{\beta}{2}$

or $x=a(1-3t^2)/(1+t^2)$ ...(iii)

And $\dfrac{y}{b}=\dfrac{\sin(3\beta/2)}{\cos(\beta/2)}=\dfrac{3\sin(\beta/2)-4\sin^3(\beta/2)}{\cos(\beta/2)}$

$$=3\tan\frac{\beta}{2}-4\tan\frac{\beta}{2}\sin^2\frac{\beta}{2}=\tan\frac{\beta}{2}\left[3-4\sin^2\frac{\beta}{2}\right]$$

$$=\tan\frac{\beta}{2}\left[\frac{3\sec^2(\beta/2)-4\tan^2(\beta/2)}{\sec^2(\beta/2)}\right],$$

multiplying num and denom. by $\sec^2(\beta/2)$

$$=\tan\frac{\beta}{2}\left[\frac{3\{1+\tan^2(\beta/2)\}-4\tan^2(\beta/2)}{1+\tan^2(\beta/2)}\right]$$

$$=t\left[\frac{3(1+t^2)-4t^2}{1+t^2}\right]=\frac{t(3-t^2)}{1+t^2}$$

or $$y = bt(3-t^2)/(1+t^2) \quad ...(iv)$$

Hence from (ii), (iii) and (iv) we get the required result.

****Ex. 13. The generators through P of the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ meets the principal elliptic section of A and B. If the median of the triangle APB through P is parallel to the fixed plane $\alpha x+\beta y+\gamma z=0$, show that P lies on the surface**

$$z(\alpha x+\beta y)+\gamma(c^2+z^2)=0.$$

Sol. Let the coordinates of P, A and B be (x_1, y_1, z_1), $(a\cos\theta, b\sin\theta, 0)$ and $(a\cos\phi, b\sin\phi, 0)$ respectively.

The values of x_1, y_1, z_1 can be found as given in (iii) and (iv) of Ex. 11 Page 13 of this chapter.

Also the coordinates of F, the mid-point of AB are

$$[\tfrac{1}{2}a(\cos\theta+\cos\phi), \tfrac{1}{2}b(\sin\theta+\sin\phi), 0]$$

or $$\left[a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\ b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\ 0\right]$$

$\therefore$ Direction ratios of the median PF through P are

$$x_1-a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\ y_1-b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\ z_1-0$$

or $$\frac{a\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}-a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

$$\frac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}-b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\qquad \frac{c\sin\frac{\theta-\phi}{2}}{\cos\frac{\theta-\phi}{2}}$$

from (iii) and (iv) of Ex. 11 Page 13

or $$\frac{a\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}\left(1-\cos^2\frac{\theta-\phi}{2}\right),\ \frac{b\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}\left(1-\cos^2\frac{\theta-\phi}{2}\right),\ \frac{c\sin\frac{\theta-\phi}{2}}{\cos\frac{\theta-\phi}{2}}$$

or $$a\cos\frac{\theta+\phi}{2}\sec\frac{\theta-\phi}{2},\ b\sin\frac{\theta+\phi}{2}\sec\frac{\theta-\phi}{2},\ c\tan\frac{\theta-\phi}{2}\operatorname{cosec}^2\frac{\theta-\phi}{2}$$

or $$x_1,\ y_1,\ z_1\operatorname{cosec}^2\frac{\theta-\phi}{2} \quad ...\text{See (iii), (iv) of Ex. 11 P. 13}$$

or $$x_1, y_1, z_1\left(1+\cot^2\frac{\theta-\phi}{2}\right),\ \text{where } \frac{z_1}{c}=\tan\frac{\theta-\phi}{2}$$

or $$x_1, y_1, z_1[1+(c^2/z_1^2)]$$

As PF is parallel to the plane $\alpha x+\beta y+\gamma z=0$

$\therefore\ \alpha x_1 + \beta y_1 + \gamma z_1 [1 + (\ldots)] = 0$ or $(\alpha x_1 + \beta y_1) z_1 + \gamma (z_1^2 + c^2) = 0$

$\therefore$ The required locus of $P(x_1, y_1, z_1)$ is $z(\alpha x + \beta y) + \gamma (z^2 + c^2) = 0$

Hence proved.

***Ex. 14. If the generators through a point P on the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ ... O, ... $z = 0$ in A and B and the volume of the tetrahedron OAPB is constant and equal to abc/6, prove that P has on one of the planes $z = \pm c$.**

Sol. Let the coordinates of P, A and B be (x_1, y_1, z_1), $(a \cos \alpha, b \sin \alpha, 0)$ and $(a \cos \beta, b \sin \beta, 0)$ respectively. The values of x_1, y_1, z_1 can be found as given in (iii) and (iv) of Ex. 11 Page 13. Also O is the origin.

$\therefore$ The volume of the tetrahedraon $OAPB$

$$= \text{“}\frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\text{”, as } O \text{ is origin}$$

$$= \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ a\cos\alpha & b\sin\alpha & 0 \\ a\cos\beta & b\sin\beta & 0 \end{vmatrix}$$, substituting the coordinates of three vertices P, A and B

$$= \frac{1}{6} z_1\, ab\,(\sin\beta\cos\alpha - \sin\alpha\cos\beta) = \frac{1}{6} ab\, z_1 \sin(\beta - \alpha)$$

$$= \frac{1}{6} ab\left[c \tan\frac{\alpha-\beta}{2}\right]\sin(\beta - \alpha), \text{ from (iv) of Ex. 11 Page 13}$$

$$= \frac{1}{6} a \tan\frac{\alpha-\beta}{2}\sin(\alpha-\beta), \text{ numerically}$$

$$= \frac{1}{6} abc, \text{ given}$$

$$\therefore\ \tan\frac{\alpha-\beta}{2}\sin(\alpha-\beta) = 1 \quad \text{or} \quad \tan\frac{\alpha-\beta}{2}\cdot\frac{2\tan\{(\alpha-\beta)/2\}}{1+\tan^2\{(\alpha-\beta)/2\}} = 1$$

or $$2\tan^2\frac{\alpha-\beta}{2} = 1 + \tan^2\frac{\alpha-\beta}{2} \quad \text{or} \quad \tan^2\frac{\alpha-\beta}{2} = 1$$

or $$\tan\frac{\alpha-\beta}{2} = \pm 1 \quad \text{or} \quad \frac{z_1}{c} = \pm 1, \text{ from (iv) of Ex. 11 Page 13}$$

or $z_1 = \pm c$.

$\therefore$ $P(x_1, y_1, z_1)$ lies on one of the planes $z = \pm c$.

***Ex. 15. Show that the perpendicular from the origin on the generator of the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ ... curve**

$$\frac{a^2(b^2+c^2)^2}{x^2} \ldots \frac{\ldots(a^2-c^2)^2}{y^2} = \frac{\ldots(a^2-b^2)^2}{z^2}$$

Sol. We know [See Ex. 3 Page 6 of this chapter] that the equations to generator of the hyperboloid through any point of the principal elliptic section

$(x^2/a^2)+(y^2/b^2)=1,\ z=0$ are $\dfrac{x-a\cos\theta}{a\sin\theta}=\dfrac{y-b\sin\theta}{-b\cos\theta}=\dfrac{z-0}{+c}$ ...(i)

Equations of any line through the origin are $\dfrac{x-0}{l}=\dfrac{y-0}{m}=\dfrac{z-0}{n}$ ...(ii)

If the line (ii) is perpendicular to the generator (i), then

$$al\sin\theta-bm\cos\theta+cn=0 \quad \text{...(iii)}$$

Also if (i) and (ii) are coplanar, then

$$\begin{vmatrix} a\cos\theta & b\sin\theta & 0 \\ a\sin\theta & -b\cos\theta & +c \\ l & m & n \end{vmatrix}=0$$

...See chapter on Straight Lines

or $a\cos\theta\,(-nb\cos\theta-mc)-b\sin\theta\,(an\sin\theta-lc)=0$

or $-anb\,(\cos^2\theta+\sin^2\theta)-amc\cos\theta+lbc\sin\theta=0$

or $bcl\sin\theta-acn\cos\theta-abn=0$ ...(iv)

Solving (iii) and (iv) simultaneously for $\sin\theta$ and $\cos\theta$, we get

$$\frac{\sin\theta}{ab^2\,nm+ac^2\,mn}=\frac{\cos\theta}{bc^2\,nl+a^2b\,ln}=\frac{1}{-a^2c\,lm+b^2c\,lm}$$

or $\dfrac{\sin\theta}{amn\,(b^2+c^2)}=\dfrac{\cos\theta}{bnl\,(c^2+a^2)}=-\dfrac{1}{-clm\,(a^2-b^2)}$

$\Rightarrow \sin\theta=\dfrac{an\,(b^2+c^2)}{-cl\,(a^2-b^2)},\ \cos\theta=\dfrac{bn\,(c^2+a^2)}{-cm\,(a^2-b^2)}$

$\Rightarrow \left[\dfrac{an\,(b^2+c^2)}{-cl\,(a^2-b^2)}\right]^2+\left[\dfrac{bn\,(c^2+a^2)}{-cm\,(a^2-b^2)}\right]^2=1,\quad \because\ \cos^2\theta+\sin^2\theta=1$

$\Rightarrow \dfrac{a^2\,(b^2+c^2)^2}{l^2}+\dfrac{b^2\,(c^2+a^2)^2}{m^2}=\dfrac{c^2\,(a^2-b^2)^2}{n^2}$

This shows that the line (ii) lies on the curve

$$\frac{a^2\,(b^2+c^2)^2}{x^2}+\frac{b^2\,(c^2+a^2)^2}{y^2}=\frac{c^2\,(a^2-b^2)^2}{z^2} \qquad \text{Proved.}$$

Ex. 16. If the generator through a point of the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ meet the principal elliptic section in two points such that the eccentric angle of one is three times that of other, prove that P lies on the curve of intersection of the hyperboloid with the cylinder $y^2\,(z^2+c^2)=4b^2z^2$.

In case the difference of the eccentric angle is 2θ, then the locus of P is the curve of intersection of the hyperboloid with the cone

$$(x^2/a^2)+(y^2/b^2)=z^2/(c^2\sin^2\theta)$$

Sol. Let P be (x_1, y_1, z_1) and let the generator through P of the given hyperbolid meet the principal elliptic section in A and B, where A is $(a\cos\alpha, b\sin\alpha, 0)$ and B is $(a\cos\beta, b\sin\beta, 0)$

Here $\alpha = 3\beta$ (given)

$\therefore$ From results (iii) and (iv) of Ex. 11 Page 13, we have

$$\frac{x_1}{a} = \frac{\cos\{(\alpha+\beta)/2\}}{\cos\{(\alpha-\beta)/2\}} = \frac{\cos 2\beta}{\cos\beta} \quad \text{...(i)}$$

$$\frac{y_1}{b} = \frac{\sin\{(\alpha+\beta)/2\}}{\cos\{(\alpha-\beta)/2\}} = \frac{\sin 2\beta}{\cos\beta} = 2\sin\beta \quad \text{...(ii)}$$

$$\frac{z_1}{c} = \pm\tan\left(\frac{\alpha-\beta}{2}\right) = \pm\tan\beta \quad \text{...(iii)}$$

From (iii), $\dfrac{z_1}{c} = \pm\left(\dfrac{\sin\beta}{\cos\beta}\right) = \pm\dfrac{(y_1/2b)}{\cos\beta}$, from (ii)

or $\cos\beta = \pm\dfrac{cy_1}{2bz_1}$ and from (ii), $\sin\beta = \dfrac{y_1}{2b}$

Squaring and adding these, β is eliminated and we get

$$\frac{c^2y_1^2}{4b^2z_1^2} + \frac{y_1^2}{4b^2} = 1 \quad \text{or} \quad y_1^2(c^2+z_1^2) = 4b^2z_1^2$$

$\therefore P(x_1, y_1, z_1)$ lies on the curve of intersection of the given hyperboloid and the cylinder $\quad y^2(c^2+z^2) = 4b^2z^2$.

Again if $\alpha - \beta = 2\theta$, then from results (iii) and (iv) of Ex. 11 Page 13, we have

$$\frac{x_1}{a} = \frac{\cos\{(\alpha+\beta)/2\}}{\cos\{(\alpha-\beta)/2\}} = \frac{\cos\{\cos(\alpha+\beta)/2\}}{\cos\theta}$$

$$\frac{y_1}{b} = \frac{\sin\{(\alpha+\beta)/2\}}{\cos\{(\alpha-\beta)/2\}} = \frac{\sin\{(\alpha+\beta)/2\}}{\cos\theta},$$

and $\quad \dfrac{z_1}{c} = \pm\tan\dfrac{\alpha-\beta}{2} = \pm\tan\theta$

These give $\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = \dfrac{1}{\cos^2\theta} = \sec^2\theta = 1 + \tan^2\theta = \left(1 + \dfrac{z_1}{c}\right)^2$

or
$$(x_1^2/a^2) + (y_1^2/b^2) - (z_1^2/c^2) = 1 \quad \text{...(iv)}$$

Also $\quad \dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = \dfrac{1}{\cos^2\theta} = \dfrac{1}{\cos^2\theta}\left(\pm\dfrac{z_1}{c\tan\theta}\right)^2 \quad$ **(Note)**

or
$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{z_1^2}{c^2\sin^2\theta}$$

$\therefore$ From (iv) and (v) we conclude that the locus of $P(x_1, y_1, z_1)$ is the curve of intersection of the hyperbolid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ and $(x^2/a^2) + (y^2/b^2) = z^2/(c^2\sin^2\theta)$, which being a homogeneous equation of second degree in x, y and z represents a cone. Hence proved.

****Ex. 17. Prove that the shortest distance between generators of the system drawn at the ends of diameters of the principal elliptic section**

of the hyperbolloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ lie on the surfaces whose equa[illegible]

S[illegible] We know that [illegible] on the elliptic section be $P(a\cos\alpha, b\sin\alpha, 0)$, then the equations of the generator through it are

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z-0}{c} \qquad \text{...(i)}$$

The extremity of the diameter through this point P is $Q(-a\cos\alpha, -b\sin\alpha, 0)$ which is obtained by putting $\alpha+\pi$ for α in the coordinates of the point P and so the equations of the generator of the same system through Q is obtained by putting $\alpha+\pi$ for α in (i) and are

$$\frac{x+a\cos\alpha}{-a\sin\alpha}=\frac{y+b\sin\alpha}{b\cos\alpha}=\frac{z-0}{c} \qquad \text{...(ii)}$$

If l, m, n be the direction cosines of the S. D. then

$$la\sin\alpha-mb\cos\alpha+nc=0$$

and

$$-la\sin\alpha+mb\cos\alpha+nc=0$$

Solving these simultaneously for l, m, n, we get

$$\frac{l}{-2bc\cos\alpha}=\frac{m}{-2ac\sin\alpha}=\frac{n}{0} \quad \text{or} \quad \frac{l}{b\cos\alpha}=\frac{m}{a\sin\alpha}=\frac{n}{0} \qquad \text{...(iii)}$$

Also equation of plane containing the generator (i) and the line of S.D. is

$$\begin{vmatrix} x-a\cos\alpha & y-b\sin\alpha & z-0 \\ a\sin\alpha & -b\cos\alpha & c \\ b\cos\alpha & a\sin\alpha & 0 \end{vmatrix}=0$$

and the equation of the plane containing the generator (ii) and the line of S.D. is

$$\begin{vmatrix} x+a\cos\alpha & y+b\sin\alpha & z-0 \\ -a\sin\alpha & b\cos\alpha & c \\ b\cos\alpha & a\sin\alpha & 0 \end{vmatrix}=0$$

Expanding the above determinants w.r. to third column, we get

$$z(a^2\sin^2\alpha+b^2\cos^2\alpha)-c[a\sin\alpha(x-a\cos\alpha)$$
$$-b\cos\alpha(y-b\sin\alpha)]=0 \qquad \text{...(iv)}$$

and

$$-z(a^2\sin^2\alpha+b^2\cos^2\alpha)-c[a\sin\alpha(x+a\cos\alpha)$$
$$-b\cos\alpha(y+b\sin\alpha)]=0 \qquad \text{...(v)}$$

Eliminating α between (iv) and (v) we can find the locus of S.D. For this adding and subtracting (iv) and (v), we [illegible]

[illegible]

and [illegible] $\cos\alpha=$ [illegible] ...(vii)

From (vii), $z(a^2\tan^2\alpha+b^2)+c(a^2-b^2)\tan\alpha=0$

or $z[a^2(b^2y^2/a^2x^2)+b^2]+c(a^2-b^2)(by/ax)=0$, from (vi)

or [illegible]

or $\quad abz(x^2+y^2)+cxy(a^2-b^2)=0$ or $\dfrac{cxy}{x^2+y^2}=\dfrac{-abz}{(a^2-b^2)}$

In a similar manner if we consider the generator of the other system, we can find that the locus of S.D. is $\dfrac{cxy}{x^2+y^2}=\dfrac{abz}{a^2-b^2}$

Hence the required locus of S. D. is $\dfrac{cxy}{x^2+y^2}=\dfrac{\pm abz}{a^2-b^2}$.

****Ex. 18. Prove that the equations of the generating lines, through the point (θ, ϕ) on the hyperboloid of one sheet are**

$$\frac{\mathbf{x-a\cos\theta\sec\phi}}{\mathbf{a\sin(\theta\pm\phi)}}=\frac{\mathbf{y-b\sin\theta\sec\phi}}{\mathbf{-b\cos(\theta\pm\phi)}}=\frac{\mathbf{z-c\tan\phi}}{\mathbf{\pm c}}$$

(*Garhwal* 93, 92; *Gorakhpur* 97)

Sol. The point $P(\theta, \phi)$ on the hyperboloid of one sheet

$$(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1 \qquad \text{...(i)}$$

is $(a\cos\theta\sec\phi,\ b\sin\theta\sec\phi,\ c\tan\phi)$ **(Remember)**

Now we know that the generating lines through P are the lines of intersection of the hyperboloid (i) with the tangent plane at P whose equation is

$$(x/a)\cos\theta\sec\phi+(y/b)\sin\theta\sec\phi-(z/c)\tan\phi=1 \qquad \text{...(ii)}$$

This plane (ii) meets the plane $z=0$ in the line given by

$$(x/a)\cos\theta\sec\phi+(y/b)\sin\theta\sec\phi=1,\ z=0$$

or
$$(x/a)\cos\theta+(y/b)\sin\theta=\cos\phi,\ z=0 \qquad \text{...(iii)}$$

Also the section of the hyperboloid (i) by the plane $z=0$ is

$$(x^2/a^2)+(y^2/b^2)=1,\ z=0 \qquad \text{...(iv)}$$

Let the line (iii) meet the section (iv) of the hyperboloid (i) in the points $A(a\cos\alpha, b\sin\alpha, 0)$ and $B(a\cos\beta, b\sin\beta, 0)$. Then the equation of AB is

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right),\ z=0 \qquad \text{...(v)}$$

Comparing (iii) and (v), we get $\theta=\dfrac{\alpha+\beta}{2}$ and $\phi=\dfrac{\alpha-\beta}{2}$

Adding and substracting, these, we get

$$\alpha=\theta+\phi \quad \text{and} \quad \beta=\theta-\phi \qquad \text{...(vi)}$$

Hence the two generators through P are AP and BP.

Now the direction ratios of AP are

$a(\cos\alpha-\cos\theta\sec\phi),\ b(\sin\alpha-\sin\theta\sec\phi),\ c(0-\tan\phi)$ **(Note)**

or $\quad a\left[\dfrac{\cos(\theta+\phi)\cos\phi-\cos\theta}{\cos\phi}\right],\ b\left[\dfrac{\sin(\theta+\phi)\cos\phi-\sin\theta}{\cos\phi}\right],\ \dfrac{-c\sin\phi}{\cos\phi},$

from (vi)

or $\quad a\left[\dfrac{\cos\theta\cos^2\phi-\sin\theta\sin\phi\cos\phi-\cos\theta}{\cos\phi}\right],$

$$b\left[\frac{\sin\theta\cos^2\phi+\cos\theta\sin\phi\cos\phi-\sin\theta}{\cos\phi}\right],\ \frac{-c\sin\phi}{\cos\phi}$$

or $a\left[\ldots\right.$ [illegible] $\sin\phi$

[illegible] by $(\cos$ [illegible]

or $a(-\ldots$ [illegible]

or $a\sin(\theta+\phi)$ [illegible]

where $\theta+\phi$ [illegible] the generator AP, whose equations [illegible]

$$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta+\phi)}=\frac{y-b\sin\theta\sec\phi}{-b\cos(\theta+\phi)}=\frac{z-c\tan\phi}{c} \quad \text{...(vii)}$$

Similarly we can show that the equations of the generator BP are

$$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta-\phi)}=\frac{y-b\sin\theta\sec\phi}{[\text{illegible}]}=\frac{z-c\tan\phi}{c} \quad \text{...(viii)}$$

where $\theta-\phi$ is [illegible] on the generator.

Combining (vii) and (viii) we get the required result.

***Ex. 19. The normals to $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ at points of a generator meet the plane $z=0$ at points lying on a straight line and for different generators of the same system this line touches a fixed conic.**

Sol. [illegible] generator through a point $(\theta,$ [illegible] the given hyperboloid [illegible]

$$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta+\phi)}=\frac{y-b\sin\theta\sec\phi}{-b\cos(\theta+\phi)}=\frac{z-c\tan\phi}{c}, \quad \text{...}$$

where $\theta+\phi=$ constant $=\alpha$, say [See Ex. 18 above] ...(ii)

The equation to the tangent plane to the given hyperboloid at the point (θ,ϕ) *i.e.* $(a\cos\theta\sec\phi,\ b\sin\theta\sec\phi,\ c\tan\phi)$ is

$$\frac{x}{a}\cos\theta\sec\phi+\frac{y}{b}\sin\theta\sec\phi-\frac{z}{c}\tan\phi=1$$

or $\quad \frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-\frac{z}{c}\sin\phi=\cos\phi$

$\therefore$ The [illegible] to the given hyperboloid are [illegible]

$\therefore$ The [illegible]

$$\frac{x-a\cos\theta\sec\phi}{[\text{illegible}]}=\frac{y-b\sin\theta\sec\phi}{[\text{illegible}]}=\frac{z-c\tan\phi}{[\text{illegible}]}$$

This [illegible]

[illegible]

$y=\frac{\sin\theta}{\ldots}$ [illegible]

Putting $\theta = \alpha - \phi$, from (ii) in above, we get

$$x = \frac{a^2 + c^2}{a} \frac{\cos(\alpha - \phi)}{\cos\phi} = \frac{a^2 + c^2}{a} [\cos\alpha + \sin\alpha \tan\phi],$$

$$y = \frac{b^2 + c^2}{b} \frac{\sin(\alpha - \phi)}{\cos\phi} = \frac{b^2 + c^2}{b} [\sin\alpha - \cos\alpha \tan\phi]$$

and $z = 0$

or
$$\frac{ax}{a^2 + c^2} = \cos\alpha + \sin\alpha \tan\phi. \quad \text{...(iii)}$$

$$\frac{by}{b^2 + c^2} = \sin\alpha - \cos\alpha \tan\phi \quad \text{...(iv)}$$

and
$$z = 0 \quad \text{...(v)}$$

Multiplying (iii) by $\cos\alpha$ and (iv) by $\sin\alpha$ and adding, ϕ is eliminated and we get
$$\frac{ax\cos\alpha}{a^2 + c^2} + \frac{by\sin\alpha}{b^2 + c^2} = 1, \; z = 0 \quad \text{...(vi)}$$

which are the equations of the required line.

The envelope of this line, for different generators of the same system is obtained by differentiating (vi) with respect to α and then eliminating α.

Differentiating (vi) w.r. to α, we get $\dfrac{-ax\sin\alpha}{a^2 + c^2} + \dfrac{by\cos\alpha}{b^2 + c^2} = 0$ ...(vii)

Squaring and adding (vi) and (vii), we get

$$\frac{a^2x^2}{(a^2 + c^2)^2} + \frac{b^2y^2}{(b^2 + c^2)^2} = 1, \quad z = 0, \text{ which is a fixed conic.}$$

****Ex. 20. Find the locus of the poing of intersection of perpendiculars of a hyperboloid of one sheet.**

Sol. As in Ex. 18 Page 20 we can obtain the equations of the generators through any point (θ, ϕ) *i.e.* $(a\cos\theta\sec\phi, b\sin\theta\sec\phi, c\tan\phi)$ or (x_1, y_1, z_1) as

$$\frac{x - a\cos\theta\sec\phi}{a\sin(\theta \pm \phi)} = \frac{y - b\sin\theta\sec\phi}{-b\cos(\theta \pm \phi)} = \frac{z - c\tan\phi}{\pm c}$$

$\therefore$ If these generators are mutually perpendicular, then

$$'l_1l_2 + m_1m_2 + n_1n_2 = 0'$$

i.e. $a\sin(\theta + \phi) \,.\, a\sin(\theta - \phi) + \{-b\cos(\theta + \phi)\}\{-b\cos(\theta - \phi)\}$
$$+ (c)(-c) = 0$$

or $a^2(\sin^2\theta - \sin^2\phi) + b^2(\cos^2\theta - \sin^2\phi) - c^2 = 0$

or $a^2(\cos^2\phi - \cos^2\theta) + b^2(\cos^2\phi - \sin^2\theta) - c^2 = 0$,

$\because \sin^2\alpha = 1 - \cos^2\alpha$, and $\cos^2\alpha = 1 - \sin^2\alpha$

or $a^2\cos^2\phi - a^2\cos^2\theta + b^2\cos^2\phi - b^2\sin^2\theta - c^2 = 0$

or $a^2\cos^2\phi - (x_2\cos\phi)^2 + b^2\cos^2\phi - (y_1\cos\phi)^2 - c^2 = 0$,

$\because x_1 = a\cos\theta\sec\phi, \; y_1 = b\sin\theta\sec\phi$

or $(x_1^2 + y_1^2) \cos^2 \phi = a^2 \cos^2 \phi + b^2 \cos^2 \phi - c^2$

or $x_1^2 + y_1^2 = a^2 + b^2 - c^2 \sec^2 \phi = a^2 + b^2 - c^2 (1 + \tan^2 \phi)$

or $x_1^2 + y_1^2 = a^2 + b^2 - c^2 - (c \tan \phi)^2$, where $z_1 = c \tan \phi$

or $x_1^2 + y_1^2 + z_1^2 = a^2 + b^2 - c^2$

$\therefore$ The required locus of the point (x_1, y_1, z_1) is curve of intersection of the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ and the director sphere

$$x^2 + y^2 + z^2 = a^2 + b^2 - c^2.$$

****Ex. 21. Prove that the angle between the generators through any point P on the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ is given by**

$$\tan \alpha = 2abc/[p\,(a^2 + b^2 - c^2 - OP^2)],$$

where p is the perpendicular from the centre on the tangent plane at P.

Hence or otherwise find the locus of the point of intersection of perpendicular generators.

Sol. Let the point P be $(a \cos \theta \sec \phi, b \sin \theta \sec \phi, c \tan \phi)$

Then $OP^2 = a^2 \cos^2 \theta \sec^2 \phi + b^2 \sin^2 \theta \sec^2 \phi + c^2 \tan^2 \phi$ ...(i)

Also equation of the tangent plane to the given hyperboloid at P is

$$\frac{x}{a} \cos \theta \sec \phi + \frac{y}{b} \sin \theta \sec \phi - \frac{z}{c} \tan \phi = 1$$

$\therefore$ If p be the length of the perpendicular from origin O on the above tangent plane, then

$$p = \frac{1}{\sqrt{[\{(\cos \theta \sec\phi)/a\}^2 + \{(\sin \theta \sec \phi)/b\}^2 + \{(\tan \phi)/c\}^2]}}$$

or $$\frac{1}{p} = \sqrt{\left[\frac{\cos^2 \theta \sec^2 \phi}{a^2} + \frac{\sin^2 \theta \sec^2 \phi}{b^2} + \frac{\sin^2 \phi \sec^2 \phi}{c^2}\right]} \quad \textbf{(Note)}$$

or $$\frac{abc \cos \phi}{p} = \sqrt{[b^2c^2 \cos^2 \theta + c^2a^2 \sin^2 \theta + a^2b^2 \sin^2 \phi]} \quad ...(ii)$$

Also as in Ex. 18 Page 20 we can find that the direction ratios of the two generators through $P(\theta, \phi)$ are

$$a \sin (\theta + \phi), - b \cos (\theta + \phi), c$$

and $$a \sin (\theta - \phi), - b \cos (\theta - \phi), - c$$

$\therefore$ If α be the angle between these generators, then

$$\tan \alpha = \frac{\sqrt{[(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2]}}{(l_1l_2 + m_1m_2 + n_1n_2)}$$

$$= \frac{\begin{array}{l}[\{- ab \sin (\theta - \phi) \cos (\theta - \phi) + ab \cos (\theta + \phi) \sin (\theta - \phi)\}^2 \\ \quad + \{bc \cos (\theta + \phi) + bc \cos (\theta - \phi)\}^2 \\ \quad + \{ac \sin (\theta - \phi) + ac \sin (\theta + \phi)^2]^{1/2}\end{array}}{a^2 \sin (\theta + \phi) \sin (\theta - \phi) + b^2 \cos (\theta + \phi) \cos (\theta - \phi) - c^2} \quad ...(iii)$$

Numerator of R. H. S. of (iii)

$= \sqrt{[a^2b^2\{\sin(\theta+\phi)\cos(\theta-\phi)-\cos(\theta+\phi)\sin(\theta-\phi)\}^2}$

$\quad + b^2c^2\{\cos(\theta+\phi)+\cos(\theta+\phi)\}^2 + a^2c^2\{\sin(\theta+\phi)+\sin(\theta-\phi)\}^2$

$= \sqrt{[a^2b^2(\sin 2\phi)^2 + b^2c^2(2\cos\theta\cos\phi)^2 + a^2c^2(2\sin\theta\cos\phi)^2]}$

$= \sqrt{[4\cos^2\phi\{a^2b^2\sin^2\phi + b^2c^2\cos^2\theta + a^2c^2\sin^2\theta\}]}$

$= (2\cos\phi)[(abc\cos\phi)/p]$, from (ii)

$= 2(abc/p)\cos^2\phi$

And denominator of R. H. S. of (iii)

$= a^2(\sin^2\theta - \sin^2\phi) + b^2(\cos^2\theta - \sin^2\phi) - c^2$

$= a^2(\cos^2\phi - \cos^2\theta) + b^2(\cos^2\phi - \sin^2\theta) - c^2, \quad \because \quad \cos^2\alpha + \sin^2\alpha = 1$

$= (a^2+b^2)\cos^2\phi - (a^2\cos^2\theta + b^2\sin^2\theta) - c^2$

$= [a^2+b^2-(a^2\cos^2\theta\sec^2\phi + b^2\sin^2\theta\sec^2\phi) - c^2\sec^2\phi]\cos^2\phi$

$= [a^2+b^2-(a^2\cos^2\theta\sec^2\phi + b^2\sin^2\theta\sec^2\phi) - c^2(1+\tan^2\phi)]\cos^2\phi$

$= [(a^2+b^2-c^2)-(a^2\cos^2\theta\sec^2\phi + b^2\sin^2\theta\sec^2\phi + c^2\tan^2\phi)]\cos^2\phi$

$= [a^2+b^2-c^2-OP^2]\cos^2\phi$, from (i) ...(v)

$\therefore$ From (iii) with the help of (iv) and (v), we get

$$\tan\alpha = \frac{2(abc/p)\cos^2\phi}{(a^2+b^2-c^2-OP^2)\cos^2\phi} = \frac{2abc}{p(a^2+b^2-c^2-OP^2)}$$ Proved.

Again if $\alpha = 90°$ (*i.e.* the generators are perpendicular), then we have $\tan\alpha = \tan 90° = \infty$ and so $p(a^2+b^2-c^2-OP^2) = 0$

or $\quad OP^2 = a^2+b^2-c^2 \quad$ or $\quad x_1^2+y_1^2+z_1^2 = a^2+b^2-c^2$,

if P be (x_1, y_1, z_1)

$\therefore$ The required locus of $P(x_1, y_1, z_1)$ in this case is

$$x^2+y^2+z^2 = a^2+b^2-c^2.$$ **Ans.**

Ex. 22. If A and A′ are the extremities of the major axis of the principal elliptic section and any generator meets two generators of the same system through A and A′ in P and P′ respectively, then prove that

$$\mathbf{AP \cdot A'P' = b^2 + c^2}$$

Sol. We know that the points of intersection of a generator of λ-system with a generator of μ-system for the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ are given by

$$x = \frac{a(1+\lambda\mu)}{\lambda+\mu},\ y = \frac{b(\lambda-\mu)}{\lambda+\mu},\ z = \frac{c(1-\lambda\mu)}{\lambda+\mu} \quad ...(i)$$

[See Ex. 9 Page 11]

The extremities of the major axis of the principal elliptic section are $A(a, 0, 0)$ and $A'(-a, 0, 0)$

$\therefore$ At A and A' from (i) we have $\lambda - \mu = 0, 1 - \lambda\mu = 0$

$\Rightarrow \lambda = \mu \quad$ and $\quad 1 - \lambda^2 = 0 \quad$ or $\quad \lambda = \pm 1$

Now consider the generator through $A\ (a, 0, 0)$ corresponding to $\lambda = +1$ and then its point of intersection P with a generator of μ-system is obtained from (i) by putting $\lambda = +1$ and is

$$\left(a, \frac{b\,(1-\mu)}{1+\mu}, \frac{c\,(1-\mu)}{1+\mu}\right) \quad \text{or} \quad (a, bt, ct), \text{ where } t = \frac{1-\mu}{1+\mu}$$

$$\therefore\ AP^2 = (a-a)^2 + (bt-0)^2 + (ct-0)^2 = (b^2+c^2)\,t^2 \qquad \text{...(ii)}$$

Again the generator through $A'\ (-a, 0, 0)$ corresponding to $\lambda = -1$ meets the generator of μ-system at P', whose coordinates are obtained from (i) by putting $\lambda = -1$ and is

$$\left(-a, \frac{b\,(1+\mu)}{1-\mu}, \frac{c\,(1+\mu)}{-(1-\mu)}\right) \quad \text{or} \quad \left(-a, \frac{b}{t}, -\frac{c}{t}\right), \text{ where } t = \frac{1-\mu}{1+\mu}$$

$$\therefore\ (A'P')^2 = (-a-a)^2 + \left(\frac{b}{t}-0\right)^2 + \left(-\frac{c}{t}-0\right)^2 = \frac{b^2+c^2}{t^2} \qquad \text{...(iii)}$$

$\therefore$ From (ii) and (iii) we get $AP^2 \,.\, (A'P')^2 = (b^2+c^2)\,t^2 \,.\, [(b^2+c^2)/t^2]$

or $\qquad AP^2 \,.\, (A'P')^2 = (b^2+c^2)^2 \quad$ or $\quad AP \,.\, A'P' = b^2+c^2 \qquad$ Proved.

***Ex. 23. Show that the equations $y - \lambda z + \lambda + 1 = 0$, $(\lambda+1)\,x + y + \lambda = 0$ represent for different values of λ generators of one system of the hyperboloid $yz + zx + xy + 1 = 0$ and find the equations to the generators of the other system.**

Sol. Given $y + 1 = \lambda\,(z-1)$ and $x + 1 = -\dfrac{x+y}{\lambda}$ ...(i)

Multiplying, λ is eliminated and we get $(y+1)(x+1) = -(z-1)(x+y)$

or $\quad yx + y + x + 1 = -(zx + zy - x - y)$

or $\quad xy + yz + zx + 1 = 0$, which is the given surface.

Also generators of the other system [with the help of (i)] are

$$y + 1 = \mu\,(x+y),\ x + 1 = -(1/\mu)\,(z-1) \qquad \textbf{(Note)}$$

or $\qquad \mu x + \mu y - y - 1 = 0,\ \mu x + \mu + z - 1 = 0 \qquad$ **Ans.**

Ex. 24. Prove that any point on the lines $x + 1 = \mu y = -(\mu+1)\,z$ lies on the surface $yz + zx + xy + y + z = 0$ and find equations to determine the other system of lines which lies on the surface.

Sol. Given $\qquad x + 1 = \mu y = -(\mu+1)\,z$

which gives $\qquad x + 1 = \mu y$ and $\mu\,(y+z) = -z$

or $\qquad x + 1 = \mu y$ and $y + z = -z/\mu$ ...(i)

Eliminating μ (by multiplying), we get the surface

$$(x+1)(y+z) = -yz \quad \text{or} \quad xy + xz + y + z = -yz$$

or $$xy + yz + zx + y + z = 0$$

Also the other system of generators with the help of (i) can be written as

$$x + 1 = \lambda z,\ \ y + z = -y/\lambda \qquad \text{(Note)}$$

or $\qquad x + 1 = \lambda z = -(\lambda+1)\,y \qquad$ **Ans.**

****Ex. 25. Obtain the conditions for the line given by equations $l_1x + m_1y + n_1z + p_1 = 0$, $l_2x + m_2y + n_2z + p_2 = 0$ to be a generator of the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$.**

Sol. If the given line is a generator, then any plane through it must be a tangent plane to the given hyperboloid. **(Remember)**

Now the equation of any plane through the given line (generator) is

$(l_1 x + m_1 y + n_1 z + p_1) + k\,(l_2 x + m_2 y + n_2 z + p_2) = 0$

or $(l_1 + kl_2)\,x + (m_1 + km_2)\,y + (n_1 + kn_2)\,z + (p_1 + kp_2) = 0$

This plane, for all values of k, will be a tangent plane to the given hyperboloid if $a\,(l_1 + kl_2)^2 + b\,(m_1 + km_2)^2 + c\,(n_1 + kn_2)^2 = (p_1 + kp_2)^2$ **(Note)**

or $k^2\,(al_2^2 + bm_2^2 + cn_2^2 - p_2^2) + 2k\,(al_1 l_2 + bm_1 m_2 + cn_1 n_2 - p_1 p_2)$

$+ (al_1^2 + bm_1^2 + cn_1^2 - p_1^2) = 0$

As this relation holds goods for all values of k, so we have

$al_2^2 + bm_2^2 + cn_2^2 = p_2^2$, $al_1 l_2 + bm_1 m_2 + cn_1 n_2 = p_1 p_2$

and $al_1^2 + bm_1^2 + cn_1^2 = p_1^2$, which are the required conditions. **Ans.**

****§ 13.06. Generating Lines of a hyperbolic paraboloid.**

The equation of a hyperbolic paraboloid is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 2z$ ...(i)

Consider any line whose equations are

$$\frac{\mathbf{x}}{\mathbf{a}} - \frac{\mathbf{y}}{\mathbf{b}} = \lambda\,\mathbf{z}, \quad \frac{\mathbf{x}}{\mathbf{a}} + \frac{\mathbf{y}}{\mathbf{b}} = \frac{\mathbf{2}}{\lambda}, \qquad \text{...(ii)}$$

where λ is a constant.

