LINEAR ALGEBRA PYQS 2010 – 2020

2020

1.

खण्ड—A / SECTION—A

1\. (a) माना समुच्चय V में सभी $n \times n$ के वास्तविक मैजिक वर्ग हैं। दिखाइए कि समुच्चय V, R पर एक सदिश समष्टि है। दो भिन्न-भिन्न 2×2 मैजिक वर्ग के उदाहरण दीजिए।

Consider the set V of all $n \times n$ real magic squares. Show that V is a vector space over R. Give examples of two distinct 2×2 magic squares. 10

(b) माना $M_2(R)$ सभी 2×2 वास्तविक आव्यूहों का सदिश समष्टि है। माना $B = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}$. माना $T : M_2(R) \to M_2(R)$ एक रैखिक रूपांतरण है, जो $T(A) = BA$ द्वारा परिभाषित है। T की कोटि (रैंक) व शून्यता (नलिटि) ज्ञात कीजिए। आव्यूह A ज्ञात कीजिए, जो शून्य आव्यूह को प्रतिचित्रित करता है।

Let $M_2(R)$ be the vector space of all 2×2 real matrices. Let $B = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}$. Suppose $T : M_2(R) \to M_2(R)$ is a linear transformation defined by $T(A) = BA$. Find the rank and nullity of T. Find a matrix A which maps to the null matrix. 10

2 2

2.

[3]

Define an $n \times n$ matrix as $A = I - 2u \cdot u^T$, where u is a unit column vector.

(i) Examine if A is symmetric.

(ii) Examine if A is orthogonal.

(iii) Show that trace $(A) = n - 2$.

(iv) Find $A_{3\times3}$, when $u = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$. 20

3.

(b) माना F सम्मिश्र संख्याओं का एक उपक्षेत्र है व $T: F^3 \to F^3$ एक ऐसा फलन है, जो निम्न रूप से परिभाषित है :

$$T(x_1, x_2, x_3) = (x_1 + x_2 + 3x_3,\ 2x_1 - x_2,\ -3x_1 + x_2 - x_3)$$

a, b, c पर क्या शर्तें हैं कि (a, b, c), T के शून्य समष्टि में है? T की शून्यता निकालिए।

Let F be a subfield of complex numbers and T a function from $F^3 \to F^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2 + 3x_3,\ 2x_1 - x_2,\ -3x_1 + x_2 - x_3)$. What are the conditions on a, b, c such that (a, b, c) be in the null space of T? Find the nullity of T. 15

1 $a + c = 0,\ b + 2c = 0$

4.

Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix}$$

(i) Find AB.

(ii) Find $\det(A)$ and $\det(B)$.

(iii) Solve the following system of linear equations : 1, 2, 1

$$x + 2z = 3,\quad 2x - y + 3z = 3,\quad 4x + y + 8z = 14$$ 15

2019

1.

(c) माना कि $T: \mathbb{R}^2 \to \mathbb{R}^2$ एक रैखिक प्रतिचित्र है, जैसा कि $T(2, 1) = (5, 7)$ एवं $T(1, 2) = (3, 3)$. अगर A मानक आधारों e_1, e_2 के सापेक्ष T के संगत आव्यूह है, तो A की कोटि ज्ञात कीजिए।

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(2, 1) = (5, 7)$ and $T(1, 2) = (3, 3)$. If A is the matrix corresponding to T with respect to the standard bases e_1, e_2, then find Rank (A). 10

2.

If

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$$

then show that $AB = 6I_3$. Use this result to solve the following system of equations :

$$2x + y + z = 5$$
$$x - y = 0$$
$$2x + y - z = 1$$

10

(b) माना कि A और B समान कोटि के दो लांबिक आव्यूह हैं तथा $\det A + \det B = 0$. दर्शाइए कि $A + B$ एक अव्युत्क्रमणीय (सिंगुलर) आव्यूह है।

Let A and B be two orthogonal matrices of same order and $\det A + \det B = 0$. Show that $A + B$ is a singular matrix. 15

3.

Let

$$A = \begin{pmatrix} 5 & 7 & 2 & 1 \\ 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \end{pmatrix}$$

(i) Find the rank of matrix A.

