Ex. 10. (*a*) *If a particle starts from rest at a given point of cycloid with its axis vertical and vertex downwards, prove that it falls 1/n of the vertical distance to the lowest point in time $2(a/g)^{1/2} \sin^{-1} (1/\sqrt{n})$, where a is the radius of the generating circle.*

[Agra 2000, 02, 05; Kanpur 1999; Purvanchal 1996]

Sol. Proceed as in Art. 11.4 upto equation (11). For the solution of this problem, do not write equations (2) and (5). With equaton (11), now proceed as follows. Re-writing (11), we get

$$\cos \{t(g/4a)^{1/2}\} = s/s_0 \quad \text{or} \quad t = 2(a/g)^{1/2} \cos^{-1} (s/s_0) \quad \ldots(12)$$

Since $s^2 = 8ay$ so $s_0^2 = 8ay_0$. Thus $s/s_0 = (y/y_0)^{1/2}$

Then, (12) reduces to $t = 2(a/g)^{1/2} \cos^{-1} (y/y_0)$ ...(13)

Let the particle take time t' in falling a distance $(1/n)y_0$ so that when $t = t'$, $y = y_0 - y_0/n = (1 - 1/n)y_0$ and so (13) gives

$$t' = 2(a/g)^{1/2} \cos^{-1} (1 - 1/n)^{1/2} = 2(a/g)^{1/2} \sin^{-1} \{1 - (1 - 1/n)\}^{1/2},$$

[using formula, $\cos^{-1} x = \sin^{-1} (1 - x^2)^{1/2}$]

or $t' = 2(a/g)^{1/2} \sin^{-1} (1/\sqrt{n})$, which is the required time.

Ex. 10. (*b*) *A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest, starting from rest at a given point of the cycloid. Prove that the time occupied in falling down the first half of the vertical height to the lowest point is equal to the time of falling the second half.*

Sol. Proceed as in Ex. 10(*a*) by replacing *n* by 2. **[Meerut 2007]**

5. *A particle oscillates in a cycloid under gravity, the amplitude of the motion being b, and period being T. Show that its velocity at any time t measured from a position of rest is*

$$\frac{2\pi b}{T} \sin \left(\frac{2\pi t}{t}\right).$$

[Meerut 1977 (S)]

Sol. Refer § 4·12 above.

The equations of motion are

$$m\,(d^2s/dt^2) = -mg \sin \psi \quad \ldots(1)$$

and

$$m\,(v^2/\rho) = R - mg \cos \psi, \quad \ldots(2)$$

For the cycloid, $s = 4a \sin \psi$. ...(3)

Using (3), (1) becomes

$$(d^2s/dt^2) = -(g/4a)\, s,$$

which represents a S.H.M

$\therefore$ the time period T of the particle is given by

$$T = 2\pi/\sqrt{(g/4a)} \quad \text{or} \quad T = 4\pi \sqrt{(a/g)}. \quad \ldots(4)$$

Integrating, $$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a} s^2 + A. \quad \ldots(5)$$

But the amplitude of the motion is b. So the actual distance of a position of rest from the vertex O is b *i.e.*, $v = 0$ when $s = b$.

$\therefore$ from (5), we get

$$A = (g/4a)\, b^2.$$

Substituting in (5), we get

$$v^2=\left(\frac{ds}{dt}\right)^2=\frac{g}{4a}(b^2-s^2). \qquad \ldots(6)$$

$$\therefore \quad \frac{ds}{dt}=-\frac{1}{2}\sqrt{\left(\frac{g}{a}\right)}\sqrt{(b^2-s^2)}$$

(−ive sign is taken because the particle is moving in the direction of s decreasing)

or $$dt=-2\sqrt{(a/g)}\,\frac{ds}{\sqrt{(b^2-s^2)}}.$$

Integrating, $t=2\sqrt{(a/g)}.\cos^{-1}(s/b)+B$.

Initially $t=0$ when $s=b$. $\therefore$ $B=0$.

$\therefore$ $t=2\sqrt{(a/g)}\cos^{-1}(s/b)$

or $$s=b\cos\left\{\frac{t}{2}\sqrt{\left(\frac{g}{a}\right)}\right\}.$$

Substituting in (6), we get

$$v^2=\frac{g}{4a}\left[b^2-b^2\cos^2\left\{\frac{t}{2}\sqrt{(g/a)}\right\}\right]=\frac{g}{4a}b^2\sin^2\left\{\frac{t}{2}\sqrt{(g/a)}\right\}$$

or $v=\frac{b}{2}\sqrt{(g/a)}\sin\left\{\frac{t}{2}\sqrt{(g/a)}\right\}$ From (4), $\sqrt{(g/a)}=4\pi/T$

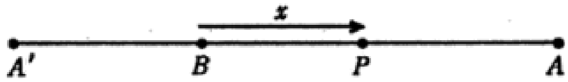
$\therefore$ the velocity of the particle at any time t measured from the position of rest is given by

$$v=\frac{b}{2}\cdot\frac{4\pi}{T}\sin\left(\frac{t}{2}\cdot\frac{4\pi}{T}\right)=\left(\frac{2\pi b}{T}\right)\sin\left(\frac{2\pi t}{T}\right).$$

Ex. 51. *A particle rests in equilibrium under the attraction of two centres of force which attract directly as the distance, their intensities being μ and μ'; the particle is displaced slightly towards one of them, show that the time of a small oscillation is*

$2\pi/\sqrt{(\mu+\mu')}$. **(Rohilkhand 1988; Agra 86)**

Sol. Supoose A and A' are the two centres of force, their intensities being μ and μ' respectively. Let a particle of mass m be in equilibrium at B under the attraction of these two centres. If $AB=a$ and $A'B=a'$, the forces of attraction at B due to the centres A and A' are $m\mu a$ and $m\mu' a'$ respectively in opposite directions. As these two forces balance, we have



$$m\mu a=m\mu' a'. \qquad \ldots(1)$$

Now suppose the particle is slightly displaced towards A and then let go. Let P be the position of the particle after time t, when $BP=x$.

The attraction at P due to the centre A is $m\mu \,.\, AP$ or $m\mu\ (a-x)$ in the direction PA *i.e.*, in the direction of x increasing. Also the attraction at P due to the centre A' is $m\mu' \,.\, A'P$ or $m\mu'\ (a'+x)$ in the direction PA' *i.e.*, in the direction of x decreasing. Hence by Newton's second law of motion, the equation of motion of the particle at P is

$$m\ (d^2x/dt^2) = m\mu\ (a-x) - m\mu'\ (a'+x), \qquad \ldots(2)$$

where the force in the direction of x increasing has been taken with +ive sign and the force in the direction of x decreasing has been taken with −ive sign.

