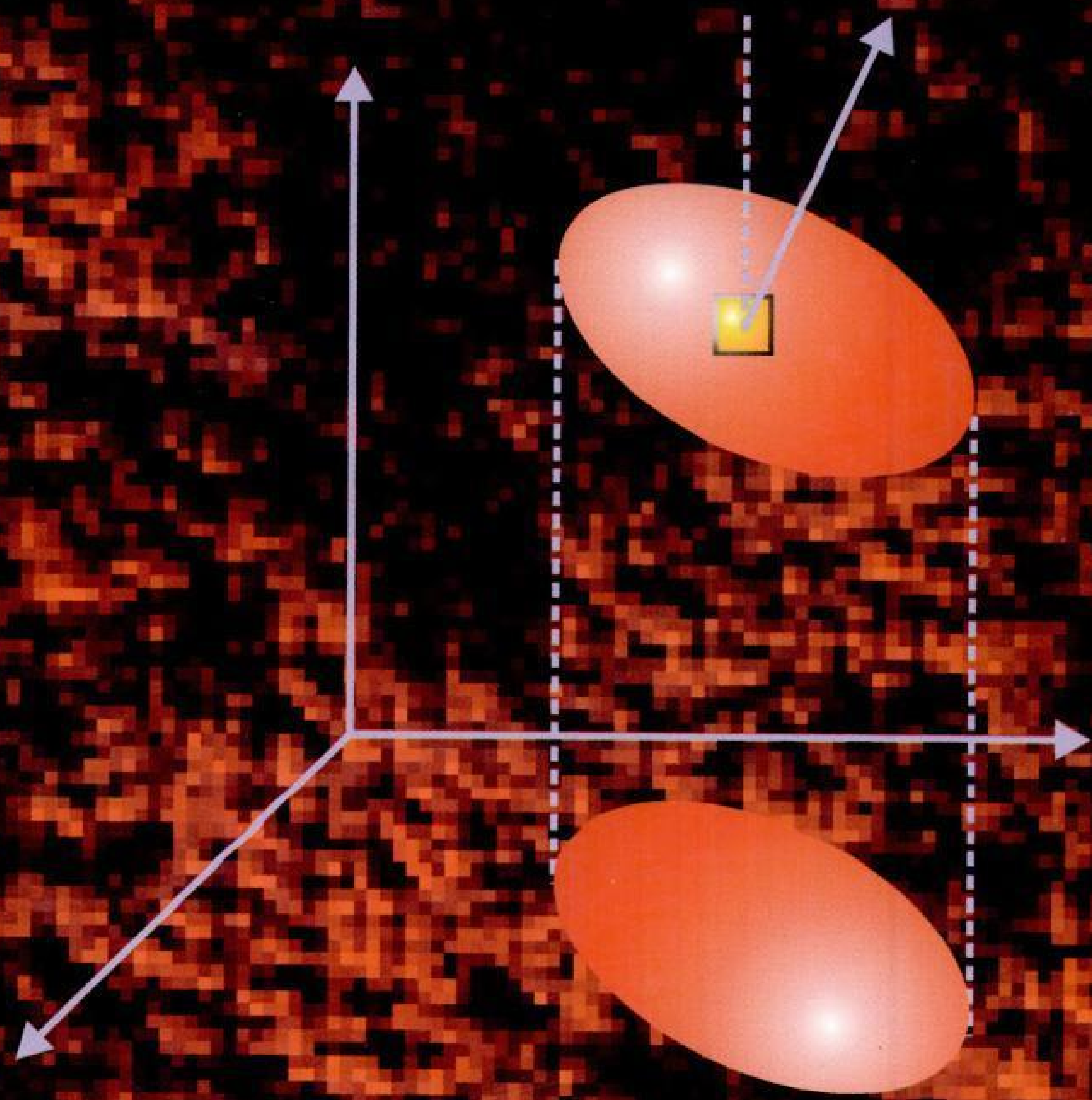
Vector Calculus



KRISHNA Prakashan Media (P) Ltd., Meerut

Krishna's

VECTOR CALCULUS

First Edition : 1975

THIRTY THIRD EDITION : 2009

Edited by : Dr. (Mrs.) Hemlata Vasishtha

Book Code No. 234-33

Price : Rs. 85.00

Published by : **KRISHNA** Prakashan Media (P) Ltd.
11, Shivaji Road Meerut - 250 001 (U.P.) India
Phones : (0121) 2642946, 2644766, Fax : 0121-2645855
Website : www. krishnaprakashan.com
e-mail : info@krishnaprakashan.com.

Printed at : Raj Printers, Meerut.

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Differentiation and Integration of Vectors

§ 1. **Vector Function.** We know that a scalar quantity possesses only magnitude and has no concern with direction. A single real number gives us a complete representation of a scalar quantity. Thus a scalar quantity is nothing but a real number.

Let D be any subset of the set of all real numbers. If to each element t of D, we associate by some rule a unique real number $f(t)$, then this rule defines a **scalar function** of the scalar variable t. Here $f(t)$ is a scalar quantity and thus f is a scalar function.

In a similar manner we define a vector function.

Let D be any subset of the set of all real numbers. If to each element t of D, we associate by some rule a unique vector $\mathbf{f}(t)$, *then this rule defines a* **vector function** *of the scalar variable t. Here* $\mathbf{f}(t)$ *is a vector quantity and thus* $\mathbf{f}$ *is a vector function.*

We know that every vector can be uniquely expressed as a linear combination of three fixed non-coplanar vectors. Therefore we may write

$$\mathbf{f}(t)=f_1(t)\,\mathbf{i}+f_2(t)\,\mathbf{j}+f_3(t)\,\mathbf{k}$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote a fixed right handed triad of three mutually perpendicular non-coplanar unit vectors.

§ 2. **Scalar Fields and Vector Fields.** If to each point $P(x, y, z)$ of a region R in space there corresponds a unique scalar $f(P)$, then f is called a **scalar point function** and we say that a **scalar field** f has been defined in R.

Examples. (1) The temperature at any point within or on the surface of earth at a certain time defines a scalar field.

(2) $f(x, y, z)=x^2y^3-3z^2$ defines a scalar field.

If to each point $P(x, y, z)$ of a region R in space there corresponds a unique vector $\mathbf{f}(P)$, then $\mathbf{f}$ is called a **vector point function** and we say that a **vector field** $\mathbf{f}$ has been defined in R.

Examples. (1) If the velocity at any point (x, y, z) of a particle moving in a curve is known at a certain time, then a vector field is defined.

(2) $\mathbf{f}(x, y, z)=xy^2\,\mathbf{i}+3yz^3\,\mathbf{j}-2x^2 z\mathbf{k}$ defines a vector field.

§ 3. Limit and Continuity of a vector function.

Definition 1. *A vector function* $\mathbf{f}(t)$ *is said to tend to a limit* $\mathbf{l}$, *when* t *tends to* t_0, *if for any given positive number* ϵ, *however small, there corresponds a positive number* δ *such that*

$$|\mathbf{f}(t)-\mathbf{l}|<\epsilon$$

whenever $\quad 0<|t-t_0|<\delta.$

If $\mathbf{f}(t)$ tends to a limit $\mathbf{l}$ as t tends to t_0, we write

$$\lim_{t\to t_0} \mathbf{f}(t)=\mathbf{l}.$$

Definition 2. *A vector function* $\mathbf{f}(t)$ *is said to be continuous for a value* t_0 *of* t *if*

(*i*) $\mathbf{f}(t_0)$ *is defined and*

(*ii*) *for any given positive number* ϵ, *however small, there corresponds a positive number* δ *such that*

$$|\mathbf{f}(t)-\mathbf{f}(t_0)|<\epsilon$$

whenever $\quad |t-t_0|<\delta.$

Further a vector function $\mathbf{f}(t)$ *is said to be continuous if it is continuous for every value of* t *for which it has been defined.*

We shall give here (without proof) some important results about the limits and continuity of a vector function.

Theorem 1. *The necessary and sufficient condition for a vector function* $\mathbf{f}(t)$ *to be continuous at* $t=t_0$ *is that*

$$\lim_{t\to t_0} \mathbf{f}(t)=\mathbf{f}(t_0).$$

Theorem 2. *If* $\mathbf{f}(t)=f_1(t)\,\mathbf{i}+f_2(t)\,\mathbf{j}+f_3(t)\,\mathbf{k}$, *then* $\mathbf{f}(t)$ *is continuous if and only if* $f_1(t), f_2(t), f_3(t)$ *are continuous.*

Theorem 3. *Let* $\mathbf{f}(t)=f_1(t)\,\mathbf{i}+f_2(t)\,\mathbf{j}+f_3(t)\,\mathbf{k}$

and $\quad \mathbf{l}=l_1\,\mathbf{i}+l_2\,\mathbf{j}+l_3\,\mathbf{k}.$

Then the necessary and sufficient conditions that $\lim_{t\to t_0} \mathbf{f}(t)=\mathbf{l}$

are $\lim_{t\to t_0} f_1(t)=l_1$, $\lim_{t\to t_0} f_2(t)=l_2$ *and* $\lim_{t\to t_0} f_3(t)=l_3$.

Theorem 4. *If* $\mathbf{f}(t)$, $\mathbf{g}(t)$ *are vector functions of scalar variable* t *and* $\phi(t)$ *is a scalar function of scalar variable* t, *then*

(*i*) $\displaystyle\lim_{t\to t_0}[\mathbf{f}(t)\pm\mathbf{g}(t)]=\lim_{t\to t_0}\mathbf{f}(t)\pm\lim_{t\to t_0}\mathbf{g}(t)$

(ii) $\lim_{t\to t_0} [\mathbf{f}(t)\cdot\mathbf{g}(t)]=\left[\lim_{t\to t_0} \mathbf{f}(t)\right]\cdot\left[\lim_{t\to t_0} \mathbf{g}(t)\right]$

(iii) $\lim_{t\to t_0} [\mathbf{f}(t)\times\mathbf{g}(t)]=\left[\lim_{t\to t_0} \mathbf{f}(t)\right]\times\left[\lim_{t\to t_0} \mathbf{g}(t)\right]$

(iv) $\lim_{t\to t_0} [\phi(t)\,\mathbf{f}(t)]=\left[\lim_{t\to t_0} \phi(t)\right]\left[\lim_{t\to t_0} \mathbf{f}(t)\right]$

(v) $\lim_{t\to t_0} \left|\mathbf{f}(t)\right|=\left|\lim_{t\to t_0} \mathbf{f}(t)\right|.$

§ 4. Derivative of a vector function with respect to a scalar.

[Banaras 61; Kolhapur 73]

Definition. *Let* $\mathbf{r}=\mathbf{f}(t)$ *be a vector function of the scalar variable t. We define* $\mathbf{r}+\delta\mathbf{r}=\mathbf{f}(t+\delta t)$.