If λ is eliminated from these, by multiplying them, we get $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 2z$, which is the equation (i) of the hyperbolic paraboloid.

$\therefore$ We conclude that all those points which satisfy the equation (ii) *i.e.* which lie on the line (ii) must also satisfy the equation (i) of the hyperbolic paraboloid.

Hence the line given by (ii) lies on the hyperbolic paraboloid given by (i).

As λ can take different values, so the equation (ii) represents an infinite number of straight lines all lying wholly on the hyperbolic paraboloid given by (i) *i.e.* these lines cover the whole of the surface (i) and are called the generators or the generating lines of the hyperbolic paraboloid given by (i).

In a similar manner we can show that the system of lines given by the equations

$$\frac{\mathbf{x}}{\mathbf{a}} - \frac{\mathbf{y}}{\mathbf{b}} = \frac{\mathbf{2}}{\mu}, \quad \frac{\mathbf{x}}{\mathbf{a}} + \frac{\mathbf{y}}{\mathbf{b}} = \mu\,\mathbf{z} \qquad \text{...(iii)}$$

lie wholly on the paraboloid (i) and so are its generators.

Then we find that as λ and μ vary, we get two families of straight lines such that every member of each system lies wholly on the paraboloid given by (i) and these two systems of lines given by (ii) and (iii) are known as the two systems of generating lines.

****§ 13.07. Properties of generating lines of hyperbolic paraboloid.**

Let the equation of the hyperbolic paraboloid be

$$(x^2/a^2)-(y^2/b^2)=2z \quad ...(i)$$

From § 13.06 above we know that the systems of generators of (i) are given by the equations

$(x/a)-(y/b)=\lambda z$ ...(ii), $(x/a)+(y/b)=2/\lambda$ ...(iii)

and $(x/a)-(y/b)=2/\mu$ ...(iv) $(x/a)+(y/b)=\mu z$ ...(v)

Prop I. *One generator of each system passes through every point of the hyperbolic paraboloid.*

Let $P(\alpha, \beta, \gamma)$ be any point on the hyperbolic paraboloid (i), then

$$(\alpha^2/a^2)-(\beta^2/b^2)=2\gamma \quad ...(vi)$$

Now the generators of λ-system of (i) given by (ii) and (iii) will pass through the point $P(\alpha, \beta, \gamma)$ if and only if λ has a value equal to each of the fractions $\dfrac{(\alpha/a)-(\beta/b)}{\gamma}, \dfrac{2}{(\alpha/a)+(\beta/b)}$ ...(vii)

obtained from (ii) and (iii).

or $\dfrac{(\alpha/a)-(\beta/b)}{\gamma}=\dfrac{2}{(\alpha/a)+(\beta/b)}$, equating above two values of λ

or $[(\alpha/a)-(\beta/b)][(\alpha/a)+(\beta/b)]=2\gamma$

or $(\alpha^2/a^2)-(\beta^2/b^2)=2\gamma$, which is true by virtue of (vi).

Thus if λ is chosen equal to the values given by either of the fractions (vii) the corresponding generator of the system of generators given by (ii) and (iii) will pass through the point $P(\alpha, \beta, \gamma)$.

In a similar manner we can show that if μ is equal to either of the fractions $\dfrac{2}{(\alpha/a)-(\beta/b)}$ or $\dfrac{(\alpha/a)+(\beta/b)}{\gamma}$ [obtained by evaluating μ from the equations given by (iv) and (v)], then a member of the μ-system of generators given by (iv) and (v) corresponding to either of equal values of μ will pass through the point $P(\alpha, \beta, \gamma)$.

Prop. II. *No two generators of the same system intersect.*

Consider two generators of the λ-system given by (ii) and (iii) corresponding to two distinct values λ_1, λ_2 of λ.

$(x/a)-(y/b)=\lambda_1 z$ ...(viii) $(x/a)+(y/b)=2/\lambda_1$ ...(ix)

and $(x/a)-(y/b)=\lambda_2 z$ ...(x) $(x/a)+(y/b)=2/\lambda_2$ ...(xi)

Substracting (x) from (viii) we get $(\lambda_1-\lambda_2)z=0$ or $z=0$, $\because \lambda_1 \neq \lambda_2$

Similarly subtracting (xi) from (ix) we get $(1/\lambda_1)-(1/\lambda_2)=0$

or $\lambda_2-\lambda_1=0$ or $\lambda_2=\lambda_1$ which contradicts $\lambda_1 \neq \lambda_2$.

Thus we find that the four equations giving two generators of the same system are inconsistent and so we conclude that the two generators of the same system do not intersect.

Prop. III. *Any two generators of the different systems intersect.*

Here we consider two generafros, one of each system, given by (ii), (iiii) and (iv), (v).

Solving (ii) and (iv), we get $\lambda z=2/\mu$ or $z=2/(\lambda\mu)$

Adding (iii) and (iv) we have $2(x/a)=(2/\lambda)+(2/\mu)$

or $x/a = (\lambda+\mu)/\lambda\mu$ or $x = a(\lambda+\mu)/\lambda\mu$

Subtracting (iv) from (iii) we get $2(y/b) = (2/\lambda) - (2/\mu)$

or $$y/b = (\mu-\lambda)/\mu\lambda \quad \text{or} \quad y = b(\mu-\lambda)/\lambda\mu$$

Hence the point of intersection of two generators, one of each system is

$$\left(\frac{a(\lambda+\mu)}{\lambda\mu}, \frac{b(\mu-\lambda)}{\lambda\mu}, \frac{2}{\lambda\mu}\right) \quad \text{...(xii)}$$

Here we observe that for all values of λ and μ the coordinates of this point satisfy the equation of the hyperbolic paraboloid $(x^2/a^2) - (y^2/b^2) = 2z$ and so the parametric equations of this hyperbolic paraboloid can be taken as

$$x = \frac{a(\lambda+\mu)}{\lambda\mu}, \quad y = \frac{b(\mu-\lambda)}{\lambda\mu}, \quad z = \frac{2}{\lambda\mu} \quad \text{...(xiii)}$$

Prop. IV. *The tangent planes at any point meet the hyperboloid in two generators through that point.*

Left as an exercise for the students.

Solved Examples on Generating lines of a hyperbolic paraboloid.

Ex. 1. Find the locus of the perpendicular from the vertex of the paraboloid $x^2/a^2 - (y^2/b^2) = 2z$ to the generators of the one system.

Sol. The equations for a generator of λ-system are given by

$$(x/a) - (y/b) = \lambda z \quad \text{and} \quad (x/a) + (y/b) = 2/\lambda$$

The symmetrical form of the above generator is

$$\frac{x-(a/\lambda)}{a\lambda} = \frac{y-(b/\lambda)}{-b\lambda} = \frac{z-0}{2} \quad \textbf{(Note)} \quad \text{...(i)}$$

Equations of any line through the origin are $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n} = r$, say ...(ii)

If (ii) is perpendicular to (i), then $a\lambda l - b\lambda m + 2n = 0$ ...(iii)

Again as (i) and (ii) intersect, so they are coplanar and the condition for the same is
$$\begin{vmatrix} 0-(a/\lambda) & 0-(b/\lambda) & 0-0 \\ a\lambda & -b\lambda & 2 \\ l & m & n \end{vmatrix} = 0$$

or $-\dfrac{a}{\lambda}(-bn\lambda - 2m) + \dfrac{b}{\lambda}(an\lambda - 2l) = 0$ or $abn + \dfrac{2ma}{\lambda} + abn - \dfrac{2bl}{\lambda} = 0$

or $2abn = 2(bl - ma)/\lambda$ or $\lambda = (bl - am)/(abn)$

Substituting this value of λ in (iii) we get $\dfrac{(al-bm)(bl-am)}{abn} + 2n = 0$

or $abl^2 - a^2lm - b^2lm + abm^2 + 2abn^2 = 0$

or $l^2 - \dfrac{alm}{b} - \dfrac{blm}{a} + m^2 + 2n^2 = 0$, dividing by ab

or $l^2 + m^2 + 2n^2 - \left(\dfrac{a}{b} + \dfrac{b}{a}\right) lm = 0$

or $l^2 + m^2 + 2n^2 - [(a^2+b^2)/ab]\, lm = 0$

Hence the locus of the line (ii) is given by

$$x^2 + y^2 + 2z^2 - [(a^2+b^2)/ab]\, xy = 0$$

Similarly if we consider the generator of μ-system [given by (iii) of § 13.06 Page 26] then the locus of the line (ii) is

$$x^2+y^2+2z^2+[(a^2+b^2)/ab]\,xy=0.$$

Ex. 2. Find the point of intersection of the generators of the paraboloid $xy=az$ and in case the generators include an angle α, their point of intersection lies on the curve of intersection of the paraboloid and $x^2+y^2-z^2\tan^2\alpha+a^2=0$.

Sol. Let the two generators of λ and μ-system be $x=\lambda z,\ y=a/\lambda$ ...(i)

and $$x=a/\mu,\ y=\mu z \quad \text{...(ii)}$$

Solving these we get $x=a/\mu,\ y=a/\lambda$ and $z=x/\lambda=a/\lambda\mu$

$\therefore$ The point of intersection of (i) and (ii) is $\left(\dfrac{a}{\mu}, \dfrac{a}{\lambda}, \dfrac{a}{\lambda\mu}\right)$ ...(iii)

The direction ratios of these two generators given by (i) and (ii) can be calculated as in Ex. 1 above to be $\lambda, 0, 1$ and $0, \mu, 1$.

$\therefore$ As α is the angle between these two generators, so

$$\cos\alpha=\frac{l_1l_2+m_1m_2+n_1n_2}{\sqrt{(\Sigma l_1^2)}\,.\,\sqrt{(\Sigma l_2^2)}}=\frac{\lambda\,.\,0+0\,.\,\mu+1\,.\,1}{\sqrt{(\lambda^2+0+1)}\,.\,\sqrt{(0+\mu^2+1)}}$$

or $$\cos\alpha=\frac{1}{\sqrt{(\lambda^2+1)}\,\sqrt{(\mu^2+1)}} \quad \text{or} \quad \sec^2\alpha=(\lambda^2+1)(\mu^2+1)$$

or $$1+\tan^2\alpha=\lambda^2\mu^2+\lambda^2+\mu^2+1 \quad \text{or} \quad \tan^2\alpha=\lambda^2\mu^2+\lambda^2+\mu^2 \quad \text{...(iv)}$$

If (x_1, y_1, z_1) be the point of intersection of the generators (i) and (ii), then from (iii) we have $x_1=a/\mu,\ y_1=a/\lambda,\ z_1=a/\lambda\mu$ ...(v)

$\therefore\ \mu=a/x_1,\ \lambda=a/y_1$ and $\lambda\mu=a/z_1$

Substituting these values in (iv), we get

$$\tan^2\alpha=(a^2/z_1^2)+(a^2/y_1^2)+(a^2/x_1^2)$$

or $$\tan^2\alpha=\frac{a^2}{(x_1y_1/a)^2}+\frac{a^2}{y_1^2}+\frac{a^2}{x_1^2}, \quad \because\ x_1y_1=az_1, \text{ as the point } (x_1, y_1, z_1)$$

lies on the given paraboloid

or $$x_1^2y_1^2\tan^2\alpha=a^4+a^2x_1^2+a^2y_1^2=a^2(a^2+x_1^2+y_1^2)$$

or $$(az_1)^2\tan^2\alpha=a^2(a^2+x_1^2+y_1^2), \because\ x_1y_1=az_1, \text{ as before}$$

or $$x_1^2+y_1^2-z_1^2\tan^2\alpha+a^2=0$$

$\therefore$ The required locus of the point of intersection (x_1, y_1, z_1) of the generators (i) and (ii) is $x^2+y^2-z^2\tan^2\alpha+a^2=0$. Proved.

****Ex. 3. Find the locus of the point of intersection of perpendicular generators of the hyperbolic paraboloid.**

Sol. Let the equation of the hyperbolic paraboloid be

$$(x^2/a^2)-(y^2/b^2)=2z$$

Its generators of λ and μ-systems are given by

$$(x/a)-(y/b)=\lambda z,\quad (x/a)+(y/b)=2/\lambda \quad \text{...(i)}$$

and $$(x/a)-(y/b)=2\mu,\quad (x/a)+(y/b)=\mu z \qquad \text{...(ii)}$$

Equations (i) can be re-written as

$$(x/a)-(y/b)-\lambda z=0,\quad (x/a)+(y/b)+0\,.\,z-(2/\lambda)=0$$

$\therefore$ If l_1, m_1, n_1 be the direction ratios of this generator, then

$$\frac{l_1}{a}-\frac{m_1}{b}-\lambda n_1=0,\ \frac{l_1}{a}+\frac{m_1}{b}+0\,.\,n_1=0$$

Solving these simultaneously, we get

$$\frac{l_1}{\lambda/b}=\frac{m_1}{-\lambda/a}=\frac{n_1}{2/ab}\quad\text{or}\quad\frac{l_1}{a\lambda}=\frac{m_1}{-b\lambda}=\frac{n_1}{2} \qquad \text{...(iii)}$$

Again equations (ii) can be rewritten as

$$(x/a)-(y/b)+0\,.\,z-(2/\mu)=0,\ (x/a)+(y/b)-\mu\,.\,z=0$$

$\therefore$ If l_2, m_2, n_2 be the direction ratios of this generator then

$$\frac{l_2}{a}-\frac{m_2}{b}+0\,.\,n_2=0,\ \frac{l_2}{a}+\frac{m_2}{b}-\mu\,.\,n_2=0$$

Solving these simultaneously, we get

$$\frac{l_2}{\mu/b}=\frac{m_2}{\mu/a}=\frac{n_2}{2/ab}\quad\text{or}\quad\frac{l_2}{a\mu}=\frac{m_2}{b\mu}=\frac{n_2}{2} \qquad \text{...(iv)}$$

As the two generators given by (i) and (ii) are perpendicular, so

$$l_1l_2+m_1m_2+n_1n_2=0\quad\text{or}\quad a\lambda.a\mu+(-b\lambda)\,b\mu+2.2=0$$

or $$a^2\lambda\mu-b^2\lambda\mu+4=0 \qquad \text{...(v)}$$

Also we know [See § 13.07 Prop. III Page 27] that the coordinates of the point of intersection (x_1, y_1, z_1) of generators (i) and (ii) are given by

$$x_1=a\,(\lambda+\mu)/\lambda\mu,\ y_1=b\,(\mu-\lambda)/\lambda\mu,\ z_1=2/\lambda\mu$$

$\therefore$ From (v) viz. $(a^2-b^2)\,\lambda\mu+4=0$ we get $(a^2-b^2)\,(2/z_1)+4=0$

$\therefore$ The locus of the point of intersection (x_1, y_1, z_1) of the generators (i) and (ii) is $(a^2-b^2)\,(2/z)+4=0$ or $a^2-b^2+2z=0$ **Ans.**

***Ex. 4. Prove that the projections of the generators of a hyperbolic paraboloid on any principal plane are tangent to the section by the plane.**

Sol. Let the hyperbolic paraboloid be $(x^2/a^2)-(y^2/b^2)=2z$.

Its generators of λ and μ-system are given by

$$(x/a)-(y/b)=\lambda z,\quad (x/a)+(y/b)=2/\lambda \qquad \text{...(i)}$$

and $$(x/a)-(y/b)=2/\mu,\quad (x/a)+(y/b)=\mu z \qquad \text{...(ii)}$$

The projection of the paraboloid and its generators on the principal plane $x=0$ are $y^2=-2b^2z$ (a parabola) and from (i) we get $(y/b)=-\lambda z$, $(y/b)=2/\lambda$

or $$2\,(y/b)=(2/\lambda)-\lambda z,\ \text{on adding}$$

or $$b\lambda^2z+2\lambda y-2b=0 \qquad \text{...(iii)}$$

And from (ii) we get $(y/b)=-2/\mu$, $y/b=\mu z$

or $$2y/b=\mu z-(2/\mu),\ \text{on adding}$$

or $$b\mu^2z-2\mu y-2b=0 \qquad \text{...(iv)}$$

Envelope of (iii) or (iv) where λ or μ are parameters is given by "$B^2-4AC=0$" *i.e.* $4y^2+8b^2z=0$ or $y^2=-2b^2z$ and $x=0$. *i.e.* the envelope of the projections of the generators (i) or (ii) on the plane $x=0$ is the projection of the paraboloid on the plane $x=0$.

Hence the projection of generators on the plane $x=0$ is the tangent to the projection of paraboloid on the plane $x=0$.

In a similar manner we can consider projections on the planes $y=0$ and $z=0$.

***Ex. 5. Find the generators of the paraboloid $(x^2/a^2)-(y^2/b^2)=4z$ drawn through the point $(\alpha, 0, \gamma)$ and prove that the angle between them is**

$$\cos^{-1}[(a-b+\gamma)/(a+b+\gamma).]$$

Sol. Equations of any line through the point $(\alpha, 0\ \gamma)$ are

$$\frac{x-\alpha}{l}=\frac{y-0}{m}=\frac{z-\gamma}{n}=r, \text{ say} \qquad \text{...(i)}$$

Any point on it is $(\alpha+lr, mr, \gamma+nr)$, so intersection of the line (i) with the given paraboloid is given by $\dfrac{(\alpha+lr)^2}{a}-\dfrac{(mr)^2}{b}=4(\gamma+nr)$

or
$$\left(\frac{l^2}{a}-\frac{m^2}{b}\right)r^2+2\left(\frac{l\alpha}{a}-2n\right)r+\left(\frac{\alpha^2}{a}-4\gamma\right)=0 \qquad \text{...(ii)}$$

If the line (i) is a generator of the given paraboloid, then the line (i) lies wholly on the paraboloid and the conditions for the same from (ii) are

$$\frac{l^2}{a^2}-\frac{m^2}{b}=0,\ \frac{l\alpha}{a}-2n=0,\ \frac{\alpha^2}{a}-4\gamma=0 \qquad \text{...(iii)}$$

$$\Rightarrow \frac{l}{\sqrt{a}}=\frac{m}{\pm\sqrt{b}}=\frac{2n\sqrt{a}}{\alpha} \Rightarrow \frac{l}{2a}=\frac{m}{\pm 2\sqrt{(ab)}}=\frac{n}{\alpha}$$

$\Rightarrow$ direction ratios of the two generators are

$$2a,\ 2\sqrt{(ab)},\ \alpha \text{ and } 2a,\ -2\sqrt{(ab)},\ \alpha$$

$\therefore$ If θ be the angle between these two generators, then

$$\cos\theta=\frac{l_1l_2+m_1m_2+n_1n_2}{\sqrt{(\Sigma l_1^2)}\,.\,\sqrt{(\Sigma l_2^2)}}=\frac{4a^2-4ab+\alpha^2}{\sqrt{(4a^2+4ab+\alpha^2)}\,.\,\sqrt{(4a^2+4ab+\alpha^2)}}$$

$$=\frac{4a^2-4ab+\alpha^2}{4a^2+4ab+\alpha^2}=\frac{4a^2-4ab+4a\gamma}{4a^2+4ab+4a\gamma}, \quad \because\ \alpha^2=4a\gamma \text{ from (iii)}$$

or $\quad \cos\theta=\dfrac{4a(a-b+\gamma)}{4a(a+b+\gamma)}$ or $\theta=\cos^{-1}\left(\dfrac{a-b+\gamma}{a+b+\gamma}\right)$ **Proved.**

Ex. 6. Prove that the angle θ between generating lines of the hyperbolic paraboloid $(x^2/a^2)-(y^2/b^2)=2z$ through (x, y, z) is given by

$$\tan\theta=ab\left(1+\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)^{1/2}\left(z+\frac{a^2-b^2}{2}\right)^{-1} \qquad (\textit{Gorakhpur } 95)$$

Sol. As in Ex. 3 Page 29 we can prove that the direction ratios of the generators, one each of λ and μ-systems, are $a\lambda, -b\lambda, 2$ and $a\mu, b\mu, 2$ and the co-ordinates of the point of intersection of these two generators [See § 13.07 Prop. III Page 27] are given by $x=\dfrac{a(\lambda+\mu)}{\lambda\mu},\ y=\dfrac{b(\mu-\lambda)}{\lambda\mu},\ z=\dfrac{2}{\lambda\mu}$ ...(i)

∴ If θ be the angle between these two generators, then

$$\tan\theta=\frac{\sqrt{\{\Sigma(m_1n_2-m_2n_1)^2\}}}{l_1l_2+m_1m_2+n_1n_2}$$

or
$$\tan\theta=\frac{\sqrt{[(-2b\lambda-2b\mu)^2+(2a\mu-2a\lambda)^2+(ab\lambda\mu+ab\lambda\mu)^2]}}{a\lambda\,.\,a\mu+(-b\lambda)(b\mu)+2.2}$$

$$=\frac{\sqrt{[4b^2(\lambda+\mu)^2+4a^2(\mu-\lambda)^2+4a^2b^2\lambda^2\mu^2]}}{(a^2-b^2)\lambda\mu+4}$$

$$=\frac{\sqrt{\left[4b^2\left(\dfrac{\lambda+\mu}{\lambda\mu}\right)^2+4a^2\left(\dfrac{\mu-\lambda}{\lambda\mu}\right)^2+4a^2b^2\right]}}{(a^2-b^2)+[4/\lambda\mu]},\ \text{dividing by } \lambda\mu$$

$$=\frac{\sqrt{[4b^2(x^2/a^2)+4a^2(y^2/b^2)+4a^2b^2]}}{(a^2-b^2)+2z},\ \text{from (i)}$$

$$=\frac{2ab\sqrt{[(x^2/a^4)+(y^2/b^4)+1]}}{2[z+\{(a^2-b^2)/2\}]}$$

$$=ab\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+1\right)^{1/2}\left(z+\frac{a^2-b^2}{2}\right)^{-1}$$ Hence proved.

***Ex. 7. Find the equation to the generating lines of the paraboloid $(x+y+z)(2x+y-z)=6z$, which pass through the point (1, 1, 1).**

Sol. The equations of the two generators of λ-μ system can be written as

$$x+y+z=6\lambda,\ 2x+y-z=z/\lambda \quad ...(i)$$

and
$$x+y+z=z/\mu,\ 2x+y-z=6\mu \quad ...(ii)$$

If these pass through the point (1, 1, 1) then

$$3=6\lambda \text{ and } 2=6\mu \Rightarrow \lambda=1/2,\ \mu=1/3$$ **(Note)**

∴ From (i) and (ii) the generators are given by

$$x+y+z=3,\ 2x+y-z=2z$$

and
$$x+y+z=3z,\ 2x+y-z=2$$

i.e.
$$x+y+z=3,\ 2x+y-3z=0 \quad ...(iii)$$

and
$$x+y-2z=0,\ 2x+y-z=2 \quad ...(iv)$$

As in Ex. 3 Page 29 we can find that the direction ratios of the generators given by (iii) and (iv) are 4, – 5, – 1 and 1, – 3, – 1 respectively and as they pass through the given point (1, 1, 1), so their equations are

$$\frac{x-1}{4}=\frac{y-1}{-5}=\frac{z-1}{-1} \text{ and } \frac{x-1}{1}=\frac{y-1}{-3}=\frac{z-1}{-1}$$ **Ans.**

***Ex. 8. Planes are drawn through the origin O and the generators through any point P of the paraboloid $x^2 - y^2 = az$. Prove that the angle between them is $\tan^{-1}(2r/a)$, where r is the length of OP.**

Sol. The equations of the two generators, of λ-μ systems can be taken as

$$x - y = \lambda z, \quad x + y = a/\lambda \qquad \text{...(i)}$$

and

$$x - y = a/\mu, \; x + y = \mu z \qquad \text{...(ii)}$$

Equation of any plane through the first generator is

$$(x - y - \lambda z) + k\,[x + y - (a/\lambda)] = 0 \qquad \text{...(iii)}$$

If its passes through the origin (0, 0, 0) then $k = 0$ and so from (iii), the equation of the plane through generator (i) is $\quad x - y - \lambda z = 0 \qquad$ ...(iv)

Similarly we can find that the equation of the plane through the generator (ii) is

$$x + y - \mu z = 0 \qquad \text{...(v)}$$

$\therefore$ If α be the angle between the planes (iv) and (v), then

$$\cos\alpha = \frac{1 \,.\, 1 - 1 \,.\, 1 + \lambda\mu}{\sqrt{(1^2 + 1^2 + \lambda^2)} \,.\, \sqrt{(1^2 + 1^2 + \mu^2)}} = \frac{\lambda\mu}{\sqrt{[(2 + \lambda^2)(2 + \mu^2)]}}$$

or

$$\sec^2\alpha = \frac{(2 + \lambda^2)(2 + \mu^2)}{\lambda^2\mu^2} = \frac{4}{\lambda^2\mu^2} + \frac{2}{\lambda^2} + \frac{2}{\mu^2} + 1$$

or

$$\tan^2\alpha = \frac{4}{\lambda^2\mu^2} + \frac{2}{\lambda^2} + \frac{2}{\mu^2} = \frac{4 + 2(\lambda^2 + \mu^2)}{\lambda^2\mu^2} \qquad \text{...(vi)}$$

Now the point of intersection P of these generators is

$$\left(\frac{a(\lambda + \mu)}{2\lambda\mu}, \frac{a(\mu - \lambda)}{2\lambda\mu}, \frac{a}{\lambda\mu}\right)$$ ...See § 13.07 Prop, III P. 27

$$\therefore r^2 = OP^2 = \frac{a^2(\lambda + \mu)^2}{4\lambda^2\mu^2} + \frac{a^2(\mu - \lambda)^2}{4\lambda^2\mu^2} + \frac{a^2}{\lambda^2\mu^2}$$

$$= \frac{a^2}{4\lambda^2\mu^2}[(\lambda + \mu)^2 + (\mu + \lambda)^2 + 4] = \frac{a^2}{4\lambda^2\mu^2}[2(\lambda^2 + \mu^2) + 4]$$

$\therefore$ From (vi), we have $r^2 = (a^2/4)\tan^2\alpha \;$ or $\; \tan^2\alpha = 4r^2/a^2$

or $\quad \tan\alpha = 2r/a \quad$ or $\quad \alpha = \tan^{-1}(2r/a) \qquad$ Proved.

****Ex. 9. Show that the equations to the generators through the point (r, θ) on the hyperbolic paraboloid $(x^2/a^2) - (y^2/b^2) = 2z$ are**

$$\mathbf{\frac{x - ar\cos\theta}{a} = \frac{y - br\sin\theta}{\pm b} = \frac{z - (1/2)\,r^2\cos 2\theta}{r(\cos\theta \mp \sin\theta)}}$$

Sol. The point (r, θ) on the given paraboloid can be taken as

$$x = ar\cos\theta, \; y = br\sin\theta, \; z = (1/2)\,r^2\cos 2\theta \qquad \textbf{(Remember)}$$

The equations of any generator through this point are

$$\frac{x - ar\cos\theta}{l} = \frac{y - br\sin\theta}{m} = \frac{z - (1/2)\,r^2\cos 2\theta}{n} = \rho, \text{ say} \qquad \text{...(i)}$$

$\therefore$ Its intersection with the given paraboloid is given by

$$\frac{1}{a^2}(l\rho + ar\cos\theta)^2 - \frac{1}{b^2}(m\rho + br\sin\theta)^2 = 2\left[n\rho + \frac{1}{2}r^2\cos 2\theta\right]$$

or
$$\left(\frac{l^2}{a^2} - \frac{m^2}{b^2}\right)\rho^2 + 2\left(\frac{lr\cos\theta}{a} - \frac{mr\sin\theta}{b} - n\right)\rho$$
$$+ r^2(\cos^2\theta - \sin^2\theta - \cos 2\theta) = 0$$

or
$$\left(\frac{l^2}{a^2} - \frac{m^2}{b^2}\right)\rho^2 + 2\left(\frac{lr\cos\theta}{a} - \frac{mr\sin\theta}{b} - n\right)\rho = 0$$

If the line (i) is a generator, it lies wholly on the paraboloid and the conditions for the same are $\dfrac{l^2}{a^2} - \dfrac{m^2}{b^2} = 0,\ \dfrac{lr\cos\theta}{a} - \dfrac{mr\sin\theta}{b} - n = 0$ **(Note)**

$$\therefore \frac{l^2}{a^2} = \frac{m^2}{b^2} \Rightarrow \frac{l}{a} = \frac{m}{\pm b} = k \text{ (say)} \quad \Rightarrow l = ak,\ m = \pm bk$$

Also from $\dfrac{lr\cos\theta}{a} - \dfrac{mr\sin\theta}{b} - n = 0$ we have

$$n = r(k\cos\theta \mp k\sin\theta) \quad \text{or} \quad n = kr(\cos\theta \mp \sin\theta)$$

$\therefore$ From (i), the required equations of the generators are

$$\frac{x - ar\cos\theta}{ak} = \frac{y - br\sin\theta}{\pm bk} = \frac{z - (1/2)r^2\cos 2\theta}{kr(\cos\theta \mp \sin\theta)}$$

or
$$\frac{x - ar\cos\theta}{a} = \frac{y - br\sin\theta}{\pm b} = \frac{z - (1/2)r^2\cos 2\theta}{r(\cos\theta \mp \sin\theta)}$$

Ex. 10. Show that the polar lines with respect to the sphere $x^2 + y^2 + z^2 = a^2$ of generators of the quadric $x^2 - y^2 = 2az$ all lie on the quadratic $x^2 - y^2 = -2az$.

Sol. Any generator of λ-system of the paraboloid $x^2 - y^2 = 2az$ is

$$x - y = \lambda z,\ x + y = 2a/\lambda$$

or
$$x - y - \lambda z = 0,\ x + y + 0.z - 2(a/\lambda) = 0$$

Let the direction ratios of this generator be l, m, n, then

$$l - m - \lambda n = 0 \text{ and } l + m + 0.n = 0$$

Solving these simultaneously, we get

$$\frac{l}{0 + \lambda} = \frac{m}{-\lambda - 0} = \frac{n}{1 + 1} \quad \text{or} \quad \frac{l}{1} = \frac{m}{-1} = \frac{n}{2/\lambda}$$

$\therefore$ The direction ratios of this generator are $1, -1$ and $2/\lambda$.

Again any point on this generator is $(a/\lambda, a/\lambda, 0)$

$\therefore$ The equations of above generator are

$$\frac{x - (a/\lambda)}{1} = \frac{y - (a/\lambda)}{-1} = \frac{z - 0}{2/\lambda} = r, \text{ (say)}$$

and so any point on it is $\left(r+\frac{a}{\lambda}, -r+\frac{a}{\lambda}, \frac{2r}{\lambda}\right)$

The polar plane of this point with respect to the given sphere $x^2+y^2+z^2=a^2$ is

$$x\left(r+\frac{a}{\lambda}\right)+y\left(-r+\frac{a}{\lambda}\right)+z\left(\frac{2r}{\lambda}\right)=a^2 \quad \text{or} \quad r\left(x-y+\frac{2z}{\lambda}\right)+a\left(\frac{x}{\lambda}+\frac{y}{\lambda}-a\right)=0$$

$\therefore$ The polar line of λ-generator is given by the planes

$$x-y+2(z/\lambda)=0 \quad \text{and} \quad x+y-a\lambda=0 \qquad \textbf{(Note)}.$$

or

$$\frac{x-y}{-2z}=\frac{1}{\lambda} \quad \text{and} \quad \frac{x+y}{a}=\lambda$$

Eliminating λ, the required locus of the polar lines is

$$\left(\frac{x-y}{-2z}\right)\cdot\left(\frac{x+y}{a}\right)=1 \quad \text{or} \quad x^2-y^2=-2az. \qquad \text{Proved.}$$

***Ex. 11. Prove that the equations**

$$\mathbf{2x=ae^{2\phi},\ y=be^{\phi}\cosh\theta,\ z=c\,e^{\phi}\sinh\theta}$$

determine a hyperbolic paraboloid, and that $\theta+\phi$ is constant for points of given generator of one system and $\theta-\phi$ is constant for a given generator of the other system. (*Gorakhpur* 96)

Sol. From the given equations, we have

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=e^{2\phi}(\cosh^2\theta-\sinh^2\theta)=e^{2\phi}(1)=\frac{2x}{a}$$

$\therefore$ On eliminating θ and ϕ from the given equations, we get

$$(y^2/b^2)-(z^2/c^2)=2(x/a), \qquad \text{...(i)}$$

which is a hyperbolic paraboloid. Proved.

The generators of λ and μ systems of the above hyperboloid are given by

$$\frac{y}{b}-\frac{z}{c}=2\lambda,\ \frac{y}{b}+\frac{z}{c}=\frac{x}{a\lambda} \qquad \text{...(ii)}$$

and

$$\frac{y}{b}-\frac{z}{c}=\frac{x}{a\mu},\ \frac{y}{b}+\frac{z}{c}=2\mu \qquad \text{...(iii)}$$

Both of these generators pass through a given point.

Again from (ii) on substituting

$$x=\frac{1}{2}a\,e^{2\phi},\ y=b\,e^{\phi}\cosh\theta,\ z=c\,e^{\phi}\sinh\theta$$

we get $\quad e^{\phi}(\cosh\theta-\sinh\theta)=2\lambda,\ e^{\phi}(\cosh\theta+\sinh\theta)=(1/2\lambda)\,e^{2\phi}$

i.e. $\quad e^{\phi}\cdot e^{-\theta}=2\lambda,\ e^{\phi}\cdot e^{\theta}=e^{2\phi}/(2\lambda)$

i.e. $\quad e^{\phi-\theta}=2\lambda$, from both the equations.

$\therefore$ For a given generator of λ-system, λ being constant, we find that $\phi-\theta$ or $\theta-\phi$ is constant. Proved.

In a similar manner from (iii), we get

$e^{\phi}(\cosh\theta - \sinh\theta) = (1/2\mu)\, e^{2\phi},\ e^{\phi}(\cosh\theta + \sinh\theta) = 2\mu$

or $\quad e^{\phi}\, e^{-\theta} = e^{2\phi}/(2\mu)\ ,\ e^{\phi}\,.\,e^{\theta} = 2\mu$

or $\quad e^{\phi+\theta} = 2\mu$, from both the equations.

∴ As before for a given generator of μ-system, μ being constant, we find that $\theta + \phi$ is constant. Proved.

Exercises on Chapter XIII

Ex. 1. A variable generator meets two generators of the same system through the extremities B and B' of the minor axis of the principal elliptic section of the hyperboloid in P and P', prove that $BP\,.\,B'P' = b^2 + c^2$.

[Hint : See Ex. 22 Page 24]

*Ex. 2. Prove that the tangent planes to the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ which are parallel to the tangent planes to

$$\frac{b^2c^2x^2}{c^2 - b^2} + \frac{c^2a^2y^2}{c^2 - a^2} + \frac{a^2b^2z^2}{a^2 + b^2} = 0$$

meet the surface in perpendicular generators.

*Ex. 3. Prove that the equations $4x = a(1 + \cos 2\theta)$, $y = b\cosh\phi\cos\theta$, $z = c\sinh\phi\cos\theta$ determine a hyperbolic paraboloid and show that the angle between the generators through (θ, ϕ) is given by

$$\sec^{-1}\left[\frac{\{(b^2 + c^2)^2 + a^4\cos^4\theta + 2a^2(b^2 + c^2)\cos^2\theta\cosh 2\phi\}^{1/2}}{a^2\cos^2\theta + b^2 - c^2}\right]$$

CHAPTER XIV

CONFOCAL CONICOIDS

§ 14.01. Confocal conicoids.

Definition. The conicoids whose principal sections have the same foci are called confocal conicoids.

Equation of confocal conicoids.

The equation $$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1 \qquad \text{...(i)}$$

for all values of λ (which is called parameter of the confocal) represents a conicoid confocal with the conicoid $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \qquad \text{...(ii)}$$

because the sections of the conicoids (i) and (ii) by the coordinate planes $x=0$, $y=0$ and $z=0$ are confocal conics.

Suppose $a>b>c$ and let λ vary from $-\infty$ to $+\infty$.

When $\lambda<0$, the surface (i) is an ellipsoid. In this case as λ increases numerically *i.e.* as $\lambda \to -\infty$, the principal axes of the surface increase and their ratios tend to unity. From this we conclude that a sphere of infinite radius is a limiting case of the confocals.