Simplifying the equation (2), we get

$$m\ (d^2x/dt^2) = m\ (\mu a - \mu x - \mu' a' - \mu' x)$$

or $$d^2x/dt^2 = -\ (\mu + \mu')\, x. \qquad [\because \text{ by (1), } m\mu a = m\mu' a']$$

This is the equation of a S.H.M. with centre at the origin. Hence the motion of the particle is simple harmonic with centre at B and its time period is $2\pi/\sqrt{(\mu+\mu')}$.

Ex. 4. *A particle of mass m, is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, show that the distance fallen through in time t is* $\dfrac{gm^2}{\mu^2}\left[e^{-(\mu/m)t} - 1 + \dfrac{\mu t}{m}\right]$.

[Meerut 1975, 79, 83, 85, 87S, 88S, 90S]

Sol. Let a particle of mass m falling under gravity be at a distance x from the starting point, after time t. If v is its velocity at this point, then the resistance on the particle is μv acting vertically upwards *i.e.*, in the direction of x decreasing. The weight mg of the particle acts vertically downwards *i.e.*, in the direction of x increasing.

∴ the equation of motion of the particle is

$$m\frac{d^2x}{dt^2}=mg-\mu v$$

or $$\frac{dv}{dt}=g-\frac{\mu}{m}v, \qquad \left[\because \frac{d^2x}{dt^2}=\frac{dv}{dt}\right]$$

or $$dt=\frac{dv}{g-(\mu/m)\,v}.$$

Integrating, we have

$$t=-\frac{m}{\mu}\log\left(g-\frac{\mu}{m}v\right)+A, \text{ where } A \text{ is a constant.}$$

But initially when $t=0$, $v=0$; $\therefore$ $A=(m/\mu)\log g$.

$$\therefore \quad t=-\frac{m}{\mu}\log\left(g-\frac{\mu}{m}v\right)+\frac{m}{\mu}\log g$$

or $$t=-\frac{m}{\mu}\log\left\{\frac{g-(\mu/m)\,v}{g}\right\}$$

or $$-\frac{\mu t}{m}=\log\left(1-\frac{\mu}{gm}v\right) \quad \text{or} \quad 1-\frac{\mu}{gm}v=e^{-\mu t/m}$$

or $$v=\frac{dx}{dt}=\frac{gm}{\mu}\left(1-e^{-\mu t/m}\right) \quad \text{or} \quad dx=\frac{gm}{\mu}\left(1-e^{-\mu t/m}\right)dt.$$

Integrating, we have

$$x=\frac{gm}{\mu}\left[t+\frac{m}{\mu}e^{-\mu t/m}\right]+B, \qquad \ldots(1)$$

where B is a constant.

But initially when $t=0$, $x=0$.

$$\therefore \quad 0=\frac{gm}{\mu}\left[\frac{m}{\mu}\right]+B. \qquad \ldots(2)$$

Subtracting (2) from (1), we have

$$x=\frac{gm}{\mu}\left\{\frac{m}{\mu}e^{-\mu t/m}-\frac{m}{\mu}+t\right\}=\frac{gm^2}{\mu^2}\left\{e^{-(\mu t/m)}-1+\frac{\mu t}{m}\right\}.$$

Ex. 28. *A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, show that the equation to its path is $r\cos(\theta/\sqrt{2}) = a$.* **[Rohilkhand 77, 81; Allahabad 78; Meerut 78; Agra 86]**

Sol. Here the central acceleration varies inversely as the cube of the distance *i.e.*, $P = \mu/r^3 = \mu u^3$, where μ is a constant.
If V is the velocity for a circle of radius a, then

$$\frac{V^2}{a} = \left[P\right]_{r=a} = \frac{\mu}{a^3}$$

or $$V = \sqrt{(\mu/a^2)}.$$

$\therefore$ the velocity of projection $v_1 = \sqrt{2}V = \sqrt{(2\mu/a^2)}$.

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^3}{u^2} = \mu u.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + A, \qquad \text{...(1)}$$

where A is a constant.

But initially when $r = a$ *i.e.*, $u = 1/a$, $du/d\theta = 0$ (at an apse), and $v = v_1 = \sqrt{(2\mu/a^2)}$.

$\therefore$ from (1), we have

$$\frac{2\mu}{a^2} = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{a^2} + A.$$

$\therefore$ $h^2 = 2\mu$ and $A = \mu/a^2$.

Substituting the values of h^2 and A in (1), we have

$$2\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + \frac{\mu}{a^2}$$

or $$2\left(\frac{du}{d\theta}\right)^2 = \frac{1}{a^2} + u^2 - 2u^2 = \frac{1 - a^2u^2}{a^2}$$

or $$\sqrt{2}\,a\frac{du}{d\theta} = \sqrt{(1 - a^2u^2)} \text{ or } \frac{d\theta}{\sqrt{2}} = \frac{a\,du}{\sqrt{(1 - a^2u^2)}}.$$

Integrating, $(\theta/\sqrt{2}) + B = \sin^{-1}(au)$, where B is a constant.

But initially, when $u = 1/a, \theta = 0$. $\quad\therefore B = \sin^{-1} 1 = \frac{1}{2}\pi$.

$\therefore$ $(\theta/\sqrt{2}) + \frac{1}{2}\pi = \frac{1}{2}\sin^{-1}(au)$ or $au = a/r = \sin\{\frac{1}{2}\pi + (\theta/\sqrt{2})\}$

or $a = r\cos(\theta/\sqrt{2})$, which is the required equation of the path.

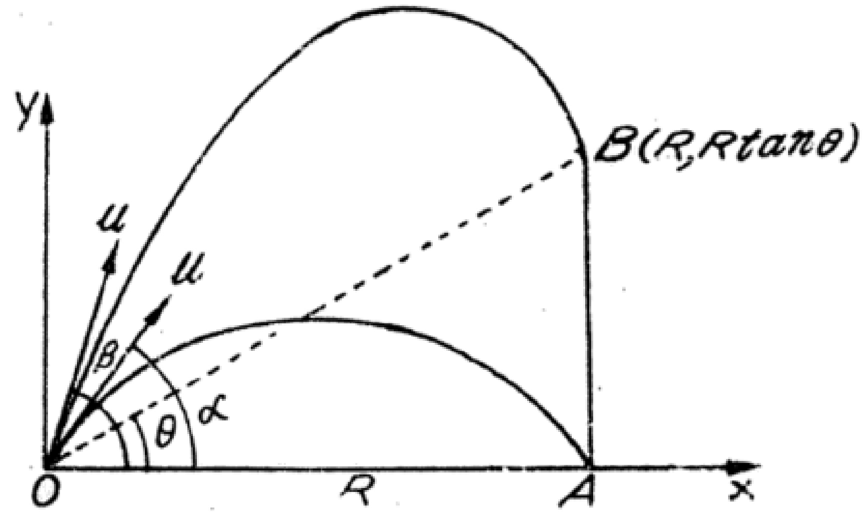
Ex. 20. *A shot fired at an elevation α is observed to strike the foot of a tower which rises above a horizontal plane through the point of projection. If θ be the angle subtended by the tower at this point, show that the elevation required to make the shot strike the top of the tower is* $\frac{1}{2}[\theta+\sin^{-1}(\sin\theta+\sin 2\alpha\cos\theta)]$.