$\therefore\quad \delta\mathbf{r}=\mathbf{f}(t+\delta t)-\mathbf{f}(t).$

Consider the vector $\dfrac{\delta\mathbf{r}}{\delta t}=\dfrac{\mathbf{f}(t+\delta t)-\mathbf{f}(t)}{\delta t}.$

If $\lim_{\delta t\to 0}\dfrac{\delta\mathbf{r}}{\delta t}=\lim_{\delta t\to 0}\dfrac{\mathbf{f}(t+\delta t)-\mathbf{f}(t)}{\delta t}$ *exists, then the value of this limit, which we shall denote by* $\dfrac{d\mathbf{r}}{dt}$, *is called the derivative of the vector function* **r** *with respect to the scalar t. Symbolically*

$$\frac{d\mathbf{r}}{dt}=\lim_{\delta t\to 0}\frac{(\mathbf{r}+\delta\mathbf{r})-\mathbf{r}}{\delta t}=\lim_{\delta t\to 0}\frac{\mathbf{f}(t+\delta t)-\mathbf{f}(t)}{\delta t}.$$

If $\dfrac{d\mathbf{r}}{dt}$ exists, then **r** is said to be differentiable. Since $\dfrac{\delta\mathbf{r}}{\delta t}$ is a vector quantity, therefore $\dfrac{d\mathbf{r}}{dt}$ is also a vector quantity.

Successive Derivatives. If **r** is a vector function of the scalar variable t, then $\dfrac{d\mathbf{r}}{dt}$ is also in general a vector function of t. If $\dfrac{d\mathbf{r}}{dt}$ is differentiable, then its derivative is denoted by $\dfrac{d^2\mathbf{r}}{dt^2}$ and is called the *second derivative of* **r**. Similarly the derivative of $\dfrac{d^2\mathbf{r}}{dt^2}$ is denoted by $\dfrac{d^3\mathbf{r}}{dt^3}$ and is called the third derivative of **r** and so on.

$\dfrac{d\mathbf{r}}{dt}, \dfrac{d^2\mathbf{r}}{dt^2}$... are also represented by $\dot{\mathbf{r}}, \ddot{\mathbf{r}}$,...respectively.

§ 5. Differentiation Formulae.

Theorem. *If* **a**, **b** *and* **c** *are differentiable vector functions of a scalar t and* ϕ *is a differentiable scalar function of the same variable t, then*

1. $\frac{d}{dt}(\mathbf{a}+\mathbf{b})=\frac{d\mathbf{a}}{dt}+\frac{d\mathbf{b}}{dt}$

2. $\frac{d}{dt}(\mathbf{a}\cdot\mathbf{b})=\mathbf{a}\cdot\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\cdot\mathbf{b}$ **[Calcutta 63]**

3. $\frac{d}{dt}(\mathbf{a}\times\mathbf{b})=\mathbf{a}\times\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\times\mathbf{b}$

[Agra 1967; Marathwada 74; Kolhapur 73]

4. $\frac{d}{dt}(\phi\mathbf{a})=\phi\frac{d\mathbf{a}}{dt}+\frac{d\phi}{dt}\mathbf{a}$

5. $\frac{d}{dt}\left[\mathbf{a}\ \mathbf{b}\ \mathbf{c}\right]=\left[\frac{d\mathbf{a}}{dt}\ \mathbf{b}\ \mathbf{c}\right]+\left[\mathbf{a}\ \frac{d\mathbf{b}}{dt}\ \mathbf{c}\right]+\left[\mathbf{a}\ \mathbf{b}\ \frac{d\mathbf{c}}{dt}\right]$

6. $\frac{d}{dt}\left\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\right\}=\frac{d\mathbf{a}}{dt}\times(\mathbf{b}\times\mathbf{c})+\mathbf{a}\times\left(\frac{d\mathbf{b}}{dt}\times\mathbf{c}\right)+\mathbf{a}\times\left(\mathbf{b}\times\frac{d\mathbf{c}}{dt}\right).$

[Rohilkhand 1978]

Proof. 1. $\frac{d}{dt}(\mathbf{a}+\mathbf{b})=\lim_{\delta t\to 0}\frac{\{(\mathbf{a}+\delta\mathbf{a})+(\mathbf{b}+\delta\mathbf{b})\}-(\mathbf{a}+\mathbf{b})}{\delta t}$

$$=\lim_{\delta t\to 0}\frac{\delta\mathbf{a}+\delta\mathbf{b}}{\delta t}=\lim_{\delta t\to 0}\left(\frac{\delta\mathbf{a}}{\delta t}+\frac{\delta\mathbf{b}}{\delta t}\right)$$

$$=\lim_{\delta t\to 0}\frac{\delta\mathbf{a}}{\delta t}+\lim_{\delta t\to 0}\frac{\delta\mathbf{b}}{\delta t}=\frac{d\mathbf{a}}{dt}+\frac{d\mathbf{b}}{dt}.$$

Thus the derivative of the sum of two vectors is equal to the sum of their derivatives, as it is also in Scalar Calculus.

Similarly we can prove that $\frac{d}{dt}(\mathbf{a}-\mathbf{b})=\frac{d\mathbf{a}}{dt}-\frac{d\mathbf{b}}{dt}$.

2. $\frac{d}{dt}(\mathbf{a}\cdot\mathbf{b})=\lim_{\delta t\to 0}\frac{(\mathbf{a}+\delta\mathbf{a})\cdot(\mathbf{b}+\delta\mathbf{b})-\mathbf{a}\cdot\mathbf{b}}{\delta t}$

$$=\lim_{\delta t\to 0}\frac{\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\delta\mathbf{b}+\delta\mathbf{a}\cdot\mathbf{b}+\delta\mathbf{a}\cdot\delta\mathbf{b}-\mathbf{a}\cdot\mathbf{b}}{\delta t}$$

$$=\lim_{\delta t\to 0}\frac{\mathbf{a}\cdot\delta\mathbf{b}+\delta\mathbf{a}\cdot\mathbf{b}+\delta\mathbf{a}\cdot\delta\mathbf{b}}{\delta t}$$

$$=\lim_{\delta t\to 0}\left\{\mathbf{a}\cdot\frac{\delta\mathbf{b}}{\delta t}+\frac{\delta\mathbf{a}}{\delta t}\cdot\mathbf{b}+\frac{\delta\mathbf{a}}{\delta t}\cdot\delta\mathbf{b}\right\}$$

$$=\lim_{\delta t\to 0}\mathbf{a}\cdot\frac{\delta\mathbf{b}}{\delta t}+\lim_{\delta t\to 0}\frac{\delta\mathbf{a}}{\delta t}\cdot\mathbf{b}+\lim_{\delta t\to 0}\frac{\delta\mathbf{a}}{\delta t}\cdot\delta\mathbf{b}$$

$=\mathbf{a}\cdot\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\cdot\mathbf{b}+\frac{d\mathbf{a}}{dt}\cdot\mathbf{0}$, since $\delta\mathbf{b}\to$ zero vector as $\delta t\to 0$

$$=\mathbf{a}\cdot\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\cdot\mathbf{b}.$$

Note. We know that $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$. Therefore while evaluating $\frac{d}{dt}(\mathbf{a}\cdot\mathbf{b})$, we should not bother about the order of the factors.

3. $\frac{d}{dt}(\mathbf{a}\times\mathbf{b})=\lim_{\delta t\to 0}\frac{(\mathbf{a}+\delta\mathbf{a})\times(\mathbf{b}+\delta\mathbf{b})-\mathbf{a}\times\mathbf{b}}{\delta t}$

$=\lim_{\delta t\to 0}\frac{\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\delta\mathbf{b}+\delta\mathbf{a}\times\mathbf{b}+\delta\mathbf{a}\times\delta\mathbf{b}-\mathbf{a}\times\mathbf{b}}{\delta t}$

$=\lim_{\delta t\to 0}\frac{\mathbf{a}\times\delta\mathbf{b}+\delta\mathbf{a}\times\mathbf{b}+\delta\mathbf{a}\times\delta\mathbf{b}}{\delta t}$

$=\lim_{\delta t\to 0}\left\{\mathbf{a}\times\frac{\delta\mathbf{b}}{\delta t}+\frac{\delta\mathbf{a}}{\delta t}\times\mathbf{b}+\frac{\delta\mathbf{a}}{\delta t}\times\delta\mathbf{b}\right\}$

$=\lim_{\delta t\to 0}\mathbf{a}\times\frac{\delta\mathbf{b}}{\delta t}+\lim_{\delta t\to 0}\frac{\delta\mathbf{a}}{\delta t}\times\mathbf{b}+\lim_{\delta t\to 0}\frac{\delta\mathbf{a}}{\delta t}\times\delta\mathbf{b}$

$=\mathbf{a}\times\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\times\mathbf{b}+\frac{d\mathbf{a}}{dt}\times\mathbf{0}$, since $\delta\mathbf{b}\to$ zero vector as $\delta t\to 0$

$=\mathbf{a}\times\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\times\mathbf{b}+\mathbf{0}=\mathbf{a}\times\frac{d\mathbf{b}}{dt}+\frac{d\mathbf{a}}{dt}\times\mathbf{b}.$

Note. We know that cross product of two vectors is not commutative because $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$. Therefore while evaluating $\frac{d}{dt}(\mathbf{a}\times\mathbf{b})$, we must maintain the order of the factors **a** and **b**.

4. $\frac{d}{dt}(\phi\mathbf{a})=\lim_{\delta t\to 0}\frac{(\phi+\delta\phi)(\mathbf{a}+\delta\mathbf{a})-\phi\mathbf{a}}{\delta t}$

$=\lim_{\delta t\to 0}\frac{\phi\mathbf{a}+\phi\delta\mathbf{a}+\delta\phi\mathbf{a}+\delta\phi\delta\mathbf{a}-\phi\mathbf{a}}{\delta t}=\lim_{\delta t\to 0}\frac{\phi\delta\mathbf{a}+\delta\phi\mathbf{a}+\delta\phi\delta\mathbf{a}}{\delta t}$

$=\lim_{\delta t\to 0}\left\{\phi\frac{\delta\mathbf{a}}{\delta t}+\frac{\delta\phi}{\delta t}\mathbf{a}+\frac{\delta\phi}{\delta t}\delta\mathbf{a}\right\}$

$=\lim_{\delta t\to 0}\phi\frac{\delta\mathbf{a}}{\delta t}+\lim_{\delta t\to 0}\frac{\delta\phi}{\delta t}\mathbf{a}+\lim_{\delta t\to 0}\frac{\delta\phi}{\delta t}\delta\mathbf{a}$

$=\phi\frac{d\mathbf{a}}{dt}+\frac{d\phi}{dt}\mathbf{a}+\frac{d\phi}{dt}\mathbf{0}$, since $\delta\mathbf{a}\to$ zero vector as $\delta t\to 0$

$=\phi\frac{d\mathbf{a}}{dt}+\frac{d\phi}{dt}\mathbf{a}+\mathbf{0}=\phi\frac{d\mathbf{a}}{dt}+\frac{d\phi}{dt}\mathbf{a}.$

Note. $\phi\mathbf{a}$ is the multiplication of a vector by a scalar. In the case of such multiplication we usually write the scalar in the first position and the vector in the second position.