(ii) Find the dimension of the subspace

$$V = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle|\; A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \right\}$$

15+5=20

2018

1. *(a)* मान लीजिये कि A एक 3×2 आव्यूह है और B एक 2×3 आव्यूह है। दर्शाइये कि $C=A\cdot B$ एक अव्युत्क्रमणीय आव्यूह है।

Let A be a 3×2 matrix and B a 2×3 matrix. Show that $C=A\cdot B$ is a singular matrix. 10

(b) आधार सदिशों $e_1=(1,0)$ और $e_2=(0,1)$ को $\alpha_1=(2,-1)$ एवं $\alpha_2=(1,3)$ के रैखिक संयोग के रूप में व्यक्त कीजिये।

Express basis vectors $e_1=(1,0)$ and $e_2=(0,1)$ as linear combinations of $\alpha_1=(2,-1)$ and $\alpha_2=(1,3)$. 10

2.

2. *(a)* अगर A और B समरूप $n\times n$ आव्यूह हैं, तो दर्शाइये कि उनके आइगेन मान एक ही हैं।

Show that if A and B are similar $n\times n$ matrices, then they have the same eigenvalues. 12

3.

For the system of linear equations

$$x+3y-2z=-1$$
$$5y+3z=-8$$
$$x-2y-5z=7$$

determine which of the following statements are true and which are false :

(i) The system has no solution.

(ii) The system has a unique solution.

(iii) The system has infinitely many solutions. 13

2017

1.

1.(a) मान लीजिए $A=\begin{pmatrix}2&2\\1&3\end{pmatrix}$ । एक व्युत्क्रमणीय आव्यूह P ज्ञात कीजिए ताकि $P^{-1}AP$ एक विकर्ण-आव्यूह हो ।

Let $A=\begin{pmatrix}2&2\\1&3\end{pmatrix}$. Find a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix. 10

1.(b) दर्शाइए कि समरूप आव्यूहों के समान अभिलक्षणिक बहुपद होते हैं ।
Show that similar matrices have the same characteristic polynomial. 10

2.

2.(d) मान लीजिए कि U व W सदिश समष्टि V के चार सुस्पष्ट विमीय उप-आकाश जहाँ पर विमा $V = 6$ । उप-आकाश $(U \cap W)$ की सम्भावित विमाएँ ज्ञात कीजिए ।

Suppose U and W are distinct four dimensional subspaces of a vector space V, where dim $V = 6$. Find the possible dimensions of subspace $U \cap W$. 10

3.

3.(a) विचारिए आव्यूह-प्रतिरूपण $A : R^4 \to R^3$ है, जहाँ पर $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{pmatrix}$ । A की प्रतिछाया की विमा व एक आधार तथा कर्नल A की विमा व एक आधार भी ज्ञात कीजिए ।

Consider the matrix mapping $A : R^4 \to R^3$, where $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{pmatrix}$. Find a basis and dimension of the image of A and those of the kernel A. 15

3.(b) सिद्ध कीजिए कि आव्यूह के विभिन्न अशून्य-अभिलक्षणिक सदिश रैखिक स्वतंत्र होते हैं ।
Prove that distinct non-zero eigenvectors of a matrix are linearly independent. 10

3.(b) सिद्ध कीजिए कि आव्यूह के विभिन्न अशून्य-अभिलक्षणिक सदिश रैखिक स्वतंत्र होते हैं ।
Prove that distinct non-zero eigenvectors of a matrix are linearly independent. 10

4.

Consider the following system of equations in x, y, z:

$$x + 2y + 2z = 1$$
$$x + ay + 3z = 3$$
$$x + 11y + az = b.$$

(i) For which values of a does the system have a unique solution ?

(ii) For which pair of values (a, b) does the system have more than one solution ? 15

2016

1.

1. (a) (i) यदि $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ है, तो प्रारम्भिक पंक्ति संक्रिया (elementary row operation) के प्रयोग से A^{-1} निकालिये।

Using elementary row operations, find the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$. 6

(ii) यदि $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ है, तो $A^{14} + 3A - 2I$ का मान निकालिये।

If $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$, then find $A^{14} + 3A - 2I$. 4

2.