When $\lambda>0$ and less than c^2, the surface (i) is an ellipsoid. But the ellipsoid becomes more and more flat at $\lambda \to c^2$ from the left *i.e.* through values less than c^2.

Thus the elliptic disc $$z=0,\ \frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1 \qquad \text{...(iii)}$$

is also a limiting case of the confocals.

When $\lambda>0$ and $c^2<\lambda<b^2$, the surface (i) is a hyperboloid of one sheet as $c^2-\lambda<0$. As $\lambda \to c^2$ from the right *i.e.* through values greater than c^2, the hyperboloid tends to coincide with that of the xy-plane which is exterior to the ellipse (iii).

As $\lambda \to b^2$ from the left *i.e.* through values less than b^2, the hyperboloid tends to coincide with that part of the zx-plane which lies betwen the two branches of the hyperbola $$y=0,\ \frac{x^2}{a^2-b^2}-\frac{z^2}{b^2-c^2}=1 \qquad \text{...(iv)}$$

and contains the centre.

When $\lambda>0$ and $b^2<\lambda<a^2$, the surface (i) is a hyperboloid of two sheets as $c^2-\lambda<0$ and $b^2-\lambda<0$. As $\lambda \to b^2$ from the right *i.e.* through values greater than b^2, the hyperboloid tends to coincide with that part of the zx-plane which does not contain the centre and is bounded by the two branches of the hyperbola given by (iv).

When $\lambda > 0$ and $\lambda \to a^2$, the surface (i) tends to the imaginary ellipse

$$x = 0, \frac{y^2}{a^2 - b^2} + \frac{z^2}{a^2 - c^2} = -1 \qquad \text{...(v)}$$

When $\lambda > 0$ and $\lambda > a^2$, the surface (i) is imaginary.

The conics (iii) and (iv) which are the boundaries of the limiting cases of the confocal conicoids are knwon as **focal conics;** the conic (iii) is called the **focal ellipse** and the conic (iv) is called the **focal hyperbola.**

The general equation of a system of confocal conicoids is also written as

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1$$

In a similar way the sections of the paraboloids

$$\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 2z - \lambda, \; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z \qquad \text{...(vi)}$$

by the coordinate planes $x = 0$, $y = 0$ consist of confocal parabolas and therefore the paraboloids given by (vi) represent a system of confocal paraboloids.

The general equation of a system of confocal paraboloids is also written as

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 2z + \lambda.$$

Also if in the equation $\quad \dfrac{x^2}{\alpha^2} + \dfrac{y^2}{\alpha^2 - b^2} + \dfrac{z^2}{\alpha^2 - c^2} = 1 \qquad$ ...(vii)

arbitrary values are assigned to α, kepping b and c as constants, we get the equation to a system of confocal conicoids.

If the form of the equation (iii) is chosen to represent a confocal conicoid, then α is known as the **primary semi-axis.**

(See Ex. 7 Page 8 Ch. XIV also)

****§ 14.02. Three confocals through a given point.**

To show that through any point there pass three conicoids confocal with a given ellipsoid, an ellipsoid, a hyperboloid of one sheet and a hyperboloid of two sheets. (*Gorakhpur* 97)

Let the given conicoid be $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$, $a > b > c$ and let the equation of a conicoid confocal to it be

$$\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} + \frac{z^2}{c^2 - \lambda} = 1$$

If this confocal passes through the given point $P(\alpha, \beta, \gamma)$ then we have

$$\frac{\alpha^2}{a^2 - \lambda} + \frac{\beta^2}{b^2 - \lambda} + \frac{\gamma^2}{c^2 - \lambda} = 1$$

or $\quad f(\lambda) \equiv \alpha^2 (b^2 - \lambda)(c^2 - \lambda) + \beta^2 (c^2 - \lambda)(a^2 - \lambda) + \gamma^2 (a^2 - \lambda)(b^2 - \lambda)$

$$-(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)=0$$

This cubic equation in λ, gives the parameters of three confocals which pass through the point $P(\alpha, \beta, \gamma)$.

If we substitute for λ, the values $+\infty, a^2, b^2, c^2, -\infty$

we find λ	=	∞,	a^2,	b^2,	c^2,	$-\infty$
$f(\lambda)$	=	+,	+,	−,	+,	−

This shows that the equation $f(\lambda)=0$ has three real roots $\lambda_1, \lambda_2, \lambda_3$ such that $a^2>\lambda_1>b^2>\lambda_2>c^2>\lambda_3$

When $\lambda=\lambda_3$, the confocal is an ellipsoid ; when $\lambda=\lambda_2$, it is a hyperboloid of one sheet and when $\lambda=\lambda_1$, it is a hyperboloid of two sheets.

Hence through a given point, three conicoids confocal with a given central conicoid pass — an ellipsoid, a hyperboloid of one sheet and a hyperboloid of two sheets.

§ 14.03. Elliptic Coordinates.

From § 14.02 above, as $\lambda_1, \lambda_2, \lambda_3$ are the roots of the equation $f(\lambda)=0$, so $f(\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$

$$\therefore \frac{\alpha^2}{a^2-\lambda}+\frac{\beta^2}{b^2-\lambda}+\frac{\gamma^2}{c^2-\lambda}-1=\frac{f(\lambda)}{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}$$
$$=\frac{(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)}{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}$$

$\therefore$ By the rule of partial fractions, we find that

$$\alpha^2=\frac{(a^2-\lambda_1)(a^2-\lambda_2)(a^2-\lambda_3)}{(b^2-a^2)(c^2-a^2)},\ \beta^2=\frac{(b^2-\lambda_1)(b^2-\lambda_2)(b^2-\lambda_3)}{(a^2-b^2)(c^2-b^2)}$$

and $$\gamma^2=\frac{(c^2-\lambda_1)(c^2-\lambda_2)(c^2-\lambda_3)}{(a^2-c^2)(b^2-c^2)}$$

These give us the coordinates α, β, γ of a point P in terms of the parameters of the confocals of a given conicoid that pass through the point P. If the parameters are given and the octant in which P lies is known, the position of P is uniquely determined.

Consequently $\lambda_1, \lambda_2, \lambda_3$ are called the elliptic coordinates of P with reference to the conicoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$.

Solved Examples on § 14.01 – § 14.03.

***Ex. 1. Three paraboloids confocal with a given paraboloid pass through a given point — two elliptic and one hyperbolic.**

Sol. Let the paraboloid be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2cz$

Then the paraboloid confocal to it is $\dfrac{x^2}{a^2-\lambda}+\dfrac{y^2}{b^2-\lambda}=2cz-\lambda$

If it passes through the point $P(\alpha, \beta, \gamma)$, then we have

$$\frac{\alpha^2}{a^2-\lambda}+\frac{\beta^2}{b^2-\lambda}=2c\gamma-\lambda \qquad \text{...(i)}$$

or $\quad f(\lambda)\equiv\alpha^2(b^2-\lambda)+\beta^2(a^2-\lambda)-(2c\gamma-\lambda)(a^2-\lambda)(b^2-\lambda)=0$

This being a cubic in λ gives the parameters of three confocals which pass through (α, β, γ).

If we substitute for λ the values $+\infty, a^2, b^2, 2c\gamma, -\infty$

we find that $\qquad \lambda=\infty,\ a^2,\ b^2,\ 2c\gamma,\ -\infty$

$\qquad\qquad f(\lambda)=+,\ -,\ +,\ +,\ -$

The equation $f(\lambda)=0$ has three real roots $\lambda_1, \lambda_2, \lambda_3$, such that

$$\lambda_1>a^2>\lambda_2>b^2>\lambda_3.$$

When $\lambda=\lambda_3$, the confocal paraboloid is elliptic

When $\lambda=\lambda_2$, the confocal paraboloid is hyperbolic

When $\lambda=\lambda_1$, the confocal paraboloid is elliptic

****Ex. 2. Prove that the equation of the confocal through the point of the focal ellipse whose eccentric angle is α is**

$$\frac{\mathbf{x}^2}{(\mathbf{a}^2-\mathbf{b}^2)\cos^2\alpha}-\frac{\mathbf{y}^2}{(\mathbf{a}^2-\mathbf{b}^2)\sin^2\alpha}+\frac{\mathbf{z}^2}{\mathbf{c}^2-\mathbf{a}^2\sin^2\alpha-\mathbf{b}^2\cos^2\alpha}=1$$

(*Avadh* 92, 91, *Gorakhpur* 95)

Sol. We know that the focal ellipse is given by

$$z=0,\ \frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1 \qquad \text{...See § 14.01 (iii) Page 1}$$

or $\quad z=0,\ \dfrac{x^2}{A^2}+\dfrac{y^2}{B^2}=1$, where $A^2=a^2-c^2,\ B^2=b^2-c^2$

$\therefore$ On this ellipse the point whose eccentric angle is α is

$(A\cos\alpha, B\sin\alpha, 0)$ *i.e.* $[\sqrt{(a^2-c^2)}\cos\alpha, \sqrt{(b^2-c^2)}\sin\alpha, 0]$

Let $$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1 \qquad \text{...(i)}$$

be the confocal through the above point, then

$$\frac{(a^2-c^2)\cos^2\alpha}{a^2-\lambda}+\frac{(b^2-c^2)\sin^2\alpha}{b^2-\lambda}+0=1$$

or $\quad (a^2-c^2)(b^2-\lambda)\cos^2\alpha+(b^2-c^2)(a^2-\lambda)\sin^2\alpha=(a^2-\lambda)(b^2-\lambda)$

or $\quad \lambda^2+\lambda[(a^2-c^2)\cos^2\alpha+(b^2-c^2)\sin^2\alpha-(a^2+b^2)]$

$$+[a^2b^2-b^2(a^2-c^2)\cos^2\alpha-a^2(b^2-c^2)\sin^2\alpha]=0$$

or $\quad \lambda^2 - \lambda\,[a^2 \sin^2 \alpha + b^2 \cos^2 \alpha + c^2] + [a^2b^2 - a^2b^2 (\cos^2 \alpha + \sin^2 \alpha) + c^2 (b^2 \cos^2 \alpha + a^2 \sin^2 \alpha)] = 0$

or $\quad \lambda^2 - \lambda\,(a^2 \sin^2 \alpha + b^2 \cos^2 \alpha + c^2) + c^2 (b^2 \cos^2 \alpha + a^2 \sin^2 \alpha) = 0$

or $\quad \lambda^2 - \lambda\,(a^2 \sin^2 \alpha + b^2 \cos^2 \alpha) - \lambda c^2 + c^2 (a^2 \sin^2 \alpha + b^2 \cos^2 \alpha) = 0$

or $\quad \lambda\,[\lambda - (a^2 \sin^2 \alpha + b^2 \cos^2 \alpha)] - c^2 [\lambda - (a^2 \sin^2 \alpha + b^2 \cos^2 \alpha)] = 0$

or $\quad \lambda = a^2 \sin^2 \alpha + b^2 \cos^2 \alpha$, neglecting $\lambda = c^2$

$\therefore\ a^2 - \lambda = a^2 - a^2 \sin^2 \alpha - b^2 \cos^2 \alpha = (a^2 - b^2) \cos^2 \alpha$

$b^2 - \lambda = b^2 - a^2 \sin^2 \alpha - b^2 \cos^2 \alpha = -(a^2 - b^2) \sin^2 \alpha$

and $\quad c^2 - \lambda = c^2 - a^2 \sin^2 \alpha - b^2 \cos^2 \alpha$

$\therefore$ From (i), the required equation of the confocal is

$$\frac{x^2}{(a^2 - b^2) \cos^2 \alpha} - \frac{y^2}{(a^2 - b^2) \sin^2 \alpha} + \frac{z^2}{c^2 - a^2 \sin^2 \alpha - b^2 \cos^2 \alpha}$$

Hence proved.

****Ex. 3. Prove that the equation to the conicoid confocal with $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$ which has a system of circular sections parallel to the plane $x = y$ is**

$$\frac{x^2}{(c^2 - a^2)(a^2 - b^2)} + \frac{y^2}{(b^2 - c^2)(a^2 - b^2)} - \frac{z^2}{2(b^2 - c^2)(c^2 - a^2)} = \frac{1}{2c^2 - a^2 - b^2}$$

(*Avadh* 95; *Garhwal* 96, 93, 92)

Sol. Let the equation of the confocal of the given ellipsoid be

$$\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} + \frac{z^2}{c^2 - \lambda} = 1 \qquad ...(i)$$

Now we are to find the condition that the section of (i) by the plane $x = y$ or $x - y = 0$ is a circle

The given equation of the confocal can be written in the form

$$\left[\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} + \frac{z^2}{c^2 - \lambda} - \mu (x^2 + y^2 + z^2)\right] + \mu (x^2 + y^2 + z^2) - 1 = 0$$

$\therefore$ The equation $\dfrac{x^2}{a^2 - \lambda} + \dfrac{y^2}{b^2 - \lambda} + \dfrac{z^2}{c^2 - \lambda} - \mu (x^2 + y^2 + z^2) = 0$ is to represent two planes one of which is $x - y = 0$

$$\therefore \frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} + \frac{z^2}{c^2 - \lambda} - \mu (x^2 + y^2 + z^2)$$

$$= (x - y)\left[\left(\frac{1}{a^2 - \lambda} - \mu\right) x - \left(\frac{1}{b^2 - \lambda} - \mu\right) y + vz\right]$$

whence comparing the coefficients of z^2, xy, xz and yz on both sides, we get

$$\frac{1}{c^2-\lambda}-\mu=0 \quad \text{or} \quad \mu=\frac{1}{c^2-\lambda} \qquad \text{...(ii)}$$

$$0=\left(\frac{1}{a^2-\lambda}-\mu\right)+\left(\frac{1}{b^2-\lambda}-\mu\right) \quad \text{or} \quad 2\mu=\frac{1}{a^2-\lambda}+\frac{1}{b^2-\lambda}. \qquad \text{...(iii)}$$

and $0=\nu$

$\therefore$ From (ii) and (iii) we get $\dfrac{2}{c^2-\lambda}=\dfrac{1}{a^2-\lambda}+\dfrac{1}{b^2-\lambda}$

or $\quad 2(a^2-\lambda)(b^2-\lambda)=(c^2-\lambda)[(b^2-\lambda)+(a^2-\lambda)]$

or $\quad 2(a^2b^2-a^2\lambda-b^2\lambda+\lambda^2)=c^2(a^2+b^2)-2c^2\lambda-(a^2+b^2)\lambda+2\lambda^2$

or $\quad \lambda(2c^2-a^2-b^2)=c^2(a^2+b^2)-2a^2b^2$

or $\quad \lambda=[c^2(a^2+b^2)-2a^2b^2]/(2c^2-a^2-b^2)$

$$\therefore a^2-\lambda=a^2-\frac{c^2(a^2+b^2)-2a^2b^2}{2c^2-a^2-b^2}=\frac{(c^2-a^2)(a^2-b^2)}{2c^2-a^2-b^2}$$

$$b^2-\lambda=b^2-\frac{c^2(a^2+b^2)-2a^2b^2}{2c^2-a^2-b^2}=\frac{(b^2-c^2)(a^2-b^2)}{2c^2-a^2-b^2}$$

$$c^2-\lambda=c^2-\frac{c^2(a^2+b^2)-2a^2b^2}{2c^2-a^2-b^2}=\frac{-2(b^2-c^2)(c^2-a^2)}{2c^2-a^2-b^2}$$

Subtituting these values in (i), the required equation of the confocal is

$$\frac{x^2}{(c^2-a^2)(a^2-b^2)}+\frac{y^2}{(b^2-c^2)(a^2-b^2)}-\frac{z^2}{2(b^2-c^2)(c^2-a^2)}=\frac{1}{2c^2-a^2-b^2}.$$

Hence proved.

***Ex. 4. If a_1, b_1, c_1 ; a_2, b_2, c_2 ; a_3, b_3, c_3 are the axes of the confocals to a given ellipsoid which pass through the point (α, β, γ), prove that**

$$\alpha^2+\beta^2+\gamma^2=a_1^2+b_2^2+c_3^2$$

Sol. Let the given ellipsoid be $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ and let the parameters of the confocals through (α, β, γ) be $\lambda_1, \lambda_2, \lambda_3$.

Then $a_1^2=a^2-\lambda_1$, $b_1^2=b^2-\lambda_1$, $c_1^2=c^2-\lambda_1$; $a_2^2=a^2-\lambda_2$, etc.

Also we know from § 14.03 Page 3 that

$$\alpha^2=\frac{(a^2-\lambda_1)(a^2-\lambda_2)(a^2-\lambda_3)}{(b^2-a^2)(c^2-a^2)},\quad \beta^2=\frac{(b^2-\lambda_1)(b^2-\lambda_2)(b^2-\lambda_3)}{(c^2-b^2)(a^2-b^2)}$$

and $\quad \gamma^2=\dfrac{(c^2-\lambda_1)(c^2-\lambda_2)(c^2-\lambda_3)}{(a^2-c^2)(b^2-c^2)}$

$$\therefore \alpha^2+\beta^2+\gamma^2=-\frac{\Sigma(b^2-c^2)(a^2-\lambda_1)(a^2-\lambda_2)(a^2-\lambda_3)}{(b^2-c^2)(c^2-a^2)(a^2-b^2)} \qquad \text{...(i)}$$

The numerator of the right hand side of (i)

$= \Sigma (b^2 - c^2)(a^2 - \lambda_1)[a^4 - a^2\lambda_2 - a^2\lambda_3 + \lambda_2\lambda_3]$

$= \Sigma (b^2 - c^2)[a^6 - a^4\lambda_2 - a^4\lambda_3 + a^2\lambda_2\lambda_3 - a^4\lambda_1 + a^2\lambda_1\lambda_2 + a^2\lambda_1\lambda_3 - \lambda_1\lambda_2\lambda_3]$

$= \Sigma (b^2 - c^2)[a^6 - a^4(\lambda_1 + \lambda_2 + \lambda_3) + a^2(\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2) - \lambda_1\lambda_2\lambda_3]$

$= \Sigma (b^2 - c^2)[a^6 - a^4(\lambda_1 + \lambda_2 + \lambda_3)]$, $\because \Sigma a^2(b^2 - c^2) = 0, \Sigma(b^2 - c^2) = 0$

Also $\Sigma a^6(b^2 - c^2) = a^6(b^2 - c^2) + b^6(c^2 - a^2) + c^6(a^2 - b^2)$

$= -(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)(a^2 + b^2 + c^2)$, on factorising

and $\Sigma a^4(b^2 - c^2) = a^4(b^2 - c^2) + b^4(c^2 - a^2) + c^4(a^2 - b^2)$

$= -(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)$, on factorising

$\therefore$ The numerator of the right hand side of (i)

$= -(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)[(a^2 + b^2 + c^2) - (\lambda_1 + \lambda_2 + \lambda_3)]$

$\therefore$ From (i), $\alpha^2 + \beta^2 + \gamma^2 = (a^2 + b^2 + c^2) - (\lambda_1 + \lambda_2 + \lambda_3)$

$= (a^2 - \lambda_1) + (b^2 - \lambda_2) + (c^2 - \lambda_3) = a_1^2 + b_2^2 + c_3^2$

Hence proved.

Ex. 5. What loci are represented by the equations in elliptic coordinates,

(i) $\lambda_1 + \lambda_2 + \lambda_3 = \text{constant}$, (ii) $\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 = \text{constant}$

and (iii) $\lambda_1\lambda_2\lambda_3 = \text{constant}$?

Sol. We know that the elliptic coordinates $\lambda_1, \lambda_2, \lambda_3$ are the roots of the equation $f(\lambda) \equiv (a^2 - \lambda)(b^2 - \lambda)(c^2 - \lambda) - \alpha^2(b^2 - \lambda)(c^2 - \lambda) - \beta^2(c^2 - \lambda)(a^2 - \lambda) - \gamma^2(a^2 - \lambda)(b^2 - \lambda) = 0$

or $\lambda^3 - \lambda^2(a^2 + b^2 + c^2 - \alpha^2 - \beta^2 - \gamma^2) + \lambda(a^2b^2 + b^2c^2 + c^2a^2 - \alpha^2b^2 - \alpha^2c^2 - \beta^2c^2 - \beta^2a^2 - \gamma^2a^2 - \gamma^2b^2) - (a^2b^2c^2 - \alpha^2b^2c^2 - \beta^2c^2a^2 - \gamma^2a^2b^2) = 0$

$\therefore \lambda_1 + \lambda_2 + \lambda_3 = a^2 + b^2 + c^2 - \alpha^2 - \beta^2 - \gamma^2$

$\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = a^2b^2 + b^2c^2 + c^2a^2 - (b^2 + c^2)\alpha^2 - (c^2 + a^2)\beta^2 - (a^2 + b^2)\gamma^2$

and $\lambda_1\lambda_2\lambda_3 = a^2b^2c^2 - \alpha^2b^2c^2 - \beta^2c^2a^2 - \gamma^2a^2b^2$

(i) $\lambda_1 + \lambda_2 + \lambda_3 = $ constant

gives $a^2 + b^2 + c^2 - \alpha^2 - \beta^2 - \gamma^2 = $ constant $= k_1^2$ (say)

$\therefore$ The locus of (α, β, γ) is $x^2 + y^2 + z^2 = a^2 + b^2 + c^2 - k_1^2$, which represents a sphere.

(ii) $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = $ constant gives $a^2b^2+b^2c^2+c^2a^2-(b^2+c^2)\,\alpha^2-(c^2+a^2)\,\beta^2-(a^2+b^2)\,\gamma^2=\text{constant}$

$=k_1^4$ (say)

$\therefore$ The locus of (α, β, γ) is $(b^2+c^2)\,x^2-(c^2+a^2)\,y^2-(a^2+b^2)\,z^2$

$=a^2b^2+b^2c^2+c^2a^2-k_1^4$

which represents an ellipsoid.

(iii) $\lambda_1\lambda_2\lambda_3=$ constant

gives $a^2b^2c^2-\alpha^2b^2c^2-\beta^2c^2a^2-\gamma^2a^2b^2=\text{constant}$

or $\quad \dfrac{\alpha^2}{a^2}+\dfrac{\beta^2}{b^2}+\dfrac{\gamma^2}{c^2}=1+\text{constant}=\text{constant}.$

$\therefore$ The locus of (α, β, γ) is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=\text{constant},$

which represents an ellipsoid.

Ex. 6. If $\lambda_1, \lambda_2, \lambda_3$ are the parameters of the paraboloids confocal to $(x^2/a)+(y^2/b)=2z$, which pass through the point (α, β, γ), prove that

$$\alpha^2=\frac{(a-\lambda_1)(a-\lambda_2)(a-\lambda_3)}{b-a};\ \beta^2=\frac{(b-\lambda_1)(b-\lambda_2)(b-\lambda_3)}{a-b}$$

and $\quad 2\gamma=(\lambda_1+\lambda_2+\lambda_3-a-b).$

Sol. The equation $\dfrac{x^2}{a-\lambda}+\dfrac{y^2}{b-\lambda}=2z-\lambda$ represents any paraboloid confocal with the given paraboloid $\quad (x^2/a)+(y^2/b)=2z.$

In this confocal passes through (α, β, γ), then $\dfrac{\alpha^2}{a-\lambda}+\dfrac{\beta^2}{b-\lambda}=2\gamma-\lambda.$

or $\quad f(\lambda)\equiv\alpha^2(b-\lambda)+\beta^2(a-\lambda)-(2\gamma-\lambda)(a-\lambda)(b-\lambda)=0$

or $\quad f(\lambda)\equiv\lambda^3-\lambda^2(2\gamma+a+b)-\lambda(\alpha^2+\beta^2-2a\gamma-2b\gamma-ab)$

$+(\alpha^2b+\beta^2a-2ab\gamma)=0 \quad \ldots(i)$

$\therefore$ $\lambda_1, \lambda_2, \lambda_3$ are the roots of $f(\lambda)=0$, so $f(\lambda)=(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$

$$\therefore \frac{\alpha^2}{a-\lambda}+\frac{\beta^2}{b-\lambda}-(2\gamma-\lambda)=\frac{f(\lambda)}{(a-\gamma)(b-\lambda)}=\frac{(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)}{(a-\lambda)(b-\lambda)}$$

$\therefore$ By the rule of partial fractions, we find that

$$\alpha^2=\frac{(a-\lambda_1)(a-\lambda_2)(a-\lambda_3)}{(b-a)},\ \beta^2=\frac{(b-\lambda_1)(b-\lambda_2)(b-\lambda_3)}{a-b}$$

Also from (i), $\lambda_1+\lambda_2+\lambda_3=$ sum of roots $=2\gamma+a+b$

or $\quad 2\gamma=\lambda_1+\lambda_2+\lambda_3-a-b \quad$ Hence proved.

****Ex. 7. If a_1, a_2, a_3 are the primary semi-axes of the confocals to $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ which pass through a point (α, β, γ),**

$$\alpha^2=\frac{a_1^2a_2^2a_3^2}{(b^2-a^2)(c^2-a^2)},\ \beta^2=\frac{(b^2-a^2+a_1^2)(b^2-a^2+a_2^2)(b^2-a^2+a_3^2)}{(c^2-b^2)(a^2-b^2)}$$

$$\gamma^2 = \frac{(c^2 - a^2 + a_1^2)(c^2 - a^2 + a_2^2)(c^2 - a^2 + a_3^2)}{(a^2 - c^2)(b^2 - c^2)}$$

Sol. We know that the equation of any confocal of the given ellipsoid is

$$\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} + \frac{z^2}{c^2 - \lambda} = 1 \qquad \text{...(i)}$$

Let $a^2 - \lambda = \mu^2$, then $-\lambda = \mu^2 - a^2$ ...(ii)

$\therefore\ b^2 - \lambda = b^2 + \mu^2 - a^2 = \mu^2 - (a^2 - b^2) = \mu^2 - B^2$, say

and $c^2 - \lambda = c^2 + \mu^2 - a^2 = \mu^2 - (a^2 - c^2) = \mu^2 - C^2$, say

where $B^2 = a^2 - b^2,\ C^2 = a^2 - c^2$ ...(iii)

Then equation (i) becomes $\dfrac{x^2}{\mu^2} + \dfrac{y^2}{\mu^2 - B^2} + \dfrac{z^2}{\mu^2 - C^2} = 1$,

where B and C are constants and μ is known as primary semi-axis.

If is passes through the point (α, β, γ), then

$$\frac{\alpha^2}{\mu^2} + \frac{\beta^2}{\mu^2 - B^2} + \frac{\gamma^2}{\mu^2 - C^2} = 1$$

or $\alpha^2 (\mu^2 - B^2)(\mu^2 - C^2) + \beta^2 \mu^2 (\mu^2 - C^2) + \gamma^2 \mu^2 (\mu^2 - B^2)$
$$= \mu^2 (\mu^2 - B^2)(\mu^2 - C^2)$$

or $\mu^6 - \mu^4 (B^2 + C^2 + \alpha^2 + \beta^2 + \gamma^2)$
$$+ \mu^2 (B^2C^2 + \alpha^2 B^2 + \alpha^2 C^2 + \beta^2 C^2 + B^2\gamma^2) - \alpha^2 B^2 C^2 = 0$$

According to the problem, its roots are a_1^2, a_2^2, a_3^2

$\therefore\ a_1^2 + a_2^2 + a_3^2 = B^2 + C^2 + \alpha^2 + \beta^2 + \gamma^2$ ...(iii)

$a_1^2 a_2^2 + a_2^2 a_3^2 + a_3^2 a_1^2 = B^2C^2 + \alpha^2 B^2 + \alpha^2 C^2 + \beta^2 C^2 + \gamma^2 B^2$ ...(iv)

and $a_1^2 a_2^2 a_3^2 = \alpha^2 B^2 C^2$ ...(v)

From (v), $\alpha^2 = \dfrac{a_1^2 a_2^2 a_3^2}{B^2C^2} = \dfrac{a_1^2 a_2^2 a_3^2}{(a^2 - b^2)(a^2 - c^2)}$, from (ii)

or $\alpha^2 = a_1^2 a_2^2 a_3^2 / [(b^2 - a^2)(c^2 - a^2)]$ ...(vi)

From (iii), $B^2 (a_1^2 + a_2^2 + a_3^2) = B^4 + B^2C^2 + \alpha^2 B^2 + \beta^2 B^2 + \gamma^2 B^2$ ...(vii)

Subtracting (iv) from (vii), we get

$a_1^2 (B^2 - a_2^2) + a_2^2 (B^2 - a_3^2) + a_3^2 (B^2 - a_1^2) = B^4 + \beta^2 (B^2 - C^2) - \alpha^2 C^2$

or $\beta^2 (B^2 - C^2) = a_1^2 (B^2 - a_2^2) + a_2^2 (B^2 - a_3^2) + a_3^2 (B^2 - a_1^2) - B^4 + \alpha^2 C^2$

or $\beta^2 [(a^2 - b^2) - (a^2 - c^2)] = a_1^2 (B^2 - a_2^2) + a_2^2 (B^2 - a_3^2)$
$$+ a_3^2 (B^2 - a_1^2) - B^4 + \frac{a_1^2 a_2^2 a_3^2}{B^2}, \text{ from (ii), (v)}$$

or $$\beta^2 (c^2 - b^2) = \frac{-a_1^2 B^2 (a_2^2 - B^2) - a_2^2 B^2 (a_3^2 - B^2) - a_3^2 B^2 (a_1^2 - B_1^2) - B^6 + a_1^2 a_2^2 a_3^2}{B^2}$$

or $$\beta^2 (c^2-b^2) = \frac{a_1^2 (a_2^2 a_3^2 - a_2^2 B^2 - a_3^2 B^2 + B^4) - B^2 (a_2^2 a_3^2 - a_2^2 B^2 - a_3^2 B^2 + B^4)}{B^2}$$

or $$\beta^2 = \frac{(a_1^2 - B^2)(a_2^2 a_3^2 - a_2^2 B^2 - a_3^2 B^2 + B^4)}{B^2 (c^2-b^2)} = \frac{(a_1^2 - B^2)(a_2^2 - B^2)(a_3^2 - B^2)}{B^2 (c^2-b^2)}$$

$$= \frac{(a_1^2 + b^2 - a^2)(a_2^2 + b^2 - a^2)(a_3^2 + b^2 - a^2)}{(a^2-b^2)(c^2-b^2)}, \text{ from (ii)}$$

or $$\beta^2 = \frac{(b^2 - a^2 + a_1^2)(b^2 - a^2 + a_2^2)(b^2 - a^2 + a_3^2)}{(c^2-b^2)(a^2-b^2)} \quad \text{...(viii)}$$

From (iii), $\gamma^2 = a_1^2 + a_2^2 + a_3^2 - B^2 - C^2 - \alpha^2 - \beta^2$

Substitute values of B, C, α^2 and β^2 to evaluate γ^2 and we can prove that

$$\gamma^2 = \frac{(c^2 - a^2 + a_1^2)(c^2 - a^2 + a_2^2)(c^2 - a^2 + a_3^2)}{(a^2-c^2)(b^2-c^2)}$$

***Ex. 8. Show that the product of eccentricities of the focal conics of an ellipsoid is unity.**

Sol. Let the ellipsoid be $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$

Its focal conics are $z = 0, \dfrac{x^2}{a^2-c^2} + \dfrac{y^2}{b^2-c^2} = 1$ (Ellipse) ...(i)

and $y = 0, \dfrac{x^2}{a^2-b^2} - \dfrac{x^2}{b^2-c^2} = 1$ (Hyperbola) ...(ii)

...See § 14.01 results (iii) & (iv) Page 1 Ch. XIV

Let e_1 and e_2 be the eccentricities of the above ellipse and hyperbola respectively.

Then from (i), $(b^2-c^2) = (a^2-c^2)(1-e_1^2)$ ... $\because b^2 = a^2(1-e^2)$ for ellipse

or $b^2 - c^2 = a^2 - c^2 - e_1^2 (a^2-c^2)$

or $(a^2-c^2) e_1^2 = a^2 - b^2$ or $e_1^2 = (a^2-b^2)/(a^2-c^2)$ ...(iii)

And from (ii), $b^2 - c^2 = (a^2-b^2)(e_2^2 - 1)$ ... $\because b^2 = a^2(e^2-1)$ for hyperbola

or $b^2 - c^2 = e_2^2 (a^2-b^2) - a^2 + b^2$

or $(a^2-b^2) e_2^2 = (a^2-c^2)$ or $e_2^2 = (a^2-c^2)/(a^2-b^2)$ ...(iv)

$\therefore$ From (iii) and (iv), we have $e_1 \cdot e_2 = \sqrt{\left(\dfrac{a^2-b^2}{a^2-c^2}\right)} \cdot \sqrt{\left(\dfrac{a^2-c^2}{a^2-b^2}\right)} = 1$

Hence proved.

****Ex. 9. Show that the locus of the umbilics of a system of confocal ellipsoids is the focal hyperbola.** (*Garhwal* 94)

Sol. Let the confocal of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ be

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1 \quad \text{or} \quad \frac{x^2}{A^2}+\frac{y^2}{B^2}+\frac{z^2}{C^2}=1, \qquad \text{...(i)}$$

where $$A^2=a^2-\lambda,\ B^2=b^2-\lambda,\ C^2=c^2-\lambda \qquad \text{...(ii)}$$

The central circular sections of the confocal (i) are given by

$$\sqrt{(A^2-B^2)}\,\frac{x}{A}\pm\sqrt{(B^2-C^2)}\,\frac{z}{C}=0 \qquad \text{...(iii)}$$

...See Ch. XI on Plane sections of conicoid

If $P(x_1, y_1, z_1)$ is an umbilic, the diameteral plane of OP is a plane through the centre parallel to the tangent plane at P and is therefore the central circular section.

Hence the equation $$\frac{xx_1}{A^2}+\frac{yy_1}{B^2}+\frac{zz_1}{C^2}=0 \qquad \text{...(iv)}$$

is identical with the equation (iii).

$\therefore$ Comparing (iii) and (iv) we get

$$\frac{x_1/A^2}{\sqrt{(A^2-B^2)}/A}=\frac{y_1/B^2}{0}=\frac{z_1/C^2}{\pm\sqrt{(B^2-C^2)}/C}$$

or $$\frac{x_1/A}{\sqrt{(A^2-B^2)}}=\frac{y_1/B}{0}=\frac{z_1/C}{\pm\sqrt{(B^2-C^2)}}=\frac{\sqrt{[(x_1/A)^2+(y_1/B)^2+(z_1/C)^2]}}{\sqrt{[(A^2-B^2)+0+(B^2-C^2)]}}$$

$$=\frac{1}{\pm\sqrt{(A^2-C^2)}}$$

or $$x_1=\frac{A\sqrt{(A^2-B^2)}}{\pm\sqrt{(A^2-C^2)}},\quad y=0,\quad z_1=\frac{C\sqrt{(B^2-C^2)}}{\pm\sqrt{(A^2-C^2)}}$$

$$\therefore \frac{x_1^2}{a^2-b^2}-\frac{z_1^2}{b^2-c^2}=\frac{A^2(A^2-B^2)}{(A^2-C^2)(a^2-b^2)}-\frac{C^2(B^2-C^2)}{(A^2-C^2)(b^2-c^2)}$$

$$=\frac{(a^2-\lambda)(a^2-b^2)}{(a^2-c^2)(a^2-b^2)}-\frac{(c^2-\lambda)(b^2-c^2)}{(a^2-c^2)(b^2-c^2)}, \text{ from (ii)}$$

$$=\frac{1}{a^2-c^2}[(a^2-\lambda)-(c^2-\lambda)]=1$$

Thus we have $y_1=0,\ \dfrac{x_1^2}{a^2-b^2}-\dfrac{z_1^2}{b^2-c^2}=1$

$\therefore$ The locus of the umbilic $P(x_1, y_1, z_1)$ is $y=0,\ \dfrac{x^2}{a^2-b^2}-\dfrac{z^2}{b^2-c^2}=1$

which is the equation of the focal hyperbola. ...See § 14.01 (iv) Page 1 Ch. XIV

§ 14.04. Confocals cut at right angles. *(Gorakhpur 95)*

The tangent planes to two confocals at any common point are at right angles.

Let (x_1, y_1, z_1) be a common point of the two confocals

$$\frac{x^2}{a^2-\lambda_1}+\frac{y^2}{b^2-\lambda_1}+\frac{z^2}{c^2-\lambda_1}=1 \quad \text{and} \quad \frac{x^2}{a^2-\lambda_2}+\frac{y^2}{b^2-\lambda_2}+\frac{z^2}{c^2-\lambda_2}=1$$

As the point (x_1, y_1, z_1) lies on both of them, so we have

$$\frac{x_1^2}{a^2-\lambda_1}+\frac{y_1^2}{b^2-\lambda_1}+\frac{z_1^2}{c^2-\lambda_1}=1 \quad \text{and} \quad \frac{x_1^2}{a^2-\lambda_2}+\frac{y_1^2}{b^2-\lambda_2}+\frac{z_1^2}{c^2-\lambda_2}=1$$

Substracting we get

$$x_1^2\left[\frac{1}{a^2-\lambda_1}-\frac{1}{a^2-\lambda_2}\right]+y_1^2\left[\frac{1}{b^2-\lambda_1}-\frac{1}{b^2-\lambda_2}\right]+z_1^2\left[\frac{1}{c^2-\lambda_1}-\frac{1}{c^2-\lambda_2}\right]=0$$

or $$\frac{x_1^2}{(a^2-\lambda_1)(a^2-\lambda_2)}+\frac{y_1^2}{(b^2-\lambda_1)(b^2-\lambda_2)}+\frac{z_1^2}{(c^2-\lambda_1)(c^2-\lambda_2)}=0 \ \because \lambda_1 \neq \lambda_2$$

which is the condition that the tangent planes at (x_1, y_1, z_1) to the confocals are at right angles.