5. $\frac{d}{dt}\left[\mathbf{a}\ \mathbf{b}\ \mathbf{c}\right]=\frac{d}{dt}\left\{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\right\}$

$=\mathbf{a}\cdot\frac{d}{dt}(\mathbf{b}\times\mathbf{c})+\frac{d\mathbf{a}}{dt}\cdot(\mathbf{b}\times\mathbf{c})$ [by rule (2)]

$=\mathbf{a}\cdot\left(\mathbf{b}\times\frac{d\mathbf{c}}{dt}+\frac{d\mathbf{b}}{dt}\times\mathbf{c}\right)+\frac{d\mathbf{a}}{dt}\cdot(\mathbf{b}\times\mathbf{c})$ [by rule (3)]

$$=\mathbf{a}\cdot\left(\mathbf{b}\times\frac{d\mathbf{c}}{dt}\right)+\mathbf{a}\cdot\left(\frac{d\mathbf{b}}{dt}\times\mathbf{c}\right)+\frac{d\mathbf{a}}{dt}\cdot(\mathbf{b}\times\mathbf{c})$$

$$=\left[\mathbf{a}\ \mathbf{b}\ \frac{d\mathbf{c}}{dt}\right]+\left[\mathbf{a}\ \frac{d\mathbf{b}}{dt}\ \mathbf{c}\right]+\left[\frac{d\mathbf{a}}{dt}\ \mathbf{b}\ \mathbf{c}\right]$$

$$=\left[\frac{d\mathbf{a}}{dt}\ \mathbf{b}\ \mathbf{c}\right]+\left[\mathbf{a}\ \frac{d\mathbf{b}}{dt}\ \mathbf{c}\right]+\left[\mathbf{a}\ \mathbf{b}\ \frac{d\mathbf{c}}{dt}\right].$$

Note. Here [**a b c**] is the scalar triple product of three vectors **a, b** and **c**. Therefore while evaluating $\frac{d}{dt}$ [**a b c**] we must maintain the cyclic order of each factor.

6. $\frac{d}{dt}\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\}=\mathbf{a}\times\frac{d}{dt}(\mathbf{b}\times\mathbf{c})+\frac{d\mathbf{a}}{dt}\times(\mathbf{b}\times\mathbf{c})$ [by rule (3)]

$$=\mathbf{a}\times\left(\frac{d\mathbf{b}}{dt}\times\mathbf{c}+\mathbf{b}\times\frac{d\mathbf{c}}{dt}\right)+\frac{d\mathbf{a}}{dt}\times(\mathbf{b}\times\mathbf{c})$$

$$=\mathbf{a}\times\left(\frac{d\mathbf{b}}{dt}\times\mathbf{c}\right)+\mathbf{a}\times\left(\mathbf{b}\times\frac{d\mathbf{c}}{dt}\right)+\frac{d\mathbf{a}}{dt}\times(\mathbf{b}\times\mathbf{c})$$

$$=\frac{d\mathbf{a}}{dt}\times(\mathbf{b}\times\mathbf{c})+\mathbf{a}\times\left(\frac{d\mathbf{b}}{dt}\times\mathbf{c}\right)+\mathbf{a}\times\left(\mathbf{b}\times\frac{d\mathbf{c}}{dt}\right).$$

§ 6 Derivative of a function of a function.

Suppose **r** is a differentiable vector function of a scalar variable s and s is a differentiable scalar function of another scalar variable t. Then **r** is a function of t.

An increment δt in t produces an increment $\delta\mathbf{r}$ in **r** and an increment δs in s. When $\delta t\to 0$, $\delta\mathbf{r}\to\mathbf{0}$ and $\delta s\to 0$.

We have $$\frac{d\mathbf{r}}{dt}=\lim_{\delta t\to 0}\frac{\delta\mathbf{r}}{\delta t}=\lim_{\delta t\to 0}\left(\frac{\delta s}{\delta t}\frac{\delta\mathbf{r}}{\delta s}\right)$$

$$=\left(\lim_{\delta t\to 0}\frac{\delta s}{\delta t}\right)\left(\lim_{\delta t\to 0}\frac{\delta\mathbf{r}}{\delta s}\right)=\frac{ds}{dt}\frac{d\mathbf{r}}{ds}.$$

Note. We can also write $\frac{d\mathbf{r}}{dt}=\frac{d\mathbf{r}}{ds}\frac{ds}{dt}$. But it should be clear that $\frac{d\mathbf{r}}{dt}$ is a vector quantity and $\frac{ds}{dt}$ is a scalar quantity. Thus $\frac{d\mathbf{r}}{ds}\frac{ds}{dt}$ is nothing but the multiplication of the vector $\frac{d\mathbf{r}}{ds}$ by the scalar $\frac{ds}{dt}$.

§ 7. Derivative of a constant vector.

A vector is said to be constant only if both its magnitude and direction are fixed. If either of these changes then the vector will change and thus it will not be constant.

Let **r** be a constant vector function of the scalar variable t. Let $\mathbf{r}=\mathbf{c}$, where **c** is a constant vector. Then $\mathbf{r}+\delta\mathbf{r}=\mathbf{c}$.

$\therefore\quad \delta\mathbf{r}=\mathbf{0}$ (zero vector).

$\therefore\quad \dfrac{\delta\mathbf{r}}{\delta t}=\dfrac{\mathbf{0}}{\delta t}=\mathbf{0}.$

$\therefore\quad \lim_{\delta t\to 0}\dfrac{\delta\mathbf{r}}{\delta t}=\lim_{\delta t\to 0}\mathbf{0}=\mathbf{0}.$

$\therefore\quad \dfrac{d\mathbf{r}}{dt}=\mathbf{0}$ (zero vector).

Thus the derivative of a constant vector is equal to the null vector.

§ 8. Derivative of a vector function in terms of its components.

Let **r** be a vector function of the scalar variable t.

Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ where the components x, y, z are scalar functions of the scalar variable t and **i, j, k** are fixed unit vectors.

We have $\mathbf{r}+\delta\mathbf{r}=(x+\delta x)\,\mathbf{i}+(y+\delta y)\,\mathbf{j}+(z+\delta z)\,\mathbf{k}$.

$\therefore\quad \delta\mathbf{r}=(\mathbf{r}+\delta\mathbf{r})-\mathbf{r}=\delta x\,\mathbf{i}+\delta y\,\mathbf{j}+\delta z\,\mathbf{k}.$

$\therefore\quad \dfrac{\delta\mathbf{r}}{\delta t}=\dfrac{\delta x}{\delta t}\,\mathbf{i}+\dfrac{\delta y}{\delta t}\,\mathbf{j}+\dfrac{\delta z}{\delta t}\,\mathbf{k}.$

$\therefore\quad \lim_{\delta t\to 0}\dfrac{\delta\mathbf{r}}{\delta t}=\lim_{\delta t\to 0}\left\{\dfrac{\delta x}{\delta t}\,\mathbf{i}+\dfrac{\delta y}{\delta t}\,\mathbf{j}+\dfrac{\delta z}{\delta t}\,\mathbf{k}\right\}.$

$\therefore\quad \dfrac{d\mathbf{r}}{dt}=\dfrac{dx}{dt}\,\mathbf{i}+\dfrac{dy}{dt}\,\mathbf{j}+\dfrac{dz}{dt}\,\mathbf{k}.$

Thus in order to differentiate a vector we should differentiate its components.

Note. If $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then sometimes we also write it as $\mathbf{r}=(x, y, z)$. In this notation

$$\frac{d\mathbf{r}}{dt}=\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right), \frac{d^2\mathbf{r}}{dt^2}=\left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right), \text{ and so on.}$$

Alternative Method.

We have $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, where **i, j, k** are constant vectors and so their derivatives will be zero.

$$\text{Now } \frac{d\mathbf{r}}{dt}=\frac{d}{dt}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=\frac{d}{dt}(x\mathbf{i})+\frac{d}{dt}(y\mathbf{j})+\frac{d}{dt}(z\mathbf{k})$$

$$=\frac{dx}{dt}\,\mathbf{i}+x\frac{d\mathbf{i}}{dt}+\frac{dy}{dt}\,\mathbf{j}+y\frac{d\mathbf{j}}{dt}+\frac{dz}{dt}\,\mathbf{k}+z\frac{d\mathbf{k}}{dt}$$

$$=\frac{dx}{dt}\,\mathbf{i}+\frac{dy}{dt}\,\mathbf{j}+\frac{dz}{dt}\,\mathbf{k}, \text{ since } \frac{d\mathbf{i}}{dt} \text{ etc. vanish.}$$

§ 9. Some important results.

Theorem 1. *The necessary and sufficient condition for the* *vector function* **a**(t) *to be constant is that* $\frac{d\mathbf{a}}{dt}=0$.

Proof. The condition is necessary. Let $\mathbf{a}(t)$ be a constant vector function of the scalar variable t. Then $\mathbf{a}(t+\delta t)=\mathbf{a}(t)$. We have $\frac{d\mathbf{a}}{dt}=\lim_{\delta t\to 0}\frac{\mathbf{a}(t+\delta t)-\mathbf{a}(t)}{\delta t}=\lim_{\delta t\to 0}\frac{0}{\delta t}=0.$

Therefore the condition is necessary.

The condition is sufficient. Let $\frac{d\mathbf{a}}{dt}=0$. Then to prove that **a** is a constant vector. Let $\mathbf{a}(t)=a_1(t)\,\mathbf{i}+a_2(t)\,\mathbf{j}+a_3(t)\,\mathbf{k}$. Then

$$\frac{d\mathbf{a}}{dt}=\frac{da_1}{dt}\,\mathbf{i}+\frac{da_2}{dt}\,\mathbf{j}+\frac{da_3}{dt}\,\mathbf{k}.$$

Therefore $\frac{d\mathbf{a}}{dt}=0$ gives, $\frac{da_1}{dt}\,\mathbf{i}+\frac{da_2}{dt}\,\mathbf{j}+\frac{da_3}{dt}\,\mathbf{k}=0.$

Equating to zero the coefficients of **i**, **j** and **k**, we get

$$\frac{da_1}{dt}=0,\ \frac{da_2}{dt}=0,\ \frac{da_3}{dt}=0.$$

Hence a_1, a_2, a_3 are constant scalars *i.e.* they are independent of t. Therefore $\mathbf{a}(t)$ is a constant vector function.

Theorem 2. *If* **a** *is a differentiable vector function of the scalar variable t and if* $|\mathbf{a}|=a$, *then*

(*i*) $\frac{d}{dt}(\mathbf{a}^2)=2a\frac{da}{dt}$; *and* (*ii*) $\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}=a\frac{da}{dt}$.

Proof. (i) We have $\mathbf{a}^2=\mathbf{a}\cdot\mathbf{a}=(a)(a)\cos 0=a^2$.

Therefore $\frac{d}{dt}(\mathbf{a}^2)=\frac{d}{dt}(a^2)=2a\frac{da}{dt}$.

(ii) We have $\frac{d}{dt}(\mathbf{a}^2)=\frac{d}{dt}(\mathbf{a}\cdot\mathbf{a})=\frac{d\mathbf{a}}{dt}\cdot\mathbf{a}+\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}=2\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}$.

Also $\frac{d}{dt}(\mathbf{a}^2)=\frac{d}{dt}(a^2)=2a\frac{da}{dt}$.

$\therefore\quad 2\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}=2a\frac{da}{dt}$ or $\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}=a\frac{da}{dt}$.