Using elementary row operations, find the condition that the linear equations

$$x - 2y + z = a$$
$$2x + 7y - 3z = b$$
$$3x + 5y - 2z = c$$

have a solution. 7

(ii) यदि

$$W_1 = \{(x, y, z) \mid x + y - z = 0\}$$
$$W_2 = \{(x, y, z) \mid 3x + y - 2z = 0\}$$
$$W_3 = \{(x, y, z) \mid x - 7y + 3z = 0\}$$

तो $\dim(W_1 \cap W_2 \cap W_3)$ तथा $\dim(W_1 + W_2)$ का मान निकालिये।

If

$$W_1 = \{(x, y, z) \mid x + y - z = 0\}$$
$$W_2 = \{(x, y, z) \mid 3x + y - 2z = 0\}$$
$$W_3 = \{(x, y, z) \mid x - 7y + 3z = 0\}$$

then find $\dim(W_1 \cap W_2 \cap W_3)$ and $\dim(W_1 + W_2)$. 3

3.

2. *(a)* *(i)* यदि $M_2(R)$, 2×2 कोटि (order) के वास्तविक आव्यूहों की समष्टि (space) तथा $P_2(x)$, वास्तविक बहुपदों (polynomials), जिनकी अधिकतम घात (degree) 2 है, की समष्टि (space) हो, तो $T: M_2(R) \to P_2(x)$, जहाँ $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+c+(a-d)x+(b+c)x^2$, का $M_2(R)$ एवं $P_2(x)$ के मानक आधारों (standard bases) के सापेक्ष आव्यूह निरूपित कीजिये। इसके अलावा T का शून्य समष्टि (null space) प्राप्त कीजिये।

If $M_2(R)$ is space of real matrices of order 2×2 and $P_2(x)$ is the space of real polynomials of degree at most 2, then find the matrix representation of $T: M_2(R) \to P_2(x)$, such that $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a+c+(a-d)x+(b+c)x^2$, with respect to the standard bases of $M_2(R)$ and $P_2(x)$. Further find the null space of T. 10

(ii) यदि $T: P_2(x) \to P_3(x)$ इस प्रकार है कि $T(f(x)) = f(x)+5\int_0^x f(t)\,dt$, तो $\{1, 1+x, 1-x^2\}$ एवं $\{1, x, x^2, x^3\}$ को क्रमशः $P_2(x)$ एवं $P_3(x)$ का आधार (bases) लेते हुए T का आव्यूह निकालिये।

If $T: P_2(x) \to P_3(x)$ is such that $T(f(x)) = f(x)+5\int_0^x f(t)\,dt$, then choosing $\{1, 1+x, 1-x^2\}$ and $\{1, x, x^2, x^3\}$ as bases of $P_2(x)$ and $P_3(x)$ respectively, find the matrix of T. 6

4.

(b) *(i)* यदि $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ है, तो A के अभिलक्षणिक मान (eigenvalues) तथा अभिलक्षणिक सदिशों (eigenvectors) को निकालिये।

If $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then find the eigenvalues and eigenvectors of A. 8

(ii) सिद्ध कीजिये कि हर्मिटी (Hermitian) आव्यूह के सभी अभिलक्षणिक मान वास्तविक हैं।

Prove that eigenvalues of a Hermitian matrix are all real. 8

(c) यदि आधारों (bases) $\{1-x, x(1-x), x(1+x)\}$ एवं $\{1, 1+x, 1+x^2\}$ के सापेक्ष रैखिक रूपांतरण (linear transformation) $T: P_2(x) \to P_2(x)$ के तहत आव्यूह निरूपण $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ हो, तो T प्राप्त कीजिये।

If $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$ is the matrix representation of a linear transformation $T: P_2(x) \to P_2(x)$ with respect to the bases $\{1-x, x(1-x), x(1+x)\}$ and $\{1, 1+x, 1+x^2\}$, then find T. 18

2015

1.

SECTION—A

Q. 1(a) दिए गए सदिश $V_1 = (1, 1, 2, 4)$, $V_2 = (2, -1, -5, 2)$, $V_3 = (1, -1, -4, 0)$ तथा $V_4 = (2, 1, 1, 6)$ रैखिकत : स्वतंत्र हैं। क्या यह सत्य है ? अपने उत्तर के पक्ष में तर्क दीजिये।

The vectors $V_1 = (1, 1, 2, 4)$, $V_2 = (2, -1, -5, 2)$, $V_3 = (1, -1, -4, 0)$ and $V_4 = (2, 1, 1, 6)$ are linearly independent. Is it true ? Justify your answer. 10

Q. 1(b) निम्नलिखित आव्यूह को पंक्ति सोपानक रूप में समानीत कीजिये और तत्पश्चात् इसकी कोटि निकालिए :

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}.$$

Reduce the following matrix to row echelon form and hence find its rank :

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}.$$

10

2.