Sol. Let AB be the tower and O the point of projection. It is given that $\angle AOB=\theta$.

Let u be the velocity of projection of the shot. When the shot is fired at an elevation α from O, it strikes the foot A of the tower AB. Let $OA=R$.

Then $$R=\frac{u^2\sin 2\alpha}{g}.$$

Referred to the horizontal and vertical lines OX and OY lying in the plane of motion as the co-ordinate axes, the co-ordinates of the top B of the tower are $(R, R\tan\theta)$.



If β be the angle of projection to hit B from O, then the point B lies on the trajectory whose equation is

$$y=x\tan\beta-\tfrac{1}{2}g\frac{x^2}{u^2\cos^2\beta}.$$

$$\therefore\quad R\tan\theta=R\tan\beta-\tfrac{1}{2}g\frac{R^2}{u^2\cos^2\beta}$$

or $$\tan\theta=\tan\beta-\tfrac{1}{2}g\frac{R}{u^2\cos^2\beta}.\qquad [\because\ R\neq 0]$$

Substituting the value of R from (1), we get

$$\tan\theta=\tan\beta-\tfrac{1}{2}g\,\frac{u^2\sin 2\alpha}{g}\cdot\frac{1}{u^2\cos^2\beta}$$

or $$\tan\theta=\tan\beta-\frac{\sin 2\alpha}{2\cos^2\beta}$$

or $$\frac{\sin\theta}{\cos\theta}=\frac{\sin\beta}{\cos\beta}-\frac{\sin 2\alpha}{2\cos^2\beta}.$$

Multiplying both sides by $2\cos^2\beta\cos\theta$, we get

$$2\cos^2\beta\sin\theta=2\sin\beta\cos\beta\cos\theta-\cos\theta\sin 2\alpha$$

or $$(1+\cos 2\beta)\sin\theta=\sin 2\beta\cos\theta-\cos\theta\sin 2\alpha$$

or $$\sin 2\beta\cos\theta-\cos 2\beta\sin\theta=\sin\theta+\cos\theta\sin 2\alpha$$

or $$\sin(2\beta-\theta)=\sin\theta+\cos\theta\sin 2\alpha$$

or $$2\beta-\theta=\sin^{-1}(\sin\theta+\cos\theta\sin 2\alpha)$$

or $$2\beta=\theta+\sin^{-1}(\sin\theta+\cos\theta\sin 2\alpha)$$

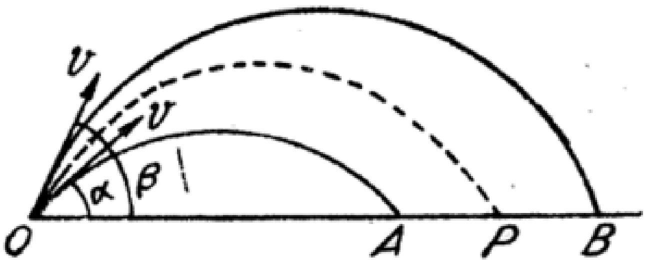
or $$\beta=\tfrac{1}{2}[\theta+\sin^{-1}(\sin\theta+\sin 2\alpha\cos\theta)].$$

Ex. 19. *A projectile aimed at a mark which is in a horizontal plane through the point of projection, falls a metres short of it when the elevation is α and goes b metres too far when the elevation is β. Show that, if the velocity of projection be the same in all cases, the proper elevation is* $\frac{1}{2}\sin^{-1}\frac{a\sin 2\beta+b\sin 2\alpha}{a+b}$.

[Gorakhpur 1976; Meerut 82, 85P, 86P]

Sol. Let O be the point of projection and v the velocity of projection in all the cases. Let P be the point in the horizontal plane through O required to be hit from O. Let θ be the correct angle of projection to hit P from O. Then

OP=the range for the angle of projection $\theta=\frac{v^2\sin 2\theta}{g}$.



When the angle of projection is α, the particle falls at A and when the angle of projection is β, it falls at B. We have

$$OA=\frac{v^2 \sin 2\alpha}{g} \text{ and } OB=\frac{v^2 \sin 2\beta}{g}.$$

According to the question,

$$AP=OP-OA=a \quad \text{and} \quad PB=OB-OP=b.$$

$$\therefore \quad a=\frac{v^2 \sin 2\theta}{g}-\frac{v^2 \sin 2\alpha}{g}=\frac{v^2}{g}(\sin 2\theta-\sin 2\alpha), \quad \ldots(1)$$

and $$b=\frac{v^2 \sin 2\beta}{g}-\frac{v^2 \sin 2\theta}{g}=\frac{v^2}{g}(\sin 2\beta-\sin 2\theta). \quad \ldots(2)$$

Dividing (1) by (2), we get

$$\frac{a}{b}=\frac{\sin 2\theta-\sin 2\alpha}{\sin 2\beta-\sin 2\theta}$$

or $$a \sin 2\beta-a \sin 2\theta=b \sin 2\theta-b \sin 2\alpha$$

or $$(a+b) \sin 2\theta=a \sin 2\beta+b \sin 2\alpha$$

or $$\sin 2\theta=\frac{a \sin 2\beta+b \sin 2\alpha}{a+b}.$$

$$\therefore \quad 2\theta=\sin^{-1}\frac{a \sin 2\beta+b \sin 2\alpha}{a+b} \text{ or } \theta=\tfrac{1}{2}\sin^{-1}\frac{a \sin 2\beta+b \sin 2\alpha}{a+b}.$$

Ex. 5. *A particle is acted only a force parallel to the axis of y whose acceleration is λy and is initially projected with a velocity $a\sqrt{\lambda}$ parallel to the axis of x at a point where $y = a$. Prove that it will describe the catenary*

$$y = a \cosh (x/a). \qquad \textbf{[Meerut 1998]}$$

Sol. Here, given that $\ddot{y} = d^2y/dt^2 = \lambda y$ ...(1)

Since there is no force parallel to the x-axis, it follows that

$$\ddot{x} = d^2x/dt^2 = 0 \qquad ...(2)$$

Integrating (2), $dx/dt = c$, where c is constant of integration.

But, given that initially, $dx/dt = a\sqrt{\lambda}$. Hence $c = a\sqrt{\lambda}$.

Thus, $dx/dt = a\sqrt{\lambda}$ ...(3)

Re-writing (1), $\dot{y}\ (d\dot{y}/dy) = \lambda y$ or $2\dot{y}d\dot{y} = 2\lambda y dy$.

Integrating, $\dot{y}^2 = \lambda y^2 + c'$, c' is constant of integration ...(4)

Since initially at $y = a$, there is no velocity parallel to y-axis, we have $\dot{y} = 0$ when $y = a$. So (4) gives $c' = -\lambda a^2$.