Theorem 3. *If* **a** *has constant length (fixed magnitude), then* **a** *and* $\frac{d\mathbf{a}}{dt}$ *are perpendicular provided* $\left|\frac{d\mathbf{a}}{dt}\right|\neq 0$.

Proof. Let $|\mathbf{a}|=a=$constant. Then $\mathbf{a}\cdot\mathbf{a}=a^2=$constant.

$\therefore\quad \frac{d}{dt}(\mathbf{a}\cdot\mathbf{a})=0$, or $\frac{d\mathbf{a}}{dt}\cdot\mathbf{a}+\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}=0$

or $\quad 2\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}=0$ or $\mathbf{a}\cdot\frac{d\mathbf{a}}{dt}=0$.

Thus the scalar product of two vectors **a** and $\frac{d\mathbf{a}}{dt}$ is zero.

Therefore **a** is perpendicular to $\frac{d\mathbf{a}}{dt}$ provided $\frac{d\mathbf{a}}{dt}$ is not null vector *i.e.* provided $\left|\frac{d\mathbf{a}}{dt}\right| \neq 0$.

Thus the derivative of a vector of constant length is perpendicular to the vector provided the vector itself is not constant.

Theorem 4. *The necessary and sufficient condition for the vector* $\mathbf{a}(t)$ *to have constant magnitude is* $\mathbf{a} \cdot \frac{d\mathbf{a}}{dt} = 0$.

[Agra 1970, 75; Allahabad 80; Kanpur 75, 78; Sambalpur 74]

Proof. Let **a** be a vector function of the scalar variable t. Let $|\mathbf{a}| = a$=constant. Then $\mathbf{a} \cdot \mathbf{a} = a^2$=constant.

$$\therefore \quad \frac{d}{dt}(\mathbf{a} \cdot \mathbf{a}) = 0 \quad \text{or} \quad \mathbf{a} \cdot \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{a}}{dt} \cdot \mathbf{a} = 0$$

or $\quad 2\mathbf{a} \cdot \frac{d\mathbf{a}}{dt} = 0 \quad$ or $\quad \mathbf{a} \cdot \frac{d\mathbf{a}}{dt} = 0.$

Therefore the condition is necessary.

Condition is sufficient. If $\mathbf{a} \cdot \frac{d\mathbf{a}}{dt} = 0$, then

$$\mathbf{a} \cdot \frac{d\mathbf{a}}{dt} + \frac{d\mathbf{a}}{dt} \cdot \mathbf{a} = 0$$

or $$\frac{d}{dt}(\mathbf{a} \cdot \mathbf{a}) = 0$$

or $$\mathbf{a} \cdot \mathbf{a} = \text{constant}$$

or $$\mathbf{a}^2 = \text{constant}$$

or $$a^2 = \text{constant}$$

or $$|\mathbf{a}| = \text{constant}.$$

Theorem 5. *If* **a** *is a differentiable vector function of the scalar variable* t, *then*

$$\frac{d}{dt}\left(\mathbf{a} \times \frac{d\mathbf{a}}{dt}\right) = \mathbf{a} \times \frac{d^2\mathbf{a}}{dt^2}.$$ **[Agra 1967]**

Proof. We have $\frac{d}{dt}\left(\mathbf{a} \times \frac{d\mathbf{a}}{dt}\right) = \frac{d\mathbf{a}}{dt} \times \frac{d\mathbf{a}}{dt} + \mathbf{a} \times \frac{d^2\mathbf{a}}{dt^2}$

$= 0 + \mathbf{a} \times \frac{d^2\mathbf{a}}{dt^2}$, since the cross product of two equal vectors $\frac{d\mathbf{a}}{dt}$ is zero

$= \mathbf{a} \times \frac{d^2\mathbf{a}}{dt^2}.$

Theorem 6. *The necessary and sufficient condition for the vector* **a** (t) *to have constant direction is*

$$\mathbf{a}\times\frac{d\mathbf{a}}{dt}=\mathbf{0}.$$

[Agra 1970; Sambalpur 74; Allahabad 80; Kolhapur 73]

Proof. Let **a** be vector function of the scalar variable t. Let **A** be a unit vector in the direction of **a**. If a be the magnitude of **a**, then $\mathbf{a}=a\mathbf{A}$.

$$\therefore \quad \frac{d\mathbf{a}}{dt}=a\frac{d\mathbf{A}}{dt}+\frac{da}{dt}\mathbf{A}.$$

Hence $\mathbf{a}\times\frac{d\mathbf{a}}{dt}=(a\mathbf{A})\times\left(a\frac{d\mathbf{A}}{dt}+\frac{da}{dt}\mathbf{A}\right)=a^2\mathbf{A}\times\frac{d\mathbf{A}}{dt}+a\frac{da}{dt}\mathbf{A}\times\mathbf{A}$

$$=a^2\mathbf{A}\times\frac{d\mathbf{A}}{dt}\ [\because\ \mathbf{A}\times\mathbf{A}=\mathbf{0}] \qquad ...(1)$$

The condition is necessary. Suppose **a** has a constant direction. Then **A** is a constant vector because it has constant direction as well as constant magnitude. Therefore $\frac{d\mathbf{A}}{dt}=\mathbf{0}$.

$\therefore$ From (1), we get $\mathbf{a}\times\frac{d\mathbf{a}}{dt}=a^2\mathbf{A}\times\mathbf{0}=\mathbf{0}$.

Therefore the condition is necessary.

The condition is sufficient.

Suppose that $\mathbf{a}\times\frac{d\mathbf{a}}{dt}=\mathbf{0}$.

Then form (1), we get $a^2\mathbf{A}\times\frac{d\mathbf{A}}{dt}=\mathbf{0}$

or $$\mathbf{A}\times\frac{d\mathbf{A}}{dt}=\mathbf{0}. \qquad ...(2)$$

Since **A** is of constant length, therefore

$$\mathbf{A}\cdot\frac{d\mathbf{A}}{dt}=0. \qquad ...(3)$$

From (2) and (3), we get $\frac{d\mathbf{A}}{dt}=\mathbf{0}$.

Hence **A** is a constant vector *i.e.* the direction of **a** is constant.

§ 10. Curves in space.

A curve in a three dimensional Euclidean space may be regarded as the intersection of two surfaces represented by two equations of the form $F_1(x, y, z)=0,\ F_2(x, y, z)=0$.

It can be easily seen that the parametric equations of the form

$$x=f_1(t),\ y=f_2(t),\ z=f_3(t),$$

where x, y, z are scalar functions of the scalar t, also represents a curve in three dimensional space. Here (x, y, z) are coordinates of a current point of the curve. The scalar variable t may range over a set of values $a \leqslant t \leqslant b$.

In vector notation an equation of the form $\mathbf{r}=\mathbf{f}(t)$, represents a curve in three dimensional space if $\mathbf{r}$ is the position vector of a current point on the curve. As t changes, $\mathbf{r}$ will give position vectors of different points on the curve. The vector $\mathbf{f}(t)$ can be expressed as $f_1(t)\,\mathbf{i}+f_2(t)\,\mathbf{j}+f_3(t)\,\mathbf{k}$.

Also if (x, y, z) are the coordinates of a current point on the curve whose position vector is $\mathbf{r}$, then $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

Therefore the single vector equation $\mathbf{r}=\mathbf{f}(t)$

i.e. $$x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=f_1(t)\,\mathbf{i}+f_2(t)\,\mathbf{j}+f_3(t)\,\mathbf{k}$$

is equivalent to the three parametric equations

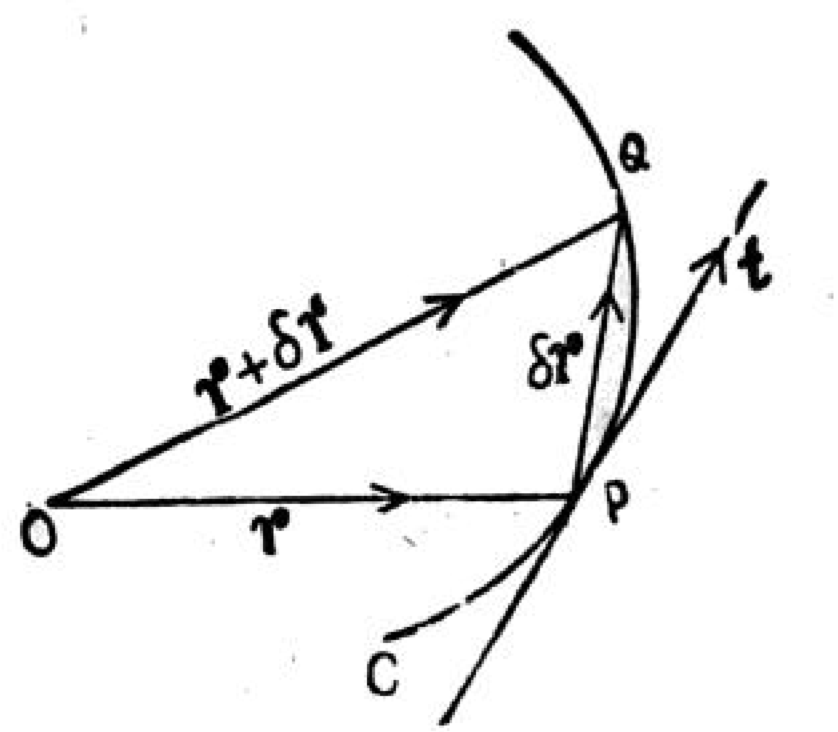
$$x=f_1(t),\ y=f_2(t),\ z=f_3(t).$$

The vector equation $\mathbf{r}=a\cos t\,\mathbf{i}+b\sin t\,\mathbf{j}+0\mathbf{k}$ represents an ellipse, as for different values of t, the end point of $\mathbf{r}$ describes an ellipse.

Similarly $\mathbf{r}=at^2\,\mathbf{i}+2at\,\mathbf{j}+0\mathbf{k}$ is the vector equation of a parabola.

Geometrical significance of $\dfrac{d\mathbf{r}}{dt}$.

Let $\mathbf{r}=\mathbf{f}(t)$ be the vector equation of a curve in space. Let $\mathbf{r}$ and $\mathbf{r}+\delta\mathbf{r}$ be the position vectors of two neighbouring points P and Q on this curve. Thus we have



$$\vec{OP}=\mathbf{r}=\mathbf{f}(t)$$

and $$\vec{OQ}=\mathbf{r}+\delta\mathbf{r}=\mathbf{f}(t+\delta t).$$

$$\therefore\quad \vec{PQ}=\vec{OQ}-\vec{OP}=(\mathbf{r}+\delta\mathbf{r})-\mathbf{r}=\delta\mathbf{r}.$$

Thus $\dfrac{\delta\mathbf{r}}{\delta t}$ is a vector parallel to the chord PQ.

As $Q\to P$ *i.e.* as $\delta t\to 0$, chord $PQ\to$ tangent at P to the curve.

$\therefore\ \lim\limits_{\delta t\to 0}\dfrac{\delta\mathbf{r}}{\delta t}=\dfrac{d\mathbf{r}}{dt}$ is a vector parallel to the tangent at P to the curve $\mathbf{r}=\mathbf{f}(t)$.