Q. 2(a) यदि आव्यूह $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ तब आव्यूह A^{30} को ज्ञात कीजिये।

If matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then find A^{30}. 12

3.

Q. 2(c) निम्नलिखित आव्यूह के आइगन मानों एवं आइगन सदिशों को ज्ञात कीजिए :

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}.$$

Find the eigen values and eigen vectors of the matrix :

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}.$$

12

4.

Let $V = \mathbb{R}^3$ and $T \in A(V)$, for all $a_i \in A(V)$, be defined by

$$T(a_1, a_2, a_3) = (2a_1 + 5a_2 + a_3, -3a_1 + a_2 - a_3, -a_1 + 2a_2 + 3a_3)$$

What is the matrix T relative to the basis

$V_1 = (1, 0, 1)$ $V_2 = (-1, 2, 1)$ $V_3 = (3, -1, 1)$? 12

5.

Q. 4(b) $\mathbb{R}^4$ की उस उपसमष्टि की विमा ज्ञात कीजिये जो समुच्चय

$$\{(1, 0, 0, 0), (0, 1, 0, 0), (1, 2, 0, 1), (0, 0, 0, 1)\}$$

द्वारा विस्तारित है। तत्पश्चात् उसका आधार निकालिए।

Find the dimension of the subspace of $\mathbb{R}^4$, spanned by the set

$$\{(1, 0, 0, 0), (0, 1, 0, 0), (1, 2, 0, 1), (0, 0, 0, 1)\}$$

Hence find its basis. 12

2014

1.

(a) एक सदिश R^3 में ज्ञात कीजिए, जो कि V तथा W के प्रतिच्छेद का जनक है, जहाँ कि V एक xy समतल है तथा W सदिश (1, 2, 3) तथा सदिश (1, −1, 1) के द्वारा जनित किया गया आकाश (स्पेस) है ।

Find one vector in R^3 which generates the intersection of V and W, where V is the xy plane and W is the space generated by the vectors (1, 2, 3) and (1, −1, 1). *10*

2.

(b) प्रारंभिक पंक्ति या स्तंभ संक्रियाओं का प्रयोग करके, आव्यूह (मैट्रिक्स)

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

की कोटि ज्ञात कीजिए ।

Using elementary row or column operations, find the rank of the matrix *10*

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}.$$

3.

Q2. (a) मान लीजिए कि V और W निम्न उपसमष्टियाँ हैं R^4 की :

$V = \{(a, b, c, d) : b - 2c + d = 0\}$ और

$W = \{(a, b, c, d) : a = d, b = 2c\}$.

(i) V, (ii) W, (iii) $V \cap W$ का एक आधार और विस्तार ज्ञात कीजिए ।

Let V and W be the following subspaces of R^4 :

$V = \{(a, b, c, d) : b - 2c + d = 0\}$ and

$W = \{(a, b, c, d) : a = d, b = 2c\}$.

Find a basis and the dimension of (i) V, (ii) W, (iii) $V \cap W$. *15*

(b) (i) λ तथा μ के मान जाँच कीजिए ताकि समीकरण $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ का (1) कोई हल नहीं है, (2) एक अद्वितीय हल है, (3) अपरिमित हल हैं ।

Investigate the values of λ and μ so that the equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have (1) no solution, (2) a unique solution, (3) an infinite number of solutions. *10*

4.

(ii) आव्यूह $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ के लिए कैली – हैमिल्टन प्रमेय सत्यापित कीजिए और अतएव इसका व्युत्क्रम ज्ञात कीजिए । साथ ही, $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10\,I$ के द्वारा निरूपित आव्यूह भी ज्ञात कीजिए ।

Verify Cayley – Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and hence find its inverse. Also, find the matrix represented by $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10\,I$. *10*

5.

(c) (i) मान लीजिए कि $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. A के आइगेन मानों और संगत आइगेन सदिशों को ज्ञात कीजिए।

Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. Find the eigen values of A and the corresponding eigen vectors. 8

(ii) सिद्ध कीजिए कि ऐकिक आव्यूह के आइगेन मानों का निरपेक्ष मान 1 होता है।

Prove that the eigen values of a unitary matrix have absolute value 1. 7

2013

... all the questions :

1.(a) Find the inverse of the matrix :

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 7 \\ 3 & 2 & -1 \end{bmatrix}$$

by using elementary row operations. Hence solve the system of linear equations

$$x + 3y + z = 10$$
$$2x - y + 7z = 21$$
$$3x + 2y - z = 4$$

10

1.(b) Let A be a square matrix and A^* be its adjoint, show that the eigenvalues of matrices AA^* and A^*A are real. Further show that trace (AA^*) = trace (A^*A). 10

1.(c) Evaluate $\int_0^1 \left(2x \sin \frac{1}{x} - \cos \frac{1}{x}\right) dx$

2.