Thus we conclude *that two confocal conicoids cut one another at right angles at their common point.*

Cor. The tangent planes at a point to three confocals which pass through this point are mutually perpendicular.

§ 14.05. Confocal touching a given plane.

Let the given plane be $lx+my+nz=p$.

If this plane touches the confocal conicoid $\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1$

then the condition is $l^2(a^2-\lambda)+m^2(b^2-\lambda)+n^2(c^2-\lambda)=p^2$, which gives one and only value of λ.

$\therefore$ We conclude that one conicoid confocal with a given conicoid touches a given plane.

****§ 14.06. Confocals touching a given line.**

To show that two conicoids confocal with a given conicoid touch a given line and the tangent planes at the points of contact are at right angles.

(*Avadh* 95, 93; *Garhwal* 96)

Let the given line be the line of intersection of the planes

$$lx+my+nz+p=0,\ l'x+m'y+n'z+p'=0 \quad \text{...(i)}$$

$\therefore$ Equation of any plane through this line is

$$(lx+my+nz+p)+k(l'x+m'y+n'z+p')=0$$

Then $(l+kl')x+(m+km')y+(n+kn')z+(p+kp')=0$

If this plane touches the confocal conicoid

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1 \quad \text{...(ii)}$$

Then $(l+kl')^2(a^2-\lambda)+(m+km')^2(b^2-\lambda)+(n+kn')^2(c^2-\lambda)=(p+kp')^2$ ...(iii)

which gives two values of k, so that there are two planes through the given line which touch the above confocal.

Now if the line be a tangent line of this confocal, the two tangent planes through it will coincide. **(Note)**

Hence the roots of the equation (iii) in k must be equal, for which the condition is "$B^2-4AC=0$"

i.e.
$$[2ll'(a^2-\lambda)+2mm'(b^2-\lambda)+2nn'(c^2-\lambda)-2pp']^2$$
$$=4\,[l'^2(a^2-\lambda)+m'^2(b^2-\lambda)+n'^2(c^2-\lambda)-p'^2]\times[l^2(a^2-\lambda)+m^2(b^2-\lambda)+n^2(c^2-\lambda)-p^2]$$

or
$$[ll'(a^2-\lambda)+mm'(b^2-\lambda)+nn'(c^2-\lambda)-pp']^2$$
$$=[l'^2(a^2-\lambda)+m'^2(b^2-\lambda)+n'^2(c^2-\lambda)-p'^2],\times[l^2(a^2-\lambda)+m^2(b^2-\lambda)+n^2(c^2-\lambda)-p^2]$$

which being a quadratic equation in λ gives two confocals which touch the given line.

Let the two confocals be given by the equations
$$\frac{x^2}{a^2-\lambda_1}+\frac{y^2}{b^2-\lambda_1}+\frac{z^2}{c^2-\lambda_1}=1 \quad\text{and}\quad \frac{x^2}{a^2-\lambda_2}+\frac{y^2}{b^2-\lambda_2}+\frac{z^2}{c^2-\lambda_2}=1$$
and let the points (x_1, y_1, z_1) and (x_2, y_2, z_2) be the points of contact of the line and the above two confocals.

Then the tangent planes to these confocals at (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively are
$$\frac{xx_1}{a^2-\lambda_1}+\frac{yy_1}{b^2-\lambda_2}+\frac{zz_1}{c^2-\lambda_1}=1 \quad\text{and}\quad \frac{xx_2}{a^2-\lambda_2}+\frac{yy_2}{b^2-\lambda_2}+\frac{zz_2}{c^2-\lambda_2}=1$$

$\therefore$ If these two tangent planes are at right angles, then we must have
$$\frac{x_1x_2}{(a^2-\lambda_1)(a^2-\lambda_2)}+\frac{y_1y_2}{(b^2-\lambda_1)(b^2-\lambda_2)}+\frac{z_1z_2}{(c^2-\lambda_1)(c^2-\lambda_2)}=0 \qquad \text{...(iv)}$$

Also as the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is a tangent line to both the above confocal conicoids, so each point must lie in the tangent plane of the other

i.e.
$$\frac{x_1x_2}{a^2-\lambda_1}+\frac{y_1y_2}{b^2-\lambda_1}+\frac{z_1z_2}{c^2-\lambda_1}=1 \quad\text{and}\quad \frac{x_1x_2}{a^2-\lambda_2}+\frac{y_1y_2}{b^2-\lambda_2}+\frac{z_1z_2}{c^2-\lambda_2}=1$$

Subtracting, we get
$$\frac{x_1x_2}{(a^2-\lambda_1)(a^2-\lambda_2)}+\frac{y_1y_2}{(b^2-\lambda_1)(b^2-\lambda_2)}+\frac{z_1z_2}{(c^2-\lambda_1)(c^2-\lambda_2)}=0$$
which is true by virtue of (iv).

Hence we conclude that *two conicoids of a confocal system touch any straight line and the tangent planes at the points of contact are at right angles.*

§ 14.07. Parameters of confocals through a point on a conicoid.

Let $P(x_1, y_1, z_1)$ by any point on the conicoid

$$(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1 \quad \text{...(i)}$$

Let $P(x_1, y_1, z_1)$ lies on its confocal whose parameter is λ, we have

$$\frac{x_1^2}{a^2-\lambda}+\frac{y_1^2}{b^2-\lambda}+\frac{z_1^2}{c^2-\lambda}=1 \quad \text{...(ii)}$$

$\therefore$ The parameters of the confocals through P are given by

$$\frac{x_1^2}{a^2-\lambda}+\frac{y_1^2}{b^2-\lambda}+\frac{z_1^2}{c^2-\lambda}=1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2} \quad \textbf{(Note)}$$

or
$$\frac{x_1^2}{a^2(a^2-\lambda)}+\frac{y_1^2}{b^2(b^2-\lambda)}+\frac{z_1^2}{c^2(c^2-\lambda)}=0, \quad \because \quad \lambda \neq 0 \quad \text{...(iii)}$$

Now the equation of the central section of the conicoid (i) parallel to the tangent plane at $P(x_1, y_1, z_1)$ is $\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}+\dfrac{zz_1}{c^2}=0$

and squares of the semi-axes of this section are given by

$$\frac{x_1^2}{a^2(a^2-r^2)}+\frac{y_1^2}{b^2(b^2-r^2)}+\frac{z_1^2}{c^2(c^2-r^2)}=0 \quad \text{...(iv)}$$

...See Ch. XI of this book.

Comparing (iii) and (iv), we conclude that the values of the parameter λ are the values of the squares of the semi-axes of this section.

Again the direction ratios l, m, n of the semi-axes of length r, are given by [See Ch. XI of this book]

$$\frac{l}{x_1/(a^2-r^2)}=\frac{m}{y_1/(b^2-r^2)}=\frac{n}{z_1/(c^2-r^2)},$$

so that the axis is parallel to the normal at $P(x_1, y_1, z_1)$ to the confocal conicoid given by (ii).

Thus we conclude that *if P is a point on a central conicoid, the parameters of the two confocals of the conicoid which pass through P are equal to the squares of the semi-axes of the central section of the conicoid which is parallel to the tangent plane at P, and the normal to the confocals at P are parallel to the axes.*

§ 14.08. Locus of poles of a plane with respect to confocals.

(*Gorakhpur* 96)

Let the equation of the confocal be $\dfrac{x^2}{a^2-\lambda}+\dfrac{y^2}{b^2-\lambda}+\dfrac{z^2}{c^2-\lambda}=1$ ...(i)

and that of the given plane be $lx+my+nz=1$ ...(ii)

If (x', y', z') be the pole of the plane (ii) with respect to the confocal (i), then the polar plane of (x', y', z') w. r. to (i) is

$$\frac{xx'}{a^2-\lambda}+\frac{yy'}{b^2-\lambda}+\frac{zz'}{c^2-\lambda}=1 \quad \text{...(iii)}$$

which must be the same as (ii).

$\therefore$ Comparing (ii) and (iii) we get $\dfrac{x'}{a^2-\lambda}=l,\ \dfrac{y'}{b^2-\lambda}=m,\ \dfrac{z'}{c^2-\lambda}=n$

or $$\frac{x'}{l}=a^2-\lambda,\ \frac{y'}{m}=b^2-\lambda,\ \frac{z'}{n}=c^2-\lambda$$

or $$\frac{x'}{l}-a^2=\frac{y'}{m}-b^2=\frac{z'}{n}-c^2=-\lambda$$

$\therefore$ The locus of the pole (x', y', z') is the straight line whose equations are

$$\frac{x-a^2l}{l}=\frac{y-b^2m}{m}=\frac{z-c^2n}{n} \quad \text{...(iv)}$$

Evidently this line (iv) is at right angles to the plane (ii).

Again, the pole of the plane with respect to that confocal which touches it is the point of contact. Hence the point of contact is on the locus which is therefore the normal to the plane at the point of contact.

Hence we conclude that *the locus of the poles of a given plane with respect to the conicoids confocal with a given coincoid is the normal to the plane at the plane of contact with that confocal which touches it.*

Solved Examples on § 14.04 to § 14.08.

****Ex. 1. Prove that the perpendiculars from the origin to the tangent plane to the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ which touch it along its curve of intersection with the confocal whose parameter is λ, lies on the cone**

$$\frac{a^2x^2}{a^2-\lambda}+\frac{b^2y^2}{b^2-\lambda}+\frac{c^2z^2}{c^2-\lambda}=0. \quad \textit{(Avadh 93)}$$

Sol. The tangent plane to the given ellipsoid $\Sigma(x^2/a^2)=1$ at the point (α, β, γ) is

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=1 \quad \text{...(i)}$$

where $$(\alpha^2/a^2)+(\beta^2/b^2)+(\gamma^2/c^2)=1 \quad \text{...(ii)}$$

Also as the point (α, β, γ) lies on the confocal conicoid

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1,$$

so we have $$\frac{\alpha^2}{a^2-\lambda}+\frac{\beta^2}{b^2-\lambda}+\frac{\gamma^2}{c^2-\lambda}=1 \quad \text{...(iii)}$$

Again the equations of the perpendicular from the origin on the tangent plane (i) are $$\frac{x}{\alpha/a^2}=\frac{y}{\beta/b^2}=\frac{z}{\gamma/c^2} \quad \text{or} \quad \frac{a^2x}{\alpha}=\frac{b^2y}{\beta}=\frac{c^2z}{\gamma} \quad \text{...(iv)}$$

$\therefore$ Eliminating α, β, γ from (ii), (iii) and (iv), we can get the required locus of the perpendicular from the origin.

From (iv), $\alpha = ka^2x,\ \beta = kb^2y,\ \gamma = kc^2z$

$\therefore$ From (ii), $$\frac{k^2a^4x^2}{a^2} + \frac{k^2b^4y^2}{b^2} + \frac{k^2c^4z^2}{c^2} = 1$$

or $$a^2x^2 + b^2y^2 + c^2z^2 = 1/k^2 \qquad \text{...(v)}$$

And from (iii), $$\frac{k^2a^4x^2}{a^2-\lambda} + \frac{k^2b^4y^2}{b^2-\lambda} + \frac{k^2c^4z^2}{c^2-\lambda} = 1$$

or $$\frac{a^4x^2}{a^2-\lambda} + \frac{b^4y^2}{b^2-\lambda} + \frac{c^4z^2}{c^2-\lambda} = \frac{1}{k^2} \qquad \text{...(vi)}$$

Equating values of $1/k^2$ from (v) and (vi), we get

$$a^2x^2 + b^2y^2 + c^2z^2 = \frac{a^4x^2}{a^2-\lambda} + \frac{b^4y^2}{b^2-\lambda} + \frac{c^2z^2}{c^2-\lambda}$$

or $$a^2x^2\left(1 - \frac{a^2}{\alpha^2-\lambda}\right) + b^2y^2\left(1 - \frac{b^2}{b^2-\lambda}\right) + c^2z^2\left(1 - \frac{c^2}{c^2-\lambda}\right) = 0$$

or $$\frac{a^2x^2}{a^2-\lambda} + \frac{b^2y^2}{b^2-\lambda} + \frac{c^2z^2}{c^2-\lambda} = 0, \quad \because \quad \lambda \neq 0.$$

which is the required locus of the perpendicular. Hence proved.

****Ex. 2. A given plane and the parallel tangent plane to a conicoid are at distances p and p_0 from the centre. Prove that the parameter of the confocal conicoid which touches the plane is $p_0^2 - p^2$.**

(Avadh 92, 91; Gorakhpur 96)

Sol. Let the equation of the conicoid be $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$ ...(i)

Let the equation of the given plane be $lx + my + nz = p$, ...(ii)

whose perpendicular distance from the centre of (i) is p.

$\therefore$ The equation of the plane parallel to (i) and at a distance p_0 from the centre of (i) is $$lx + my + nz = p_0 \qquad \text{...(iii)}$$

If (iii) is a tangent plane to the conicoid (i), then

$$l^2a^2 + m^2b^2 + n^2c^2 = p_0^2 \qquad \text{...(iv)}$$

Now let the equation of any confocal to (i) be

$$\frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} + \frac{z^2}{c^2-\lambda} = 1$$

If the plane (ii) touches this confocal, then

$$l^2(a^2-\lambda) + m^2(b^2-\lambda) + n^2(c^2-\lambda) = p^2$$

or $$a^2l^2 + b^2m^2 + c^2n^2 - p^2 = \lambda(l^2 + m^2 + n^2)$$

or $\quad p_0^2 - p^2 = \lambda$ (1), from (iv) and l, m, n are the d.c.'s of the normal to the plane (ii).

or $$\lambda = p_0^2 - p^2$$ Hence proved.

Ex. 3. Prove that the difference of the squares of the perpendiculars from the centre on any two parallel tangent planes to two given confocal conicoids is constant.

Sol. Let the two confocal conicoids of $\Sigma (x^2/a^2) = 1$ be

$$\frac{x^2}{a^2 - \lambda_1} + \frac{y^2}{b^2 - \lambda_1} + \frac{z^2}{c^2 - \lambda_1} = 1 \quad \text{...(i)}$$

and $$\frac{x^2}{a^2 - \lambda_2} + \frac{y^2}{b^2 - \lambda_2} + \frac{z^2}{c^2 - \lambda_2} = 1 \quad \text{...(ii)}$$

where λ_1 and λ_2 are constants.

Now if p_1 and p_2 be lengths of perpendiculars from the centre on two parallel planes, then their equations can be taken as

$lx + my + nz = p_1$ ....(iii) and $lx + my + nz = p_2$, ...(iv)

where l, m, n are direction cosines of the normals to these planes.

$\therefore$ If (iii) is a tangent plane of (i), we have

$$l^2 (a^2 - \lambda_1) + m^2 (b^2 - \lambda_1) + n^2 (c^2 - \lambda_1) = p_1^2 \quad \text{...(v)}$$

and if (iv) is a tangent plane of (ii), we have

$$l^2 (a^2 - \lambda_2) + m^2 (b^2 - \lambda_2) + n^2 (c^2 - \lambda_2) = p_2^2 \quad \text{...(vi)}$$

On subtracting (v) from (vi), we have

$$p_2^2 - p_1^2 = l^2 (\lambda_2 - \lambda_1) + m^2 (\lambda_2 - \lambda_1) + n^2 (\lambda_2 - \lambda_1)$$

$$= (\lambda_2 - \lambda_1)(l^2 + m^2 + n^2) = (\lambda_2 - \lambda_1)(1) = \lambda_2 - \lambda_1 = \text{constant.}$$

Ex. 4. Show that two confocal paraboloids cut everywhere at right angles.

Sol. Let the paraboloid be $\quad (x^2/a^2) + (y^2/b^2) = 2z \quad$ ...(i)

Let $P(x_1, y_1, z_1)$ be a common point of the two confocals to (i) viz.

$$\frac{x^2}{a^2 - \lambda_1} + \frac{y^2}{b^2 - \lambda_1} = 2z - \lambda_1 \quad \text{...(ii)}$$

and $$\frac{x^2}{a^2 - \lambda_2} + \frac{y^2}{b^2 - \lambda_2} = 2z - \lambda_2 \quad \text{...(iii)}$$

$\because \quad P(x_1, y_1, z_1)$ lies on both (ii) and (iii), so we have

$$\frac{x_1^2}{a^2 - \lambda_1} + \frac{y_1^2}{b^2 - \lambda_1} = 2z_1 - \lambda_1 \quad \text{and} \quad \frac{x_1^2}{a^2 - \lambda_2} + \frac{y_1^2}{b^2 - \lambda_2} = 2z_1 - \lambda_2$$

Substracting, we get

$$x_1^2\left(\frac{1}{a^2-\lambda_1}-\frac{1}{a^2-\lambda_2}\right)+y_1^2\left(\frac{1}{b^2-\lambda_1}-\frac{1}{b^2-\lambda_2}\right)=\lambda_2-\lambda_1$$

or $$\frac{x_1^2}{(a^2-\lambda_1)(a^2-\lambda_2)}+\frac{y_1^2}{(b^2-\lambda_1)(b^2-\lambda_2)}+1=0, \quad \because \lambda_1 \neq \lambda_2 \qquad \text{...(iv)}$$

Again the tangent planes at $P(x_1, y_1, z_1)$ to (ii) and (iii) are respectively

$$\frac{xx_1}{a^2-\lambda_1}+\frac{yy_1}{b^2-\lambda_1}-(z+z_1)+\lambda_1=0$$

and $$\frac{xx_1}{a^2-\lambda_2}+\frac{yy_1}{b^2-\lambda_2}-(z+z_1)+\lambda_2=0$$

If these two tangent planes are perpendicular, then

$$\left(\frac{x_1}{a^2-\lambda_1}\right)\left(\frac{x_1}{a^2-\lambda_2}\right)+\left(\frac{y_1}{b^2-\lambda_1}\right)\left(\frac{y_1}{b^2-\lambda_2}\right)+(-1)(-1)=0$$

or $$\frac{x_1^2}{(a^2-\lambda_1)(a^2-\lambda_2)}+\frac{y_1^2}{(b^2-\lambda_1)(b^2-\lambda_2)}+1=0,$$

which is true by virtue of (iv) for all values of x_1, y_1, z_1.

Hence two confocal paraboloids cut everywhere at right angles.

EXERCISES

Ex. 1. If two concentric and coaxial conicoids cut one another everywhere at right angles, they must be confocal.

Ex. 2. P and Q are points on two confocals such that the tangent planes at P and Q are at right angles, show that plane through the centre and the line of intersection of the tangent planes bisects PQ.

§ 14.09. Normals to the confocals through a point.

Let the confocals of the conicoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ ...(i) passing through the point $P(\alpha, \beta, \gamma)$ have parameters $\lambda_1, \lambda_2, \lambda_3$ and let p_1, p_2, p_3 be the perpendicular from the centre of the conicoid to the tangent planes at $P(\alpha, \beta, \gamma)$ to the confocals.

Now the equations of the normal at $P(\alpha, \beta, \gamma)$ to the confocal of parameters λ_1 are $$\frac{x-\alpha}{p_1\,\alpha/(a^2-\lambda_1)}=\frac{y-\beta}{p_1\,\beta/(b^2-\lambda_1)}=\frac{z-\gamma}{p_1\,\gamma/(c^2-\lambda_1)}$$

...See Ch IX on Conicoids

so that the coordinates of any point Q on it, such that $PQ=r$, are

$$\left(\alpha+\frac{p_1\,\alpha r}{a^2-\lambda_1},\quad \beta+\frac{p_1\,\beta r}{b^2-\lambda_1},\quad \gamma+\frac{p_1\,\gamma r}{c^2-\lambda_1}\right)$$

or $$\left[\alpha\left(1+\frac{p_1 r}{a^2-\lambda_1}\right), \beta\left(1+\frac{p_1 r}{b^2-\lambda_1}\right), \gamma\left(1+\frac{p_1 r}{c^2-\lambda_1}\right)\right] \quad \text{...(ii)}$$

Also the polar plane of $P(\alpha, \beta, \gamma)$ with respect to the concoid (i) is

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=1$$

If Q lies on it, then from (ii) we have

$$\frac{\alpha^2}{a^2}\left(1+\frac{p_1 r}{a^2-\lambda_1}\right)+\frac{\beta^2}{b^2}\left(1+\frac{p_1 r}{b^2-\lambda_1}\right)+\frac{\gamma^2}{c^2}\left(1+\frac{p_1 r}{c^2-\lambda_1}\right)=1 \quad \text{...(iii)}$$

Also as $P(\alpha, \beta, \gamma)$ lies on the confocal of parameter λ_1, so we have

$$\frac{\alpha^2}{a^2-\lambda_1}+\frac{\beta^2}{b^2-\lambda_1}+\frac{\gamma^2}{c^2-\lambda_1}=1 \quad \text{...(iv)}$$

Substracting (iv) from (iii), we get

$$\frac{\alpha^2}{a^2-\lambda_1}\left[\frac{a^2-\lambda_1+p_1 r}{a^2}-1\right]+\frac{\beta^2}{b^2-\lambda_1}\left[\frac{b^2-\lambda_1+p_1 r}{b^2}-1\right]$$
$$+\frac{\gamma^2}{c^2-\lambda_1}\left[\frac{c^2-\lambda_1+p_1 r}{c^2}-1\right]=0$$

or $$\left[\frac{\alpha^2}{a^2(a^2-\lambda_1)}+\frac{\beta^2}{b^2(b^2-\lambda_1)}+\frac{\gamma^2}{c^2(c^2-\lambda_1)}\right](p_1 r-\lambda_1)=0$$

which gives $\quad p_1 r-\lambda_1=0 \quad$ or $\quad r=PQ=\lambda_1/p_1$

In a similar manner we can prove that if the normals at P to the other two confocals (with parametrs λ_2 and λ_3) meet the polar plane of $P(\alpha, \beta, \gamma)$ with respect to the conicoid (i) in R and S, then $PR=\lambda_2/p_2$ and $PS=\lambda_3/p_3$

***§ 14.10. An Important Example.**

To prove that in § 14.09 above, the tetrahedron PQRS is self-polar with respect to the given conicoid $\quad (x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$

From § 14.09 (ii) above, we know that the coordinates of the point Q we

$$\left[\alpha\left(1+\frac{p_1 r}{a^2-\lambda_1}\right), \beta\left(1+\frac{p_1 r}{b^2-\lambda_1}\right), \gamma\left(1+\frac{p_1 r}{c^2-\lambda_1}\right)\right], \text{ where } p_1 r=\lambda_1$$

or $$\left[\alpha\left(1+\frac{\lambda_1}{a^2-\lambda_1}\right), \beta\left(1+\frac{\lambda_1}{b^2-\lambda_1}\right), \gamma\left(1+\frac{\lambda_1}{c^2-\lambda_1}\right)\right]$$

or $$\left[\frac{a^2\alpha}{a^2-\lambda_1}, \frac{b^2\beta}{b^2-\lambda_1}, \frac{c^2\gamma}{c^2-\lambda_1}\right]$$

Now the polar plane of Q with respect to the given conicoid is

$$\frac{a^2\alpha}{(a^2-\lambda_1)}\cdot\frac{x}{a^2}+\frac{b^2\beta}{(b^2-\lambda_1)}\cdot\frac{y}{b^2}+\frac{c^2\gamma}{c^2-\lambda_1}\cdot\frac{z}{c}=1$$

or
$$\frac{\alpha x}{a^2-\lambda_1}+\frac{\beta y}{b^2-\lambda_1}+\frac{\gamma z}{c^2-\lambda_1}=1$$

which is also the tangent plane at $P(\alpha, \beta, \gamma)$ to the confocal with parameter λ_1 *i.e.* plane perpendicular to PQ.

But the normals to the three confocals (with parameters $\lambda_1, \lambda_2, \lambda_3$) through P viz. PQ, PR and PS are mutually perpendicular.

$\therefore$ The tangnet plane at $P(\alpha, \beta, \gamma)$ to the first confocal, with parameter λ_1, is the plane PRS.

$\therefore$ It follows that the polar plane of Q is PRS and similarly those of R and S are PSQ and PQR, while polar plane of P is QRS.

Hence the tetrahedron $PQRS$ is self-polar with respect to the given conicoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$

§ 14.11. Axes of the envelopoing cone.

In § 14.10 above we have proved that the tetrahderon $PQRS$ is self-polar with respect to the conicoid $\quad (x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1 \quad$ ...(i)

$\therefore$ The triangle QRS is seil-polar with respect to the conic which is the section of the conicoid (i) by the polar plane of P. But this conic is also the section of the enveloping cone of the conicoid (i) whose vertex is P.

It follows that PQ, PR, PS are a set of conjugate diameters of the enveloping cone and being mutually perpendicular are the principal axes.

Hence we conclude that *the normals to the three confocals through the point P are the axes of the enveloping cone whose vertex is P.*

***§ 14.12. Equation of the enveloping cone.**

To find the equation to the enveloping cone whose vertex is P referred to its principal axes. (*Gorakhpur* 96)

Take P as origin, the tangent planes at P to three confocals as the coordinate planes and the normals PQ, PR and PS as the coordinate axes.

Then the equation of the enveloping cone will be of the form

$$Ax^2+By^2+Cz^2=0 \qquad \text{...(i)}$$

The centre C of the conicoid is $(-p_1, -p_2, -p_3)$ and so the equations to the line PC are

$$\frac{x}{p_1}=\frac{y}{p_2}=\frac{z}{p_3} \qquad \text{...(ii)}$$

Now the centre of the section of the cone or conicoid by the plane QRS lies on PC and therefore its coordinates are of the form (kp_1, kp_2, kp_3) and then the equation of the plane QRS is $(x-kp_1)\,Ap_1+(y-kp_2)\,Bp_2+(z-kp_3)\,Cp_3=0$

Again as the normals are the axes of reference, so the plane QRS makes intercepts $\lambda_1/p_1, \lambda_2/p_2, \lambda_3/p_3$ on them. ...See § 14.09 P. 18

$\therefore$ The equation of the plane QRS is also given by

$$\frac{x}{\lambda_1/p_1}+\frac{y}{\lambda_2/p_2}+\frac{z}{\lambda_3/p_3}=1 \quad \text{or} \quad \frac{p_1 x}{\lambda_1}+\frac{p_2 y}{\lambda_2}+\frac{p_3 z}{\lambda_3}=1$$

$\therefore$ We obtain $\dfrac{A}{(1/\lambda_1)}=\dfrac{B}{(1/\lambda_2)}=\dfrac{C}{(1/\lambda_3)}$ and equation (i) becomes

$$(x^2/\lambda_1)+(y^2/\lambda_2)+(z^2/\lambda_3)=0$$

****§ 14.13. Equation to conicoid.**

To find the equation to the conicoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ *referred to the normals to its confocals through any point P as coordinate axes.*

(*Avadh* 94; *Gorakhpur* 95)

We know (See § 14.12 above) that the equation of the enveloping cone whose vertex is P is $(x^2/\lambda_1)+(y^2/\lambda_2)+(z^2/\lambda_3)=0$ and the equation of polar plane of P, *i.e.* of the plane of contact, is $\dfrac{p_1 x}{\lambda_1}+\dfrac{p_2 y}{\lambda_2}+\dfrac{p_3 z}{\lambda_3}=1$

$\therefore$ The required equation of the conicoid referred to the normals to its confocals through P must be of the form

$$\frac{x^2}{\lambda_1}+\frac{y^2}{\lambda_2}+\frac{z^2}{\lambda_3}=k\left(\frac{p_1 x}{\lambda_1}+\frac{p_2 y}{\lambda_2}+\frac{p_3 z}{\lambda_3}-1\right)^2 \qquad \text{...(i)}$$

The centre of the conicoid is the point $(-p_1, -p_2, -p_3)$ and the equations of the chord of the conicoid parallel to the x-axis drawn through the centre are

$$\frac{x+p_1}{1}=\frac{y+p_2}{0}=\frac{z+p_3}{0}=r, \text{ say.}$$

This meets the conicoid (i), where

$$\frac{(r-p_1)^2}{\lambda_1}+\frac{(-p_2)^2}{\lambda_2}+\frac{(-p_3)^2}{\lambda_3}=k\left[\frac{p_1(r-p_1)}{\lambda_1}+\frac{p_2(-p_2)}{\lambda_2}+\frac{p_3(-p_3)}{\lambda_3}-1\right]^2$$

i.e.
$$\frac{(r-p_1)^2}{\lambda_1}+\frac{p_2^2}{\lambda_2}+\frac{p_3^2}{\lambda_3}=k\left[\frac{p_1(r-p_1)}{\lambda_1}-\frac{p_2^2}{\lambda_2}-\frac{p_3^2}{\lambda_3}-1\right]^2 \qquad \text{...(ii)}$$

As the chord is bisected at the centre, the above equation gives two equal and opposite values of r *i.e.* the sum of two values of r in the above equation is zero *i.e.* the coefficient of r in (ii) must be zero.

$$\therefore \frac{-2p_1}{\lambda_1}=-k\frac{2p_1}{\lambda_1}\left(\frac{p_1^2}{\lambda_1}+\frac{p_2^2}{\lambda_2}+\frac{p_3^2}{\lambda_3}+1\right) \quad \text{or} \quad \frac{1}{k}=\frac{p_1^2}{\lambda_1}+\frac{p_2^2}{\lambda_2}+\frac{p_3^2}{\lambda_3}+1$$

Hence from (i) the required equation of the conicoid becomes

$$\left(\frac{x^2}{\lambda_1}+\frac{y^2}{\lambda_2}+\frac{z^2}{\lambda_3}\right)\left(\frac{p_1^2}{\lambda_1}+\frac{p_2^2}{\lambda_2}+\frac{p_3^2}{\lambda_3}+1\right)=\left(\frac{p_1 x}{\lambda_1}+\frac{p_2 y}{\lambda_2}+\frac{p_3 z}{\lambda_3}-1\right)^2$$

EXERCISES.

Ex. 1. If $\lambda_1, \lambda_2, \lambda_3$ are the parameters of three confocals to the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ that pass through P, prove that the perpendicular from the centre to the tangent plane at P are

$$\sqrt{\left[\frac{(a^2-\lambda_1)(a^2-\lambda_2)(a^2-\lambda_3)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}\right]}, \text{ etc}$$

Ex. 2. If λ and μ are the parameters of the confocal hyperboloid through a point P on the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$, prove that the perpendicular from the centre to the tangent plane at P to the ellipsoid is $abc/\sqrt{(\lambda\mu)}$. Prove also that the perpendiculars to the tangent planes to the hyperboloid are

$$\sqrt{\left[\frac{(a^2-\lambda)(b^2-\lambda)(c^2-\lambda)}{\lambda(\lambda-\mu)}\right]},\quad \sqrt{\left[\frac{(a^2-\mu)(b^2-\mu)(c^2-\mu)}{\mu(\mu-\lambda)}\right]}$$

§ 14.14. Corresponding Points.

Two points (x, y, z) and $(\xi\ \eta,\ \zeta)$ on each of two coaxial conicoids whose equations are $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$, $(x^2/\alpha^2)+(y^2/\beta^2)+(z^2/\gamma^2)=1$ are said to correspond, when $\dfrac{x}{a}=\dfrac{\xi}{\alpha}, \dfrac{y}{b}=\dfrac{\eta}{\beta}, \dfrac{z}{c}=\dfrac{\zeta}{\gamma}$

Note : If the real points on one conicoid is to correspond to real points on the other, the two surfaces must be of the same nature and must be similarly placed.

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points on the first conicoid and $P'(\xi_1, \eta_1, \zeta_1)$ and $Q'(\xi_2, \eta_2, \zeta_2)$ be the corresponding points on the other conicoid which is confocal to the first then $PQ'=P'Q$.

Let P and Q lie on $\quad (x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1 \quad$ ...(i)

and let P' and Q' lie on its confocal $\quad \dfrac{x^2}{a^2-\lambda}+\dfrac{y^2}{b^2-\lambda}+\dfrac{z^2}{c^2-\lambda}=1 \quad$ ...(ii)

Now here from the definition of corresponding points, we have

$$\frac{x_1}{a}=\frac{\xi_1}{\sqrt{(a^2-\lambda)}},\ \frac{y_1}{b}=\frac{\eta_1}{\sqrt{(b^2-\lambda)}},\ \frac{z_1}{c}=\frac{\zeta_1}{\sqrt{(c^2-\lambda)}};$$

$$\frac{x_2}{a}=\frac{\xi_2}{\sqrt{(a^2-\lambda)}},\ \frac{y_2}{b}=\frac{\eta_2}{\sqrt{(b^2-\lambda)}},\ \frac{z_2}{c}=\frac{\zeta_2}{\sqrt{(c^2-\lambda)}}$$

and $\quad a^2-(a^2-\lambda)=\lambda=b^2-(b^2-\lambda)=c^2-(c^2-\lambda)$

Now $\quad (PQ')^2=(x_1-\xi_2)^2+(y_1-\eta_2)^2+(z_1-\zeta_2)^2$

$$=\left[\frac{a\xi_1}{\sqrt{(a^2-\lambda)}}-\frac{x_2\sqrt{(a^2-\lambda)}}{a}\right]^2+\left[\frac{b\eta_1}{\sqrt{(b^2-\lambda)}}-\frac{y_2\sqrt{(b^2-\lambda)}}{b}\right]^2$$

$$+\left[\frac{c\zeta_1}{\sqrt{(c^2-\lambda)}}-\frac{z_2\sqrt{(c^2-\lambda)}}{c}\right]^2$$

And $(P'Q)^2=(\xi_1-x_2)^2+(\eta_1-y_2)^2+(\zeta_1-z_2)^2$

$\therefore \quad (PQ')^2-(P'Q)^2$

$$=\sum\left[\left\{\frac{a\xi_1}{\sqrt{(a^2-\lambda)}}-\frac{x_2\sqrt{(a^2-\lambda)}}{a}\right\}^2-(\xi_1-x_2)^2\right]$$

$$=\sum\left[\left\{\frac{a^2\xi_1^2}{(a^2-\lambda)}+\frac{x_2^2(a^2-\lambda)}{a^2}-2x_2\xi_1\right\}-\left\{\xi_1^2+x_2^2-2\xi_1x_2\right\}\right]$$

$$=\sum\left[\left(\frac{a^2}{a^2-\lambda}-1\right)\xi_1^2+\left(\frac{(a^2-\lambda)}{a^2}-1\right)x_2^2\right]$$

$$=\sum\left[\frac{\lambda\xi_1^2}{a^2-\lambda}-\frac{\lambda x_2^2}{a^2}\right]=\lambda\left[\sum\frac{\xi_1^2}{a^2-\lambda}-\sum\frac{x_2^2}{a^2}\right]$$

$$=\lambda\left[\left(\frac{\xi_1^2}{a^2-\lambda}+\frac{\eta_1^2}{b^2-\lambda}+\frac{\zeta_1^2}{c^2-\lambda}\right)-\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}\right)\right]$$

$=\lambda\,[1-1]$, $\because$ P' lie on (ii) and Q lies on (i)

$=0$

$\therefore \quad PQ'=P'Q$

From here we conclude that *the distance between two points one on each of the two confocal conicoids is equal to the distance between the two corresponding points.*

Solved Examples on § 14.14.

Ex. 1. If P is a point on an ellipsoid and P′ is the corresponding point on a confocal whose parameter is λ, show that $(OP)^2-(OP')^2=\lambda$

Sol. Let $P(x_1, y_1, z_1)$ be a point on the ellipsoid

$$(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$$

and $P'(\xi_1, \eta_1, \zeta_1)$ be the corresponding point on its confocal

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1$$

Then $$\frac{\xi_1^2}{a^2-\lambda}+\frac{\eta_1^2}{b^2-\lambda}+\frac{\zeta_1^2}{c^2-\lambda}=1 \quad \text{...(i)}$$

Also from the definition of corresponding points, we have

$$\frac{x_1}{a}=\frac{\xi_1}{\sqrt{(a^2-\lambda)}},\ \frac{y_1}{b}=\frac{\eta_1}{\sqrt{(b^2-\lambda)}},\ \frac{z_1}{c}=\frac{\zeta_1}{\sqrt{(c^2-\lambda)}}, \quad \text{...(ii)}$$

$\therefore (OP)^2-(OP')^2=(x_1^2+y_1^2+z_1^2)-(\xi_1^2+\eta_1^2+\zeta_1^2)$ **(Note)**

$$=\left(\frac{a^2\xi_1^2}{a^2-\lambda}+\frac{b^2\eta_1^2}{b^2-\lambda}+\frac{c^2\zeta_1^2}{c^2-\lambda}\right)-(\xi_1^2+\eta_1^2+\zeta_1^2)$$

$$=\left(\frac{a^2}{a^2-\lambda}-1\right)\xi_1^2+\left(\frac{b^2}{b^2-\lambda}-1\right)\eta_1^2+\left(\frac{c^2}{c^2-\lambda}-1\right)\zeta_1^2$$

$$=\lambda\left[\frac{\xi_1^2}{a^2-\lambda}+\frac{\eta_1^2}{b^2-\lambda}+\frac{\zeta_1^2}{c^2-\lambda}\right]=\lambda\,(1), \text{ from (i)}$$

$=\lambda$

Hence proved.