Unit tangent vector to a curve. **[Allahabad 1979]**

Suppose in place of the scalar parameter t, we take the parameter as s where s denotes the arc length measured along the curve from any convenient fixed point C on the curve. Thus arc $CP=s$ and arc $CQ=s+\delta s$.

In this case $\frac{d\mathbf{r}}{ds}$ will be a vector along the tangent at P to the curve and in the direction of s increasing. Also we have

$$\left|\frac{d\mathbf{r}}{ds}\right|=\lim_{\delta s\to 0}\left|\frac{\delta\mathbf{r}}{\delta s}\right|=\lim_{Q\to P}\frac{|\delta\mathbf{r}|}{\text{arc } PQ}=\lim_{Q\to P}\frac{\text{chord } PQ}{\text{arc } PQ}=1.$$

Thus $\frac{d\mathbf{r}}{ds}$ is a unit vector along the tangent at P in the direction of s increasing. We denote it by **t**.

§ 11. **Velocity and Acceleration.** If the scalar variable t be the time and **r** be the position vector of a moving particle P with respect to the origin O, then $\delta\mathbf{r}$ is the displacement of the particle in time δt.

The vector $\frac{\delta\mathbf{r}}{\delta t}$ is the average velocity of the particle during the interval δt. If **v** represents the velocity vector of the particle at P, then

$$\mathbf{v}=\lim_{\delta t\to 0}\frac{\delta\mathbf{r}}{\delta t}=\frac{d\mathbf{r}}{dt}.$$

Since $\frac{d\mathbf{r}}{dt}$ is a vector along the tangent at P to the curve in which the particle is moving, therefore the direction of velocity is along the tangent.

If $\delta\mathbf{v}$ be the change in the velocity **v** during the time δt, then $\frac{\delta\mathbf{v}}{\delta t}$ is the average acceleration during that interval. If **a** represents the acceleration of the particle at time t, then

$$\mathbf{a}=\lim_{\delta t\to 0}\frac{\delta\mathbf{v}}{\delta t}=\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)=\frac{d^2\mathbf{r}}{dt^2}.$$

SOLVED EXAMPLES

Ex. 1. *If* $\mathbf{r}=(t+1)\,\mathbf{i}+(t^2+t+1)\,\mathbf{j}+(t^3+t^2+t+1)\,\mathbf{k}$, *find* $\frac{d\mathbf{r}}{dt}$ *and* $\frac{d^2\mathbf{r}}{dt^2}$.

Solution. Since **i, j, k** are constant vectors, therefore

$$\frac{d\mathbf{i}}{at}=0=\frac{d\mathbf{j}}{dt}=\frac{d\mathbf{k}}{dt}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt}=\frac{d}{dt}(t+1)\,\mathbf{i}+\frac{d}{dt}(t^2+t+1)\,\mathbf{j}+\frac{d}{dt}(t^3+t^2+t+1)\,\mathbf{k}$$

$$=\mathbf{i}+(2t+1)\,\mathbf{j}+(3t^2+2t+1)\,\mathbf{k}.$$

Again, $\dfrac{d^2\mathbf{r}}{dt^2}=\dfrac{d}{dt}\left(\dfrac{d\mathbf{r}}{dt}\right)=\dfrac{d\mathbf{i}}{dt}+\dfrac{d}{dt}(2t+1)\,\mathbf{j}+\dfrac{d}{dt}(3t^2+2t+1)\,\mathbf{k}$

$$=0+2\mathbf{j}+(6t+2)\,\mathbf{k}=2\mathbf{j}+(6t+2)\,\mathbf{k}.$$

Ex. 2. *If* $\mathbf{r}=\sin t\,\mathbf{i}+\cos t\,\mathbf{j}+t\,\mathbf{k}$, *find*

(*i*) $\dfrac{d\mathbf{r}}{dt}$, (*ii*) $\dfrac{d^2\mathbf{r}}{dt^2}$, (*iii*) $\left|\dfrac{d\mathbf{r}}{dt}\right|$, (*iv*) $\left|\dfrac{d^2\mathbf{r}}{dt^2}\right|$. **[Agra 78]**

Solution. Since **i, j, k** are constant vectors, therefore $\dfrac{d\mathbf{i}}{dt}=0$ etc. Therefore

(i) $\dfrac{d\mathbf{r}}{dt}=\dfrac{d}{dt}(\sin t)\,\mathbf{i}+\dfrac{d}{dt}(\cos t)\,\mathbf{j}+\dfrac{d}{dt}(t)\,\mathbf{k}=\cos t\,\mathbf{i}-\sin t\,\mathbf{j}+\mathbf{k}.$

(ii) $\dfrac{d^2\mathbf{r}}{dt^2}=\dfrac{d}{dt}\left(\dfrac{d\mathbf{r}}{dt}\right)=\dfrac{d}{dt}(\cos t)\,\mathbf{i}-\dfrac{d}{dt}(\sin t)\,\mathbf{j}+\dfrac{d\mathbf{k}}{dt}$

$$=-\sin t\,\mathbf{i}-\cos t\,\mathbf{j}+0=-\sin t\,\mathbf{i}-\cos t\,\mathbf{j}.$$

(iii) $\left|\dfrac{d\mathbf{r}}{dt}\right|=\sqrt{[(\cos t)^2+(-\sin t)^2+(1)^2]}=\sqrt{2}$

(iv) $\left|\dfrac{d^2\mathbf{r}}{dt^2}\right|=\sqrt{[(-\sin t)^2+(-\cos t)^2]}=1.$

Ex. 3. *If* $\mathbf{r}=(\cos nt)\,\mathbf{i}+(\sin nt)\,\mathbf{j}$, *where n is a constant and t varies, show that* $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}=n\mathbf{k}.$ **[Utkal 1973]**

Solution. We have

$$\frac{d\mathbf{r}}{dt}=\frac{d}{dt}(\cos nt)\,\mathbf{i}+\frac{d}{dt}(\sin nt)\,\mathbf{j}=-n\sin nt\,\mathbf{i}+n\cos nt\,\mathbf{j}.$$

$$\therefore \quad \mathbf{r}\times\frac{d\mathbf{r}}{dt}=(\cos nt\,\mathbf{i}+\sin nt\,\mathbf{j})\times(-n\sin nt\,\mathbf{i}+n\cos nt\,\mathbf{j})$$

$$=-n\cos nt\sin nt\,\mathbf{i}\times\mathbf{i}+n\cos^2 nt\,\mathbf{i}\times\mathbf{j}$$
$$-n\sin^2 nt\,\mathbf{j}\times\mathbf{i}+n\cos nt\sin nt\,\mathbf{j}\times\mathbf{j}$$

$$=n\cos^2 nt\,\mathbf{k}+n\sin^2 nt\,\mathbf{k}$$

$$[\because \mathbf{i}\times\mathbf{i}=0,\ \mathbf{j}\times\mathbf{j}=0,\ \mathbf{i}\times\mathbf{j}=\mathbf{k},\ \mathbf{j}\times\mathbf{i}=-\mathbf{k}]$$

$$=n(\cos^2 nt+\sin^2 nt)\,\mathbf{k}=n\mathbf{k}.$$

Ex. 4. *If* **a, b** *are constant vectors, ω is a constant, and* **r** *is a vector function of the scalar variable t given by*

$$\mathbf{r}=\cos \omega t\,\mathbf{a}+\sin \omega t\,\mathbf{b},$$

show that

(i) $\frac{d^2\mathbf{r}}{dt^2}+\omega^2\mathbf{r}=0$, *and* (ii) $\mathbf{r}\times\frac{d\mathbf{r}}{dt}=\omega\mathbf{a}\times\mathbf{b}$. **[Madras 1983]**

Solution. Since **a**, **b** are constant vectors, therefore

$$\frac{d\mathbf{a}}{dt}=0,\ \frac{d\mathbf{b}}{dt}=0.$$

(i) $\frac{d\mathbf{r}}{dt}=\frac{d}{dt}(\cos\omega t)\,\mathbf{a}+\frac{d}{dt}(\sin\omega t)\,\mathbf{b}$

$$=-\omega\sin\omega t\,\mathbf{a}+\omega\cos\omega t\,\mathbf{b}.$$

$$\therefore\quad \frac{d^2\mathbf{r}}{dt^2}=-\omega^2\cos\omega t\,\mathbf{a}-\omega^2\sin\omega t\,\mathbf{b}$$

$$=-\omega^2(\cos\omega t\,\mathbf{a}+\sin\omega t\,\mathbf{b})=-\omega^2\mathbf{r}.$$

$$\therefore\quad \frac{d^2\mathbf{r}}{dt^2}+\omega^2\mathbf{r}=0.$$

(ii) $\mathbf{r}\times\frac{d\mathbf{r}}{dt}=(\cos\omega t\,\mathbf{a}+\sin\omega t\,\mathbf{b})\times(-\omega\sin\omega t\,\mathbf{a}+\omega\cos\omega t\,\mathbf{b})$

$=\omega\cos^2\omega t\,\mathbf{a}\times\mathbf{b}-\omega\sin^2\omega t\,\mathbf{b}\times\mathbf{a}$ $\quad[\because\ \mathbf{a}\times\mathbf{a}=0,\ \mathbf{b}\times\mathbf{b}=0]$

$=\omega\cos^2\omega t\,\mathbf{a}\times\mathbf{b}+\omega\sin^2\omega t\,\mathbf{a}\times\mathbf{b}$

$$=\omega(\cos^2\omega t+\sin^2\omega t)\,\mathbf{a}\times\mathbf{b}=\omega\mathbf{a}\times\mathbf{b}.$$

Ex. 5. *If* $\mathbf{r}=(\sinh t)\,\mathbf{a}+(\cosh t)\,\mathbf{b}$, *where* **a** *and* **b** *are constant vectors, then show that* $\frac{d^2\mathbf{r}}{dt^2}=\mathbf{r}$.