$\therefore$ From (4), $(dy/dt)^2 = \lambda(y^2 - a^2)$ or $dy/dt = \sqrt{\lambda}\,(y^2 - a^2)^{1/2}$ ...(5)

where the positive sign is put on R.H.S. because the particle is moving in the direction of y increasing.

Dividing (5) by (3), $\dfrac{dy}{dx} = \dfrac{(y^2 - a^2)^{1/2}}{a}$ or $\dfrac{dy}{(y^2 - a^2)^{1/2}} = \dfrac{dx}{a}$.

Integrating, $\cosh^{-1}(y/a) = (x/a) + c''$, where c'' is a constant. ...(6)

Let us take $x = 0$ when $y = a$. Then (6) gives $c'' = 0$. So (6) gives

$$\cosh^{-1}(y/a) = x/a \quad \text{or} \quad y = a\cosh(x/a), \text{ which is a catenary.}$$

Ex. 6. *A particle is acted on by a force parallel to the axis of y whose acceleration (always towards the x-axis) is μy^{-2} and when $y = a$, it is projected parallel to the x-axis with velocity $(\mu/a)^{1/2}$. Prove that it will describe a cycloid.*

Sol. Given that $\ddot{y} = d^2y/dt^2 = -\mu y^{-2}$ ... (1)

where the negative sign is put on the R.H.S. because the acceleration is always towards the x-axis (*i.e.*, in the direction of y decreasing).

Since there is no force parallel to x-axis, it follows that

$$\ddot{x} = d^2x/dt^2 = 0 \qquad ...(2)$$

Integrating (2), $dx/dt = c$, where c is constant of integration.

But, given that initially, $dx/dt = (\mu/a)^{1/2}$. Hence $c = (\mu/a)^{1/2}$

Thus, $dx/dt = (\mu/a)^{1/2}$ ...(3)

Re-writing (1), $\dot{y}(d\dot{y}/dy) = -\mu/y^2$ or $2\dot{y}\,d\dot{y} = -2\mu\, y^{-2}\, dy$

Integrating, $\dot{y}^2 = (2\mu/y) + c'$, c' being constant of integration ...(4)

Since initially at $y = a$, there is no velocity parallel to the y-axis, we have $\dot{y} = 0$ when $y = a$. So (4) gives $c' = -(2\mu/a)$. Hence (4) gives

$$\left(\frac{dy}{dt}\right)^2 = \frac{2\mu}{y} - \frac{2\mu}{a} = \frac{2\mu}{a}\left(\frac{a-y}{y}\right) \quad \text{or} \quad \frac{dy}{dt} = -\left(\frac{2\mu}{a}\right)^{1/2}\left(\frac{a-y}{y}\right)^{1/2} \qquad ...(5)$$

where the negative sign is put on R.H.S. because the particle is moving in the direction of y decreasing.

Dividing (5) by (3), $\dfrac{dy}{dx} = -\left(\dfrac{a-y}{y}\right)$ or $dx = -\left(\dfrac{y}{a-y}\right)^{1/2} dy$ ...(6)

Let $y = a\cos^2\theta$ so that $dy = -2a\cos\theta\sin\theta\, d\theta$ ...(7)

Using (7), (6) gives $dx = 2a\cos^2\theta\, d\theta = a(1 + \cos 2\theta)\, d\theta$.

Integrating $x = a[\theta + (1/2)\sin 2\theta] + c'' = (a/2)(2\theta + \sin 2\theta) + c''$ ...(8)

Initially, when $y = a$, let us take $x = 0$. Now, when $y = a$ then $y = a\cos^2\theta \Rightarrow a = a\cos^2\theta \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$. So putting $x = 0$ and $\theta = 0$ in (8), we get $c'' = 0$. Hence (8) reduces to $x = (a/2)(2\theta + \sin 2\theta)$ ...(9)

Rewriting (7), $y = (a/2)(1 + \cos 2\theta)$ ...(10)

(9) and (10) give parametric equation of a cycloid. Hence the required path of the particle is a cycloid.

Ex. 16. *A particle falls towards the earth from infinity; show that its velocity on reaching the surface of the earth is the same as that which it would have acquired in falling with constant acceleration g through a distance equal to the earth's radius.*

(Agra 1987)

Sol. Let a be the radius of the earth and O be the centre of the earth taken as origin. Let the vertical line through O meet the earth's surface at A. [Draw figure as in Ex. 15].

A particle falls from rest from infinity towards the earth. Let P be the position of the particle at any time t, where $OP = x$. [**Note** that O is the origin and OP is the direction of x increasing.] According to Newton's law of gravitation the acceleration of the particle at P is μ/x^2 towards O *i.e.*, in the direction of x decreasing. Hence the equation of motion of the particle at P is $\dfrac{d^2x}{dt^2} = -\dfrac{\mu}{x^2}$. ...(1)

The equation (1) holds good for the motion of the particle upto A. At A (*i.e.*, on the surface of the earth),

$$x = a \text{ and } \frac{d^2x}{dt^2} = -g.$$

$$\therefore \quad -g = -\mu/a^2 \quad \text{or} \quad \mu = a^2 g.$$

Thus the equation (1) becomes $\dfrac{d^2x}{dt^2} = -\dfrac{a^2 g}{x^2}$.

Multiplying both sides by $2\,(dx/dt)$ and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x} + C.$$

But initially when $x = \infty$, the velocity $dx/dt = 0$. Therefore $C = 0$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x}. \qquad ...(2)$$

Putting $x = a$ in (2), the velocity V at the earth's surface is given by

$$V^2 = 2a^2 g/a = 2ag \quad \text{or} \quad V = \sqrt{(2ag)}. \qquad ...(3)$$

If v_1 is the velocity acquired by the particle in falling a distance equal to the earth's radius a with constant acceleration g, then $v_1^2 = 0 + 2ag$ or $v_1 = \sqrt{(2ag)}$. ...(4)

From (3) and (4), we have $V = v_1$, which proves the required result.

Ex. 19. *A particle is projected vertically upwards from the surface of earth with a velocity just sufficient to carry it to the infinity. Prove that the time it takes to reach a height h is*

$$\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right],$$

where a is the radius of the earth.

(Meerut 1988, 2004; Kanpur 87; Agra 88; Rohilkhand 88)

Sol. [Refer fig. of Ex. 17]

Let O be the centre of the earth and A the point of projection on the earth's surface.

If P is the position of the particle at any time t, such that $OP = x$, then the acceleration at $P = \mu/x^2$ directed towards O.

$\therefore$ the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}. \qquad \ldots(1)$$

But at the point A, on the surface of the earth, $x = a$ and $d^2x/dt^2 = -g$.

$\therefore \quad -g = -(\mu/a^2)$ or $\mu = a^2 g$.

$$\therefore \quad \frac{d^2x}{dt^2} = -\frac{a^2 g}{x^2}.$$

Multiplying by $2(dx/dt)$ and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x} + C, \text{ where } C \text{ is a constant.}$$

But when $x \to \infty$, $dx/dt \to 0$.