2.(a)(i) Let P_n denote the vector space of all real polynomials of degree atmost n and $T: P_2 \to P_3$ be a linear transformation given by

$$T(p(x)) = \int_0^x p(t)dt, \qquad p(x) \in P_2.$$

Find the matrix of T with respect to the bases $\{1, x, x^2\}$ and $\{1, x, 1+x^2, 1+x^3\}$ of P_2 and P_3 respectively. Also, find the null space of T. 10

2.(a)(ii) Let V be an n-dimensional vector space and $T: V \to V$ be an invertible linear operator. If $\beta = \{X_1, X_2, \ldots, X_n\}$ is a basis of V, show that $\beta' = \{TX_1, TX_2, \ldots, TX_n\}$ is also a basis of V. 8

2.(b)(i) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$ where $\omega (\neq 1)$ is a cube root of unity. If $\lambda_1, \lambda_2, \lambda_3$ denote the eigenvalues of A^2, show that $|\lambda_1| + |\lambda_2| + |\lambda_3| \leq 9$. 8

3.

2.(b)(ii) Find the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 8 & 12 \\ 3 & 5 & 8 & 12 & 17 \\ 5 & 8 & 12 & 17 & 23 \\ 8 & 12 & 17 & 23 & 30 \end{bmatrix}$$

8

2.(c)(i) Let A be a Hermetian matrix having all distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. If $X_1, X_2, \ldots, X_n$ are corresponding eigenvectors then show that the $n \times n$ matrix C whose k^{th} column consists of the vector X_k is non singular. 8

2.(c)(ii) Show that the vectors $X_1 = (1, 1+i, i)$, $X_2 = (i, -i, 1-i)$ and $X_3 = (0, 1-2i, 2-i)$ in C^3 are linearly independent over the field of real numbers but are linearly dependent over the field of complex numbers. 8

2012

1

(c) Prove or disprove the following statement : 12

If $B = \{b_1, b_2, b_3, b_4, b_5\}$ is a basis for $\mathbb{R}^5$ and V is a two-dimensional subspace of $\mathbb{R}^5$, then V has a basis made of just two members of B.

(d) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(\alpha, \beta, \gamma) = (\alpha + 2\beta - 3\gamma,\ 2\alpha + 5\beta - 4\gamma,\ \alpha + 4\beta + \gamma)$$

Find a basis and the dimension of the image of T and the kernel of T. 12

2\. (a) (i) Let V be the vector space of all 2×2 matrices over the field of real numbers. Let W be the set consisting of all matrices with zero determinant. Is W a subspace of V? Justify your answer. 8

(ii) Find the dimension and a basis for the space W of all solutions of the following homogeneous system using matrix notation : 12

$$\begin{aligned} x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 &= 0 \\ 2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 &= 0 \\ 3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 &= 0 \end{aligned}$$

(b) (i) Consider the linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x, y) = (3x + 4y, 2x - 5y)$$

Find the matrix A relative to the basis $\{(1, 0), (0, 1)\}$ and the matrix B relative to the basis $\{(1, 2), (2, 3)\}$. 12

(ii) If λ is a characteristic root of a non-singular matrix A, then prove that $\frac{|A|}{\lambda}$ is a characteristic root of Adj A. 8

(c) Let

$$H = \begin{pmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{pmatrix}$$

be a Hermitian matrix. Find a non-singular matrix P such that $D = P^T H \bar{P}$ is diagonal. 20

2011

1.

SECTION—A

1. (a) Let A be a non-singular, $n \times n$ square matrix. Show that $A \,.\, (\text{adj } A) = |A| \,.\, I_n$. Hence show that $|\text{adj}\,(\text{adj } A)| = |A|^{(n-1)^2}$. 10

(b) Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & 6 & 7 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$.

Solve the system of equations given by

$AX = B$

Using the above, also solve the system of equations $A^T X = B$ where A^T denotes the transpose of matrix A. 10

2.