Solution. Since **a**, **b** are constant vectors, therefore

$$\frac{d\mathbf{a}}{dt}=0,\ \frac{d\mathbf{b}}{dt}=0.$$

$$\therefore\quad \frac{d\mathbf{r}}{dt}=\frac{d}{dt}(\sinh t)\,\mathbf{a}+\frac{d}{dt}(\cosh t)\,\mathbf{b}$$

$$=(\cosh t)\,\mathbf{a}+(\sinh t)\,\mathbf{b}.$$

$$\therefore\quad \frac{d^2\mathbf{r}}{dt^2}=(\sinh t)\,\mathbf{a}+(\cosh t)\,\mathbf{b}=\mathbf{r}.$$

Ex. 6. *If* $\mathbf{r}=a\cos t\,\mathbf{i}+a\sin t\,\mathbf{j}+at\tan\alpha\,\mathbf{k}$, *find*

$$\left|\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right| \text{ and } \left[\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2},\frac{d^3\mathbf{r}}{dt^3}\right].$$ **[Agra 1977]**

Solution. We have

$$\frac{d\mathbf{r}}{dt}=-a\sin t\,\mathbf{i}+a\cos t\,\mathbf{j}+a\tan\alpha\,\mathbf{k}$$

$$\frac{d^2\mathbf{r}}{dt^2}=-a\cos t\,\mathbf{i}-a\sin t\,\mathbf{j},\ \left[\because\ \frac{d\mathbf{k}}{dt}=0\right]$$

$$\frac{d^3\mathbf{r}}{dt^3}=a\sin t\,\mathbf{i}-a\cos t\,\mathbf{j}$$

$$\therefore \quad \frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$=a^2\sin t\tan\alpha\,\mathbf{i}-a^2\cos t\tan\alpha\mathbf{j}+a^2\mathbf{k}.$$

$$\therefore \quad \left|\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right|=\sqrt{\left(a^4\sin^2 t\tan^2\alpha+a^4\cos^2 t\tan^2\alpha+a^4\right)}$$

$$=a^2\sec\alpha.$$

Also $\left[\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \frac{d^3\mathbf{r}}{dt^3}\right]=\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)\cdot\frac{d^3\mathbf{r}}{dt^3}$

$=(a^2\sin t\tan\alpha\,\mathbf{i}-a^2\cos t\tan\alpha\,\mathbf{j}+a^2\mathbf{k})\cdot(a\sin t\,\mathbf{i}-a\cos t\,\mathbf{j})$

$=a^3\sin^2 t\tan\alpha\,\mathbf{i}\cdot\mathbf{i}+a^3\cos^2 t\tan\alpha\,\mathbf{j}\cdot\mathbf{j}$ [$\because$ $\mathbf{i}\cdot\mathbf{j}=0$ etc.]

$=a^3\tan\alpha(\sin^2 t+\cos^2 t)$ [$\because$ $\mathbf{i}\cdot\mathbf{i}=1=\mathbf{j}\cdot\mathbf{j}$]

$=a^3\tan\alpha.$

Ex. 7. *If* $\frac{d\mathbf{u}}{dt}=\mathbf{w}\times\mathbf{u}, \frac{d\mathbf{v}}{dt}=\mathbf{w}\times\mathbf{v}$, *show that*

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v})=\mathbf{w}\times(\mathbf{u}\times\mathbf{v}).$$

[Meerut 1975; Kanpur 77]

Solution. We have

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v})=\frac{d\mathbf{u}}{dt}\times\mathbf{v}+\mathbf{u}\times\frac{d\mathbf{v}}{dt}=(\mathbf{w}\times\mathbf{u})\times\mathbf{v}+\mathbf{u}\times(\mathbf{w}\times\mathbf{v})$$

$$=(\mathbf{v}\cdot\mathbf{w})\,\mathbf{u}-(\mathbf{v}\cdot\mathbf{u})\,\mathbf{w}+(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}-(\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}$$

$$=(\mathbf{v}\cdot\mathbf{w})\,\mathbf{u}-(\mathbf{u}\cdot\mathbf{w})\,\mathbf{v} \quad [\because \ \mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}]$$

$$=(\mathbf{w}\cdot\mathbf{v})\,\mathbf{u}-(\mathbf{w}\cdot\mathbf{u})\,\mathbf{v}=\mathbf{w}\times(\mathbf{u}\times\mathbf{v}).$$

Ex. 8. *If* $\mathbf{R}$ *be a unit vector in the direction of* $\mathbf{r}$, *prove that*

$$\mathbf{R}\times\frac{d\mathbf{R}}{dt}=\frac{1}{r^2}\,\mathbf{r}\times\frac{d\mathbf{r}}{dt},\ \textit{where } r=|\,\mathbf{r}\,|.$$

[Kanpur 1979; Agra 74]

Solutiou. We have $\mathbf{r}=r\mathbf{R}$; so that $\mathbf{R}=\frac{1}{r}\,\mathbf{r}$.

$$\therefore \quad \frac{d\mathbf{R}}{dt}=\frac{1}{r}\frac{d\mathbf{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\mathbf{r}.$$

Hence $\mathbf{R}\times\frac{d\mathbf{R}}{dt}=\frac{1}{r}\,\mathbf{r}\times\left(\frac{1}{r}\frac{d\mathbf{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\,\mathbf{r}\right)$

$$=\frac{1}{r^2}\,\mathbf{r}\times\frac{d\mathbf{r}}{dt}-\frac{1}{r^3}\frac{dr}{dt}\,\mathbf{r}\times\mathbf{r}$$

$$=\frac{1}{r^2}\,\mathbf{r}\times\frac{d\mathbf{r}}{dt}. \qquad [\because \ \mathbf{r}\times\mathbf{r}=0]$$

Ex. 9. *If* $\mathbf{r}$ *is a vector function of a scalar t and* $\mathbf{a}$ *is a constant vector, m a constant, differentiate the following with respect to t :—*

(*i*) $\mathbf{r}\cdot\mathbf{a}$, (*ii*) $\mathbf{r}\times\mathbf{a}$, (*iii*) $\mathbf{r}\times\frac{d\mathbf{r}}{dt}$, (*iv*) $\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}$,

(*v*) $\mathbf{r}^2+\frac{1}{\mathbf{r}^2}$, (*vi*) $m\left(\frac{d\mathbf{r}}{dt}\right)^2$, (*vii*) $\frac{\mathbf{r}+\mathbf{a}}{\mathbf{r}^2+\mathbf{a}^2}$, (*viii*) $\frac{\mathbf{r}\times\mathbf{a}}{\mathbf{r}\cdot\mathbf{a}}$.

Solution. (i) Let $R=\mathbf{r}\cdot\mathbf{a}$. [Note $\mathbf{r}\cdot\mathbf{a}$ is a scalar]

Then $\frac{dR}{dt}=\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}+\mathbf{r}\cdot\frac{d\mathbf{a}}{dt}$

$=\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}+\mathbf{r}\cdot\mathbf{0}$ $\left[\because \frac{d\mathbf{a}}{dt}=\mathbf{0}, \text{ as } \mathbf{a} \text{ is constant}\right]$

$=\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}+0=\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}$.

(ii) Let $\mathbf{R}=\mathbf{r}\times\mathbf{a}$.

Then $\frac{d\mathbf{R}}{dt}=\frac{d\mathbf{r}}{dt}\times\mathbf{a}+\mathbf{r}\times\frac{d\mathbf{a}}{dt}$

$=\frac{d\mathbf{r}}{dt}\times\mathbf{a}+\mathbf{r}\times\mathbf{0}$ $\left[\because \frac{d\mathbf{a}}{dt}=\mathbf{0}\right]$

$=\frac{d\mathbf{r}}{dt}\times\mathbf{a}+\mathbf{0}=\frac{d\mathbf{r}}{dt}\times\mathbf{a}$.

(iii) Let $\mathbf{R}=\mathbf{r}\times\frac{d\mathbf{r}}{dt}$.

Then $\frac{d\mathbf{R}}{dt}=\frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}+\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}$

$=\mathbf{0}+\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}$ $\left[\because \frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}=\mathbf{0}\right]$

$=\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}$.

(iv) Let $R=\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}$.

Then $\frac{dR}{dt}=\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}+\mathbf{r}\cdot\frac{d^2\mathbf{r}}{dt^2}=\left(\frac{d\mathbf{r}}{dt}\right)^2+\mathbf{r}\cdot\frac{d^2\mathbf{r}}{dt^2}$.

(v) Let $R=\mathbf{r}^2+\frac{1}{\mathbf{r}^2}$.

Then $\frac{dR}{dt}=\frac{d}{dt}(\mathbf{r}^2)+\frac{d}{dt}\left(\frac{1}{\mathbf{r}^2}\right)$

$=\frac{d}{dt}(r^2)+\frac{d}{dt}\left(\frac{1}{r^2}\right)$, where $r=|\mathbf{r}|$

$=2r\frac{dr}{dt}-\frac{2}{r^3}\frac{dr}{dt}$.

(vi) Let $R=m\left(\frac{d\mathbf{r}}{dt}\right)^2$.

Then $\frac{dR}{dt}=m\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)^2$

$$=2m\,\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2} \qquad \left[\text{Note}\quad \frac{d\mathbf{r}^2}{dt}=2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right]$$

$$=2m\,\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}.$$

(vii) Let $\mathbf{R}=\dfrac{\mathbf{r}+\mathbf{a}}{\mathbf{r}^2+\mathbf{a}^2}$.

Then $\dfrac{d\mathbf{R}}{dt}=\dfrac{1}{(\mathbf{r}^2+\mathbf{a}^2)}\dfrac{d}{dt}(\mathbf{r}+\mathbf{a})+\left\{\dfrac{d}{dt}\left(\dfrac{1}{\mathbf{r}^2+\mathbf{a}^2}\right)\right\}(\mathbf{r}+\mathbf{a})$

[Note that $\mathbf{r}^2+\mathbf{a}^2$ is a scalar]

$$=\frac{1}{\mathbf{r}^2+\mathbf{a}^2}\left(\frac{d\mathbf{r}}{dt}+\frac{d\mathbf{a}}{dt}\right)-\left\{\frac{1}{(\mathbf{r}^2+\mathbf{a}^2)^2}\frac{d}{dt}(\mathbf{r}^2+\mathbf{a}^2)\right\}(\mathbf{r}+\mathbf{a})$$

$$=\frac{1}{(\mathbf{r}^2+\mathbf{a}^2)}\frac{d\mathbf{r}}{dt}-\frac{2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}}{(\mathbf{r}^2+\mathbf{a}^2)^2}(\mathbf{r}+\mathbf{a}).$$

$$\left[\because\quad \frac{d\mathbf{a}}{dt}=0,\ \frac{d}{dt}\mathbf{r}^2=2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt},\ \frac{d}{dt}\mathbf{a}^2=0\right]$$

(viii) Let $\mathbf{R}=\dfrac{\mathbf{r}\times\mathbf{a}}{\mathbf{r}\cdot\mathbf{a}}$.

Then $\dfrac{d\mathbf{R}}{dt}=\dfrac{1}{\mathbf{r}\cdot\mathbf{a}}\dfrac{d}{dt}(\mathbf{r}\times\mathbf{a})+\left\{\dfrac{d}{dt}\left(\dfrac{1}{\mathbf{r}\cdot\mathbf{a}}\right)\right\}(\mathbf{r}\times\mathbf{a})$

[Note that $\mathbf{r}\cdot\mathbf{a}$ is a scalar quantity]

$$=\frac{1}{\mathbf{r}\cdot\mathbf{a}}\left(\frac{d\mathbf{r}}{dt}\times\mathbf{a}+\mathbf{r}\times\frac{d\mathbf{a}}{dt}\right)-\left\{\frac{1}{(\mathbf{r}\cdot\mathbf{a})^2}\frac{d}{dt}(\mathbf{r}\cdot\mathbf{a})\right\}(\mathbf{r}\times\mathbf{a})$$

$$=\frac{\frac{d\mathbf{r}}{dt}\times\mathbf{a}}{\mathbf{r}\cdot\mathbf{a}}-\left\{\frac{1}{(\mathbf{r}\cdot\mathbf{a})^2}\left(\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}+\mathbf{r}\cdot\frac{d\mathbf{a}}{dt}\right)\right\}(\mathbf{r}\times\mathbf{a})$$

$$=\frac{\frac{d\mathbf{r}}{dt}\times\mathbf{a}}{\mathbf{r}\cdot\mathbf{a}}-\frac{\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}}{(\mathbf{r}\cdot\mathbf{a})^2}(\mathbf{r}\times\mathbf{a}). \qquad \left[\because\quad \frac{d\mathbf{a}}{dt}=0\right]$$

Ex. 10. *If* **r** *is a vector function of a scalar t, r its module, and* **a**, **b** *are constant vectors, differentiate the following with respect to t :*

(*i*) $r^3\mathbf{r}+\mathbf{a}\times\dfrac{d\mathbf{r}}{dt}$, (*ii*) $r^2\mathbf{r}+(\mathbf{a}\cdot\mathbf{r})\,\mathbf{b}$, (*iii*) $r^n\mathbf{r}$, (*iv*) $(a\mathbf{r}+r\mathbf{b})^2$.