$\therefore \quad C = 0.$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x} \quad \text{or} \quad \frac{dx}{dt} = \frac{a\sqrt{(2g)}}{\sqrt{x}} \qquad \ldots(2)$$

[Here +ive sign is taken because the particle is moving in the direction of x increasing.]

Separating the variables, we have

$$dt = \frac{1}{a\sqrt{(2g)}}\sqrt{(x)}\,dx.$$

Integrating between the limits $x = a$ to $x = a + h$, the required time t to reach a height h is given by

$$t = \frac{1}{a\sqrt{(2g)}}\int_a^{a+h}\sqrt{(x)}\,dx = \frac{1}{a\sqrt{(2g)}}\left[\frac{2}{3}x^{3/2}\right]_a^{a+h}$$

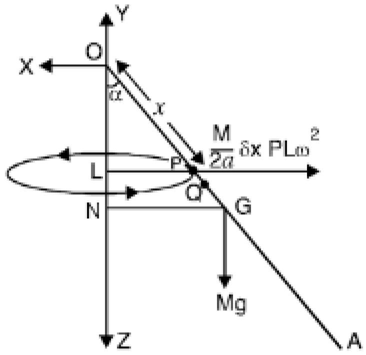
$$= \frac{1}{3a}\sqrt{\left(\frac{2}{g}\right)}\left[(a+h)^{3/2} - a^{3/2}\right] = \frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right].$$

Ex. 5. (*a*) *A uniform rod OA, of length 2a, free to turn about its end O, revolves with uniform angular velocity ω about the vertical OZ through O, and is inclined at a constant angle α to OZ, show that the value of α is either zero or* $\cos^{-1}(3g/4a\omega^2)$.

[Agra 2009, 11; Guwahati 2007; Kanpur 2006, 2008]

(*b*) *A rod, of length 2a, revolves with uniform angular velocity ω about a vertical axis through a smooth joint at one extremity of the rod so that it describes a cone of semi-vertical angle α, show that* $\omega^2 = 3g/(4a\cos\alpha)$. *Prove also that the direction of reaction at the hinge makes with the vertical an angle* $\tan^{-1}\{(3/4)\tan\alpha\}$. **[Meerut 2000, 2011; Kanpur 2011]**

Sol. (*a*) Take an element $PQ(=\delta x)$ at a distance x from O, such that $OP = x$. The mass of the element PQ is $(M/2a)\,\delta x$. Draw PL perpendicular to OZ. Then element PQ will describe a circle of radius $PL(= x\sin\alpha)$ about L. Hence the effective force on this element PQ is $(M/2a)\,\delta x\cdot PL\omega^2$ along PL. So the reversed effective force on the element PQ is $(M/2a)\,\delta x\cdot x\sin\alpha\cdot\omega^2$ along LP as shown in the figure.



Now by D'Alembert's principle all the reversed effective forces acting at different points of the rod, and the external forces, namely weight Mg and reaction at O are in equilibrium. To avoid reaction at O, taking moments about O, we have

$$Mg\cdot a\sin\alpha - \{\Sigma(M/2a)\,\delta x\cdot\omega^2 x\sin\alpha\}\cdot x\cos\alpha = 0$$

or
$$Mg\, a\sin\alpha - \frac{M\omega^2\sin\alpha\cos\alpha}{2a}\int_0^{2a} x^2\,dx = 0$$

or
$$Mg\, a\sin\alpha - (M/2a)\omega^2\sin\alpha\cos\alpha\,\{(2a)^3/3\} = 0$$

or
$$Mg\, a\sin\alpha\,\{1 - (4a/3g)\omega^2\cos\alpha\} = 0$$

giving either $\sin\alpha = 0$ *i.e.*, $\alpha = 0$ or $\cos\alpha = 3g/4a\omega^2$ ...(1)

Hence, the rod is inclined at an angle zero or $\cos^{-1}(3g/4a\,\omega^2)$

Remark. If $\omega^2 < 3g/4a$, then $\cos\alpha > 1$ and so in this case second value of α is not possible and hence $\alpha = 0$ is the only possible value.

(*b*) For first part refer part (*a*). To find the direction of reaction at the hinge O, let X, Y be the horizontal and vertical components of reaction at O. Then resolving the forces horizontally and vertically, we get

$$X = \Sigma\frac{M}{2a}\delta x\,\omega^2 x\sin\alpha = \int_0^{2a}\frac{M\omega^2\sin\alpha}{2a}x\,dx = Ma\omega^2\sin\alpha$$

and
$$Y = Mg$$

Let the reaction at O make an angle θ with vertical. Then

$$\tan\theta = \frac{X}{Y} = \frac{Maw^2\sin\alpha}{Mg} = \frac{a\sin\alpha}{g}\cdot\frac{3g}{4\cos\alpha}, \text{ using (1)}$$

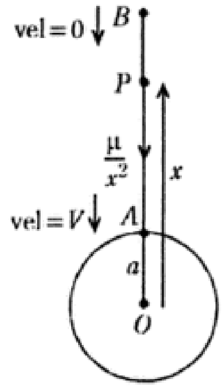
or $\tan\theta = (3/4)\tan\alpha$ so that $\theta = \tan^{-1}\{(3/4)\tan\alpha\}$

Ex. 21. ***Assuming that a particle falling freely under gravity can penetrate the earth without meeting any resistance, show that a particle falling from rest at a distance b (b > a) from the centre of the earth would on reaching the centre acquire a velocity*** $\sqrt{[ga(3b-2a)/b]}$ ***and the time to travel from the surface to the centre of the earth is*** $\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\sqrt{\left[\frac{b}{3b-2a}\right]}$, ***where a is the radius of the earth and g is the acceleration due to gravity on the earth's surface.*** **(Meerut 1982; Agra 84, 86)**

Sol. Let the particle fall from rest from the point *B* such that $OB = b$, where *O* is the centre of the earth. Let *P* be the position of the particle at any time *t* measured from the instant it starts falling from *B* and let $OP = x$.

Acceleration at $P = \mu/x^2$ towards O. The equation of motion of P is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2},$$



which holds good for the motion from B to A *i.e.*, outside the surface of the earth.

But at the point A (on the earth's surface) $x = a$ and $d^2x/dt^2 = -g$.

$$\therefore \qquad -g = -\mu/a^2 \quad \text{or} \quad \mu = a^2 g.$$

$$\therefore \qquad \frac{d^2x}{dt^2} = -\frac{a^2 g}{x^2}. \qquad \ldots(1)$$

Multiplying both sides of (1) by 2 (dx/dt) and then integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x} + A, \text{ where } A \text{ is a constant of integration.}$$

But at B, $x = OB = b$ and $dx/dt = 0$.

$$\therefore \qquad 0 = \frac{2a^2 g}{b} + A \quad \text{or} \quad A = -\frac{2a^2 g}{b}.$$

$$\therefore \qquad \left(\frac{dx}{dt}\right)^2 = 2a^2 g\left(\frac{1}{x} - \frac{1}{b}\right). \qquad \ldots(2)$$

If V is the velocity of the particle at the point A, then at A, $x = OA = a$ and $(dx/dt)^2 = V^2$.

$$\therefore \qquad V^2 = 2a^2 g\left(\frac{1}{a} - \frac{1}{b}\right). \qquad \ldots(3)$$

Now the particle starts moving through a hole from A to O with velocity V at A.