2. (a) (i) Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of a $n \times n$ square matrix A with corresponding eigen vectors $X_1, X_2, \ldots, X_n$. If B is a matrix similar to A show that the eigen values of B are same as that of A. Also find the relation between the eigen vectors of B and eigen vectors of A. 10

(ii) Verify the Cayley-Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}.$$

Using this, show that A is non-singular and find A^{-1}. 10

3.

(b) (i) Show that the subspaces of $\mathbb{R}^3$ spanned by two sets of vectors $\{(1, 1, -1), (1, 0, 1)\}$ and $\{(1, 2, -3), (5, 2, 1)\}$ are identical. Also find the dimension of this subspace. 10

(ii) Find the nullity and a basis of the null space of the linear transformation $A : R^{(4)} \to R^{(4)}$ given by the matrix

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}.$$ 10

(c) (i) Show that the vectors (1, 1, 1), (2, 1, 2) and (1, 2, 3) are linearly independent in $R^{(3)}$. Let $T : \mathbb{R}^{(3)} \to \mathbb{R}^{(3)}$ be a linear transformation defined by

$$T(x, y, z) = (x + 2y + 3z, x + 2y + 5z, 2x + 4y + 6z).$$

Show that the images of above vectors under T are linearly dependent. Give the reason for the same. 10

(ii) Let $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ and C be a non-singular matrix of order 3×3. Find the eigen values of the matrix B^3 where $B = C^{-1} AC$. 10

2010

1.

1. Attempt any *five* of the following :

(a) If $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of the matrix

$$A = \begin{pmatrix} 26 & -2 & 2 \\ 2 & 21 & 4 \\ 4 & 2 & 28 \end{pmatrix}$$

show that

$$\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \leqq \sqrt{1949}$$ 12

(b) What is the null space of the differentiation transformation

$$\frac{d}{dx} : P_n \rightarrow P_n$$

where P_n is the space of all polynomials of degree $\leqq n$ over the real numbers? What is the null space of the second derivative as a transformation of P_n? What is the null space of the kth derivative? 12

2.

2. *(a)* Let $M = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Find the unique linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ so that M is the matrix of T with respect to the basis

$$\beta = \{v_1 = (1, 0, 0), v_2 = (1, 1, 0), v_3 = (1, 1, 1)\}$$

of $\mathbb{R}^3$ and

$$\beta' = \{w_1 = (1, 0), w_2 = (1, 1)\}$$

of $\mathbb{R}^2$. Also find $T(x, y, z)$. 20

3.

4. (a) (i) In the n-space $\mathbb{R}^n$, determine whether or not the set

$$\{e_1 - e_2, e_2 - e_3, \cdots, e_{n-1} - e_n, e_n - e_1\}$$

is linearly independent.

(ii) Let T be a linear transformation from a vector space V over reals into V such that $T - T^2 = I$. Show that T is invertible. 20

2009

1.

1. Attempt any *five* of the following :

(a) Find a Hermitian and a skew-Hermitian matrix each whose sum is the matrix

$$\begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix}$$

12

(b) Prove that the set V of the vectors (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ which satisfy the equations $x_1 + x_2 + 2x_3 + x_4 = 0$ and $2x_1 + 3x_2 - x_3 + x_4 = 0$, is a subspace of $\mathbb{R}^4$. What is the dimension of this subspace? Find one of its bases. 12

2.

2. (a) Let $\mathscr{B} = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ and $\mathscr{B}' = \{(2, 1, 1), (1, 2, 1), (-1, 1, 1)\}$ be the two ordered bases of $\mathbb{R}^3$. Then find a matrix representing the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ which transforms $\mathscr{B}$ into $\mathscr{B}'$. Use this matrix representation to find $T(\bar{x})$, where $\bar{x} = (2, 3, 1)$. 20

3.

(c) Find a 2×2 real matrix A which is both orthogonal and skew-symmetric. Can there exist a 3×3 real matrix which is both orthogonal and skew-symmetric? Justify your answer. 20

3. (a) Let $L: \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined by

$$L((x_1, x_2, x_3, x_4)) = (x_3 + x_4 - x_1 - x_2,\ x_3 - x_2,\ x_4 - x_1)$$

Then find the rank and nullity of L. Also, determine null space and range space of L. 20

4.

4. (a) Prove that the set V of all 3×3 real symmetric matrices forms a linear subspace of the space of all 3×3 real matrices. What is the dimension of this subspace? Find at least one of the bases for V. 20