****Ex. 2. OP, OQ, OR are conjugate diameters of an ellipsoid and P′, Q′, R′ are the points of a concentric sphere corresponding to P, Q, R. Prove that OP′, OQ′, OR′ are mutually perpendicular.**

Sol. Let the coordinates of P, Q and R be (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). Now if OP, OQ and OR are conjugate diameters of the ellipsoid

$$(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1 \quad \text{...(i)}$$

then we must have

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1,\ \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}=1,\ \frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2}=1$$

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0,\ \frac{x_2x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}=0,\ \frac{x_3x_1}{a^2}+\frac{y_3y_1}{b^2}+\frac{z_3z_1}{c^2}=0 \quad \text{...(ii)}$$

Now any sphere concentric with ellipsoid (i) can be taken as

$$x^2+y^2+z^2=r^2 \quad \text{or} \quad (x^2/r^2)+(y^2/r^2)+(z^2/r^2)=1 \quad \text{...(iii)}$$

Let $P'(\xi_1, \eta_1, \zeta_1)$, $Q'(\xi_2, \eta_2, \zeta_2)$ and $R'(\xi_3, \eta_3, \zeta_3)$ be the points of the sphere (iii) corresponding to the points P, Q and R respectively. Then according to definition we must have $x_1/a=\xi_1/r,\ y_1/b=\eta_1/r,\ z_1/c=\zeta_1/r$, etc.

Now direction ratios of OP', OQ' and OR' are

ξ_1, η_1, ζ_1 ; ξ_2, η_2, ζ_2 and ξ_3, η_3, ζ_3 respectively

Now $$\xi_1\xi_2+\eta_1\eta_2+\zeta_1\zeta_2=\frac{rx_1}{a}\cdot\frac{rx_2}{a}+\frac{ry_1}{b}\cdot\frac{ry_2}{b}+\frac{rz_1}{c}\cdot\frac{rz_2}{c}$$

$$=r^2\left(\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}\right)=\frac{r^2}{a^2}(0), \text{ from (ii)}$$

or $\xi_1 \xi_2 + \eta_1 \eta_2 + \zeta_1 \zeta_2 = 0$ *i.e.* OP' is perpendicular to OQ'

Similarly we can prove that OP' is perpendicular to OR' and OQ' is perpendicular to OR'.

Hence OP', OQ' and OR' are mutually perpendicular. Proved.

***Ex. 3. If the points P, Q, R on an ellipsoid correspond to the points P′, Q′, R′ on a coaxal ellipsoid and OP, OQ, OR are conjugate diameters, prove that OP′, OQ′ and OR′ are also conjugate diameters.**

Sol. Choose the coordinates of the points P, Q, R on the ellipsoid

$$(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1 \qquad ...(i)$$

as (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) respectively.

Let the coordinates of the points P', Q', R' on a coaxal ellipsoid

$$(x^2/\alpha^2)+(y^2/\beta^2)+(z^2/\gamma^2)=1 \qquad ...(ii)$$

be (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) and (ξ_3, η_3, ζ_3) respectively.

Now as OP, OQ and OR are conjugate diameters of (i), then

$$\Sigma (x_1^2/a^2)=1 \text{ etc. and } \Sigma (x_1 x_2/a^2)=0, \qquad ...(iii)$$

as in Ex. 2 above.

Also as P', Q', R' correspond to P, Q, R respectively, so by definition we have

$$x_1/a=\xi_1/\alpha,\ y_1/b=\eta_1/\beta,\ z_1/c=\zeta_1/\gamma \qquad ...(iv)$$

Now as P', Q', R' lie on (ii), so $\dfrac{\xi_1^2}{\alpha^2}+\dfrac{\eta_1^2}{\beta^2}+\dfrac{\zeta_1^2}{\gamma^2}=1$, etc

Also $\dfrac{\xi_1\xi_2}{\alpha^2}+\dfrac{\eta_1\eta_2}{\beta^2}+\dfrac{\zeta_1\zeta_2}{\gamma^2}$

$$=\frac{1}{\alpha^2}\left[\frac{\alpha x_1}{a}\cdot\frac{\alpha x_2}{a}\right]+\frac{1}{\beta^2}\left[\frac{\beta y_1}{b}\cdot\frac{\beta y_2}{b}\right]+\frac{1}{\gamma^2}\left[\frac{\gamma z_1}{c}\cdot\frac{\gamma z_2}{c}\right], \text{ from (iv)}$$

$$=\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0, \text{ from (iii)}$$

Similarly we can prove that

$$\frac{\xi_2\xi_3}{\alpha^2}+\frac{\eta_2\eta_3}{\beta^2}+\frac{\zeta_2\zeta_3}{\gamma^2}=0,\quad \frac{\xi_3\xi_1}{\alpha^2}+\frac{\eta_3\eta_1}{\beta^2}+\frac{\zeta_3\zeta_1}{\gamma^2}=0$$

These prove that OP', OQ' and OR' are conjugate diameters of (ii).

Hence proved.

****Ex. 4. Show that a umbilic on an ellipsoid corresponds to an umbilic on a confocal ellipsoid.**

Sol. Let the equations of an ellipsoid and its confocal be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad ...(i) \quad \text{and} \quad \frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}+\frac{z^2}{c^2-\lambda}=1 \quad ...(ii)$$

respectively.

Let $P(x_1, y_1, z_1)$ be an umbilic of the ellipsoid (i) and $P'(\xi_1, \eta_1, \zeta_1)$ be the point corresponding to P on the confocal (ii).

Then $$\frac{x_1}{a}=\frac{\xi_1}{\sqrt{(a^2-\lambda)}},\ \frac{y_1}{b}=\frac{\eta_1}{\sqrt{(b^2-\lambda)}},\ \frac{z_1}{c}=\frac{\zeta_1}{\sqrt{(c^2-\lambda)}} \quad ...(iii)$$

Also if $P(x_1, y_1, z_1)$ is an umbilic of (i), then

$$x_1=\frac{a\sqrt{(a^2-b^2)}}{\sqrt{(a^2-c^2)}},\quad y_1=0,\quad z_1=\frac{c\sqrt{(b^2-c^2)}}{\sqrt{(a^2-c^2)}} \quad ...(iv)$$

...See Ch. XI on Plane Sections.

Now from (iii) we have

$$\xi_1=\frac{x_1\sqrt{(a^2-\lambda)}}{a}=\frac{a\sqrt{(a^2-b^2)}}{\sqrt{(a^2-c^2)}}\cdot\frac{\sqrt{(a^2-\lambda)}}{a},\text{ from (iv)}.$$

$$=\frac{\sqrt{(a^2-\lambda)}\sqrt{(a^2-b^2)}}{\sqrt{(a^2-c^2)}}=\frac{\sqrt{(a^2-\lambda)}\sqrt{[(a^2-\lambda)-(b^2-\lambda)]}}{\sqrt{[(a^2-\lambda)-(c^2-\lambda)]}} \quad ...(v)$$

$$\eta_1=\frac{y_1\sqrt{(b^2-\lambda)}}{b}=0,\quad \because\ y_1=0 \text{ from (ii)} \quad ...(vi)$$

And $$\zeta_1=\frac{z_1\sqrt{(c^2-\lambda)}}{c}=\frac{c\sqrt{(b^2-c^2)}}{\sqrt{(a^2-c^2)}}\cdot\frac{\sqrt{(c^2-\lambda)}}{c},\text{ from (iv)}$$

$$=\frac{\sqrt{(c^2-\lambda)}\sqrt{(b^2-c^2)}}{\sqrt{(a^2-c^2)}}=\frac{\sqrt{(c^2-\lambda)}\sqrt{[(b^2-\lambda)-(c^2-\lambda)]}}{\sqrt{[(a^2-\lambda)-(c^2-\lambda)]}} \quad ...(vii)$$

From (v), (vi) and (vii) it is evident that $P'(\xi_1, \eta_1, \zeta_1)$ is an umbilic of the confocal (ii). Hence proved.

****Ex. 5. P and Q are any points on a generator of a hyperboloid and P′ and Q′ are the corresponding points on a second hyperboloid. Prove that P′ and Q′ lie on a generator of the latter.**

Sol. Let P and Q be the points (x_1, y_1, z_1) and (x_2, y_2, z_2) on the hyperboloid $$(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1 \quad ...(i)$$

Let P' and Q' be the points corresponding to P and Q on the hyperboloid

$$(x^2/\alpha^2)+(y^2/\beta^2)-(z^2/\gamma^2)=1$$

and let the coordinates of P' and Q' be (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2) respectively.

Then from definition of the corresponding points we have

$$\frac{x_1}{a}=\frac{\xi_1}{\alpha},\ \frac{y_1}{b}=\frac{\eta_1}{\beta},\ \frac{z_1}{c}=\frac{\zeta_1}{\gamma} \quad ...(iii)$$

and $$\frac{x_2}{a}=\frac{\xi_2}{\alpha},\ \frac{y_2}{b}=\frac{\eta_2}{\beta},\ \frac{z_2}{c}=\frac{\zeta_2}{\gamma} \quad ...(iv)$$

Now as P and Q lie on a generator of the hyperboloid (i), each lies on the tangent plane to (i) at the other and the condition for the same is

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-\frac{z_1z_2}{c^2}=1.$$

which with the help of (iii) and (iv) transforms into

$$\frac{\xi_1\xi_2}{\alpha^2}+\frac{\eta_1\eta_2}{\beta^2}-\frac{\zeta_1\zeta_2}{\gamma^2}=1$$

This shows that the point P' (or Q') lies on the tangent plane to the hyperboloid (ii) at Q' (or P').

Hence P' and Q' lie on a generator of the hyperbolid given by (ii).

Hence proved.

EXERCISE

Ex. P,Q are two points on a generator of hyperboloid and P', Q' are the corresponding points on a confocal hyperboloid, show that P', Q' is a generator of the latter and that $PQ = P'Q'$.

****§ 14.15. Focus and Directrix.**

There are two definitions of a conicoid which correspond to the focus and directrix definition of a conic.

Definition. I. This definition is due to **Salmon** and is as follows :

The conicoid is the locus of a point such that the square on its distance from a given point is in a constant ratio to the rectangle contained by its distances from two fixed planes.

The given (fixed) point is called the **focus** and the line of intersection of the given fixed planes is called the **directrix.**

The equation of the locus is evidently of the form

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2$$
$$=k^2\left[(lx+my+nz+p)(l'x+m'y+n'z+p')\right],$$

which is the equation of a conicoid.

This is of the form $\lambda\,\varphi - uv = 0$,

where $\varphi=(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2$ and $u=0,\ v=0$ represent planes.

Definition II. This definition according to MacCullagh is as follows :

The conicoid is the locus of a point where distance from a fixed pointt is in a constant ratio to its distance from a given straight line, measured parallel to a given plane.

The fixed point is called the **focus** and the given line the **directrix.**

Choose rectangular axes such that the given plane be the xy- plane *i.e.* the plane $z=0$ and the point of intersection of the given line and the given plane be the origin. Let the fixed point be (α, β, γ) and the equations of the given line be

$$x/l = y/m = z/n.$$

Now if (ξ, η, ζ) be any point on the locus, then the plane through it parallel to the plane $z=0$ meets the given line in the point $(l\zeta/n,\ m\zeta/n,\ \zeta)$.

$\therefore$ The distance of (ξ, η, ζ) from the line measured parallel to the given plane

$$=\sqrt{\left[\left(\xi-\frac{l\zeta}{n}\right)^2+\left(\eta-\frac{m\zeta}{n}\right)^2+(\zeta-\zeta)^2\right]}=\left[\left(\xi-\frac{l\zeta}{n}\right)^2+\left(\eta-\frac{m\zeta}{n}\right)^2\right]$$

Hence, according to definition, the equation of the locus is

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=k^2\left[\left(x-\frac{lz}{n}\right)^2+\left(y-\frac{mz}{n}\right)^2\right]$$

which represents a conicoid.

This is of the form $\lambda\,\varphi-(u^2+v^2)=0$.

where $\varphi\equiv(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2$ and $u=0$, $v=0$ represent planes.

Now in either of the two above cases if $S=0$ represents the locus of the point, then $S-\lambda\,\varphi=0$ represents a pair of planes which are real in I and imaginary in II but their line of intersection $u=0=v$ is real in both the cases.

Rule for finding the focus and directrix of a conicoid.

From above we find that if $S=0$ is the equation to a conicoid and $\lambda, \alpha, \beta, \gamma$ are constants such that the equation $S-\lambda\,\varphi=0$ represents two planes (real or imaginary), then (α, β, γ) is a focus and the line of intersection of the two planes is the corresponding directrix.

Lemma. If the equation $F(x, y, z)=0$ represents a pair of planes, the equations $\dfrac{\partial F}{\partial x}=0$, $\dfrac{\partial F}{\partial y}=0$, $\dfrac{\partial F}{\partial z}=0$ represents three planes passing through their lines of intersection.

****§ 14.16. Foci of the conicoid $ax^2+by^2+cz^2=1$.**

From § 14.15 above, if (α, β, γ) is the focus, then $S-\lambda\,\varphi$

i.e. $(ax^2+by^2+cz^2-1)-\lambda\,[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]$ ...(i)

must be the product of two linear factors.

$\therefore$ λ must be equal to a or b or c (See Lemma above)

Case I. When $\lambda=a$, then (i) becomes

$$(ax^2+by^2+cz^2-1)-a\,[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]$$

or $$(b-a)\,y^2+(c-a)\,z^2+2a\alpha x+2a\beta y+2a\gamma z-a\,(\alpha+\beta^2+\gamma^2)-1$$

or $$(b-a)\left[y-\frac{a\beta}{b-a}\right]^2+(c-a)\left[z+\frac{a\gamma}{c-a}\right]^2-\frac{ab\beta^2}{b-a}-\frac{ac\gamma^2}{c-a}-1+2a\alpha x-a\alpha^2$$

Now if this may be resolved into two linear factors, we must have

$$\alpha=0 \quad\text{and}\quad \frac{ab\beta^2}{b-a}+\frac{ac\gamma^2}{c-a}+1=0$$

i.e. $$\alpha=0 \quad\text{and}\quad \frac{\beta^2}{(1/b)-(1/a)}+\frac{\gamma^2}{(1/c)-(1/a)}=1$$

Case II. When $\lambda=b$, then as in Case I above we must have

$$\beta=0 \quad\text{and}\quad \frac{\gamma^2}{(1/c)-(1/b)}+\frac{\alpha^2}{(1/a)-(1/b)}=1$$

Case III. When $\lambda=c$, then as in Case I above we must have

$$\gamma = 0 \quad \text{and} \quad \frac{\alpha^2}{(1/a)-(1/c)} + \frac{\beta^2}{(1/b)-(1/c)} = 1$$

Thus there are three conics, one in each principal plane on which the foci lie.

§ 14.17. Foci of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1,\ a>b>c$

As in § 14.15 Page 27 of this chapter, if (α, β, γ) is the focus, then $S - \lambda\varphi$

i.e.
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)-\lambda\,[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2] \qquad \text{...(i)}$$

must be the product of two linear factors.

$\therefore$ λ must be equal to $1/a^2$ or $1/b^2$ or $1/c^2$ ...See Lemma § 14.15

Case I. When $\lambda = 1/a^2$, then (i) becomes

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)-\frac{1}{a^2}[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]$$

or
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1-\frac{1}{a^2}[x^2+y^2+z^2-2\alpha x-2\beta y-2\gamma z+\alpha^2+\beta^2+\gamma^2]$$

or
$$y^2\left[\frac{1}{b^2}-\frac{1}{a^2}\right]+z^2\left[\frac{1}{c^2}-\frac{1}{a^2}\right]+\frac{2\alpha x}{a^2}+\frac{2\beta y}{a^2}+\frac{2\gamma z}{a^2}-\frac{1}{a^2}(\alpha^2+\beta^2+\gamma^2)-1$$

or
$$\left(\frac{a^2-b^2}{a^2b^2}y^2+\frac{2\beta y}{a^2}\right)+\left(\frac{a^2-c^2}{a^2c^2}z^2+\frac{2\gamma z}{a^2}\right)+\frac{2\alpha x}{a^2}-\frac{(\alpha^2+\beta^2+\gamma^2)}{a^2}-1$$

or
$$\frac{a^2-b^2}{a^2b^2}\left[y+\frac{b^2\beta}{a^2-b^2}\right]^2+\frac{a^2-c^2}{a^2c^2}\left[z+\frac{c^2\gamma}{a^2-c^2}\right]^2$$
$$-\frac{b^2\beta^2}{a^2(a^2-b^2)}-\frac{c^2\gamma^2}{a^2(a^2-c^2)}+\frac{2\alpha x}{a^2}-\frac{\alpha^2+\beta^2+\gamma^2}{a^2}-1$$

or
$$\frac{a^2-b^2}{a^2b^2}\left[y+\frac{b^2\beta}{a^2-b^2}\right]^2+\frac{a^2-c^2}{a^2c^2}\left[z+\frac{c^2\gamma}{a^2-c^2}\right]^2$$
$$-\frac{\beta^2}{a^2}\left[\frac{b^2}{a^2-b^2}+1\right]-\frac{\gamma^2}{a^2}\left[\frac{c^2}{a^2-c^2}+1\right]+\frac{2\alpha x}{a^2}-\frac{\alpha^2}{a^2}-1$$

or
$$\frac{a^2-b^2}{a^2b^2}\left[y+\frac{b^2\beta}{a^2-b^2}\right]^2+\frac{a^2-c^2}{a^2c^2}\left[z+\frac{c^2\gamma}{a^2-c^2}\right]^2$$
$$-\frac{\beta^2}{a^2-b^2}-\frac{\gamma^2}{a^2-c^2}+\frac{2\alpha x}{a^2}-\frac{\alpha^2}{a^2}-1$$

Now if this may be resolved into two linear factors, we have

$$\alpha=0 \quad \text{and} \quad \frac{\beta^2}{a^2-b^2}+\frac{\gamma^2}{a^2-c^2}+1=0 \qquad \text{...(ii)}$$

Case II. When $\lambda=1/b^2$, then as in Case I above we get

$$\beta=0 \quad \text{and} \quad \frac{\gamma^2}{b^2-c^2}+\frac{\alpha^2}{b^2-a^2}+1=0 \qquad \text{...(iii)}$$

Case III. When $\lambda=1/c^2$, then as in Case I above, we get

$$\gamma=0 \quad \text{and} \quad \frac{\alpha^2}{c^2-a^2}+\frac{\beta^2}{c^2-b^2}+1=0 \qquad \text{...(iv)}$$

$\therefore$ From (ii), (iii) and (iv) we find that if the surface is the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$, the conics on which the focus (α, β, γ) lies are

$$x=0, -\frac{y^2}{a^2-b^2}-\frac{z^2}{a^2-c^2}=1 \qquad \text{...(v)}$$

$$y=0, \frac{x^2}{a^2-b^2}-\frac{z^2}{b^2-c^2}=1 \qquad \text{...(vi)}$$

$$z=0, \frac{x^2}{a^2-c^2}-\frac{y^2}{b^2-c^2}=1 \qquad \text{...(vii)}$$

(v), (vi) and (vii) are known as **focal conics,** of which (v) is an **imaginary ellipse,** (vi) is a **real hyperbola** and (vii) is a **real ellipse.**

Note : It is obvious that confocal conicoids have the same focal conics.

Solved Examples on Focal Conics.

Ex. 1. Find the focal conics of the cone $ax^2+by^2+cz^2=0$.

Sol. If the point (α, β, γ) is a focus of the given cone, then the equation

$$S-\lambda\varphi=0$$

or $$(ax^2+by^2+cz^2)-\lambda\,[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]=0 \qquad \text{...(i)}$$

represents a pair of planes.

Hence λ must be equal to a or b or c. ...See § 14.16 P. 28

Case I. When $\lambda=a$, then (i) becomes

$$ax^2+by^2+cz^2-a\,[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]=0$$

or $$(b-a)\,y^2+(c-a)\,z^2+2a\alpha x+2a\beta y+2a\gamma z-a\,(\alpha^2+\beta^2+\gamma^2)=0.$$

or $$(b-a)\left[y+\frac{a\beta}{b-a}\right]^2+(c-a)\left[z+\frac{a\gamma}{c-a}\right]^2-\frac{ab\beta^2}{b-a}-\frac{ac\gamma^2}{c-a}+2a\alpha x-a\alpha^2=0$$

If this represents two planes, then we must have

$$\alpha=0 \quad \text{and} \quad \frac{b\beta^2}{b-a}+\frac{c\gamma^2}{c-a}=0$$

Hence the focus (α, β, γ) lies on the locus given by

$$x = 0 \quad \text{and} \quad \frac{y^2}{(1/a)-(1/b)} + \frac{z^2}{(1/a)-(1/c)} = 0$$

which represents a pair of lines, real or imaginary; depending on a, b and c.

Case II. When $\lambda = b$, the locus of the focus (α, β, γ) as in Case I above can be found as $$y = 0, \frac{z^2}{(1/c)-(1/b)} + \frac{x^2}{(1/a)-(1/b)} = 0,$$

which also represents a pair of lines, real or imaginary, depending on a, b and c.

Case III. When $\lambda = c$, the locus of the focus (α, β, γ) as in Case I above can be found as $$z = 0, \frac{x^2}{(1/c)-(1/a)} + \frac{y^2}{(1/b)-(1/c)} = 0,$$

which also represents a pair of lines, real or imaginary, depending on a, b and c.

It will be seen that whatever be the relation, of magintude between a, b and c only one of these three pairs of lines is real and are called the **focal lines** of the given cone.

Ex. 2. Find the focal conics of the paraboloid $ax^2 + by^2 = 2z$.

Sol. If the point (α, β, γ) is a focus of the given paraboloid, then the equation $S - \lambda\varphi = 0$

or $$(ax^2 + by^2 - 2z) - \lambda [(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2] = 0 \quad \text{...(i)}$$

represents a pair of planes.

Hence λ must be equal to a and b. ...See § 14.16 Page 28.

Case I. When $\lambda = a$, the equation (i) becomes

$$ax^2 + by^2 - 2z - a [(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2] = 0$$

or $$(b-a) y^2 + 2a\beta y + 2a\alpha x - a\alpha^2 - a\beta^2 + 2(a\gamma - 1) z - a\gamma^2 = 0$$

or $$(b-a)\left[y + \frac{a\beta}{b-a}\right]^2 - a\left[z - \frac{a\gamma-1}{a}\right]^2 - \frac{ab\beta^2}{b-a} - \frac{2a\gamma-1}{a} + 2a\alpha x - a\alpha^2 = 0$$

If this represents a pair of planes, then we must have

$$\alpha = 0 \quad \text{and} \quad \frac{ab\beta^2}{b-a} + \frac{2a\gamma - 1}{a} = 0$$

or $$\alpha = 0 \quad \text{and} \quad \frac{\beta^2}{(1/a)-(1/b)} + 2\gamma - \frac{1}{a} = 0 \quad \text{...(ii)}$$

This shows that the locus of the focus (α, β, γ) is

$$x = 0, \frac{y^2}{(1/a)-(1/b)} + 2z - \frac{1}{a} = 0,$$

which represents a parabola known as **focal parabola.**

Case II. When $\lambda = b$, the equation (i) becomes

$$(a-b) x^2 + 2b\alpha x - b\alpha^2 + 2b\beta y - b\beta^2 - bz^2 + 2(b\gamma - 1) z - b\gamma^2 = 0$$

or $\quad (a-b)\left[x+\frac{b\alpha}{a-b}\right]^2-b\left[z-\frac{b\gamma-1}{b}\right]^2-\frac{ab\alpha^2}{a-b}-\frac{2b\gamma-1}{b}+2b\beta y-b\beta^2=0$

If this represents two planes, then we must have

$$\beta=0 \quad \text{and} \quad \frac{ab\alpha^2}{a-b}+\frac{2b\gamma-1}{b}=0$$

This shows that the locus of the focus (α, β, γ) is

$$y=0, \frac{x^2}{(1/b)-(1/a)}+2z-\frac{1}{b}=0,$$

which also represents a parabola known as **focal parabola.**

Exercises on Chapter XIV

Ex. 1. Prove that the locus of the point of intersection of three planes mutually at right angles, each of which touches a confocal of ellipsoid is a sphere. *(Gorakhpur 97)*

Ex. 2. If λ and μ are the parameters of the confocal hyperboloid through a point P of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$, prove that the perpendicular from the centre is the tangent plane at P to the ellipsoid is $abc/\sqrt{(\lambda\mu)}$. *(Gorakhpur 97)*

(A) VERY SHORT AND SHORT ANSWER TYPE QUESTIONS

Ch. I. System of Coordinates.

1. Name the different systems of coordinates you have read in Analytical Geometry (Solid Geometry) of three dimensions.

2. Find the distance of the point (1, 2, 3) from the axes.
Ans. $\sqrt{(13)}, \sqrt{(10)}, \sqrt{5}$

3. Find the spherical polar coordinates of the point whose cartesian coordinates are (1, 2, 3). [See Ex. 1 (a) Page 5]

4. Find the cylindrical coordinates of the point whose cartesian coordinates are (2, 3, 5). [See Ex. 1 (b) Page 5]

5. Find the distance between the points (2, 0, – 5) and (0, 7, – 3).
(Kanpur 2001) **Ans.** $\sqrt{(57)}$

6. Find the locus of a point which is at a distance r from the point (a, b, c).
[See Ex. 1 (b) Page 8]

7. Find the coordinates of a point which is equidistant from the four points O, A, B and C, where O is the origin and A, B, C are the points on the axes of x, y, z respectively at distances a, b, c from the origin. [See Ex. 4 Page 9]

8. Find the centroid of the tetrahedron having the points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) for its vertices. *(Kanpur 2000)*

Ch. II. Direction-cosines and Projections.

9. Define the direction-cosines of a line. *(Purvanchal 97)*

10. If α, β, γ be the angles which a line makes with the positive directions of the axes, prove that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$. *(Kanpur 2001, 98)*

11. If l, m, n be the direction-cosines of a line OP, O being the origin and $OP = r$, then find the coordinates of P. [See § 2·03 cor. 1 P. 20]

12. Find the direction-cosines of the coordinates axes. Also find the direction-cosines of a line which is equally inclined to the positive direction of the axes. *(Kanpur 2000)*

13. If α, β, γ be the angles which a given line makes with the positive directions of the axes, then prove that $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$.
(Gorakhpur 2001)

14. Write down the direction-cosines of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2). [See § 2·08 Page 23]

15. What is the projection of a line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) on another line whose direction-cosines are l, m, n.
[See § 2·09 P. 23]

16. Find the direction cosines of the line joining (1, 2, 3) and (– 2, 3, 1).
(Purvanchal 98) **Ans.** –3, 1, – 2

17. Find the direction-cosines of the line joining (1, 1, 0) and (0, 1, 0).

(Purvanchal 99) **Ans.** – 1, 0, 0

18. What is the value of $\cos\theta$ if θ be the angle between two lines whose direction-cosines are l_1, m_1, n_1 and l_2, m_2, n_2? [See § 2·10 P. 27]

19. Write down the condition of perpendicularity of two lines whose direction cosines are given.

20. What is the condition of parallelism of two lines ?

21. Find the direction cosines of the line parpendicular to the two lines whose direction ratios are 1, – 2, – 2 and 0, 2, 1. [See Ex. 11 (b) P. 37]

22. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube. Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = 4/3$. *(Kanpur 98)*

23. If l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of two mutually perpendicular lines, what are the direction cosines of the line perpendicular to both of them ? *(Purvanchal 2000)*

Ans. $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1$, where $l_1l_2 + m_1m_2 + n_1n_2 = 0$

24. Prove that three concurrent lines with direction cosines (l_1, m_1, n_1), (l_2, m_2, n_2) and (l_3, m_3, n_3) are coplanar, if $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$.

(B) OBJECTIVE TYPE QUESTIONS

(I) MULTIPLE CHOICE TYPE QUESTIONS :

Select (a), (b), (c) or (d) whichever is correct :

Ch. I. Systems of Coordinates

1. The point (2, – 3, – 4) lies in the octant :

(i) $OX'YZ'$ (ii) $OXY'Z'$ (iii) $OX'Y'Z'$ (iv) $OX'Y'Z$

2. The ratio in which the line joining the points (3, 5, – 7) and (– 2, 1, 8) is divided by the yz-plane is

(i) 4 : 3 (ii) 3 : 2 (iii) 2 : 3 (iv) 3 : 4

3. Given three collinear points A (3, – 4, 5), B (– 3, 4, – 5) and C (0, 0, 0), then B divides AC in the ratio

(i) 1 : 1 (ii) 1 : 2 (iii) 2 : 1 (iv) – 2 : 1

4. If the following equations, which of the equation is the equation of xy-plane

(i) $z = 0$ (ii) $x = 0$ (iii) $y = 0$ (iv) $x = 0$ and $y = 0$

(Kanpur 2001)

5. The distance between the points (0, 7, 10) and (– 1, 6, 6) is

(i) $2\sqrt{3}$ (ii) $4\sqrt{3}$ (iii) $3\sqrt{2}$ (iv) none of these

6. The four points (5, –1, 1), (7, – 4, 7), (1, – 6, 10) and (–1, – 3, 4) are the vertices of a

(i) square (ii) rhombus (iii) rectangle (iv) parallelogram

7. The point equidistant from the four points (0, 0, 0), $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is

(i) (a, b, c) (ii) $(a/2, b/2, c/2)$

(iii) $(a/3, b/3, c/3)$ (iv) none of these

8. The x-coordinate of the centroid of the tetrahedron, having the points (x_i, y_i, z_i), $i = 1, 2, 3, 4$ for its vertices is

(i) $\frac{x_1 + x_2 + x_3}{3}$ (ii) $\frac{x_1 + x_2 + x_3 + x_4}{4}$

(iii) $\frac{x_1 + x_2 + x_3 + x_4}{3}$ (iv) $\frac{x_1 + x_2 + x_3}{4}$

9. Which of the following equations is the equations of z-axis

(i) $x = 0, y = 0$ (ii) $x = 0, z = 0$

(iii) $y = 0, z = 0$ (iv) $x = 0, y = 0, z = 0$

10. The points $(1, -3, 4)$, $(2, 4, -1)$ and $(3, 11, -6)$ are

(i) vertices of a rhombus (ii) vertices of a square

(iii) collinear (iv) none of these

11. In how many compartments the coordinate planes divide the whole space ?

(i) 3 (ii) 4 (iii) 6 (iv) 8

Ch. II. Direction cosines and projections

12. In the following options which of the options is the direction cosine of the x-axis ?

(i) 1, 0, 0 (ii) 0, 1, 0 (iii) 0, 0, 1 (iv) 0, 1, 1

13. If l, m, n be the direction-cosines of a line, then $l^2 + m^2 + n^2 =$

(Kanpur 2001)

(i) 0 (ii) 1 (iii) 2 (iv) none of these

14. The direction-cosines of a line which is equally inclined to the positive direction of the axes are

(i) 1, 1, 1 (ii) $1/\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2}$

(iii) $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ (iv) none of these

15. Which of the following triplets give the direction cosines of a line ?

(i) 1, 1, 1 (ii) 1, 0, 2

(iii) 0, 0, 0. (iv) $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$

16. If the direction cosines of a line are $1/a, 1/a, 1/a$, then

(i) $a = 1$ (ii) $0 < a < 1$ (iii) $a = \pm\sqrt{3}$ (iv) none of these

17. If α, β, γ be the angles which a line makes with the positive directions of the axes, then $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma =$

(i) 0 (ii) 1 (iii) 2 (iv) none of these

18. If $\alpha, \beta, \gamma, \delta$ be the angles made by a line with four diagonals of a cube, then $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta =$

(i) 0 (ii) 1 (iii) 2/3 (iv) 4/3

19. The direction cosines of a line which makes equal angles with the three mutually perpendicular lines having direction cosines l_1, m_1, n_1; l_2, m_2, n_2 and l_3, m_3, n_3 are

(i) $\frac{l_1+m_1+n_1}{\sqrt{3}}, \frac{l_2+m_2+n_2}{\sqrt{3}}, \frac{l_3+m_3+n_3}{\sqrt{3}}$

(ii) $\frac{l_1+l_2+l_3}{\sqrt{3}}, \frac{m_1+m_2+m_3}{\sqrt{3}}, \frac{n_1+n_2+n_3}{\sqrt{3}}$

(iii) $l_1+l_2+l_3, m_1+m_2+m_3, n_1+n_2+n_3$

(iv) none of these

20. The angle between the lines whose direction ratios are $\sqrt{3}-1, \sqrt{3}+1, 4$ and $1, -1, 2$ is

(i) $\pi/6$ **(ii)** $\pi/4$ **(iii)** $\pi/3$ **(iv)** $\pi/2$

21. If the vertices of a triangle are $A(-1, 3, 2)$, $B(2, 3, 5)$ and $C(3, 5, -2)$, show that $\angle A =$

(i) $\pi/6$ **(ii)** $\pi/4$ **(iii)** $\pi/3$ **(iv)** $\pi/2$

22. If l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of two lines, then these lines are parallel if

(i) $(l_1+l_2)+(m_1+m_2)+(n_1+n_2)=0$

(ii) $l_1l_2+m_1m_2+n_1n_2=0$

(iii) $\frac{l_1}{m_1}+\frac{l_2}{m_2}+\frac{l_3}{m_3}=0$

(iv) $\frac{l_1}{l_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2}$

23. The direction cosines of a line equally inclined to the axes are

(i) 1, 0, 1 **(ii)** 1, 1, 1

(iii) $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ **(iv)** none of these

24. If l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of two mutually perpendicular lines, then

(i) $l_1+m_1+n_1=l_2+m_2+n_2$ **(ii)** $l_1l_2+m_1m_2+n_1n_2=0$

(iii) $l_1l_2+m_1m_2+n_1n_2=1$ **(iv)** $l_1/l_2=m_1/m_2=n_1/n_2$

25. The acute angle between the lines where direction cosines are given by the relations $l+m+n=0$ and $l^2+m^2-n^2=0$ is

(i) $\pi/6$ **(ii)** $\pi/4$ **(iii)** $\cos^{-1}(3/5)$ **(iv)** $\pi/3$

26. If AB be any segment, PQ be any line and θ be the angle between AB and PQ, then the projection of AB on PQ is

(i) $AB \sin\theta$ **(ii)** $AB \cos\theta$ **(iii)** $AB \tan\theta$ **(iv)** none of these

27. If $A(1, 2, 3)$ and $B(a, -2, 1)$ be two points, such that OA and OB are mutually perpendicular, where O is the origin, then $a =$

(i) 0 **(ii)** -1

(iii) 1 **(iv)** no value of a exists

(II) TRUE & FALSE TYPE QUESTIONS :

Write 'T' or 'F' according as the following statement is True or False :

Ch. I. System of Coordinates.

1. The equation of yz-plane is $y = 0$ and $z = 0$.

2. The distance of the point (x, y, z) from yz-plane is x.

3. The distance of the point $(1, 2, 3)$ from the origin is 6.

4. The point $(-3, 4, 5)$ lies in the octant $OX'YZ$.

5. The four points $(5, -1, 1)$, $(7, -4, 7)$, $(1, -6, 10)$ and $(-1, -3, 4)$ are the vertices of a square.

6. The point $(a/3, b/3, c/3)$ is equidistant from the four points $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$

7. The coordinate planes divide the whole space in six compartments.

Ch. II. Direction-cosines and Projections.

8. The direction cosines of y-axis are $(1, 0, 1)$.

9. The direction cosines of two parallel lines are proportional.

10. The direction cosines and direction ratios are the same.

11. The direction cosines of two perpendicular lines are equal.

12. The direction ratios of a line which is equally inclined to the positive direction of axes are 1, 1, 1.

13. If α, β, γ be the angles which a line makes with positive directions of axes, then $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 1$.

14. If l, m, n be the direction-cosines of a line, then $l^2 + m^2 + n^2 = 0$

15. If $\alpha, \beta, \gamma, \delta$ be the angles which a line makes with the four diagonals of a cube, then

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = 1$$

16. The projections of a line joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2) on another line whose direction cosnies are l, m, n is

$$(x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n.$$

(III) FILL IN THE BLANKS TYPE QUESTIONS :

Fill in the blanks in the following :

Ch. I. System of Coordinates :

1. The coordinates planes divide the whole space in .... compartments calld octants.

2. The equation of the xy-plane is ....

3. The y-coordinate of any point on x-axis is ....

4. The x-coordinate of the centroid of the triangle whose vertices are (x_i, y_i, z_i), $i = 1, 2, 3$ is ....

5. The point $(-2, -3, -7)$ lies in the octant ....

6. The points $(1, -3, 4)$, $(2, 4, -1)$ and $(3, 11, -6)$ are ....

7. The four points $(5, -1, 1)$, $(7, -4, 7)$, $(1, -6, 10)$ and $(-1, -3, 4)$ are the vertices of a ....

Ch. II. Direction-cosines and Projections.

8. If O be the origin, $OP = r$ and the direction cosines of the line OP be l, m, n then the coordinates of P are ....

9. If a, b, c be the direction ratios of a line, then its direction cosines are obtained by dividing each of these ratios by ....

10. If l, m, n be the direction cosines of a line, then $l^2 + m^2 + n^2 =$ .....

11. The direction cosines of a line equally inclined to the axes are ....

12. If l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of two parallel lines, then ....

13. If a_1, b_1, c_1 and a_2, b_2, c_2 be the direction ratios of two perpendicular lines, then .... $= 0$.

14. The angle θ between two lines whose direction-cosines are l_1, m_1, n_1 and l_2, m_2, n_2 is given by .... $= l_1l_2 + m_1m_2 + n_1n_2$.

15. The projections of a line joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2) on another line whose direction cosines are l, m, n is ....

16. If $\alpha, \beta, \gamma, \delta$ be the angles which a line makes with the four diagonals of a cube, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta =$ ....