Let x, $(x < a)$, be the distance of the particle from the centre of the earth at any time t measured from the instant the particle starts penetrating the earth at A. The acceleration at this point will be λx towards O, where λ is constant.

The equation of motion (inside the earth) is $d^2x/dt^2 = -\lambda x$, which holds good for the motion from A to O.

At A, $x = a$ and $d^2x/dt^2 = -g$. Therefore, $\lambda = g/a$.

$$\therefore \qquad \frac{d^2x}{dt^2} = -\frac{g}{a}x.$$

Multiplying both sides by 2 (dx/dt) and then integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + B, \text{ where } B \text{ is a constant.} \qquad \ldots(4)$$

But at A, $x = OA = a$ and $\left(\frac{dx}{dt}\right)^2 = V^2 = 2a^2 g\left(\frac{1}{a} - \frac{1}{b}\right)$, from (3).

$$\therefore \qquad 2a^2 g\left(\frac{1}{a} - \frac{1}{b}\right) = -\frac{g}{a}a^2 + B \quad \text{or} \quad B = ag\left(\frac{3b - 2a}{b}\right).$$

Substituting the value of B in (4), we have

$$\left(\frac{dx}{dt}\right)^2 = ag\left(\frac{3b-2a}{b}\right) - \frac{g}{a}x^2. \qquad \ldots(5)$$

Putting $x = 0$ in (5), we get the velocity on reaching the centre of the earth as $\sqrt{[ga(3b-2a)/b]}$.

Again from (5), we have

$$\left(\frac{dx}{dt}\right)^2 = \frac{g}{a}\left[a^2\frac{(3b-2a)}{b} - x^2\right] = \frac{g}{a}(c^2 - x^2),$$

where $\quad c^2 = \frac{a^2}{b}(3b-2a).$

$\therefore \quad \frac{dx}{dt} = -\sqrt{\left(\frac{g}{a}\right)} \cdot \sqrt{(c^2 - x^2)},$ the $-$ive sign being taken because the particle is moving in the direction of x decreasing

or $\quad dt = -\sqrt{\left(\frac{a}{g}\right)} \cdot \frac{dx}{\sqrt{(c^2 - x^2)}},$ separating the variables.

Integrating from A to O, the required time t is given by

$$t = -\sqrt{\left(\frac{a}{g}\right)}\int_{x=a}^{0}\frac{dx}{\sqrt{(c^2-x^2)}}$$

$$= \sqrt{\left(\frac{a}{g}\right)}\int_0^a\frac{dx}{\sqrt{(c^2-x^2)}} = \left[\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\frac{x}{c}\right]_0^a$$

$$= \sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\frac{a}{c} = \sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\left[\frac{a}{a\sqrt{\{!(3b-2a)/b\}}}\right],$$

[Substituting for c]

$$= \sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\sqrt{\left(\frac{b}{3b-2a}\right)}.$$

Ex. 15. *A rod is movable in a vertical plane about a smooth hinge at one end, and at the other end is fastened a weight $W/2$, the weight of the rod being W. This end is fastened by a string of length l to a point at a height c vertically over the hinge. Show that the tension of the string is lW/c.*

Sol. Let a rod AB of length $2a$ (say) be movable in a vertical plane about a smooth hinge at the end A. A weight $W/2$ is attached at the other end B of the rod and this end is fastened by a string BC of length l to a point C at a height $AC=c$ vertically over the hinge at A. The rod is in equilibrium under the action of the following forces :

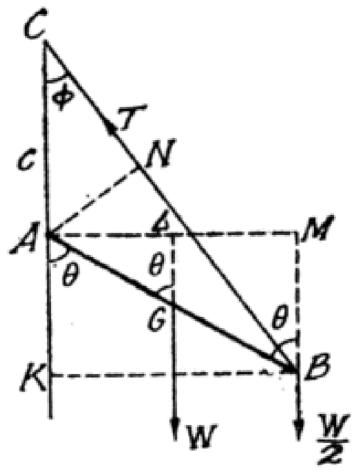
(i) W, weight of the rod at its mid-point G, acting vertically down-wards,

(ii) $W/2$, weight attached at the end B, acting vertically down-wards,

(iii) T, tension in the string along BC,

and (iv) the reaction at the hinge at A.

Let θ and ϕ be the angles of inclination of the rod and the string respectively to the vertical.



(Fig. 2·18)

To avoid reaction at A, taking moments about the point A, we have

$$T.AN = W\ AL + \tfrac{1}{2}W\ \ AM$$

or $$T.AC \sin\phi = W.AG \sin\theta + \tfrac{1}{2}W.AB \sin\theta$$

or $$T.c \sin\phi = W.a \sin\theta + \tfrac{1}{2}W.2a \sin\theta \quad [\because \quad AB = 2a]$$

or $$T = W\frac{2a \sin\theta}{c \sin\phi}. \qquad \ldots(1)$$

Now from the $\triangle CBK$, $BK = BC \sin\phi = l \sin\phi$

and from the $\triangle ABK$, $BK = AB \sin\theta = 2a \sin\theta$.

$$\therefore \quad l \sin\phi = 2a \sin\theta. \qquad \ldots(2)$$

$\therefore$ from (1) and (2), we get

$$T = Wl/c.$$

Example 25:

A uniform straight rod of length 2a. has two small rings at its ends which can respectively slide on thin smooth horizontal and vertical wires OX and OY. The rods starts at an angle α, to the horizontal with angular velocity **[{3g (1 – sin α)}/2a]$^{1/2}$ and downwards. Show that it will strike the horizontal wire at the end of time**

$$2\sqrt{\left(\frac{a}{3g}\right)}\log\left[\left\{\cot\left(\frac{\pi}{8}-\frac{\alpha}{4}\tan\right)\right\}\frac{\pi}{8}\right].$$

Solution:

At any time t, let the rod AB be inclined at an angle θ to the horizontal al. If G be the C.G. of the rod AB, then co-ordinates of G are given by x = a cos θ, y = a sin θ, so that

$\dot{x} = -a\sin\theta\dot{\theta},\ \dot{y} = a\cos\theta\dot{\theta},$

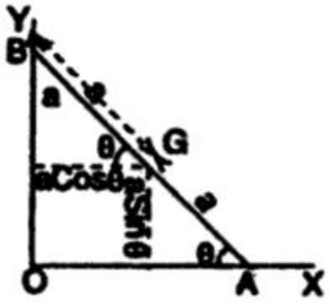


Fig. 1.7

$\therefore$ velocity of $G = \sqrt{\{(\dot{x}^2 + \dot{y}^2)\}}$

$$= \sqrt{[(-a\sin\theta\dot{\theta})^2 + (a\cos\theta\ddot{\theta}^2]} = \dot{\theta}$$

$\therefore$ kinetic energy of the rod at time t

$$= \frac{1}{2}m\left[\frac{a^2}{3}\dot{\theta}^2 + a^2\dot{\theta}^2\right] = \frac{2}{3}ma^2\dot{\theta}^2 \qquad \text{...(1)}$$

Initial angular velocity is

$$= \left[\frac{3a}{2a}(1-\sin\alpha)\right]^{1/2} \quad \text{(given)}$$

$\therefore$ Initial kinetic energy $= \frac{2}{3}ma^2 \cdot \frac{3g}{2a}(a - \sin\alpha)$

[from (1)] = mga (1 − sin α).