Solution. (i) Let $\mathbf{R}=r^3\,\mathbf{r}+\mathbf{a}\times\dfrac{d\mathbf{r}}{dt}$.

Then $\dfrac{d\mathbf{R}}{dt}=\dfrac{d}{dt}(r^3\,\mathbf{r})+\dfrac{d}{dt}\left\{\mathbf{a}\times\dfrac{d\mathbf{r}}{dt}\right\}$

$$=3r^2\frac{dr}{dt}\mathbf{r}+r^3\frac{d\mathbf{r}}{dt}+\frac{d\mathbf{a}}{dt}\times\frac{d\mathbf{r}}{dt}+\mathbf{a}\times\frac{d^2\mathbf{r}}{dt^2}$$

$$=3r^2\frac{dr}{dt}\mathbf{r}+r^3\frac{d\mathbf{r}}{dt}+\mathbf{a}\times\frac{d^2\mathbf{r}}{dt^2} \qquad \left[\because \frac{d\mathbf{a}}{dt}=0\right]$$

(ii) Let $\mathbf{R}=r^2\mathbf{r}+(\mathbf{a}\cdot\mathbf{r})\,\mathbf{b}$.

Then $\frac{d\mathbf{R}}{dt}=\frac{d}{dt}(r^2\,\mathbf{r})+\left\{\frac{d}{dt}(\mathbf{a}\cdot\mathbf{r})\right\}\mathbf{b}+(\mathbf{a}\cdot\mathbf{r})\frac{d\mathbf{b}}{dt}$

$$=2r\frac{dr}{dt}\mathbf{r}+r^2\frac{d\mathbf{r}}{dt}+\left(\frac{d\mathbf{a}}{dt}\cdot\mathbf{r}+\mathbf{a}\cdot\frac{d\mathbf{r}}{dt}\right)\mathbf{b} \qquad \left[\because \frac{d\mathbf{b}}{dt}=0\right]$$

$$=2r\frac{dr}{dt}\mathbf{r}+r^2\frac{d\mathbf{r}}{dt}+\left(\mathbf{a}\cdot\frac{d\mathbf{r}}{dt}\right)\mathbf{b} \qquad \left[\because \frac{d\mathbf{a}}{dt}=0\right]$$

(iii) Let $\mathbf{R}=r^n\,\mathbf{r}$.

Then $\frac{d\mathbf{R}}{dt}=\left(\frac{d}{dt}r^n\right)\mathbf{r}+r^n\frac{d\mathbf{r}}{dt}=\left(nr^{n-1}\frac{dr}{dt}\right)\mathbf{r}+r^n\frac{d\mathbf{r}}{dt}.$

(iv) Let $R=(a\mathbf{r}+r\mathbf{b})^2$. Then

$$\frac{dR}{dt}=2\,(a\mathbf{r}+r\mathbf{b})\cdot\frac{d}{dt}(a\mathbf{r}+r\mathbf{b}) \qquad \left[\text{Note } \frac{d}{dt}\mathbf{r}^2=2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right]$$

$$=2(a\mathbf{r}+r\mathbf{b})\cdot\left(\frac{da}{dt}\mathbf{r}+a\frac{d\mathbf{r}}{dt}+\frac{dr}{dt}\mathbf{b}+r\frac{d\mathbf{b}}{dt}\right)$$

$$=2\,(a\mathbf{r}+r\mathbf{b})\cdot\left(a\frac{d\mathbf{r}}{dt}+\frac{dr}{dt}\mathbf{b}\right) \qquad \left[\because \frac{da}{dt}=0,\frac{d\mathbf{b}}{dt}=0\right]$$

Ex. 11. *Find*

(*i*) $\frac{d}{dt}\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2}\right]$; (*ii*) $\frac{d^2}{dt^2}\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2}\right]$;

(*iii*) $\frac{d}{dt}\left[\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)\right]$.

Solution. (i) Let $R=\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2}\right]$. Then R is the scalar triple product of three vectors $\mathbf{r}$, $\frac{d\mathbf{r}}{dt}$ and $\frac{d^2\mathbf{r}}{dt^2}$. Therefore using the rule for finding the derivative of a scalar triple product, we have

$$\frac{dR}{dt}=\left[\frac{d\mathbf{r}}{dt},\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2}\right]+\left[\mathbf{r},\frac{d^2\mathbf{r}}{dt^2},\frac{d^2\mathbf{r}}{dt^2}\right]+\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right]$$

$=\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right]$, [since scalar triple products having two equal vectors vanish.

(ii) Let $R=\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2}\right]$. Then as in part (i)

$$\frac{dR}{dt}=\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right].$$

Differentiating again, we get

$$\frac{d^2R}{dt^2}=\left[\frac{d\mathbf{r}}{dt},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right]+\left[\mathbf{r},\frac{d^2\mathbf{r}}{dt^2},\frac{d^3\mathbf{r}}{dt^3}\right]+\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^4\mathbf{r}}{dt^4}\right]$$
$$=\left[\mathbf{r},\frac{d^2\mathbf{r}}{dt^2},\frac{d^3\mathbf{r}}{dt^3}\right]+\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^4\mathbf{r}}{dt^4}\right].$$

(iii) Let $\mathbf{R}=\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)$. Then **R** is the vector triple product of three vectors. Therefore using the rule for finding the derivative of a vector triple product, we have

$$\frac{d\mathbf{R}}{dt}=\frac{d\mathbf{r}}{dt}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)+\mathbf{r}\times\left(\frac{d^2\mathbf{r}}{dt^2}\times\frac{d^2\mathbf{r}}{dt^2}\right)+\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right)$$
$$=\frac{d\mathbf{r}}{dt}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)+\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right),$$

since $\frac{d^2\mathbf{r}}{dt^2}\times\frac{d^2\mathbf{r}}{dt^2}=0$, being vector product of two equal vectors.

Ex. 12. *If* $\mathbf{a}=\sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{j}+\theta\mathbf{k}$, $\mathbf{b}=\cos\theta\,\mathbf{i}-\sin\theta\,\mathbf{j}-3\mathbf{k}$ *and* $\mathbf{c}=2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$, *find* $\frac{d}{d\theta}\left\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\right\}$ *at* $\theta=\frac{\pi}{2}$. **[Rohilkband 1979]**

Solution. We have

$$\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\theta & -\sin\theta & -3\\ 2 & 3 & -3\end{vmatrix}=(3\sin\theta+9)\,\mathbf{i}+(3\cos\theta-6)\,\mathbf{j}+(3\cos\theta+2\sin\theta)\,\mathbf{k}.$$

$$\therefore\quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \sin\theta & \cos\theta & \theta\\ 3\sin\theta+9 & 3\cos\theta-6 & 3\cos\theta+2\sin\theta\end{vmatrix}$$
$$=(3\cos^2\theta+2\sin\theta\cos\theta-3\theta\cos\theta+6\theta)\,\mathbf{i}+(3\theta\sin\theta+9\theta-3\sin\theta\cos\theta-2\sin^2\theta)\,\mathbf{j}+(-6\sin\theta-9\cos\theta)\,\mathbf{k}.$$

$$\therefore\quad \frac{d}{d\theta}\left\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\right\}$$
$$=(-6\cos\theta\sin\theta+2\cos^2\theta-2\sin^2\theta-3\cos\theta+3\theta\sin\theta+6)\,\mathbf{i}+(3\sin\theta+3\theta\cos\theta+9-3\cos^2\theta+3\sin^2\theta-4\sin\theta\cos\theta)\,\mathbf{j}+(-6\cos\theta+9\sin\theta)\,\mathbf{k}.$$

Putting $\theta=\pi/2$, we get the required derivative

$$=(4+\tfrac{3}{2}\pi)\,\mathbf{i}+15\mathbf{j}+9\mathbf{k}.$$

Ex. 13. *Show that if* **a**, **b**, **c** *are constant vectors, then* $\mathbf{r}=\mathbf{a}\,t^2+\mathbf{b}\,t+\mathbf{c}$ *is the path of a particle moving with constant acceleration.* **[Delhi 1962]**

Solution. The velocity of the particle$=\frac{d\mathbf{r}}{dt}=2t\mathbf{a}+\mathbf{b}$.

The acceleration of the particle$=\frac{d^2\mathbf{r}}{dt^2}=2\mathbf{a}$.

Thus the point whose path is $\mathbf{r}=\mathbf{a}\,t^2+\mathbf{b}\,t+\mathbf{c}$ is moving with constant acceleration.

Ex. 14. *A particle moves along the curve $x=4\cos t$, $y=4\sin t$, $z=6t$. Find the velocity and acceleration at time $t=0$ and $t=\frac{1}{2}\pi$. Find also the magnitudes of the velocity and acceleration at any time t.* **[Kanpur 1980]**

Solution. Let $\mathbf{r}$ be the position vector of the particle at time t.

Then $\mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}=4\cos t\,\mathbf{i}+4\sin t\,\mathbf{j}+6t\,\mathbf{k}$. If $\mathbf{v}$ is the velocity of the particle at time t and $\mathbf{a}$ its acceleration at that time then

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=-4\sin t\,\mathbf{i}+4\cos t\,\mathbf{j}+6\mathbf{k},$$

$$\mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=-4\cos t\,\mathbf{i}-4\sin t\,\mathbf{j}.$$

Magnitude of the velocity at time $t=|\mathbf{v}|$

$$=\sqrt{(16\sin^2 t+16\cos^2 t+36)}=\sqrt{(52)}=2\sqrt{(13)}.$$

Magnitude of the acceleration

$$=|\mathbf{a}|=\sqrt{(16\cos^2 t+16\sin^2 t)}=4.$$

At $t=0$, $\mathbf{v}=4\mathbf{j}+6\mathbf{k}$, $\mathbf{a}=-4\mathbf{i}$.

At $t=\frac{1}{2}\pi$, $\mathbf{v}=-4\,\mathbf{i}+6\,\mathbf{k}$, $\mathbf{a}=-4\,\mathbf{j}$.

Ex. 15. *A particle moves along the curve $x=t^3+1$, $y=t^2$, $z=2t+5$ where t is the time. Find the components of its velocity and acceleration at $t=1$ in the direction* $\mathbf{i}+\mathbf{j}+3\mathbf{k}$.