ANSWERS TO OBJECTIVE TYPE QUESTIONS (Chs. I, II) :

(I) Multiple choice Type :

1.	(ii);	2.	(iii)	3.	(iv);	4.	(i);	5.	(iii);	6.	(ii);
7.	(ii);	8.	(ii);	9.	(i);	10.	(iii);	11.	(iv);	12.	(i);
13.	(ii);	14.	(iii);	15.	(iv);	16.	(iii).	17.	(iii);	18.	(iii)
19.	(ii);	20.	(iii)	21.	(iv);	22.	(iv);	23.	(iii);	24.	(ii);
25.	(iv);	26.	(ii);	27.	(iii).						

(II) True & False Type :

1.	F;	2.	T;	3.	F;	4.	T;	5.	F;	6.	F;	7.	F;
8.	F;	9.	T;	10.	F;	11.	F;	12.	T;	13.	F;	14.	F;
15.	F;	16.	T.										

(III) Fill in the blanks Type :

1. eight; 2. $z = 0$; 3. 0; 4. $(x_1 + x_2 + x_3)/3$; 5. $OX'Y'Z'$; 6. collinear;

7. rhombus; 8. (lr, mr, nr); 9. $\sqrt{(a^2 + b^2 + c^2)}$; 10. 1;

11. $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$; 12. $l_1/l_2 = m_1/m_2 = n_1/n_2$;

13. $a_1a_2 + b_1b_2 + c_1c_2$; 14. $\cos\theta$;

15. $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$; 16. 4/3.

(A) VERY SHORT AND SHORT ANSWER TYPE QUESTIONS

Ch. III The Plane :

1. Define a plane.
2. Write down the general equation of a plane through a given point.
3. What is the normal form of the equation of a plane.
4. Reduce the general equation of a plane to its normal form.
5. Find the intercepts made on the coordinate axes by the plane
$5x-3y+2z=8$ **Ans.** $8/5, -8/3, 4$ *(Kanpur 2001)*
6. A plane meets the co-ordinate axes in A, B and C such that the centroid of the triangle ABC is the point (p, q, r), show that the equation of the plane is $(x/p)+(y/q)+(z/r)=3$. [See Ex. 3 Page 4]
7. Find out the equation of the plane passing through $P(3, 4, 5)$ perpendicular to the line OP, where O is the origin. *(Purvanchal 99)*
Ans. $3x+4y+5z=50$ [See Ex. 5 Page 5]
8. Find the angle between the planes $3x+5y-2z+1=0$ and $2x+4y+9z+7=0$. **Ans.** $\cos^{-1}[8\sqrt{(3 \cdot 38)}]$
[See Ex. 1 P. 7] *(Kanpur 2001)*
9. Find the equation of the plane through the point $(1, 2, 3)$ and parallel to the plane $3x+5y-7z=9$. *(Purvanchal 98)*
[See Ex. 2 (b) Page 7] **Ans.** $3x+5y-7z+8=0$
10. Write the equation of the plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). *(Purvanchal 2000)*
[See § 3·06 Page 8]
11. Find the equation of the plane through the points $(0, -1, 0)$, $(1, 1, 1)$ and $(2, 1, -1)$. **Ans.** $4x-3y+2z=2$
12. What is the equation of the plane parallel to yz-plane at a distance c from it ? **Ans.** $x=c$
13. Write down the general equation of the plane perpendicular to zx-plane.
Ans. $ax+cy+d=0$
[See § 3·07 (c) Page 11]
14. Find the equation of the plane passing through the line of intersection of the planes $2x+3y-4z=1$, $3x-y+z+2=0$ and the point $(0, 1, 1)$.
[See Ex. 1 (b) Page 12]
Ans. $5x+2y-3z+1=0$
15. Find the equation of the plane through the line of intersection of the planes $ax+by+cz+d=0$, $a'x+b'y+c'z+d'=0$ and parallel to x-axis.
[See Ex. 5 Page 15] **Ans.** $a'(by+cz+d)=a(b'y+c'z+d')$
16. Find the equation of system of planes perpendicular to the line with direction ratios a, b, c. **Ans.** $ax+by+cz+k=0$ *(Purvanchal 97)*

17. What is the distance of the point $P(x_1, y_1, z_1)$ from the plane $x \cos \alpha + y \cos \beta + z \cos \gamma = p$? **Ans.** $x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p$

18. Find the distance between the planes $2x - 2y + z + 3 = 0$ and $4x - 4y + 2z + 5 = 0$? **Ans.** $1/6$ *(Purvanchal 2001)*

(See Ex. 5 (b) Page 23)

19. Write down the equations of the planes bisecting the angles between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.

[See § 3·12 Page 26]

20. The plane $lx + my = 0$ is rotated through an angle α about its line of intersection with the plane $z = 0$. Prove that the equation to the plane in its new position is $lx + my \pm z \sqrt{(l^2 + m^2)} \tan \alpha = 0$. [See Ex. 2 Page 29]

21. Prove that the points $(1, 2, 3), (0, 5, 1)$ are on the same side of the plane $y + z - 4 = 0$. *(Kanpur 2001)*

22. A variable plane is at a constant distance $3p$ from the origin and meets the axes in A, B and C. Prove that the locus of the centroid of the triangle ABC is $x^{-2} + y^{-2} + z^{-2} = p^{-2}$. *(Meerut 2001)* [See Ex. 4 (d) Page 41]

23. What is the condition that the general homogeneous equation of second degree in x, y and z viz $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ represents a pair of planes ? [See § 3·14 Page 30]

24. Find the angle between the lines represented by $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$. [See § 3·14 Page 30]

25. If the projections of an area A on the co-ordinate planes be A_x, A_y and A_z respectively, then show that $A^2 = A_x^2 + A_y^2 + A_z^2$. [See Th. I § 3·15 P. 33]

26. A plane makes intercepts $OA = a$, $OB = b$ and $OC = c$ respectively on the co-ordinate axes. Find the area of the triangle ABC. [See Ex. 2, P. 35]

27. Find the co-ordinates of the point of intersection of the plane $(x/a) + (y/b) = (z/c) = 1$ with the co-ordinate axes. *(Purvanchal 96)*

Ch. IV. The Straight Line.

28. Write down the equations of a line in the symmerical form.

29. What are the co-ordinates of any point on the straight line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$, say, where l, m, n are the direction cosines of the line.

Ans. $(\alpha + lr, \beta + mr, \gamma + nr)$

30. Write the equations of the straight line through $(3, 5, 7)$ and parallel to the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}$. **Ans.** $\frac{x-3}{2} = \frac{y-5}{3} = \frac{z-7}{4}$

31. Find where the line $\frac{x-1}{2} = \frac{2-y}{3} = \frac{z+3}{4}$ meets the plane $2x + 4y - z = 1$

Ans. $(3, -1, 1)$ [See Ex. 3 (a) P. 49]

32. Find the image of the point (3, 5, 7) in the plane $2x+y+z=6$.

Ans. $(-1, 3, 5)$ [See Ex. 7 (a) Page 51]

33. Find the equations of the line through the point (α, β, γ) at right angles to the lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2.

[See Ex. 9 Page 53]

34. Find the point where the line $\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{3}$ meets the plane $x+y+z=2$.

Ans. $(4/3, 2/3, 0)$ *(Kanpur 2001)*

35. Find the symmetric form of the equations of the line given by $x=ay+b, z=cy+d$.

[See Ex. 1 (b) P. 59] **Ans.** $\frac{x-b}{a}=\frac{y}{1}=\frac{z-d}{c}$

36. Find the symmetric form of the equation to the line given by $x+y+z=1, 4x+y-2z+2=0$. *(Kanpur 2001)*

[See Ex. 2 Page 60] **Ans.** $\frac{3x+1}{3}=\frac{3y-2}{-6}=\frac{z}{1}$

37. Are the two lines $\frac{x-2}{3}=\frac{y-3}{2}=\frac{z+4}{3}$ and $\frac{x+1}{5}=\frac{y-2}{-6}=\frac{z+3}{2}$ at right angles to each other.

Ans. No.

38. Find the co-ordinates of a point bying on the straight line $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{4}$ which is at a distance of 5 units from the point $(1, -2, 3)$.

Ans. $(11, 13, 23)$ *(Purvanchal 98)*

39. Write the co-ordinates of any point on the straight line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2). *(Purvanchal 2000)*

Ans. $[rx_2+(1-r)x_1, ry_2+(1-r)y_1, zr_2+(1-r)z_1]$

40. Find the equations of a line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

Ans. $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$ *(Purvanchal 97)*

41. Find the conditions for the line $(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n$ to lie on the plane $ax+by+cz+d=0$. [See § 4·05 (iii) Page 65]

42. Write the conditions for the line $(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n$ to be perpendicular to the plane $ax+by+cz+d=0$. [See § 4·05 (1) P. 65]

43. Write the general equation of a plane containing the line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

[See § 4·06 P. 69] *(Purvanchal 96)*

44. Show that the plane through the point (α, β, γ) and the line $x=py+q$ $=rz+s$ is given by $\begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+q & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix}=0$

(Meerut 2001)

[See Ex. 9 Page 76]

45. Find the equation of a system of planes perpendicular to a line with direction ratios a, b, c. [See Ex. 15 Page 79]

46. Find the distance of the point $(3, 7, 2)$ from the line which passes through $(-1, 2, 3)$ and whose direction cosines are proportional to 5, 3, 1. [See Ex. 2 (c) Page 83] (*Kanpur* 2001)

47. If the lines $(x-1)/2 = (y-2)/3 = (z-3)/k$ and $4x - 3y + 1 = 0$ and $5x - 3z + 2 = 0$ are coplanar, show that the value of k will be 4. (*Purvanchal 2001*) [See Ex. 10 Page 102]

48. Prove that the lines $\frac{x-a}{a'} = \frac{y-b}{b'} = \frac{z-c}{c'}$ and $\frac{x-a'}{a} = \frac{y-b'}{b} = \frac{z-c'}{c}$ intersect. (See Ex. 3. Page 98)

Ch. V. Volume of Tetrahedron .

49. Find the volume of the tetrahedron whose vertices are $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. **Ans.** $(abc)/6$

50. $A\ (a, 0, 0)$, $B\ (0, b, 0)$ and $C\ (0, 0, c)$ are three fixed points and $P\ (x, y, z)$ moves such that the volume of the tetrahedron $PABC$ is constant. Find the locus of P. **Ans.** $(x/a) + (y/b) + (z/c) =$ constant. [See Ex. 2 P. 5]

51. Find the volume of the tetrahechron formed by planes whose equations are $x + y = 0$, $y + z = 0$, $z + x = 0$ and $x + y + z = 1$. [See Ex. 3 (d) Page 8]

Ch. VI. Skew Lines and Change of Axes.

52. Write the equations of two skew lines in the simplest form. [See § 6·01 Page 16]

53. Find the locus of a line intersecting three lines. [See § 6·02 P. 16]

54. Find the locus of a point which moves so that the ratio of its distance from two given lines is constant. [See Ex. 8 (a) Page 21]

55. A variable line interesets the x-axis and the curve $x = y$, $y^2 = cz$ and is parallel to the plane $x = 0$. Show that it generate the poraboloid $xy = cz$. [See Ex. 10 (a) Page 23]

56. Find the surface generated by the lines which intersect the lines $y = mx$, $z = c$; $y = -mx$, $z = -c$ and x-axis. [See Ex. 10 (c) P. 25]

57. What transformations are required to change the direction of co-ordinate axes without changing the origin. [See § 6·04 Page 31]

58. What are the invariants if by any change of rectangular axes the expression $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ be transformed ? [See § 6·06 Page 33]

(B) OBJECTIVE TYPE QUESTIONS

(I) Multiple Choice Type Questions :

Select (i), (ii), (iii) or (iv) whichever is correct :—

Ch. III. The Plane.

1. The equation of a plane in the normal form is

(i) $(x/l) + (y/m) + (z/n) = 1$ (ii) $lx + my + nz = p$

(iii) $ax + by + cz + d = 0$ (iv) none of these

2. The number of arbitrary constants in the general equation $ax + by + cz + d = 0$ of a plane is

(i) 1 (ii) 2

(iii) 3 (iv) 4

3. The intercepts made by the plane $3x + 5y - 7z = 9$ on the axes are

(i) 3, 5, 7 (ii) 3, 5, −7

(iii) 3, 9/5, 9/7 (iv) 3, 9/5, − 9/7.

4. The general equation of a plane perpendicular to yz plane is

(i) $Ax + By + D = 0$ (ii) $Ax + Cz + D = 0$

(iii) $By + Cz + D = 0$ (iv) none of these

5. The equation of the plane through the origin and parallel to the plane $3x - 5y + 7z = 9$ is

(i) $3x + 5y + 7z = 9$ (ii) $3x - 5y + 7z = 0$

(iii) $3x - 5y + 7z + 9 = 0$ (iv) none of these

6. The equation of the plane through the points (0, − 1, 0), (1, 1, 1) and (2, 1, − 1) is

(i) $2x + 3y + 4z = 5$ (ii) $4x - 3y + 2z = 2$

(iii) $4x + 3y + 2z = 5$ (iv) none of these

7. The equation of any plane passing through the line of intersection of the planes $P = 0$ and $Q = 0$ is

(i) $P + \lambda Q = 0$ (ii) $PQ = 0$

(iii) $P/Q = 1$ (iv) none of these

8. The planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d = 0$ are mutually perpendicular, if

(i) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (ii) $\frac{a_1}{a_2} + \frac{b_1}{b_2} + \frac{c_1}{c_2} = 0$

(iii) $a_1a_2 + b_1b_2 + c_1c_2 = 0$ (iv) none of these

9. The planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are parallel, if

(i) $\frac{a_1}{a_2} + \frac{b_1}{b_2} + \frac{c_1}{c_2} = 0$ (ii) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iii) $a_1a_2 + b_1b_2 + c_1c_2 = 0$ (iv) none of these

10. The angle θ between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is given by

(i) $\cos\theta = a_1a_2 + b_1b_2 + c_1c_2$

(ii) $\tan\theta = a_1a_2 + b_1b_2 + c_1c_2$

(iii) $\cos\theta = \dfrac{-\ a_1a_2 + b_1b_1 + c_1c_2}{\sqrt{(a_1^2+b_1^2+c_1^2)}\sqrt{(a_2^2+b_2^2+c_2^2)}}$

(iv) none of these

11. The distance between the planes $2x-2y+z+3=0$ and $4x-4y+2z+5=0$ is

(i) 2/3 (ii) 1/2

(iii) 1/3 (iv) 1/6

12. A plane meets the co-ordinates axes at A, B, C such that the centroid of the triangle is (1, 1, 1), then the equation of the plane is

(i) $x+y+z=1$ (ii) $x+y+z=3$

(iii) $x+y+z=1/3$ (iv) none of these

13. The angle between the planes $2x-y+z=6$ and $x+y+2z=3$ is

(i) $\pi/6$ (ii) $\pi/4$

(iii) $\pi/3$ (d) none of these

14. The equation to the plane through $P\,(a, b, c)$ and perpendicular to OP is

(i) $ax+by+cz=a+b+c$ (ii) $ax+by+cz=a^2+b^2+c^2$

(iii) $x+y+z=a+b+c$ (iv) none of these

15. If the plane $lx+my=0$ is rotated through an angle $\pi/4$ about the line of intersection with the plane $z=0$, then the equation of the plane in the new position is

(i) $lx+my+z=0$ (ii) $lx+my\pm z\sqrt{(l^2+m^2)}=0$

(iii) $lx+my+(l^2+m^2)z=0$ (iv) none of these

16. The distance between the planes $ax+by+cz+d=0$ and $ax+by+cz+e=0$ is

(i) $\dfrac{|d-e|}{\sqrt{(a^2+b^2+c^2)}}$ (ii) $\dfrac{|d+e|}{\sqrt{(a^2+b^2+c^2)}}$

(iii) $|d-e|$ (iv) none of these

17. The plane parallel to the plane $x+y+z=0$ and through (a, b, c) is

(i) $ax+by+cz=0$ (ii) $x+y+z=a+b+c$

(iii) $x+y+z+a+b+c=0$ (iv) none of these

18. The equation of the plane through the intersection of the planes $2x-y=0$ and $3x-y=0$ and perpendicular to the plane $4x+5y-3z=8$ is

(i) $28x-17y-9z=0$ (ii) $28x+17y+9z=3$

(iii) $28x-17y+9z=0$ (iv) none of these

19. The equation of the plane perpendicular to xy-plane and passing through the points (1, 0, 5) and (0, 3, 1) is

(i) $3x+y=-3$ (ii) $3x+y=3$

(iii) $3x-y=3$ (iv) $3x-y=-3$

20. The distance of the plane $8x+5y-11y+9=0$ from the origin is

(i) -9 (ii) 8

(iii) 9 (iv) 2

21. The equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ will represent a pair of perpendicular planes if

(i) $f + g + h = a + b + c$ (ii) $f + g + h = 0$
(iii) $a + b + c = 0$ (iv) none of these

22. A plane is parallel to the z-axis and having intercepts 2 and 5 on x and y axes, then its equation is

(i) $\frac{x}{2} + \frac{y}{5} + \frac{z}{\lambda} = 1, \lambda \in I$ (ii) $\frac{x}{2} + \frac{y}{5} + \frac{z}{1} = 0$
(iii) $\frac{x}{2} + \frac{y}{5} = 0$ (iv) $\frac{x}{2} + \frac{y}{5} = 1$

23. If A_x, A_y, A_z be the area of projections of an area A on the three co-ordinates planes, then

(i) $A = A_x + A_y + A_z$ (ii) $A^3 = A_x . A_y . A_z$
(iii) $A^2 = A_x^2 + A_y^2 + A_z^2$ (iv) none of these

24. The bisector of the acute angle between the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$ is

(i) $5x - y + 32z + 45 = 0$ (ii) $23x - 13y + 32z + 45 = 0$
(iii) $5x - 3y + 8z + 11 = 0$ (iv) none of these

25. If $d < 0$, the length of the perpendicular from the origin to the plane $ax + by + cz + d = 0$ is

(i) $\frac{ax + by + cz + d}{\sqrt{(a^2 + b^2 + c^2)}}$ (ii) $\frac{ax + by + cz - d}{\sqrt{(a^2 + b^2 + c^2)}}$
(iii) $\frac{d}{\sqrt{(a^2 + b^2 + c^2)}}$ (iv) $\frac{-d}{\sqrt{(a^2 + b^2 + c^2)}}$

Ch. IV. The Straight Line.

26. Co-ordinates of any point on the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$, say are given by

(i) $\alpha - lr, \beta - mr, \gamma - nr$ (ii) lr, mr, nr
(iii) $\alpha + lr, \beta + mr, \gamma + nr$ (iv) none of these

27. The equations of the y-axis are

(i) $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$ (ii) $\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$
(iii) $\frac{x}{0} = \frac{y}{0} = \frac{z}{1}$ (iv) $\frac{x}{0} = \frac{y}{1} = \frac{z}{1}$

28. The angle between the lines $\frac{x-2}{3} = \frac{y+1}{-2}, z = 2$ and $\frac{x-1}{1} = \frac{2y+3}{3} = \frac{z+5}{2}$ is

(i) $\pi/2$ (ii) $\pi/3$

(iii) $\pi/6$ (iv) none of these

(Kanpur 2001)

29. The value of λ for which the lines $\frac{x-2}{\lambda}=\frac{y-3}{-1}=\frac{z-4}{1}$ and $\frac{x+5}{2}=\frac{y+6}{1}=\frac{z+7}{-\lambda}$ are mutually perpendicular is

(i) -1 (ii) 0

(iii) 1 (iv) none of these

30. The point where the line $\frac{x-1}{2}=\frac{2-y}{3}=\frac{z+3}{4}$ meets the plane $2x+4y-z=1$ is

(i) $(-3, 1, -1)$ (ii) $(3, -1, 1)$

(iii) $(-1, 2, -3)$ (iv) none of these

31. The equations of the line joining the points $(1, 2, 3)$ and $(-5, 8, 9)$ are

(i) $\frac{x+5}{1}=\frac{y-8}{2}=\frac{z-9}{3}$ (ii) $\frac{x-1}{-5}=\frac{y-2}{8}=\frac{z-3}{9}$

(iii) $\frac{x-1}{-1}=\frac{y-2}{1}=\frac{z-3}{1}$ (iv) $\frac{x+1}{-1}=\frac{y+2}{1}=\frac{z+3}{1}$

32. The equations of the line through $(3, -4, 5)$ and equally inclined to the co-odinate axes are

(i) $\frac{x-3}{2}=\frac{y+4}{3}=\frac{z-5}{4}$ (ii) $\frac{x-3}{1}=\frac{y+4}{1}=\frac{z-5}{1}$

(iii) $\frac{x}{3}=\frac{y}{-4}=\frac{z}{5}$ (iv) none of these

33. The equations of the line through $(3, -4, 5)$ and parallel to the line $\frac{x+1}{2}=\frac{y-2}{5}=\frac{z+3}{7}$ are

(i) $\frac{x-3}{2}=\frac{y+4}{5}=\frac{z-5}{7}$ (ii) $\frac{x-3}{-1}=\frac{y+4}{2}=\frac{z-5}{-3}$

(iii) $\frac{x+1}{3}=\frac{y-2}{-4}=\frac{z+3}{7}$ (iv) none of these

34. The image of the point $(3,5,7)$ on the plane $2x+y+z=6$ is

(i) $(-3, -5, -7)$ (ii) $(7, 5, 3)$

(iii) $(-1, 3, 5)$ (iv) $(1, -3, 5)$

35. If the line joining the points (a, b, c) and (a', b', c') passes through the origin and r, r' be the distance of these points from the origin then $rr' =$

(i) $(a+b+c)-(a'+b'+c')$ (ii) $(a/a')+(b/b')+(c/c')$

(iii) $aa'+bb'+cc'$ (iv) none of these

36. The symmetric form of the equations of the line given by $x=ay+b$, $z=cy+d$ is

(i) $\frac{x-b}{a}=\frac{y}{1}=\frac{z-d}{c}$ (ii) $\frac{x-a}{c}=\frac{y-b}{d}=\frac{z-d}{a}$

(iii) $\frac{x-a}{b}=\frac{y}{1}=\frac{z-c}{d}$ (iv) none of these

37. The equations of the line through $(-2, 3, 4)$ and parallel to the line of intersection of the planes $2x+3y+4z=5$ and $3x+4y+5z=6$ are

(i) $\frac{x+2}{1}=\frac{y-3}{1}=\frac{z-4}{1}$ (ii) $\frac{x+2}{1}=\frac{y-3}{-2}=\frac{z-4}{1}$

(iii) $\frac{x+2}{3}=\frac{y-3}{6}=\frac{z-4}{10}$ (iv) none of these

38. The lines $x=ay+b, z=cy+d$ and $x=a'y+b', z=c'y+d'$ are perpendicular if $aa'+cc'=$

(i) 2 (ii) 1

(iii) 0 (iv) -1

39. The angle between the lines $3x+2y+z=0=x+y-2z$ and $2x-y-z=0=7x+10y-8z$ is

(i) $\pi/2$ (ii) $\pi/3$

(iii) $\pi/4$ (iv) $\pi/6$

40. The lines $\frac{x-2}{2}=\frac{y+3}{3}=\frac{z-7}{4}$ and $\frac{x-2}{1}=\frac{y+3}{-2}=\frac{z-7}{1}$ are

(i) parallel (ii) coincident

(iii) perpendicular (iv) none of these

41. The condition that the line $(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n$ is perpendicular to the plane $ax+by+cz+d=0$ is

(i) $a/l=b/m=c/n$ (ii) $l/c=m/b=n/a$

(iii) $al=bm=cn$ (iv) none of these

42. The conditions that the line $(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n$ lies on the plane $ax+by+cz+d=0$ are

(i) $al+bm+cn=0, a\alpha+b\beta+c\gamma+d\neq 0$

(ii) $al+bm+cn=0, a\alpha+b\beta+c\gamma+d=0$

(iii) $al+bm+cn=0$

(iv) none of these

43. The line $\frac{x-3}{2}=\frac{y-4}{3}=\frac{z-5}{4}$ is parallel to the plane

(i) $3x+4y-5z=7$ (ii) $2x+3y+4z=5$

(iii) $4x+4y-5z=0$ (iv) none of these

44. The direction cosines of the lines perpendicular to two lines with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 respectively are

(i) $l_1+l_2, m_1+m_2, n_1+n_2$

(ii) $l_1-l_2, m_1-m_2, n_1-n_2$

(iii) $m_1n_2-m_2n_1, n_1l_2-n_2l_1, l_1m_2-l_2m_1$

(iv) none of these

45. The equation of the plane through the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ is

$A(x-\alpha)+B(y-\beta)+C(z-\gamma)=0$, where

(i) $A+B+C=0$ (ii) $A+B+C=l+m+n$

(iii) $Al+Bm+Cn=0$ (iv) none of these

46. The equation of the plane containing the line $(y/b)+(z/c)=1, x=0$ and parallel to the line $(x/a)-(z/c)=1, x=0$ is

(i) $(x/a)+(y/b)+(z/c)=1$ (ii) $(x/a)-(y/b)-(z/c)=1$

(iii) $-(x/a)+(y/b)-(z/c)=1$ (iv) $(x/a)+(y/b)-(z/c)=-1$

47. The perpendicular distance of the point $(1, 2, 3)$ from the line $(x-6)/3=(y-7)/2=(7-z)/2$ is

(i) 1 (ii) 3

(iii) 4 (iv) 7

48. The distance of the point $(4, 3, 5)$ from the axis of y is

(i) $\sqrt{(50)}$ (ii) 5

(iii) $\sqrt{(41)}$ (iv) $\sqrt{(34)}$

49. The lines $\dfrac{x+1}{-3}=\dfrac{y-3}{2}=\dfrac{z+2}{1}$ and $\dfrac{x}{1}=\dfrac{y-7}{-3}=\dfrac{z+7}{2}$ are

(i) parallel (ii) coplanar

(iii) perpendicular (iv) none of these

50. The lines $x/\alpha=y/\beta=z/\gamma$, $ax/\alpha=by/\beta=cz/\gamma$ and $x/a\alpha=y/b\beta=z/c\gamma$ are coplanar, if

(i) $a+b=c$ (ii) $ab=c$

(iii) $a=b$ (iv) none of these

51. In the case of three planes, which one of the following is not possible

(i) intersecting in a point

(ii) intersecting in two points

(iii) intersecting in a common line

(iv) forming a triangular prism.

52. The planes $2x+4y+2z=7, 5x+y-z=9, x-y-z=6$

(i) intersect in a point (ii) pass through one line

(iii) from a triangular prism (iv) none of these

53. The shortest distance between the lines $(x-1)/2=(y-2)/3=(z-3)/4$ and $(x-2)/3=(y-4)/4=(z-5)/5$ is

(i) $\sqrt{6}$ (ii) $\sqrt{3}$

(iii) $1/\sqrt{3}$ (iv) $1/\sqrt{6}$

54. The shortest distance between the lines $x+a=2y=-12z$ and $x=y+2a=6z-6a$ is

(i) a (ii) $2a$

(iii) $3a$ (iv) $4a$

55. The symmetrical form of the equations to the line given by

$x=ay+b, z=cy+d$ is

(i) $\dfrac{x+b}{a}=\dfrac{y}{1}=\dfrac{z+d}{c}$ (ii) $\dfrac{x+b}{a}=\dfrac{y}{c}=\dfrac{z+d}{1}$

(iii) $\frac{x-b}{a}=\frac{y}{1}=\frac{z-d}{c}$ (iv) none of these

Ch. V. Volume of Tetrahedron.

56. The volume of tetrahedron formed by the planes $y+z=0, z+x=0,$ $x+y=0, x+y+z=1$ is

(i) 1/2 (ii) 2/3

(iii) 3/4 (iv) 1

57. The volume of the tetrahedron whose vertices are (0, 0, 0), $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is

(i) abc (ii) $(abc)/2$

(iii) $(abc)/3$ (iv) $(abc)/6$

Ch. VI. Skew Lines and Change of Axes.

58. The lines $y=x\tan\alpha, z=c$ and $y=-x\tan\alpha, z=-c$ are

(i) parallel (ii) perpendicular

(iii) skew (iv) none of these

(II) TRUE & FALSE TYPE QUESTIONS :

Write 'T' or 'F' according as the following statement is true or false :—

Ch. III. The Plane.

1. The general equation of first degree always represents a plane.
2. The normal form of the equation of a plane is $lx+my+nz=p$, where l, m, n are the direction cosines of the normal to the plane.
3. The intercepts made by the plane $5x-3y+2z=8$ on the co-ordinate axes are 5, 3, 2.
4. The distance of any point (x, y, z) from yz-plane is x.
5. The equation of any plane parallel to xy-plane is $x+y=0$.
6. The angle between the planes $2x+4y+9z+7=0$ and $3x+5y-2z+1=0$ is $\pi/3$.
7. The equation of the plane passing through three non-collinear points $(x_i, y_i, z_i), i=1, 2, 3$ is $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0$
8. The equation of the plane parallel to zx-plane at a distance c from it is $y=c$.
9. The equation of system of planes perpendicular to the line with direction ratios l, m, n is $lx+my+nz=0$.
10. The distance between the planes $2x-2y+z+3=0$ and $4x-4y+2z+5=0$ is 2.
11. The points (1, 2, 3) and (0, 5, 1) are on the opposite sides of the plane $y+z=4$.

12. The equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ represents a pair of planes.

13. The planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ are perpendicular if $a/a' = b/b' = c/c'$. *(Kanpur 2001)*

Ch. IV. The Straight Line.

14. The symmetric form of the equations of a line given by $x = ay + b$, $z = cy + d$ is $\frac{x-b}{a} = \frac{y}{1} = \frac{z-d}{c}$.

15. The line $(x-\alpha)/l = (y-\beta)/m = (z-\gamma)/n$ will be parallel to the plane $ax + by + cz + d = 0$ if $al + bm + cn = 0$ and $a\alpha + b\beta + c\gamma + d = 0$.

16. The equation of the plane through the line $(x-\alpha_1)/l_1 = (y-\beta_1)/m_1 = (z-\gamma_1)/n_1$ and parallel to the line $(x-\alpha_2)/l_2 = (y-\beta_2)/m_2 = (z-\gamma_2)/n_2$ is

$$\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

17. The lines $\frac{x-\alpha_1}{l_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1}$ and $\frac{x-\alpha_2}{l_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2}$ will intersect if $\begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

18. The planes $x = cy + bz$, $y = az + cx$, $z = bx + ay$ pass through one line if $a^2 + b^2 + c^2 = 1$.

19. The locus of a point whose distance from x-axis is twice its distance from the zy-plane is $4x^2 = y^2 + z^2$.

20. The S.D. between the line $x + a = 2y = -12z$ and $x = y + 2a = 6z - 2a$ is $2a$.

Ch. V. The Volume of Tetrahedron

21. The volume of the tetrahedron whose vertices are $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is $(abc)/6$.

22. The volume of the tetrahedron formed by the planes $y + z = 0$, $z + x = 0$, $x + y = 0$ and $x + y + z = 1$ is 4.

Ch. VI. Skew Lines and Change of Axes.

23. By proper of choice of axes the equations of two skew lines can be written in the form $y = x \tan\alpha$, $z = c$ and $y = -x \tan\alpha$, $z = -c$.

24. The surface generated by the lines which intersect the lines $y = mx$, $z = c$, $y = -mx$, $z = -c$ and x-axis is $cy = mzx$.

(III) FILL IN THE BLANKS TYPE QUESTIONS :

Fill in the blanks in the following :

Ch. III. The Plane.

1. $2x+3y+4z=0$ is the equation of a plane which passes through .... *(Meerut 2001)*

2. $\alpha(x-l)+\beta(y-m)+\gamma(z-n)=0$ represents a plane passing through the point .... *(Meerut 2001)*

3. Every linear equation in x, y, z represents a ....

4. The equation of any plane parallel to xy-plane is $z=$ ....

5. If two planes are parallel, then direction ratios of their normals are ......

6. If a plane is parallel to x-axis, the coefficient of x in its equation must be ....

7. The equation of the plane through the point $(1, 2, 3)$ and parallel to the plane $3x+5y-7z=9$ is $3x+5y-7z=$ .....

8. The equation of any plane passing through the line of intersection of the planes $x+y+z=0$ and $2x-3y+4z+5=0$ is ....

9. The distance between the parallel planes $2x-2y+z+3=0$ and $4x-4y+2z+5=0$ is ....

10. The points $(1, 2, 3)$ and $(0, 5, 1)$ lie on the .... side of the plane $y+z-4=0$.

11. If the projections of the area A on the co-ordinates planes be A_x, A_y and A_z, then $A^2=$ ....

12. The general equation of second degree viz. $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ will represent a pair of planes if ....

Ch. IV. The Straight Line.

13. The symmetric form of the equations of a line given by $x=ay+b$, $z=cy+d$ is $(x-b)/a=y/$..........$=(z-d)/c$.

14. The line $(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n$ is perpendicular to the plane $ax+by+cz+d=0$ if ......

15. Conditions for the line $(x-\alpha)/l=(y-\beta)/m=(z-\gamma)/n$ to lie on the plane $ax+by+cz+d=0$ are $a\alpha+b\beta+c\gamma+d=0$ and ....

16. The equation of the plane containing the line $(y/b)+(z/c)=1$, $x=0$ and parallel to the line $(x/a)-(z/c)=1$, $x=0$ is ....

17. The equation of the plane in which the lines $(x-\alpha_1)/l_1=(y-\beta_1)/m_1=(z-\gamma_1)/n_1$ and

$(x-\alpha_2)/l_2=(y-\beta_2)/m_2=(z-\gamma_2)/n_2$ lie is ....

18. The planes $2x-3y-7z=0$, $2x-14y-13z=0$, $8x-31y-33z=0$ pass through one ....

19. Prove that the planes $x=cy+bz$, $y=az+cx$, $z=bx+ay$ pass through one line if ....

20. Prove that the planes $ny-mz=\lambda$, $lz-nx=\mu$, $mx-ny=\nu$ have a common line if and only if ....

21. The S.D. between the lines $(x-3)=(5-y)/2=z-2$ and $(x-1)/7=(y+1)/-6=z+1$ is ....

22. The S.D. between the lines $x+a=2y=-12z$ and $x=y+2a=6z-6a$ is ....

23. The equations of the straight line passing through the two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) is .... *(Meerut 2001)*

24. The equations of the straight line parallel to

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z}{4} \text{ is } \frac{x+1}{2}=\frac{y+2}{\cdots\cdots}=\frac{z-3}{4}$$ *(Meerut 2001)*

25. The line $x/l=y/m=z/n$ is perpendicular to the plane $ax+by+cz+d=0$, if .... *(Meerut 2001)*

26. The distance of the point (a, b, c) from x-axis is .... *(Kanpur 2001)*

Ch. V. Volume of Tetrahedron.

27. The volume of the tetrahedron whose vertices are $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is ....

28. The volume of the tetrahedron formed by the planes $y+z=0$, $z+x=0$, $x+y=0$ and $x+y+z=1$ is ....

Ch. VI. Skew lines and Change of Axes.

29. A variable line which intersects the x-axis and the curve $x=y$, $y^2=cz$ and is parallel to the plane $x=0$ generates the surface ....

30. Without change of direction of axes, if the origin is shifted to the point (f, g, h) and the point (x, y, z) is shifted to the point (x', y', z'), then $z'=$ ....

ANSWERS TO OBJECTIVE QUESTIONS

(I) Multiple Choice Type :

1.	(ii);	2.	(iii)	3.	(iv);	4.	(iii);	5.	(ii);	6.	(ii);
7.	(i);	8.	(iii);	9.	(ii);	10.	(iii);	11.	(iv);	12.	(ii);
13.	(iii);	14.	(ii);	15.	(ii);	16.	(i);	17.	(ii);	18.	(iii);
19.	(ii);	20.	(iii);	21.	(iii);	22.	(iv);	23.	(iii);	24.	(ii);
25.	(iv).	26.	(iii);	27.	(ii);	28.	(i);	29.	(iii);	30.	(ii);
31.	(iii);	32.	(ii);	33.	(i);	34.	(iii);	35.	(iii);	36.	(i);

37.	(ii);	**38.**	(iv);	**39.**	(i);	**40.**	(iii);	**41.**	(i);	**42.**	(ii);		
43.	(iii);	**44.**	(iii);	**45.**	(iii);	**46.**	(ii).	**47.**	(iv);	**48.**	(iii);		
49.	(ii);	**50.**	(iii);	**51.**	(ii);	**52.**	(iii);	**53.**	(iv);	**54.**	(ii);		
55.	(iii);	**56.**	(ii)	**57.**	(iv);	**58.**	(iii).						

(II) True & False Type :

1.	T;	**2.**	T;	**3.**	F;	**4.**	T;	**5.**	F;	**6.**	F;	**7.**	T;
8.	T;	**9.**	F;	**10.**	F;	**11.**	F;	**12.**	T;	**13.**	F;	**14.**	T;
15.	F;	**16.**	T;	**17.**	T;	**18.**	F;	**19.**	T;	**20.**	T;	**21.**	T;
22.	F;	**23.**	T;	**24.**	T.								

(III) Fill in the blanks Type :

1. origin;
2. (l, m, n);
3. plane;
4. constant;
5. proportional;
6. zero;
7. -8
8. $(x+y+z)+\lambda\,(2x-3y+4z+5)=0$;
9. $1/6$;
10. same;
11. $A_x^2+A_y^2+A_z^2$;
12. $abc+2fgh-af^2-bg^2-ch^2=0$;
13. 1;
14. $l/a=m/b=n/c$;
15. $al+bm+cn=0$;
16. $(x/a)-(y/b)-(z/c)+1=0$.
17. $\begin{vmatrix} x-\alpha_1 & y-\beta_1 & z-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$;
18. line;
19. $a^2+b^2+c^2+2abc=1$;
20. $l\lambda+m\mu+n\nu=0$;
21. $34/\sqrt{(29)}$;
22. $2a$;
23. $\dfrac{x-x_1}{x_2-x_1}=\dfrac{y-y_1}{y_2-y_1}=\dfrac{z-z_1}{z_2-z_1}$;

24. 3;
25. $al + bm + cn = 0$;
26. $\sqrt{(b^2 + c^2)}$;
27. $(abc)/6$;
28. $2/3$;
29. $xy = cz$;
30. $z + h$.