Hence energy equation gives

$$\frac{2}{3}ma^2\dot{\theta}^2 = mga(1-\sin\alpha) = mg(a\sin\alpha) - mg(a\sin\theta)$$

i.e. $\frac{2}{3}a^2\dot{\theta}^2 - ga(1-\sin\alpha) = ga(\sin\alpha - \sin\theta)$

i.e. $\frac{2}{3}a^2\dot{\theta}^2 = ga\,(1-\sin\theta)$

or $\dot{\theta} = \frac{d\theta}{dt} = -\sqrt{\left(\frac{3g}{2a}\right)}\sqrt{(1-\sin\theta)}$

(negative sign is taken with radical sign, since θ decreases)

or $dt = -\left(\frac{2a}{3g}\right)^{1/2}\frac{1}{\sqrt{1-\sin\theta}}d\theta.$

Hence the required time

$$= -\left(\frac{2a}{3g}\right)^{1/2}\int_{\alpha}^{0}\frac{1}{\sqrt{1-\sin\theta}}d\theta = \left(\frac{2a}{3g}\right)^{1/2\alpha}\int_{0}^{\alpha}\frac{1}{\sqrt{(1-\sin\theta)}}d\theta$$

$$= \left(\frac{2a}{3g}\right)^{1/2}\int_{a}^{\alpha}\frac{d\theta}{[\{\cos^2(\theta/2)+\sin^2(\theta/2)-2\sin(\theta/2)\cos(\theta/2)]^{1/2}}$$

$$= \left(\frac{2a}{3g}\right)^{1/2}\int_{0}^{\alpha}\frac{d\theta}{\{\cos(\theta/2)-\sin(\theta/2)\}}$$

$$= \left(\frac{2a}{3g}\right)^{1/2}\int_{0}^{\alpha}\frac{d\theta}{\sqrt{2}\left[\frac{1}{\sqrt{2}}\cos(\theta/2)-\frac{1}{\sqrt{2}}\sin(\theta/2)\right]}$$

$$= \left(\frac{2a}{3g}\right)^{1/2}\frac{1}{\sqrt{2}}\int_{0}^{\alpha}\frac{d\theta}{\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)} = \left(\frac{a}{3g}\right)^{1/2}\int_{9}^{\alpha}\operatorname{cosec}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)d\theta$$

$$= -2\sqrt{\left(\frac{a}{3g}\right)}\left[\log\tan\left(\frac{\pi}{8}-\frac{\theta}{4}\right)\right]_0^{\alpha}$$

$$= 2\sqrt{\left(\frac{a}{3g}\right)}\left[\log\tan(\pi/8)-\log\tan\left(\frac{\pi}{8}-\frac{\alpha}{4}\right)\right]$$

$$= 2\sqrt{\left(\frac{a}{3g}\right)}\frac{\tan(\pi/8)}{\tan[(\pi/8)-(\alpha/4)]}$$

$$= 2\sqrt{\left(\frac{a}{3g}\right)}\log\left[\tan\frac{\pi}{8}\cot\left(\frac{\pi}{8}-\frac{\alpha}{4}\right)\right].$$

Ex. 7. *Discuss the motion of a particle falling under gravity in a medium whose resistance varies as the velocity.* **(Meerut 1992S, 93S)**

Sol. Suppose a particle of mass m starts at rest from a point O and falls vertically downwards in a medium whose resistance on the particle is mk times the velocity of the particle. Let P be the position of the particle at any time t, where $OP = x$ and let v be the velocity of the particle at P.

The forces acting on the particle at P are

(i) The force mkv due to the resistance acting vertically upwards *i.e.*, against the direction of motion of the particle, and

(ii) the weight mg of the particle acting vertically downwards.

By Newton's second law of motion the equation of motion of the particle at time t is

$$m\frac{d^2x}{dt^2} = mg - mkv$$

or
$$\frac{d^2x}{dt^2} = g - kv. \qquad ...(1)$$

If V is the terminal velocity of the particle during its downward motion, then from (1)

$$0 = g - kV \quad \text{or} \quad k = g/V.$$

Putting $k = g/V$ in (1), we get

$$\frac{d^2x}{dt^2} = g - \frac{g}{V}v = \frac{g}{V}(V - v). \qquad ...(2)$$

Relation between v and x.

The equation (2) can be written as

$$v\frac{dv}{dx} = \frac{g}{V}(V - v)$$

or
$$dx = \frac{V}{g}\frac{v}{V - v}dv = -\frac{V}{g}\frac{-v}{V - v}dv$$

$$= -\frac{V}{g}\frac{(V - v) - V}{V - v}dv = -\frac{V}{g}\left[1 - \frac{V}{V - v}\right]dv.$$

Integrating, $x = -\dfrac{V}{g}[v + V\log(V - v)] + A$, where A is a constant.

But initially at $O, x = 0$ and $v = 0$.

$$\therefore \quad A = \frac{V^2}{g} \log V.$$

$$\therefore \quad x = -\frac{V}{g} v - \frac{V^2}{g} \log (V - v) + \frac{V^2}{g} \log V$$

or $$x = -\frac{V}{g} v + \frac{V^2}{g} \log \frac{V}{V - v}, \qquad \text{...(3)}$$

which gives the velocity of the particle at any position.

Relation between v and t.

The equation (2) can also be written as

$$\frac{dv}{dt} = \frac{g}{V}(V - v).$$

$$\therefore \quad dt = \frac{V}{g} \frac{dv}{V - v} \cdot$$

Integrating, we have

$$t = -\frac{V}{g} \log (V - v) + B, \text{ where } B \text{ is a constant.}$$

Initially at $O, t = 0$ and $v = 0$.

$$\therefore \quad B = \frac{V}{g} \log V.$$

$$\therefore \quad t = -\frac{V}{g} \log (V - v) + \frac{V}{g} \log V$$

or $$t = \frac{V}{g} \log \frac{V}{V - v}, \qquad \text{...(4)}$$

which gives the velocity of the particle at any time t.

Relation between x and t.