(A) VERY SHORT AND SHORT TYPE QUESTIONS

Ch. VII Sphere

1. Define a sphere. *(Purvanchal 96)* [See § 7·01 Page 1]

2. Write down the general equations of a sphere, co-ordinates of its centre and length of its radius. [See § 7·02 (b) Page 2]

3. Write down the conditions for the general equation of second degree to represent a sphere. [See § 7·02 Page 2]

4. Find out the radius of the sphere $x^2+y^2+z^2-kx+z+8=0$

(Purvanchal 94) **Ans.** $\sqrt{[(k^2/4)+(1/4)-8]}$

5. Find the radius of the sphere $\lambda x^2+\mu y^2+8z^2+\nu\, xy+16x+24y+32z+40=0$ *(Purvanchal 98)*

[**Hint :** This equation does not represent a sphere. See § 7·02 Page 2]

6. Find the centre and radius of the sphere $2x^2+2y^2+2z^2-6x+8y-8z=1.$ **Ans.** $(3/2,-2,2),(1/2)\sqrt{(43)}$ *(Kanpur 2001)*

7. Find the equation of the sphere whose centre is $(1, 2, 1)$ and radius $\sqrt{6}$.

Ans. $x^2+y^2+z^2-2x-4y-2z=0$ *(Kanpur 2001)*

8. Write the equation to a sphere on the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2). [See § 7·04 Page 3] *(Purvanchal 2001)*

9. Write down in the determinant form the equation of a sphere through four given points (x_r, y_r, z_r), $r=1, 2, 3, 4$. [See § 7·03 Page 2]

10. Find the equation of a sphere passing through origin and making intercepts a, b, c with the axes respectively. *(Meerut 2001)*

Ans. $x^2+y^2+z^2-ax-by-cz=0.$ [See Ex. 5 Page 4]

11. Find the equation of the sphere circumscribing the tetrahedron whose faces are $x=0, y=0, z=0$ and $(x/a)+(y/b)+(z/c)=1.$

Ans. $x^2+y^2+z^2-ax-by-cz=0$ [See Ex. 9 (a) Page 6]

12. Obtain the equation of the sphere with centre at $(2,3,-4)$ and touching the plane $2x+6y-3z+15=0.$ [See Ex. 11 (a) Page 7]

13. A sphere of radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2+y^2+z^2)=4k^2.$ [See Ex. 12 Page 7]

14. A plane passes through a fixed point (p, q, r) and cuts the axes in A, B, C. Find the locus of the centre of the sphere $OABC$, where O is the origin.

[See Ex. 13 Page 8]

15. Find the curve of intersection of two spheres. *(Kanpur 2001)*

16. If r be the radius of the circle $x^2+y^2+z^2+2ux+2vy+2wz+d=0$, $lx+my+nz=0$, then prove that

$$(r^2+d)(l^2+m^2+n^2)=(mw-nv)^2+(nu-lw)^2+(lv-mu)^2$$

[See Ex. 3 Page 17]

17. Find the equation of the sphere through the circle $x^2+y^2+z^2=9$, $2x+3y+4z=5$ and the point $(1, 2, 3)$. *(Purvanchal 97)*

Ans. $3(x^2+y^2+z^2)-2x-3y-4z-22=0.$ [See Ex. 4 (b) P. 18]

18. Find the equation of the sphere for which the circle $x^2+y^2+z^2+7y-2z+2=0,\ 2x+3y+4z=8$ is a great circle.

Ans. $x^2+y^2+z^2-2x+4y-6z+10=0.$ [See Ex. 8 (a) Page 20]

19. Write the equation of the tangent plane to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ at the point (α, β, γ). [See § 7·09 P. 32]

20. Write the condition for the plane $lx+my+nz=p$ to touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0.$ [See § 7·10 P. 34]

21. Find the point of contact where $2x-2y+z+12=0$ touches $x^2+y^2+z^2-2x-4y+2z-3=0.$ Ans. $(-1, 4, -2)$ *(Purvanchal 99)*

22. If one end of a diameter of the sphere $x^2+y^2+z^2-2x+4y-6z-7=0$ be $(-1, 2, 4)$, find the co-ordinates of the other end. *(Purvanchal 2001)*

Ans. $(3, -6, 2)$ [See Ex. 6 Page 43]

23. Write the equation of the polar of the point (α, β, γ) with respect to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0.$ [See § 7·13 Page 47]

24. What is the condition of orthogonality of the spheres $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ and $x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0$?

[See § 7·16 Page 51]

25. Find the length of the tangent from the point (x_1, y_1, z_1) to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0.$ [See § 7·17 Page 57]

26. Define radical plane. [See § 7·18 Page 58]

27. Define coaxial system of spheres. [See § 7·20 Page 59]

28. Write down the equation of coaxial spheres in the simplest from.

[See § 7·21 Page 60]

29. Define Limiting points. [See § 7·22 Page 60]

Ch. VIII The Cone and Cylinder

30. Define a cone. [See § 8·01 Page 1]

31. Find the general equation of a cone of second degree which passes through the coordinate axes. [See § 8·03 Page 2]

32. Find the equation of the cone with vertex at the origin and which passes through the curve $ax^2+by^2+cz^2=1,\ \alpha x^2+\beta y^2=2z.$ [See Ex. 2(a) P. 2]

33. Find the equation of the cone with vertex as origin and guiding curve $f(x, y)=0$ and $z=c.$ *(Purvanchal 2001)* [See Ex. 5 Page 4]

34. Find the equation to the cone which has vertex at the origin and passes through the curve given by $ax^2+by^2=2z,\ lx+my+nz=p.$

[See Ex. 3 (b) Page 4]

35. Find the equation of the cone whose vertex is $(0, 0, 0)$ and whose base is the circle through the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. [See Ex. 6 P. 5]

36. Find out the number of independent constants in the equation of the cone with vertex at the origin. *(Purvanchal 2000)* Ans. 5 [See § 8·02 Page 1]

37. Find the equation of the cone whose vertex is the origin and which

passes through the curve of intersection of the plane $lx + my + nz = p$ and the surface $\lambda x^2 + \mu y^2 + \nu z^2 = 1$. *(Purvanchal 98)* [See Ex. 3 (a) Page 4]

38. Prove that a line which passes through (α, β, γ) and intersects the parabola $z^2 = 4ax, y = 0$ lies on the cone $(\beta z - \gamma y)^2 = 4b\ (\beta - y)\ (\beta x - dy)$.
[See Ex. 1 (c) Page 12]

39. Prove that the equation $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$ represents a cone if $(u^2/a) + (v^2/b) + (w^2/c) = d$. [See Ex. 11 (a) Page 20]

40. Prove that the plane $ax + by + cz = 0$ cuts the cone $yz + zx + xy = 0$ in perpendicular lines if $(1/a) + (1/b) + (1/c) = 0$. [See Ex. 2 Page 24]

41. What is the condition for the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ to have three mutually perpendicular generators ? *(Purvanchal 99)*
[See § 8·07 Page 34]

42. Define Reciprocal cone. [See § 8·10 Page 40]

43. Prove that the cones $ax^2 + by^2 + cz^2 = 0$ and $(x^2/a) + (y^2/b) + (z^2/c) = 0$ are reciprocal to each other. *(Meerut 2001)* [See Ex. 1 (a) Page 41]

44. Find the reciprocal cone of the cone $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 0$.
[See Ex. 1 (b) Page 42]

45. Find the condition that the plane $ux + vy + wz = 0$ may touch the cone $ax^2 + by^2 + cz^2 = 0$. [See Ex. 4 (a) Page 42]

46. Define an enveloping cone. *(Purvanchal 97)* [See § 8·11 Page 46]

47. Find the equation of the enveloping cone of the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ with vertex at (0, 0, 0). [See Ex. 3 (a) P. 49]

48. Define a right circular cone. *(Purvanchal 96)* [See § 8·12 Page 53]

49. Find the equation of the right circular cone whose vertex is (0, 0, 0), axis x-axis and semivertical angle is α. [See Ex. 1 (c) Page 55]

50. Define a cylinder. [See § 8·14 Page 62]

51. Find the equation of the cylinder with generators parallel to the axis of x and passing through the curve $ax^2 + by^2 + cz^2 = 1, lx + my + nz = p$.
[See Ex. 1 (d) Page 64]

52. Find the equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 = a^2$ whose generators are parallel to the line $x/l = y/m = z/n$.
[See § 8·18 (a) Page 76]

53. What is the guiding circle of a right circular cylinder ?
(Purvanchal 2000)

(B) OBJECTIVE TYPE QUESTIONS

(I) Multiple Type Questions :

Choose part (i), (ii), (iii) or (iv) whichever is correct :—

Ch. VII Sphere

1. The radius of the sphere $x^2 + y^2 + z^2 - 2x - 4y - 6z = 11$ is

(i) 2 **(ii)** 3 **(iii)** 4 **(iv)** 5

2. The equation $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ represents a real sphere if $u^2+v^2+w^2-d$ is

(i) negative (ii) zero (iii) positive (iv) none of these

3. The centre of the sphere $x^2+y^2+z^2-4x+4y-6z+5=0$ is

(i) $(-2, 2, -3)$ (ii) $(2, -2, 3)$
(iii) $(4, -4, 6)$ (iv) none of these

4. The centre of the sphere $2x^2+2y^2+2z^2-6x+8y-8z=1$ is

(i) $(-3, 4, -4)$ (ii) $(3, -4, 4)$
(iii) $(3/2, -2, 2)$ (iv) none of these

5. A sphere passes through the points $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. The co-ordinates of its centre are

(i) $(0, 0, 0)$ (ii) (a, b, c)
(iii) $(a/2, b/2, c/2)$ (iv) $(a/3, b/3, c/3)$

6. The equation to the sphere on the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) as diameter is

(i) $x(x_1-x_2)+y(y_1-y_2)+z(z_1-z_2)=0$
(ii) $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$
(iii) $[(x-x_1)/(x-x_2)]+[(y-y_1)/(y-y_2)]+[(z-z_1)/(z-z_2)]=0$
(iv) none of these

7. Centre of the sphere $\Sigma[(x-x_1)(x-x_2)]=0$ is

(i) $\left(\frac{x_1-x_2}{2}, \frac{y_1-y_2}{2}, \frac{z_1-z_2}{2}\right)$ (ii) $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$
(iii) (x_1x_2, y_1y_2, z_1z_2) (iv) none of these

8. The equation of the sphere which passes through $(1, 1, 0)$ and which has its centre at $(0, 0, 0)$ is

(i) $x^2+y^2+z^2=1$ (ii) $x^2+y^2+z^2=2$
(iii) $x^2+y^2+z^2-2x-2y=0$ (iv) none of these

9. A plane passes through a fired point (p, q, r) and cuts the axes in A, B, C. Then the locus of the centre of the sphere $OABC$ is

(i) $(x/p)+(y/q)+(z/r)=2$ (ii) $(p/x)+(q/y)+(r/z)=2$
(iii) $x^2+y^2+z^2=p^2+q^2+r^2$ (iv) none of these

10. The two equations $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ and $lx+my+nz=p$ taken together represent a

(i) sphere (ii) pair of planes
(iii) circle (iv) none of these

11. The equation of the sphere with centre at $(2, 3, -4)$ and touching the plane $2x+6y-3z+15=0$ is

(i) $x^2+y^2+z^2-4x-6y+8z-20=0$
(ii) $x^2+y^2+z^2-4x-6y+8z+20=0$
(iii) $x^2+y^2+z^2+4x+6y-8y-20=0$

(iv) none of these

12. The equation of the sphere passing through (0, 0, 0), (a, 0, 0), (0, b, 0) and (0, 0, c) is

(i) $x^2 + y^2 + z^2 + ax + by + cz = 0$

(ii) $x^2 + y^2 + z^2 - ax - by - cz = 0$

(iii) $x^2 + y^2 + z^2 + 2ax + 2by + 2cz = 0$

(iv) none of these

13. The equation of the sphere concentric with the sphere $x^2 + y^2 + z^2 - 4x - 6y + 8z - 20 = 0$ and passing through origin is

(i) $x^2 + y^2 + z^2 = 20$

(ii) $x^2 + y^2 + z^2 - 4x - 6y + 8z = 0$

(iii) $x^2 + y^2 + z^2 - 2x - 3y + 4z = 0$

(iv) none of these

14. The radius of the circle $x^2 + y^2 + z^2 - 8x + 4y + 8z = 45$, $x - 2y + 2z = 3$ is

(i) $4\sqrt{5}$ (ii) $2\sqrt{3}$ (iii) $3\sqrt{2}$ (iv) none of these

15. The centres of all sections of the sphere $x^2 + y^2 + z^2 = r^2$ by planes through the point (x', y', z') lie on the sphere.

(i) $\Sigma x (x + x') = 0$ (ii) $\Sigma x' (x - x') = 0$

(iii) $\Sigma x (x - x') = 0$ (iv) none of these

16. The equation of the sphere whose great circle is $x^2 + y^2 + z^2 + 10y - 4z = 8$, $x + y + z = 3$ is

(i) $x^2 + y^2 + z^2 + x + 11y - 3z = 11$

(ii) $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$

(iii) $x^2 + y^2 + z^2 - x + 9y - 5z = 5$

(iv) none of these

17. The equation of the tangent plane to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ at the point (x_1, y_1, z_1) is

(i) $xx_1 + yy_1 + zz_1 + ux_1 + vy_1 + wz_1 + d = 0$

(ii) $xx_1 + yy_1 + zz_1 + ux + vy + wz + d = 0$

(iii) $xx_1 + yy_1 + zz_1 + u (x + x_1) + v (y + y_1) + w (z + z_1) + d = 0$

(d) none of these

18. The plane $2x + y - z = 12$ touches the sphere $x^2 + y^2 + z^2 = 24$ at the point

(i) (– 4, – 2, 2) (ii) (4, 2, – 2)

(iii) (4, – 2, 2) (iv) (– 4, 2, – 2)

19. If one end of a diameter of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z - 7 = 0$ be (–1, 2, 4), then the co-ordinates of the other end are

(i) (3, – 6, 2) (ii) (– 3, 6, – 2)

(iii) (2, 3, – 6) (iv) none of these

20. The spheres $x^2+y^2+z^2-2x=3$ and $x^2+y^2+z^2+6x+6y+9=0$

(i) touch internally (ii) do not touch each other

(iii) touch externally (iv) intersect in a circle

21. If two spheres of radius r_1 and r_2 cut orthogonally, then the radius of the common circle is

(i) $\frac{r_1+r_2}{\sqrt{(r_1^2+r_2^2)}}$ (ii) $\frac{r_r-r_1}{\sqrt{(r_1^2+r_2^2)}}$

(iii) $\frac{r_1 r_2}{\sqrt{(r_1^2+r_2^2)}}$ (iv) $r_1 r_2$

22. The radical plane of two spheres $S_1=0$ and $S_2=0$ is given by

(i) $S_1+S_2=0$ (ii) $S_1-S_2=0$

(iii) $S_1 \bullet S_2=0$ (iv) none of these

23. The spheres cutting two given spheres along a great circle pass through some fixed points whose number is

(i) 2 (ii) 3 (iii) 4 (iv) 5

Ch. VIII The Cone and Cylinder

24. The number of independent constants in the equation of the cone with vertex at the origin is

(i) 4 (ii) 5 (iii) 6 (iv) 7

25. General equation of the cone which passes through the co-ordinate axes is

(i) $ax^2+by^2+cz^2=0$ (ii) $ax^2+by^2+cz^2=1$

(iii) $fyz+gzx+hxy=0$ (iv) $fyz+gzx+hxy=1$

26. The equation to the cone whose vertex is the point (0, 0, 0) and whose base is the curve $x^2+y^2=4, z=2$ is

(i) $x^2+y^2+z^2=0$ (ii) $x^2+y^2+z^2=2z$

(iii) $x^2+y^2=z^2$ (iv) none of these

27. The equation of the cone with vertex at the origin and base the curve $z=k, f(x,y)=0$ is

(i) $f\left(\frac{xy}{k}, \frac{yz}{k}, \frac{zx}{k}\right)=0$ (ii) $f\left(\frac{xk}{z}, \frac{yk}{z}\right)=0$

(iii) $f(k/x, k/y)$ (iv) $f(x/k, y/k)=0$

28. If the plane $ux+vy+wz=0$ cuts the cone $f(x,y,z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ in two perpendicular generators then $(a+b+c)(u^2+v^2+w^2)=$

(i) $f(a,b,c)$ (ii) $a^2+b^2+c^2$

(iii) $f(u,v,w)$ (iv) none of these

29. The plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines if

(i) $a+b+c=0$ (ii) $a+b+c=1$

(iii) $(1/a)+(1/b)+(1/c)=0$ (iv) none of these

30. The condition for the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ to have three mutually perpendicular generators is

(i) $ab+bc+ca=0$ (ii) $a+b+c=0$

(iii) $f+g+h=0$ (iv) $a^{-1}+b^{-1}+c^{-1}=0$

31. How many sets of three mutually perpendicular generators of the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ are possible

(i) three (ii) six

(iii) infinite (iv) none of these

32. The locus of the lines through the vertex at right angles to the tangent planes of the given cone is called the .... of the given cone.

(i) enveloping cone (ii) reciprocal cone

(iii) right circular cone (iv) none of these

33. If $S \equiv x^2+y^2+z^2-a^2=0$ be a cone with vertex at (α, β, γ) and $S_1=\alpha^2+\beta^2+\gamma^2-a^2$, $T=\alpha x+\beta y+\gamma z-a^2$, then the equation of the enveloping cone of $S=0$ is

(i) $S_1=T$ (ii) $S_1T=S^2$

(iii) $SS_1=T^2$ (iv) none of these

34. The section of a right circular cone by a plane perpendicular to its axis is

(i) a pair of straight lines (ii) a circle

(iii) an ellipse (iv) none of these

35. Equation to the right circular cone whose vertex is at the origin, the axis along z-axis and semi-vertical angle α, is

(i) $x^2+y^2+z^2=xy\tan\alpha$ (ii) $x^2+y^2=z^2\tan^2\alpha$

(iii) $y^2+z^2=x^2\tan^2\alpha$ (iv) $x^2+z^2=y^2\tan^2\alpha$

36. If a right circular cone $x^2+y^2=z^2\tan^2\alpha$ has three mutually perpendicular generators, then its semi-vertical angle is

(i) $\pi/3$ (ii) $\tan^{-1}1$ (iii) $\tan^{-1}\sqrt{2}$ (iv) $\tan^{-1}2$

37. Every equation of the form $f(x, y)=0$ represents a cylinder whose generators are parallel to

(i) x-axis (ii) y-axis (iii) z-axis (iv) none of these

38. The equation $\sqrt{(fx)}+\sqrt{(gy)}+\sqrt{(hz)}=0$ represents a .... which touches the co-ordinates planes.

(i) sphere (ii) cone (iii) cylinder (iv) none of these

39. $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$ represents a cone it

(i) $(u^2/a)+(v^2/b)+(w^2/c)=d$

(ii) $-(u^2/a)+(v^2/b)+(w^2/c)=d$

(iii) $(u^2/a)-(v^2/b)+(w^2/c)=d$

(iv) $(u^2/a)+(v^2/b)-(w^2/c)=d$

40. The equation of the cone whose vertex is the origin and base the circle $y^2+z^2=b^2, x=a$ is

(i) $b^2(z^2+x^2)=a^2y^2$ (ii) $a^2(y^2+z^2)=b^2x^2$

(iii) $a^2(x^2+y^2)=b^2y^2$ (iv) none of these

41. The equation $SS_1=T^2$ represents the enveloping cone of the sphere $S\equiv x^2+y^2+z^2-a^2=0$, with vertex at the point (α,β,γ), then T stands for

(i) $\alpha^2+\beta^2+\gamma^2-a^2$ (ii) $\alpha x+\beta y+\gamma z$

(iii) $\alpha x+\beta y+\gamma z-a^2$ (iv) none of these

42. The locus of lines drawn parallel to a given line and touching a given surface is

(i) right circular cylinder (ii) enveloping cylinder

(iii) enveloping cone (iv) none of these

(II) TRUE & FALSE TYEP QUESTIONS

Write 'T' or 'F' according as the following statement is True or False :—

Ch. VII Sphere

1. The centre and radius of the sphere $x^2+y^2+z^2-2x+4y-6z=11$ are $(1,-2,3)$ and 5.

2. The equation of the sphere through (0, 0, 0), $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ is $x^2+y^2+z^2+ax+by+cz=0$.

3. The equation of the sphere circumscribing the tetrahedron whose faces are $(y/b)+(z/c)=0$, $(z/c)+(x/a)=0$, $(x/a)+(y/b)=0$ and $(x/a)+(y/b)+(z/c)+1$ is $\dfrac{x^2+y^2+z^2}{a^2+b^2+c^2}=\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}$.

4. A plane passes through a fixed point (p,q,r) and cuts the axes in A,B,C; then the locus of the centre of the sphere $OABC$ is

$$(x/p)+(y/q)+(z/r)=2.$$

5. The equation of the sphere on the join of $(2,-3,4)$ and $(-5,6,-7)$ as diameter is $x^2+y^2+z^2+3x-3y+3z-56=0$.

6. The locus of a point equidistant from a fixed point in space is sphere.

7. The curve of intersection of two spheres is a circle.

8. The equation of the sphere through the circle $S=0, P=0$ is $S\bullet P=0$.

9. The general equation of a sphere contains four independent constants.

10. The radius of the circle $x^2+y^2+z^2-8x+4y+8z=45$, $x-2y+2z=3$ is $4\sqrt{5}$.

11. The equation of the sphere passing through the circles $y^2+z^2=9, x=4$ and $y^2+z^2=36, x=1$ is $x^2+y^2+z^2+6x=40$.

12. Condition for the plane $lx+my+nz=p$ to touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is

$(lu + mv + nw + p)^2 = (u^2 + v^2 + w^2)(l^2 + m^2 + n^2)$

13. The plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z = 3$ at the point $(1, -2, -4)$.

14. Two spheres touch externally, if the distance between their centres is equal to the sum of their radii.

15. If the line joining the centre of the sphere and point P meets the polar plane of P in Q, then $OP \bullet OQ = (\text{radius})^2$.

16. Two sphere of radii r_1 and r_2 cut orthogonally, if the radius of the common circle is $(r_1 - r_2)/\sqrt{(r_1^2 + r_2^2)}$.

17. Radical plane of two spheres is the locus of the point from where the square of the lengths of the tangents to the two spheres are equal.

Ch. VIII The Cone and Cylinder

18. The equation of the cone with vertex at the origin is a homogeneous second degree equation in x, y and z.

19. The general equation of a cone of second degree which passes through the co-ordinate axes is $ax^2 + by^2 + cz^2 = 0$.

20. The equation to the cone which has vertex at the origin and passes through the curve given by $ax^2 + by^2 = 2z$, $lx + my + nz = p$ is

$$p(ax^2 + by^2) = 2z(lx + my + nz)$$

21. The equation of the cone with vertex at (α, β, γ) and base $ax^2 + by^2 = 1$, $z = 0$ is $a(\alpha z - \gamma x)^2 + b(\beta z - \gamma y)^2 = (z - \gamma)^2$.

22. $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$ represents a cone if

$$au^2 + bv^2 + cw^2 = d.$$

23. If the plane $ux + vy + wz = 0$ cuts the cone $f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ in two perpendicular generators, then

$$f(u, v, w) = (a + b + c)(u^2 + v^2 + w^2).$$

24. The angle between the lines given by $x + y + z = 0$ and

$$\frac{yz}{q - r} + \frac{zx}{r - p} + \frac{xy}{p - q} = 0 \text{ is } \frac{\pi}{3}.$$

25. The angle between the lines in which the plane $x + y + z = 0$ cuts the cone $ayz + bzx + cxy = 0$ will be $\pi/2$ if $a + b + c = 0$.

26. The locus of the line of intersection of perpendicular tangent planes to the cone $ax^2 + by^2 + cz^2 = 0$ is the cone $bcx^2 + cay^2 + abz^2 = 0$.

27. The general equation to a cone which tonches the coordinate planes is

$$a^2x^2 + b^2y^2 + c^2z^2 = 2bc\,yz + 2ca\,zx + 2ab\,xy.$$

28. The lines drawn from $(0, 0, 0)$ so as to touch the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ lie on the cone $(ux + vy + wz)^2 = d(x^2 + y^2 + z^2)$.

29. The equation of the right circular cylinder through the circle of intersection of $x^2 + y^2 + z^2 = 1$ and $x + y + z = 1$ is $x^2 + y^2 + z^2 - yz - zx - xy = 0$.

30. The enveloping cylinder of the conicoid $ax^2 + by^2 + cz^2 = 1$ with generators perpendicular to x-axis meets the plane $z = 0$ in ellipses.

(III) FILL IN THE BLANKS TYPE QUESTIONS

Fill in the blanks in the following :—

Ch. VII Sphere

1. The centre of the sphere $2x^2 + 2y^2 + 2z^2 - 2x + 4y - 6z = 15$ is .... *(Meerut 2001)*

2. The equation of a sphere when end points of its diameter are (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \ldots = 0.$$ *(Meerut 2001)*

3. The radius of the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z = 11$ is ....

4. The equation of the sphere through $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is $x^2 + y^2 + z^2$ .... $= 0$.

5. A sphere of radius k passes through the origin and meets the axes in A, B, C. Then the centroid of the triangle ABC lies on the sphere $9(x^2 + y^2 + z^2) =$ ....

6. The equation of the sphere through the circle $x^2 + y^2 + z^2 = 9$, $x + y - 2z + 4 = 0$ and origin is $4(x^2 + y^2 + z^2) + 9$ (........) $= 0$.

7. Tangent line at any point is .... to the radius through that point.

8. If the plane $lx + my + nz = p$ touches the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, then $(lu + mv + nw + p)^2 = (\ldots)(u^2 + v^2 + w^2 - d)$.

9. The plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z = 3$ at the point ....

10. Two spheres touch ......, if the distance between their centres is equal to the difference of their radii.

11. The spheres $x^2 + y^2 + z^2 - 2x = 3$ and $x^2 + y^2 + z^2 + 6x + 6y + 9 = 0$ touch ....

12. The equation of polar of (α, β, γ) with respect to the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is $\alpha x + \beta y + \gamma z + \ldots + d = 0$.

13. Two lines which are such that the polar of any point on any one passes through the other, are known as......... .

14. The two spheres $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ and $x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$ intersect orthogonally provided $2uu' + 2vv' + 2ww' =$ ....

15. The radius of the common circle of two spheres of radii r_1 and r_2 and cutting orthogonally is ....

16. The square of the length of the tangent from the point (x_1, y_1, z_1) outside the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ is $x_1^2 + y_1^2 + z_1^2 +$ ....

Ch. VIII The Cone and Cylinder

17. The equation of the cone with vertex at the origin is a .... second degree equation in x, y and z.

18. The general equation of a cone of second degree which passes through the co-ordinate axes is .... $= 0$.

19. The equation of the cone with vertex at $(0, 0, 0)$ and whose base is the curve $x^2 + y^2 = 4, z = 2$ is .... $= z^2$.

20. The equation of the cone with vertex at $(0, 0, 0)$ and whose base is the circle through the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is .... $= 0$.

21. The equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ represents a cone with its vertex at (α, β, γ) if ....

22. The vertex of the cone $4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12x - 11y + 6z + 4 = 0$ is ....

23. The equation $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$ represents a cone if .... $= d$.

24. If the plane $ux + vy + wz = 0$ cuts the cone $f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ in two perpendicular generators, then

$$f(u, v, w) = (a + b + c)\,(\text{.............}).$$

25. The angle between the lines given by $x + y + z = 0$, $ayx + bzx + cxy = 0$ is $\pi/3$, if .... $= 0$.

26. The cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ has three mutually perpendicular generators if $a + b + c =$ .... *(Meerut 2001)*

27. The cones $ax^2 + by^2 + cz^2 = 0$ and $(x^2/a) + (y^2/b) = (z^2/c) = 0$ are .... to each other.

28. The equation of the cone which touches the co-ordinate planes is .... $= 0$.

29. The equation of the right circular cone with vertex at $(0, 0, 0)$ and z-axis as its axis is .... $= z^2 \tan^2 \theta$, where θ is the semi-vertical angle of the cone.

30. The equation $f(x, y) = 0$ represents a cylinder, whose generators are parallel to the axis of ....

31. The equation of the cone whose vertex is origin and which passes through the curve given by $ax^2 + by^2 = 2z$, $lx + my + nz = p$ is $ax^2 + by^2 =$ .... *(Meerut 2001)*

32. The equation of the enveloping cylinder of the sphere $x^2 + y^2 + z^2 = a^2$ and whose generators are parallel to the line $x/l = y/m = z/n$ is

$$(lx + my + nz)^2 = (l^2 + m^2 + n^2)\,(\text{......})$$ *(Meerut 2001)*

ANSWERS TO OBJECTIVE TYPE QUESTIONS

(I) Multiple Choice Type :

1. (iv); 2. (iii) 3. (ii); 4. (iii); 5. (iii); 6. (ii);

7.	(ii);	**8.**	(ii);	**9.**	(ii);	**10.**	(iii);	**11.**	(i);	**12.**	(ii);
13.	(ii);	**14.**	(i);	**15.**	(iii);	**16.**	(ii);	**17.**	(iii);	**18.**	(ii);
19.	(i);	**20.**	(iii);	**21.**	(iii);	**22.**	(ii);	**23.**	(i);	**24.**	(ii);
25.	(iii).	**26.**	(iii);	**27.**	(ii);	**28.**	(iii);	**29.**	(iii);	**30.**	(ii);
31.	(iii);	**32.**	(ii);	**33.**	(iii);	**34.**	(ii);	**35.**	(ii);	**36.**	(iii);
37.	(iii);	**38.**	(ii);	**39.**	(i);	**40.**	(ii);	**41.**	(iii);	**42.**	(ii).

(II) True & False Type :

1.	T;	**2.**	F;	**3.**	T;	**4.**	F;	**5.**	T;	**6.**	T;	**7.**	T;
8.	F;	**9.**	T;	**10.**	T;	**11.**	F;	**12.**	F;	**13.**	F;	**14.**	T.
15.	T;	**16.**	F;	**17.**	T;	**18.**	T;	**19.**	F;	**20.**	T;	**21.**	T;
22.	F;	**23.**	T	**24.**	T;	**25.**	T;	**26.**	F;	**27.**	T;	**28.**	T;
29.	F;	**30.**	T										

(III) Fill in the blanks Type :

1. $(1/2, -1, 3/2)$; **2.** $(z - z_1)(z - z_2)$; **3.** 5; **4.** $-ax - by - cz$; **5.** $4k^2$;

6. $x + y - 2z$; **7.** perpendicular; **8.** $l^2 + m^2 + n^2$; **9.** $(-1, 4, -2)$; **10.** internally;
11. externally; **12.** $u(x + \alpha) + v(y + \beta) + w(z + \gamma)$; **13.** polar lines;

14. $d + d'$; **15.** $r_1 r_2/\sqrt{(r_1^2 + r_2^2)}$; **16.** $2ux_1 + 2vy_1 + 2wz_1 + d$;

17. homogeneous; **18.** $fyz + gzx + hxy$; **19.** $x^2 + y^2$; **20.** $\Sigma a(b^2 + c^2)yz$;

21. $\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$

22. $(-1, -2, -3)$; **23.** $(u^2/a) + (v^2/b) + (w^2/c)$; **24.** $u^2 + v^2 + w^2$;

25. $a^{-1} + b^{-1} + c^{-1}$; **26.** 0; **27.** reciprocal; **28.** $\sqrt{(fx)} + \sqrt{(gy)} + \sqrt{(hz)} = 0$;

29. $x^2 + y^2$; **30.** z; **31.** $2z(lx + my + nz)/p$; **32.** $x^2 + y^2 + z^2 - a^2$.

(A) VERY SHORT AND SHORT ANSWERS TYPE QUESTIONS

CH. IX Conicoids.

1. Define conicoid. [See § 9·01 Page 1]

2. What surface is represented by $ax^2 + by^2 + cz^2 = 1$? [See § 9·05 Page 5]

3. What does equation $3x^2 + 4y^2 + z^2 = 1$ represent ? *(Purvanchal 2001)*

4. Write down the equation of the tangent plane to the conicoid $ax^2 + by^2 + cz^2 = 1$ at the point (α, β, γ). [See § 9·06 Page 5]

5. Write down the condition that the plane $lx + my + nz = p$ may be tangent plane to the conicoid $ax^2 + by^2 + cz^2 = 1$. *(Purvanchal 2001, 2000)* [See § 9·07 (a) Page 6]

6. Write down the condition that the plane $lx + my + nz = p$ may touch the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$. [See § 9·07 (b) Page 7]

7. Define director sphere of a central conicoid. *(Purvanchal 98)* [See § 9·08 Page 7]

8. (a) Write down the equation of the director sphere of conicoid $ax^2 + by^2 + cz^2 = 1$. *(Purvanchal 2001, 2000)* [See § 9·08 Page 7]

8. (b) Find the the locus of the point of intersector of three mutually perpendicular tangents to the concoid $ax^2 + by^2 + cz^2 = 1$. *(Purvanchal 99)* [See § 9·08 Page 7]

9. Define the polar lines. *(Purvanchal 2001, 2000)* [See § 9·11 Page 21]

10. Write the equation of the polar plane of the point (2, – 3, 4) with respect to the conicoid $x^2 + 2y^2 + z^2 = 4$. [See Ex. 1 Page 22]

11. Find the locus of chords bisector at a given point *(Purvanchal 2000, 99)* [See § 9·11 Page 27]

12. Write the equation of polar lines of the given line $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$ with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$. *(Purvanchal 99)* [See § 9·12 Page 21]

13. Find the equation to the plane which cuts $2x^2 - 3y^2 + 5z^2 = 1$ in a conic whose centre is at the point (2, 1, 3). [See Ex. 1 Page 27]

14. Show that the section of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$ whose centre is at the point $(a/3, b/3, c/3)$ passes through the extremiters of the axes. [See Ex. 8 Page 31]

15. Write down the equation of the normal line to the surface $F(x, y, z) = 0$ at the point (α, β, γ). *(Purvanchal 2001)*

16. Write down the equation of the normal to the conicoid at the point

(α, β, γ). *(Purvanchal 99)* [See § 9·13 Page 34]

17. How many normals can be drawn on an ellipsoid from a given point (α, β, γ)? *(Purvanchal 2001)* [See § 9·14 Page 42]

18. Define protate and oblate spheroids. *(Purvanchal 2000, 99, 98)*

19. Define conjugate diameters and conjugate diametral planes of a central conicoid. *(Puvanchal 99, 98)* [See § 9·18 Page52]

20. Prove that the sum of squares of the lengths of three conjugate semi-diameters of conicoid $ax^2 + by^2 + cz^2 = 1$ is constant. *(Purvancahl 99, 98)* [See § 9·19 Page 54]

21. Find diametral plane of the conicoid $ax^2 + by^2 + cz^2 = 1$. *(Purvanchal 98)* [See § 9·17 Page 51]

22. Define diametral plane of a conicoid *(Purvanchal 98)* [See § 9·17 P. 51]

23. Define principle plane of a conicoid. *(Purvanchal 98)* [See § 9·02 P. 2]

Ch. X Parabolids

24. Write down the equation of an elliptic paraboloid. [See § 10·01 P. 1]

25. Write down the equation of the tangent plane to the parabolid $ax^2 + by^2 = 2cz$ at the point (α, β, γ). *(Purvanchal 2001)* [See § 10·04 Page 3]

26. What is the condition that the plane $lx + my + nz = p$ may touch the paraboloid $ax^2 + by^2 = 2cz$. *(Purvanchal 99)* [See § 10·05 Page 4]

27. Define tangent lines and tangent planes for paraboloid. *(Purvanchal 2000)*

28. Find the locus of the point of intersection of three mutually perpendicular tangent planes. [See § 10·07 Page 9]

29. Find the pole of the plane $lx + my + nz = p$ with respect to the paraboloid $ax^2 + by^2 = 2cz$.

30. Write down the equation of the locus of chords with a given mid-point of the paraboloid $ax^2 + by^2 = 2cz$. [See § 10·10 Page 12]

31. Write the equations of the normal to the paraboloid $ax^2 + by^2 = 2cz$ at the point (α, β, γ). *(Purvanchal 99, 98)* [See § 10·11 Page 14]

32. How many normals can be drown to a paraboloid from any fixed point? [See § 10·12 Page 14]

33. Define diametral plane of a paraboloid and write down its equation. [See § 10·15 Page 20]

34. Define conjugate diametral planes of a paraboloid. [See § 10·16 Page 21]

35. Write down the equation of the enveoloping cone of the paraboloid $ax^2 + by^2 = 2cz$ with vertex at the point (α, β, γ). *(Purvanchal 98)* [See § 10·17 Page 22]

36. Find the enveloping cylinder of the paraboloid $ax^2 + by^2 = 2cz$.
(Purvanchal 98) [See § 10·18 P. 24]

37. Show that the equation $xy = 2cz$ represents a hyperbolic paraboloid.
(Purvanchal 98)

38. Define diameters of a paraboloid. [See § 10·06 Page 9]

(B) OBJECTIVE TYPE QUESTIONS

(I) Multiple Choice Type Questions :

Choose (i), (ii), (iii) or (iv) whichever is coorect :—

Ch. IX Conicoids

1. The equation $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ represents
(i) an ellipsoid
(ii) a hyperbolic paraboloid
(iii) a hyperboloid of two sheets
(iv) a hyperboloid of one sheet

2. The equation $(x^2/a^2) - (y^2/b^2) = 2z/c$ represents
(i) an ellipsoid **(ii)** an elliptic paraboloid
(iii) a hyperbolic paraboloid **(iv)** a hyperboloid of one sheet

3. The equation $ax^2 + by^2 + cz^2 = 1$ represents a
(i) cylinder **(ii)** conicoid
(iii) hyperboloid **(iv)** none of these

4. The equation of the tangent plane to the conicoid $ax^2 + by^2 + cz^2 = 1$ at the point (α, β, γ) is
(i) $a\alpha x + b\beta y + c\gamma z = 0$ **(ii)** $\alpha x + \beta y + \gamma z = a + b + c$
(iii) $a\alpha x + b\beta y + c\gamma z = 1$ **(iv)** none of these

5. The condition that the plane $lx + my + nz = p$ touches the conicoid $ax^2 + by^2 + cz^2 = 1$ is
(i) $(l/a) + (m/b) + (n/c) = p$
(ii) $(l/a^2) + (m/b^2) + (n/c^2) = p^2$
(iii) $(l^2/a) + (m^2/b) + (n^2/c) = p$
(iv) $(l^2/a) + (m^2/b) + (n^2/c) = p^2$

6. The condition that the plane $lx + my + nz = p$ touches the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$ is
(i) $al + bm + cn = p^2$ **(ii)** $a^2l^2 + b^2m^2 + c^2n^2 = p^2$.
(iii) $al + bm + cn = p$ **(iv)** $a^2l^2 + b^2m^2 + c^2n^2 = p^{-2}$

7. The equation of the direction sphere of $ax^2 + by^2 + cz^2 = 1$ is
(i) $x^2 + y^2 + z^2 = a^2 + b^2 + c^2$
(ii) $x^2 + y^2 + z^2 = ab + bc + ca$

(iii) $x^2+y^2+z^2=a^{-1}+b^{-1}+c^{-1}$

(iv) $x^2+y^2+z^2=a+b+c$

8. The tangent planes at the extremities of any diameter of an ellipsoid are

(i) coincident (ii) parallel

(iii) perpendicular (iv) none of these

9. The pole of the plane $lx+my+nz=p$ with respect to the conicoid $ax^2+by^2+cz^2=1$ is

(i) $(ap/l, bp/m, cp/n)$ (ii) $(p/al, p/bm, p/cn)$

(iii) $(l/ap, m/bp, n/cp)$ (iv) none of these

10. The straight line which interrsects a central conicoid in two conicident points, is called

(i) a diameter (ii) a chord of contact

(iii) a tangent line (iv) none of these

11. The locus of the poles of the tangent plane to $a^2x^2+b^2y^2-c^2z^2=1$ with respect to $\alpha^2x^2+\beta^2y^2+\gamma^2z^2=1$ is

(i) an ellipsoid (ii) a hyperboloid of one sheet

(iii) a sphere (iv) a hyperboloid of two sheets

12. The locus of the chord of $ax^2+by^2+cz^2=1$ with (α, β, γ) as mid-point is

(i) a conic (ii) a circle

(iii) a plane (iv) none of these

13. The equation of the normal to $ax^2+by^2+cz^2=1$ at (α, β, γ) are

(i) $\frac{x-\alpha}{\alpha}=\frac{y-\beta}{\beta}=\frac{z-\gamma}{\gamma}$ (ii) $\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{c\gamma}$

(iii) $\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\gamma}{c}$ (iv) none of these

14. The number of normals which can be drawn from a given point to an ellipsoid is

(i) 2 (ii) 3 (iiii) 4 (iv) 6

15. The locus of the mid-point of a system of parallel chords of the conicoid $ax^2+by^2+cz^2=1$ is

(i) tangent plane (ii) polar plane

(iii) diametral plane (iv) none of these

16. The volume of the parallelopiped having three conjugate semi-diametions of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(b^2/c^2)=1$ as coterminus edges is

(i) $\frac{1}{6}a^2b^2c^2$ (ii) $\frac{1}{6}abc$ (iii) abc (iv) none of these

17. Sum of the squares of any three conjugate semi-diameters of the ellipsoid $\Sigma(x^2/a^2)=1$ is

(i) $a+b+c$ (ii) $ab+bc+ca$

(iii) $a^2+b^2+c^2$ (iv) none of these

18. The equation of the plane which cuts $2x^2-3y^2+5z^2=1$ in a conic whose centre is at the point (2, 1, 3) is

(i) $4x-3y+15z=50$ (ii) $4x+3y-15z=50$

(iii) $4x+3y+15z=50$ (iv) none of these

Ch. X Paraboloids.

19. The equation of the tangent plane to the paraboloid $ax^2+by^2=2cz$ at the point (α, β, γ) is

(i) $a\alpha x+b\beta y=c+z$ (ii) $a\alpha x+b\beta y=c(z+\gamma)$

(iii) $a\alpha x+b\beta y=\gamma(c+z)$ (iv) none of these

20. The plane $2x-4y-z+3=0$ touches $x^2-2y^2=3z$ at the point

(i) $(3,-3,3)$ (ii) $(-3,3,3)$

(iii) $(3,3,-3)$ (iv) none of these

21. The pole of the plane $lx+my+nz=p$ with respect to the paraboloid $ax^2+by^2=2cz$ is

(i) $\left(-\frac{lc}{an}, \frac{-mc}{bn}, \frac{-p}{n}\right)$ (ii) $\left(\frac{lc}{an}, \frac{mc}{bn}, \frac{p}{n}\right)$

(iii) $(l, m, -n)$ (iv) none of these

22. How many normals can be down to a paraboloid from a given point ?

(i) 3 (ii) 4

(iii) 5 (iv) 6

(II) TRUE AND FALSE TYPE QUESTIONS :

Write 'T' or 'F' accordind as the following statement is true or false :

Ch. IX Conicoids.

1. The equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}$ represents a hyperboloid of one sheet.

2. The surface represented by $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$ is symmetrical with respect to coordinate planes which are known as principal planes of the above surface.

3. The condition that the plane $lx+my+nz=p$ may touch the ellipsoid $\Sigma(x^2/a^2)=1$ is $(l^2/a)+(m^2/b)+(n^2/c)=p^2$.

4. The planes $lx+my+nz=\pm\sqrt{(a^2l^2+b^2m^2+c^2n^2)}$ always touch the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$.

5. $x^2+y^2+z^2=a^2+b^2+c^2$ is the equation of the director sphere of the ellipsoid $(x^2/a^2)+(y^2/b^2)+(z^2/c^2)=1$

6. The tangent planes at the extermities of any diameter of an ellipsoid are perpendicular.

7. Locus of the chords with (x_1, y_1, z_1) as mid-point of the conicoid $ax^2 + by^2 + cz^2 = 1$ is $ax_1^2 + by_1^2 + cz_1^2 = a\alpha x + b\beta y + c\gamma z$.

8. The equations of the normal to $ax^2 + by^2 + cz^2 = 1$ at (α, β, γ) is

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$$

9. The equation of the locus of the mid-point of a system of parallel chords with direction cosines l, m, n of the conicoid $ax^2 + by^2 + cz^2 = 1$ is

$$alx + bmy + cnz = 0$$

10. Sum of the squares of the projections of three conjugate semi-diameters of the ellipsoid $\Sigma (x^2/a^2) = 1$ on any line is not constant.

Ch. X Paraboloids

11. The general equation of a paraboloid is $ax^2 + by^2 = 2cz$.

12. The plane $lx + my + nz = p$ touches the paraboloid $ax^2 + by^2 = 2cz$ provided $(a^2/l) + (b^2/m) + (2cp/n) = 0$.

13. The plane $2(lx + my + nz)\, n + c\, [(l^2/a) + (m^2/b)] = 0$ always touches the paraboloid $ax^2 + by^2 = 2cz$.

14. $a\alpha x + b\beta y = c(z + \gamma)$ is the equation of the polar plane of the point (α, β, γ) with respect to the paraboloid $ax^2 + by^2 = 2cz$.

15. The equations of the normal to the paraboloid $ax^2 + by^2 = 2cz$ at the point (α, β, γ) are $\dfrac{x-\alpha}{a\alpha} = \dfrac{y-\beta}{b\beta} = \dfrac{z-\gamma}{c\gamma}$.

16. The five feet of the normals that can be drawn to a paraboloid from a given point are the intersection of a certain cubic curve with the paraboloid.

17. Enveloping cylinder of the paraboloid $ax^2 + by^2 = 2cz$ with its generators parallel to the line $x/l = y/m = z/n$ is

$$(alx + bmy - cn)^2 = (al^2 + bm^2)(ax^2 + by^2 - 2cz)$$

18. The locus of the point from which three mutually perpendicular tangents can be drawn to the paraboloid $(x^2/a^2) - (y^2/b^2) = 2z$.

is $$x^2 + y^2 - 2(a^2 - b^2)\, z + a^2 + b^2 = 0.$$

(III) FILL IN THE BLANKS TYPE QUESTIONS :

Fill in the blanks in the following :

Ch. IX. Conicoids.

1. The equation $(x^2/a^2) - (y^2/b^2) - (z^2/c^2) = 1$ represents a hyperboloid of ....

2. $ax^2 + by^2 + cz^2 = 1$ is the .... equation of the conicoid.

3. The equation of the tangent plane of $ax^2 + by^2 + cz^2 = 1$ at (α, β, γ) is

.... = 1.

4. The plane $lx + my + nz = p$ will touch the conicoid $ax^2 + by^2 + cz^2 = 1$; provided .... $= p^2$.

5. The equation of the director sphere to $ax^2 + by^2 + cz^2 = 1$ is

$$x^2 + y^2 + z^2 = \text{..........}$$

6. The locus of the foot of the central perpendicular on varying tangent planes to the ellipsoid $\Sigma (x^2/a^2) = 1$ is $(x^2 + y^2 + z^2)^2 =$ ....

7. The pole of the plane $lx + my + nz = p$ with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$ is ....

8. The diection ratios of the normal to the ellipsoid $\Sigma (x^2/a^2) = 1$ at (α, β, γ) are ....

9. .... normals can be drawn to an ellipsoid from a given point.

10. The locus of the asymptotic line drawn from the origin to the conicoid $ax^2 + by^2 + cz^2 = 1$ is .... $= 0$

Ch. X. Parboloids.

11. The general equation of a hyperbolic paraboloid is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} =$ ....

12. The equation of the tangent plane to the paraboloid $(x^2/a^2) + (y^2/b^2) = 2z/c$ at the point (α, β, γ) is $(\alpha x/a^2) + (\beta y/b^2) =$ ....

13. The plane $lx + my + nz = p$ touches the paraboloid $ax^2 + by^2 = 2cz$ provided $(l^2/a^2) + (m^2/b) + (\text{......}) = 0$

14. The locus of point of intersection of three mutually perpendicular tangent planes to the paraboloid $ax^2 + by^2 = 2cz$ is $2z + c\,[\text{...........}] = 0$

15. The equation of the locus of chords of the paraboloid $ax^2 + by^2 = 2cz$ with (α, β, γ) as mid-point is $a\alpha x + b\beta y - c\,(\text{.....}) = a\alpha^2 + b\beta^2 - c\gamma$.

16. The equations of the normal to the paraboloid $ax^2 + by^2 = 2cz$ at the point (α, β, γ) are $\dfrac{x - \alpha}{a\alpha} = \dfrac{y - \beta}{b\beta} = \dfrac{z - \gamma}{\text{.......}}$

17. .... normals can be drawn to a paraboloid from any fixed point.

18. The enveloping cone of the paraboloid $ax^2 + by^2 = 2cz$ with vertex at the point (α, β, γ) is

$$(ax^2 + by^2 - 2cz)(a\alpha^2 + b\beta^2 - 2c\gamma) = [\text{......}]^2$$

ANSWERS TO OBJECTIVE TYPE QUESTIONS

(I) Multiple Choice Type :

1. (iv); 2. (iii) 3. (ii); 4. (iii); 5. (iv); 6. (ii);

7. (iii); 8. (ii); 9. (iii); 10. (iii); 11. (ii); 12. (iii);
13. (ii); 14. (iv); 15. (iii); 16. (iii); 17. (ii); 18. (i);
19. (ii); 20. (iii); 21. (i); 22. (iii).

(II) True & False Type :

1. F; 2. T; 3. F; 4. T; 5. T; 6. F; 7. T;
8. F; 9. T; 10. F; 11. T; 12. F; 13. T; 14. T.
15. F; 16. T; 17. T; 18. F.

(III) Fill in the blanks Type :

1. two sheets; **2.** standard; **3.** $a\,\alpha x + b\,\beta y + c\,\gamma z$; **4.** $(l^2/a) + (m^2/b) + (n^2/c)$;
5. $a^{-1} + b^{-1} + c^{-1}$; **6.** $a^2x^2 + b^2y^2 + c^2z^2$; **7.** $(l/ap, m/bp, n/cp)$;
8. $\alpha/a^2, \beta/b^2, \gamma/c^2$; **9.** six; **10.** $ax^2 + by^2 + cz^2$; **11.** $2z/c$; **12.** $(z + \gamma)/c$;
13. $2np/c$; **14.** $a^{-1} + b^{-1}$; **15.** $z + \gamma$; **16.** $-c$; **17.** five;
18. $a\alpha x + b\beta y - c\,(z + \gamma)$.

(A) VERY SHORT AND SHORT ANSWER TYPE QUESTIONS

Ch. XI. Plane Sections of a Conicoid.

1. What are the conditions for the plane section of the conicoid $ax^2 + by^2 + cz^2 = 1$ by the plane $lx + my + nz = p$ to be an ellipse, a parabola or a hyperbola ? [See § 11·02 Page 2]

2. What is the area of the plane central section of the conicoid $ax^2 + by^2 + cz^2 = 1$? *(Purvanchal 2000)* [See § 11·04 Cor. 2 Page 5]

3. State the condition for the plane section to be a rectangular hyperbola. *(Purvanchal 2000)* [See § 11·04 Cor. 3 Page 5]

4. State the condition for circular sections of any central conicoid. *(Purvanchal 2001, 2000)* [Sec § 11·08 Page 53]

5. Find the circular section of the paraboloid $x^2 + 10z^2 = 2y$. *(Purvanchal 99)* [See Ex. 1 Page 58]

6. Find the real circular sections of paraboloid $13y^2 + 4z^2 = 2x$. *(Purvanchal 98)*

7. Define umbilics for a conicoid. *(Purvanchal 2001)* [See § 11·10 P. 58]

8. Find the umbilics of the ellipsoid $2x^2 + 3y^2 + 6z^2 = 6$. *(Purvanchal 98)*
Ans. $(\pm \frac{1}{2}\sqrt{6}, 0, \pm \frac{1}{2}\sqrt{2})$

9. What are the umbilics of the paraboloid $ax^2 + by^2 = 2cz, a > b > 0$. [See § 11·12 Page 60]

Ch. XII. Reduction of General Equation of Second Degree.

10. Write down the conditions that equation $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ may represent a sphere. *(Purvanchal 2001)* **Ans.** $f = 0 = g = h$

11. Write down the formulae of finding the centre of the surface $F(x, y, z) = 0$. [See § 12·02 Page 2]

12. What does the equation $Ax^2 + By^2 + 2kz = 0$ represent ?
Ans. Hyperbolic paraboloid [See § 12·04 Page 4]

13. Find the centre of the conicoid $3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy - 4x - 8z + 5 = 0$. **Ans.** $(1/3, -1/3, 4/3)$ [See Ex. 1 Page 9]

14. What does the equation $x^2 + y^2 + z^2 - 2yz + 2zx - 2xy + x - 4y - z + 1 = 0$ represent ? [See Ex. 1 Page 28]

15. What is represented by $a(x^2 + y^2) + b = 0$? [See § 12·15 P. 30]

16. Show that $xy = 2cz$ represents a hyperbolic paraboloid. *(Purvanchal 98)*

Ch. XIII. Generating Lines.

17. Define a ruled surface. [See § 13·01 Page 1]

18. Write down equations of generating lines of $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ in λ and μ systems. *(Purvanchal 2001)* [See § 13·02 Page 1]

19. Write down equations of the generators of the hyperbolic paraboloid. *(Purvanchal 2000)* [See § 13·02 Page 1]

20. Prove that no two generators of the same system (of a hyperboloid) intersect. *(Purvanchal 98)* [See § 13·03 Prop. II Page 2]

21. Show that generators of λ-system intersect the generators of μ-system of a hyperboloid. *(Purvanchal 99)* [See § 13·03 Prop. III Page 3]

22. Show that if three points of any straight line lie on the conicoid $F(x, y, z) = 0$, then the line wholly lies on the conicoid. [See § 13·04 Page 3]

23. Define the asymptotic cone of the central conicoid. *(Purvanchal 2000, 99, 98)* [See § 13·05 Page 4]

24. Find the locus of the point of intersection of two mutually perpendicular generators of a hyperboloid of one sheet. [See Ex. 20 P. 22]

25. Write down the equations of generators through a point of the paraboloid. *(Purvanchal 99)* [See § 13·06 Page 26]

26. Show that no two generators of the same system of a hyperboloic paraboloid intersect. [See § 13·07 Prop. II Page 27]

27. Show that generator of λ-system intersect the generator of μ-system of the hyperbolic paraboloid. [See § 13·07 Prop. III Page 27]

Ch. XIV. Confocal Conicoids.

28. Define confocal conicoids. [See § 14·01 Page 1]

29. What is the nature of three conicoids confocal to a given ellipsoid ? [See § 14·02 Page 2]

30. Show that three paraboloids confocal with a given paraboloid pass through a given point, two of which are elliptic and one hyperbolic. [See Ex. 1 Page 3]

31. Show that confocals cut at right angles. [See § 14·04 Page 11]

32. Show that only one conicoid confocal with a given conicoid touches a given plane. [See § 14·05 Page 12]

(B) OBJECTIVE TYPE QUESTIONS

(I) MULTIPLE CHOICE TYPE QUESTIONS :

Choose (i), (ii), (iii) or (iv) whichever is correct :

Ch. XI. Plane Sections of a Conicoid.

1. The section of the conicoid $ax^2 + by^2 + cz^2 = 1$ by the plane $lx + my + nz = p$ will be an ellipse if $bcl^2 + cam^2 + abn^2$

(i) >0 (ii) $=0$ (iii) <0 (iv) none of these

2. The area of the central plane section of the conicoid $ax^2 + by^2 + cz^2 = 1$ by the plane $lx + my + nz = p$ is

(i) $\pi/\sqrt{(abc)}$ (ii) $\pi\sqrt{(abc)}$ (iii) $\pi/p\sqrt{(abc)}$ (iv) none of these

3. The section of the conicoid $ax^2 + by^2 + cz^2 = 1$ by a tangent plane to the cone $\dfrac{x^2}{b+c} + \dfrac{y^2}{c+a} + \dfrac{z^2}{a+b} = 0$ is

(i) an ellipse (ii) a parabola
(iii) a circle (iv) a rectangular hyperbola

4. All plane sections of the conicoid $ax^2 + by^2 + cz^2 = 1$ which pass through the point (α, β, γ) and are rectangular hyperbolas touch

(i) a sphere (ii) a cone
(iii) a cylinder (iv) none of these

5. Section of the surface $yz + zx + xy = a^2$ by the plane $lx + my + nz = p$ is a parabola, if

(i) $\sqrt{l} + \sqrt{m} + \sqrt{n} > 0$ (ii) $\sqrt{l} + \sqrt{m} + \sqrt{n} = 0$
(iii) $\sqrt{l} + \sqrt{m} + \sqrt{n} < 0$ (iv) none of these

6. The centre of the section of the conicoid $ax^2 + by^2 + cz^2 = 1$ by the non-central plane $lx + my + nz = p$ is

(i) $\left(\dfrac{lp_0^2}{ap}, \dfrac{mp_0^2}{bp}, \dfrac{np_0^2}{cp}\right)$ (ii) $\left(\dfrac{lp^2}{ap_0}, \dfrac{mp^2}{bp_0}, \dfrac{np^2}{cp_0}\right)$
(iii) $\left(\dfrac{lp}{ap_0^2}, \dfrac{mp}{bp_0^2}, \dfrac{np}{cp_0^2}\right)$ (iv) none of these

7. The co-ordinates of the centre of the section of the ellipsoid $3x^2 + 3y^2 + 6z^2 = 10$ by the plane $x + y + z = 1$ are

(i) $\left(\dfrac{2}{5}, \dfrac{2}{5}, \dfrac{1}{5}\right)$ (ii) $\left(\dfrac{2}{5}, -\dfrac{2}{5}, \dfrac{1}{5}\right)$
(iii) $\left(\dfrac{2}{5}, \dfrac{2}{5}, -\dfrac{1}{5}\right)$ (iv) $\left(-\dfrac{2}{5}, \dfrac{2}{5}, \dfrac{1}{5}\right)$

8. The area of the section of the ellipsoid $\Sigma(x^2/a^2) = 1$ by the plane $\Sigma(x/a) = 1$ is

(i) $\dfrac{2\pi}{3\sqrt{3}}\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}$ (ii) $\dfrac{3\pi}{2\sqrt{2}}\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}$
(iii) $\pi\sqrt{(b^2c^2 + c^2a^2 + a^2b^2)}$ (iv) none of these

9. The real central circular sections of the ellipsoid $x^2 + 2y^2 + 6z^2 = 8$ are given by

(i) $2x \neq z = 0$ **(ii)** $x \pm 2z = 0$

(iii) $2x \pm 3z = 0$ **(iv)** $3x \pm 2z = 0$

10. The plane $x + y - z = 0$ cuts the conicoid $4x^2 + 2y^2 + z^2 + 3yz + zx = 1$ in a circle, whose radius is

(i) $1/2$ **(ii)** $1/\sqrt{2}$ **(iii)** $1/3$ **(iv)** $1/\sqrt{3}$

11. All sections of the paraboloid $(x^2/a^2) + (y^2/b^2) = 2z/c$ perpendicular to the axis of the surface are

(i) parabolas **(ii)** hyperbolas

(iii) ellipses **(iv)** none of these

12. All sections of a paraboloid parallel to its axis are

(i) circles **(ii)** parabolas

(iii) ellipses **(iv)** none of these

13. Number of real umbilics of a hyperboloid of one sheet is

(i) 4 **(ii)** 2 **(iii)** 1 **(iv)** none

14. Number of real umbilics of an ellipsoid is

(i) 0 **(ii)** 1 **(iiii)** 2 **(iv)** 4

15. The umbilics of the paraboloid $5x^2 + 4y^2 = 40z$ are

(i) $\left(0, \frac{1}{10}, \pm\frac{1}{40}\right)$ **(ii)** $\left(0, \pm\frac{1}{10}, \frac{1}{40}\right)$

(iii) $\left(0, \pm\frac{1}{10}, \pm\frac{1}{40}\right)$ **(iv)** none of these

16. Condition for the section of the paraboloid $ax^2 + by^2 = 2cz$ by the plane $lx + my + nz = p$ to be a rectangular hyperbola is

(i) $al^2 + bm^2 + cn^2 = 0$ **(ii)** $al^2 + bm^2 + cn^2 = p^2$

(iii) $am^2 + bl^2 + (a/b)\,n^2 = 0$ **(iv)** none of these

17. The perpendicular distance from centre to the tangent plane at an umbilic of the ellipsoid $\Sigma\,(x^2/a^2) = 1$ is

(i) ab/c **(ii)** bc/a **(iii)** ac/b **(iv)** none of these

Ch. XII. Reduction of General Equation of Second Degree.

18. Centre of the surface $3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy - 4x - 8z + 5 = 0$ is

(i) $(1/3, 1/3, 4/3)$ **(ii)** $(1/3, -1/3, 4/3)$

(iii) $(1/3, 1/3, -4/3)$ **(iv)** none of these

19. The standard form of the surface $2x^2 - 7y^2 + 2z^2 - 10yz - 8zx - 10xy + 6x + 12y - 6z + 5 = 0$ is

(i) $x^2 + 2y^2 + 4z^2 + 5 = 0$ **(ii)** $x^2 + 2y^2 - 4z^2 = 0$

(iii) $2x^2 + y^2 - 4z^2 = 0$ (iv) none of these

20. What is represented by the equation $x^2 + y^2 + z^2 - 6yz - 2zx - 2xy - 6x - 2y - 2z + 2 = 0$?

(i) a cone

(ii) an ellipsoid

(iii) a hyperboloid of two sheets

(iv) a paraboloid

21. What is represented by the equation $2x^2 + 2y^2 + z^2 + 2yz - 2zx - 4xy + x + y = 0$?

(i) a cone (ii) an elliptic paraboloid

(iii) a hyperbolic paraboloid (iv) none of these

22. What surface is represented by $2y^2 - 2yz + 2zx - 2xy - x - 2y + 3z = 2$?

(i) a pair of planes (ii) an elliptic cylinder

(iii) a hyperbolic cylinder (iv) none of these

23. Name the surface represented by $5x^2 - 4y^2 + 5z^2 + 4yz - 14zx + 4xy + 16x + 16y + 32z + 8 = 0$

(i) a hyperbolic cylinder (ii) a pair of planes

(iii) an elliptic cylinder (iv) none of these

24. What surface is represented by $x^2 + 4y^2 + 9z^2 + 12yz + 6zx + 4xy - 54x - 52y + 62z + 113 = 0$?

(i) a pair of planes (ii) an elliptic cylinder

(iii) a parabolic cylinder (iv) a hyperbolic cylinder

25. What is represented by the equation $x^2 + y^2 + z^2 + yz + zx + xy + 3x + y + 4z + 4 = 0$?

(i) an ellipsoid of revolution (ii) a parabolid of revolution

(iii) a hyperboloid of revolution (iv) none of these

Ch. XIII. Generating Lines.

26. How many systems of generators are there for a hyperboloid of one sheet ?

(i) one (ii) two (iii) three (iv) four

27. One generator of λ-system and one of μ-system of a hyperboloid

(i) are parallel (ii) intersect

(iii) conicide (iv) none of these

28. No two generators of the same system of a hyperboloid of one sheet.

(i) are parallel (ii) coincide

(iii) intersect (iv) none of these

29. The locus of the point of intersection of perpendicular generators of

$(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ is given by $x^2+y^2+z^2=$

(i) $a^2+b^2+c^2$ (ii) $a+b+c$

(iii) abc (iv) none of these

30. No two generators of the same system of a hyperbolic paraboloid

(i) are parallel (ii) intersect

(iii) coincide (iv) none of these

31. The locus of the point of intersection of perpendicular generators of the hyperbolic paraboloid $(x^2/a^2)-(y^2/b^2)=2z$ is given by $a^2-b^2=$

(i) $2z$ (ii) $2y$ (iii) $2x$ (iv) $-2z$

Ch. XIV. Confocal Conicoids.

32. How many conicoids confocal with a given ellipsoid pass through any point ?

(i) one (ii) two (iii) three (iv) four

33. How many paraboloids confocal with a given paraboloid pass through a given point ?

(i) four (ii) three (iii) two (iv) one

34. What locus is represented by the equation $\lambda_1+\lambda_2+\lambda_3=$ constant, where λ's are elliptic co-ordinates ?

(i) sphere (ii) paraboloid (iii) ellipsoid (iv) none of these

35. The tangent planes to two confocals at any common point are

(i) parallel (ii) coincident

(iii) at right angles (iv) none of these

36. The two confocal paraboloids cut everywhere at an angle

(i) $\pi/4$ (ii) $\pi/3$ (iii) $\pi/2$ (iv) none of these

(II) TRUE AND FALSE TYPE QUESTIONS

Write 'T' or 'F' according as the following statement is true or false :

Ch. XI. Plane Sections of a conicoid.

1. The plane section of the conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=p$ will be an ellipse if $bcl^2+cam^2+abn^2<0$.

2. Area of the central plane section of the conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=p$ is $\pi/[p\sqrt{(abc)}]$.

3. The condition for the central section of the conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=p$ to be a rectangular hyperbola is $bc\,l^2+ca\,m^2+ab\,n^2=0$.

4. The condition for the section of the paraboloid $ax^2+by^2=2cz$ by the plane $lx+my+nz=p$ to be a rectangular hyperbola is

$$bl^2 + am^2 + n^2 = 0.$$

5. The central section of an ellipsoid whose area is constant touches a cone of second degree.

6. The section of the paraboloid $ax^2 + by^2 = 2cz$ by a tangent plane to the cone $(x^2/b) + (y^2/a) + [z^2/(a+b)] = 0$ is a rectangular hyperbola.

7. The co-ordinates of the centre of the section of the ellipsoid $3x^2 + 3y^2 + 6z^2 = 10$ by the plane $x + y + z = 1$ are (1/5, 2/5, 2/5).

8. The condition for the section of the ellipsoid $\Sigma (x^2/a^2) = 1$ by the plane $lx + my + nz = p$ to be a circle is

$$\frac{al}{\sqrt{(a^2 - b^2)}} = \frac{m}{0} = \frac{cn}{\pm \sqrt{(b^2 - c^2)}}.$$

9. An umbilic is a circular section of unit radius on the surface of a hyperboloid of two sheets.

10. A hyperboloid of one sheet has no real umbilics.

Ch. XII. Reduction of General Equation of Second Degree.

11. The centre of the surface $F(x, y, z) = 0$ is obtained by solving

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0 \text{ for } x, y, z.$$

12. By rotation of axes, the expression $f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ transforms to $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0$, where λ's are the roots of the cubic

$$\begin{vmatrix} a-\lambda & h & f \\ h & b-\lambda & g \\ b & g & c-\lambda \end{vmatrix} = 0.$$

13. The equation $Ax^2 + By^2 + C = 0$ represents an elliptic cylinder.

14. The equation $x^2 + z^2 = a^2$ represents a circular cylinder whose axis is parallel to y-axis.

15. The equation $A(x^2 + y^2) + cz^2 = 1$ represents a paraboloid of revolution.

16. The equation of a diametral plane of the conicoid $F(x, y, z) = 0$ is

$$l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z} = 0.$$

17. The planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ will be parallel but not the same, if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}.$$

18. The equation $2y^2 - 2yz - 2zx - 2xy - x - 2y + 3z = 2$ represents a

hyperbolic cylinder.

Ch. XIII. Generating Lines.

19. One generator of each system passes through every point of the hyperboloid of one sheet.

20. Two generators of the same system of a hyperboloid of one sheet always intersect.

21. If three points of any straight line lie on a hyperboloid of one sheet, then the line wholly lies on it.

22. The locus of the point of intersection of perpendicular generators of a hyperboloid of one sheet viz. $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ is

$$x^2 + y^2 + z^2 = ab + bc + ca.$$

23. Any two generators of different systems of a hyperbolic paraboloid do not intersect.

24. The locus of the point of intersection of perpendicular generators of the hyperbolic paraboloid $(x^2/a^2) - (y^2/b^2) = 2z$ is $a^2 + b^2 = 2z$.

Ch. XIV. Confocal Conicoids.

25. The conicoids whose principal sections have the some foci are called confocal conicoids.

26. Through any point there pass three conicoids confocal with a given ellipsoid viz. one ellipsoid and two hyperboloids of one sheet.

27. Three paraboloids confocal with a given paraboloid pass through a given point–two elliptic and one hyperbolic.

28. The locus represented by $\lambda_1 + \lambda_2 + \lambda_3 =$ constant, where λ's are elliptic co-ordinates is a hyperboloid of one sheet.

29. The product of eccentricities of the focal conics of an ellipsoid is unity.

30. The tangent planes to two confocals at any common point are always parallel.

(III) FILL IN THE BLANKS TYPE QUESTIONS

Fill in the blanks in the following :

Ch. XI. Plane Sections of a conicoid.

1. The plane section of the conicoid $ax^2 + by^2 + cz^2 = 1$ by the plane $lx + my + nz = p$ will be a hyperbola if $bcl^2 + cam^2 + abn^2$ .....

2. The condition for the central section of the conicoid $ax^2 + by^2 + cz^2 = 1$ by the plane $lx + my + nz = p$ to be a rectangular hyperbola is

$$(b + c)\, l^2 + (c + a)\, m^2 + ... = 0$$

3. Area of the central plane section of the conicoid $ax^2 + by^2 + cz^2 = 1$ by the plane $lx + my + nz = p$ is $\pi/$.........

4. The central section of an ellipsoid whose area is constant touches a ......

5. The condition for the section of the paraboloid $ax^2 + by^2 + cz^2 = 1$ by the plane $lx + my + nz = p$ to be a rectangular hyperbola is

$$bl^2 + am^2 + (.....) n^2 = 0$$

6. The co-ordinates of the centre of the non-central plane srection of the conicoid $ax^2 + by^2 + cz^2 = 1$ by the plane $lx + my + nz = 0$ are

$$(lp/ap_0^2, ..., np/cp_0^2)$$

7. The real central circular sections of the ellipsoid $x^2 + 2y^2 + 6z^2 = 8$ are given by .... $\pm 2z = 0$.

8. A hyperboloid of two sheets has .... umbilics.

9. A hyperboloid of one sheet has .... real umbilic.

10. An umbilic, if it exists, is a circular section of .... radius on the surface of a conicoid.

Ch. XII. Reduction of General Equation of Second Degree.

11. The centre of the surface $F(x, y, z) = 0$ is obtained by solving for x, y, z the equations ....

12. The equation $x^2 + z^2 = a^2$ represents a ...., whose axis is parallel to y-axis.

13. The equation $Ax^2 - By^2 - Cz^2 = 1$ represents a ....

14. The equation $y^2 = Ax$ represents a .... cylinder.

15. The equation of the .... of the surface (conicoid) $F(x, y, z) = 0$ is

$$l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z} = 0.$$

16. The planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ will be parallel but not ...., provided $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \neq \frac{d_1}{d_2}$.

17. The equation

$5x^2 - 4y^2 + 5z^2 + 4yz - 14xz + 4xy + 16x + 16y + 32z + 8 = 0$ represents a ....

18. The equation $A(x^2 + y^2) + Bz = 0$ represents a .... of revolution.

Ch. XIII Generating Lines.

19. No two generators of the same system of a hyperboloid ....

20. One generator of each system passes through every .... of a hyperboloid.

21. If three points of any straight line lie on the conicoid $F(x, y, z) = 0$, then the line .... on the conicoid.

22. The locus of the point of intersection of perpendicular generators of

$(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ is $x^2+y^2+z^2=$ ....

23. Any two generators of different systems of a hyperbolic paraboloid ......

24. The locus of the point of intersection of perpendicular generators of the hyperbolic paraboloid $(x^2/a^2)-(y^2/b^2)=2z$ is ...... $+2z=0$.

Ch. XIV. Confocal Conicoids.

25. The conicoids whose principal sections have the .... are known as confocal conicoids.

26. Three conicoids confocal with a given ellispsoid pass through any point–one ......, one a hyperboloids of one sheet and one other a hyperboloid of two sheets.

27. .... paraboloids confocal with a given paraboloid pass through a given point.

28. The locus represented by $\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2=$ constant, where λ's are elliptic co-ordinates is an......... .

29. The product of eccentricities of the .... of an ellipsoid is unity.

30. The tangent planes to two confocals at any common point are .... .

ANSWERS TO OBJECTIVE QUESTIONS

(I) Multiple Choice Type :

1. (i);	**2.** (iii)	**3.** (iv);	**4.** (ii);	**5.** (ii);	**6.** (iii);
7. (i);	**8.** (i);	**9.** (ii);	**10.** (iv);	**11.** (iii);	**12.** (ii);
13. (iv);	**14.** (iv);	**15.** (ii);	**16.** (iii);	**17.** (iii);	**18.** (ii);
19. (ii);	**20.** (iii);	**21.** (ii);	**22.** (iii);	**23.** (ii);	**24.** (iii);
25. (i);	**26.** (ii);	**27.** (ii);	**28.** (iii);	**29.** (i);	**30.** (ii);
31. (iv);	**32.** (iii);	**33.** (ii);	**34.** (i);	**35.** (iii);	**36.** (iii).

(II) True & False Type :

1. F;	**2.** T;	**3.** F;	**4.** F;	**5.** T;	**6.** T;	**7.** F;
8. T;	**9.** F;	**10.** T;	**11.** T;	**12.** F;	**13.** T;	**14.** T;
15. F;	**16.** T;	**17.** F;	**18.** T;	**19.** T;	**20.** F;	**21.** T;
22. F;	**23.** F;	**24.** F;	**25.** T;	**26.** F;	**27.** T;	**28.** F;
29. T;	**30.** F					

(III) Fill in the blanks Type :

1. <0; **2.** $(a+b)\,n^2$; **3.** $p\sqrt{(abc)}$; **4.** cone; **5.** $a+b$; **6.** mp/bp_0^2; **7.** x; **8.** four; **9.** no; **10.** zero; **11.** $\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial z}=0$; **12.** circular cylinder; **13.** hyperboloid of two sheets; **14.** parabolic; **15.** diametral plane; **16.** the same; **17.** pair of planes; **18.** paraboloid; **19.** intersect; **20.** point; **21.** wholly lies; **22.** $a^2+b^2+c^2$; **23.** intersect; **24.** a^2-b^2; **25.** same foci; **26.** ellipsoid; **27.** Three; **28.** ellipsoid; **29.** focal conics; **30.** perpendicular;