From (4), we have

$$\log \frac{V}{V - v} = \frac{gt}{V} \quad \text{or} \quad \frac{V}{V - v} = e^{gt/V}$$

or $$V - v = Ve^{-gt/V}$$

or $$v = V[1 - e^{-gt/V}]$$

or $$\frac{dx}{dt} = V[1 - e^{-gt/V}]$$

or $$dx = V[1 - e^{-gt/V}]\, dt.$$

Integrating, we get

$$x = Vt + \frac{V^2}{g} e^{-gt/V} + C, \text{ where } C \text{ is a constant.}$$

Initially at $O, x = 0$ and $t = 0$.

$$\therefore \quad C = -\frac{V^2}{g}.$$

$$\therefore \quad x = Vt + \frac{V^2}{g} e^{-gt/V} - \frac{V^2}{g}$$

or

$$x = Vt + \frac{V^2}{g} (e^{-gt/V} - 1), \qquad ...(5)$$

which gives the distance fallen through in time t.

Example 26. *A particle moves in a straight line, its acceleration is directed towards a fixed point O in the line and is always equal to* $\mu\left(\frac{a^5}{x^2}\right)^{1/3}$ *when it is at a distance x from O. If it starts from rest at a distance a from O, show that it will arrive at O with a velocity* $a\sqrt{6\mu}$ *after time* $\frac{8}{15}\sqrt{\frac{6}{\mu}}$.

(Meerut, 1986, 87, 90 S; Agra, 1981, 84; Bhopal, 1982 ; Rohilkhand, 1987)

Sol. The equation of motion of the particle is

$$v\frac{dv}{dx} = -\mu\left(\frac{a^5}{x^2}\right)^{1/3}$$

or
$$v dv = -\mu a^{5/3} x^{-2/3} dx$$

Integrating,
$$\frac{v^2}{2} = -3\mu a^{5/3} x^{1/3} + c \qquad ...(1)$$

Initially, $x = a$, $v = 0$ $\therefore$ $c = 3\mu a^{5/3} \cdot a^{1/3}$

$\therefore$ From (1) $v^2 = 6\mu a^{5/3}(a^{1/3} - x^{1/3})$ ...(2)

If the particle arrives at O $(x = 0)$ with velocity V, then, from (2) we have

$$V^2 = 6\mu a^{5/3}(a^{1/3} - 0) \quad \text{or} \quad V^2 = 6\mu a^2$$

$\therefore$
$$V = a\sqrt{6\mu}$$

Also, from (2),

$$v = \frac{dx}{dt} = -\sqrt{6\mu a^{5/3}}\sqrt{\left(a^{1/3} - x^{1/3}\right)}$$

or
$$dt = -\frac{1}{\sqrt{6\mu a^{5/3}}} \cdot \frac{dx}{\sqrt{a^{1/3} - x^{1/3}}}$$

$\therefore$ Required time T from $x = a$ to O $(x = 0)$ is given by

$$\int_0^T dt = -\frac{1}{\sqrt{6\mu a^{5/3}}}\int_a^0 \frac{dx}{\sqrt{a^{1/3} - x^{1/3}}}$$

$$\Rightarrow \qquad T = \frac{1}{\sqrt{6\mu a^{5/3}}} \int_0^a \frac{dx}{\sqrt{a^{1/3} - x^{1/3}}}$$

Putting $x = a \sin^6 \theta$ so that $dx = 6a \sin^5 \theta \cos \theta \, d\theta$.

When $x = 0,\ \theta = 0$ and when $x = a,\ \theta = \frac{\pi}{2}$

$$\therefore \qquad T = \frac{1}{\sqrt{6\mu a^{5/3}}} \int_0^{\frac{\pi}{2}} \frac{6a \sin^5 \theta \cos \theta \, d\theta}{\sqrt{a^{1/3}(1 - \sin^2 \theta)}}$$

$$= \frac{6a}{\sqrt{6\mu a^2}} \int_0^{\frac{\pi}{2}} \sin^5 \theta \, d\theta = \sqrt{\frac{6}{\mu}} \cdot \frac{4 \cdot 2}{5 \cdot 3 \cdot 1}$$

$$= \frac{8}{15} \sqrt{\frac{6}{\mu}}.$$

Example 25. *A particle whose mass is 'm' is acted upon by a force* $m \mu x^{-5/3}$ *towards then centre. If it starts from rest at a distance 'a' from this centre, then show that it will arrive at the centre after time* $\frac{2a^{4/3}}{\sqrt{3\mu}}$ *with infinite velocity.* (Bhopal, 1981)

Sol. The equation of motion is

$$m v \frac{dv}{dx} = -m \mu x^{-5/3}$$

or $$vdv = -\mu x^{-5/3}dx$$

Integrating, $$\frac{v^2}{2} = \frac{3}{2}\mu x^{-\frac{2}{3}} + c \qquad \text{...(1)}$$

Initially, $x = a$ and $v = 0$

$\therefore$ $0 = \frac{3}{2}\mu a^{-\frac{2}{3}} + c$ or $c = -\frac{3}{2}\mu a^{-\frac{2}{3}}$

From (1), $$\frac{v^2}{2} = \frac{3}{2}\mu\left(x^{-2/3} - a^{-2/3}\right) \qquad \text{...(2)}$$

or $$v = \frac{dx}{dt} = -\sqrt{3\mu}\sqrt{x^{-2/3} - a^{-2/3}}$$

(x decreases as t increases)

or $$dt = -\frac{1}{\sqrt{3\mu}}\cdot\frac{dx}{\sqrt{x^{-2/3} - a^{-2/3}}}$$

$\therefore$ Required time of moving from $x = a$ to $x = 0$ is

$$-\frac{1}{\sqrt{3\mu}}\int_a^0 \frac{dx}{\sqrt{x^{-2/3} - a^{-2/3}}} = -\frac{1}{\sqrt{3\mu}}\int_a^0 \frac{x^{1/3}a^{1/3}}{\sqrt{a^{2/3} - x^{2/3}}}\,dx$$

$$= \frac{a^{1/3}}{\sqrt{3\mu}}\int_0^a \frac{x^{1/3}}{\sqrt{a^{2/3} - a^{2/3}}}dx$$

$$= \frac{a^{1/3}}{\sqrt{3\mu}}\int_0^{\frac{\pi}{2}} \frac{a^{1/3}\sin\theta}{a^{1/3}\sqrt{1 - \sin^2\theta}}\cdot 3a\sin^2\theta\cos\theta\,d\theta$$

Putting $x = a\sin^3\theta$

$$= \frac{3a^{4/3}}{\sqrt{3\mu}}\int_0^{\frac{\pi}{2}} \sin^3\theta\,d\theta.$$

$$= \frac{3a^{4/3}}{\sqrt{3\mu}}\cdot\frac{2}{3} = \frac{2a^{4/3}}{\sqrt{3\mu}}$$

Also from (2), as $x \to 0$, $v \to \infty$

Hence the particle will arrive at the centre after time $\frac{2a^{4/3}}{\sqrt{3\mu}}$ with infinite velocity.