[Agra 1979, Rohilkhand 81]

Solution. If $\mathbf{r}$ is the position vector of any point (x, y, z) on the given curve, then

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=(t^3+1)\,\mathbf{i}+t^2\,\mathbf{j}+(2t+5)\,\mathbf{k}.$$

$$\text{Velocity}=\mathbf{v}=\frac{d\mathbf{r}}{dt}=3t^2\,\mathbf{i}+2t\mathbf{j}+2\mathbf{k}=3\mathbf{i}+2\mathbf{j}+2\mathbf{k}\quad\text{at } t=1.$$

$$\text{Acceleration}=\mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)=6t\,\mathbf{i}+2\mathbf{j}=6\mathbf{i}+2\mathbf{j}\quad\text{at } t=1.$$

Now the unit vector in the given direction $\mathbf{i}+\mathbf{j}+3\mathbf{k}$

$$=\frac{\mathbf{i}+\mathbf{j}+3\mathbf{k}}{|\mathbf{i}+\mathbf{j}+3\mathbf{k}|}=\frac{\mathbf{i}+\mathbf{j}+3\mathbf{k}}{\sqrt{(11)}}=\mathbf{b}, \text{ say}.$$

$\therefore$ the component of velocity in the given direction

$$=\mathbf{v}\cdot\mathbf{b}=\frac{(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+3\mathbf{k})}{\sqrt{(11)}}=\frac{11}{\sqrt{(11)}}=\sqrt{(11)};$$

and the component of acceleration in the given direction

$$=\mathbf{a}\cdot\mathbf{b}=\frac{(6\mathbf{i}+2\mathbf{j})\cdot(\mathbf{i}+\mathbf{j}+3\mathbf{k})}{\sqrt{(11)}}=\frac{8}{\sqrt{(11)}}.$$

Ex. 16. *A particle moves so that its position vector is given by* $\mathbf{r} = \cos \omega t\, \mathbf{i} + \sin \omega t\, \mathbf{j}$ *where* ω *is a constant; show that (i) the velocity of the particle is perpendicular to* $\mathbf{r}$, *(ii) the acceleration is directed towards the origin and has magnitude proportional to the distance from the origin, (iii)* $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt}$ *is a constant vector.*

Solution. (i) Velocity $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -\omega \sin \omega t\, \mathbf{i} + \omega \cos \omega t\, \mathbf{j}$.

We have $\mathbf{r} \cdot \dfrac{d\mathbf{r}}{dt} = (\cos \omega t\, \mathbf{i} + \sin \omega t\, \mathbf{j}) \cdot (-\omega \sin \omega t\, \mathbf{i} + \omega \cos \omega t\, \mathbf{j})$

$$= -\omega \cos \omega t \sin \omega t + \omega \sin \omega t \cos \omega t = 0.$$

Therefore the velocity is perpendicular to $\mathbf{r}$.

(ii) Acceleration of the particle

$$= \mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = -\omega^2 \cos \omega t\, \mathbf{i} - \omega^2 \sin \omega t\, \mathbf{j}$$

$$= -\omega^2 (\cos \omega t\, \mathbf{i} + \sin \omega t\, \mathbf{j}) = -\omega^2 \mathbf{r}.$$

$\therefore$ acceleration is a vector opposite to the direction of $\mathbf{r}$ *i.e.* acceleration is directed towards the origin. Also magnitude of acceleration $= |\mathbf{a}| = |-\omega^2 \mathbf{r}| = \omega^2 r$ which is proportional to r *i.e.* the distance of the particle from the origin.

(iii) $\mathbf{r} \times \dfrac{d\mathbf{r}}{dt} = (\cos \omega t\, \mathbf{i} + \sin \omega t\, \mathbf{j}) \times (-\omega \sin \omega t\, \mathbf{i} + \omega \cos \omega t\, \mathbf{j})$

$= \omega \cos^2 \omega t\, \mathbf{i} \times \mathbf{j} - \omega \sin^2 \omega t\, \mathbf{j} \times \mathbf{i}$ $[\because \mathbf{i} \times \mathbf{i} = \mathbf{0}, \mathbf{j} \times \mathbf{j} = \mathbf{0}]$

$= \omega \cos^2 \omega t\, \mathbf{k} + \omega \sin^2 \omega t\, \mathbf{k}$ $[\because \mathbf{i} \times \mathbf{j} = \mathbf{k} = -\mathbf{j} \times \mathbf{i}]$

$= \omega (\cos^2 \omega t + \sin^2 \omega t)\, \mathbf{k} = \omega \mathbf{k}$, a constant vector.

Ex. 17. *Find the unit tangent vector to any point on the curve*

$$x = a \cos t,\ y = a \sin t,\ z = bt.$$

Solution. If $\mathbf{r}$ is the position vector of any point (x, y, z) on the given curve, then

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = a \cos t\, \mathbf{i} + a \sin t\, \mathbf{j} + bt\, \mathbf{k}.$$

The vector $\dfrac{d\mathbf{r}}{dt}$ is also the tangent at the point (x, y, z) to the given curve.

We have $\dfrac{d\mathbf{r}}{dt} = -a \sin t\, \mathbf{i} + a \cos t\, \mathbf{j} + b\, \mathbf{k}$.

$$\therefore \left| \frac{d\mathbf{r}}{dt} \right| = \sqrt{(a^2 \sin^2 t + a^2 \cos^2 t + b^2)} = \sqrt{(a^2 + b^2)}.$$

Hence the unit tangent vector **t**

$$=\frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|}=\frac{-a\sin t\,\mathbf{i}+a\cos t\,\mathbf{j}+b\mathbf{k}}{\sqrt{(a^2+b^2)}}$$

$$=\frac{1}{\sqrt{(a^2+b^2)}}(-a\sin t\,\mathbf{i}+a\cos t\,\mathbf{j}+b\mathbf{k}).$$

Exercises

1. If $\mathbf{r}$ is the position vector of a moving point and r is the modulus of $\mathbf{r}$, show that

$$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=r\frac{dr}{dt}.$$

Interpret the relations $\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=0$ and $\mathbf{r}\times\frac{d\mathbf{r}}{dt}=0$.

[Sambalpur 1974]

2. If $\mathbf{r}$ is a unit vector, then prove that

$$\left|\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right|=\left|\frac{d\mathbf{r}}{dt}\right|.$$

[Rajasthan 1974]

3. If $\mathbf{r}=t^3\,\mathbf{i}+\left(2t^3-\frac{1}{5t^2}\right)\mathbf{j}$, show that $\mathbf{r}\times\frac{d\mathbf{r}}{dt}=\mathbf{k}$.

[Utkal 1973]

4. If $\mathbf{r}=e^{nt}\,\mathbf{a}+e^{-nt}\,\mathbf{b}$, where $\mathbf{a}$, $\mathbf{b}$ are constant vectors, show that $\frac{d^2\mathbf{r}}{dt^2}-n^2\mathbf{r}=0$.

[Agra 1976]

5. If $\mathbf{r}=\mathbf{a}\sin\omega t+\mathbf{b}\cos\omega t+\frac{\mathbf{c}t}{\omega^2}\sin\omega t$, prove that

$$\frac{d^2\mathbf{r}}{dt^2}+\omega^2\mathbf{r}=\frac{2\mathbf{c}}{\omega}\cos\omega t,$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are constant vectors and ω is a constant scalar.

[Marathwada 1974]

6. Show that $\mathbf{r}=\mathbf{a}e^{mt}+\mathbf{b}e^{mt}$ is the solution of the differential equation $\frac{d^2\mathbf{r}}{dt^2}-(m+n)\frac{d\mathbf{r}}{dt}+mn\,\mathbf{r}=0.$

Hence solve the equation

$$\frac{d^2\mathbf{r}}{dt^2}-\frac{d\mathbf{r}}{at}-2\mathbf{r}=0,\text{ where}$$

$$\mathbf{r}=\mathbf{i}\text{ and }\frac{d\mathbf{r}}{dt}=\mathbf{j}\text{ for }t=0.$$

[Kanpur 1977]

Ans. $\mathbf{r}=\frac{1}{3}(e^{2t}+2e^{-t})\,\mathbf{i}+\frac{1}{3}(e^{2t}-e^{-t})\,\mathbf{j}$.

7. A particle moves along the curve $x=e^{-t}$, $y=2\cos 3t$, $z=2\sin 3t$. Determine the velocity and acceleration at any time t and their magnitudes at $t=0$.

Ans. $|\mathbf{v}|=\sqrt{(37)}$; $|\mathbf{a}|=\sqrt{(325)}$.

8. If $\mathbf{A}=5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\,\mathbf{k}$ and $\mathbf{B}=\sin t\,\mathbf{i}-\cos t\,\mathbf{j}$, find

(a) $\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{B})$; (b) $\dfrac{d}{dt}(\mathbf{A}\times\mathbf{B})$; (c) $\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{A})$.

Ans. (a) $(5t^2-1)\cos t+11t\sin t$;

(b) $(t^3\sin t-3t^2\cos t)\,\mathbf{i}-(t^3\cos t+5t^2\sin t)\,\mathbf{j}$
$+(5t^2\sin t-11t\cos t-\sin t)\,\mathbf{k}$.

(c) $100t^3+2t+6t^5$.

9. Prove the following :

(a) $\dfrac{d}{dt}\left[\mathbf{a}\cdot\dfrac{d\mathbf{b}}{dt}-\dfrac{d\mathbf{a}}{dt}\cdot\mathbf{b}\right]=\mathbf{a}\cdot\dfrac{d^2\mathbf{b}}{dt^2}-\dfrac{d^2\mathbf{a}}{dt^2}\cdot\mathbf{b}$.

(b) $\dfrac{d}{dt}\left[\mathbf{a}\times\dfrac{d\mathbf{b}}{dt}-\dfrac{d\mathbf{a}}{dt}\times\mathbf{b}\right]=\mathbf{a}\times\dfrac{d^2\mathbf{b}}{dt^2}-\dfrac{d^2\mathbf{a}}{dt^2}\times\mathbf{b}$.

§ 12. Integration of Vector Functions.

We shall define *integration as the reverse process of differentiation.* Let $\mathbf{f}(t)$ *and* $\mathbf{F}(t)$ be two vector functions of the scalar t such that $\dfrac{d}{dt}\mathbf{F}(t)=\mathbf{f}(t)$.

Then $\mathbf{F}(t)$ is called the *indefinite integral* of $\mathbf{f}(t)$ with respect to t and symbolically we write $\int\mathbf{f}(t)\,dt=\mathbf{F}(t)$. ...(1)

The function $\mathbf{f}(t)$ to be integrated is called the *integrand.*

If $\mathbf{c}$ is any ***arbitrary constant vector*** independent of t, then

$$\frac{d}{dt}\left\{\mathbf{F}(t)+\mathbf{c}\right\}=\mathbf{f}(t).$$

This is equivalent to $\int\mathbf{f}(t)\,dt=\mathbf{F}(t)+\mathbf{c}$. ...(2)

From (2) it is obvious that the integral $\mathbf{F}(t)$ of $\mathbf{f}(t)$ is indefinite to the extent of an additive arbitrary constant $\mathbf{c}$. Therefore $\mathbf{F}(t)$ is called the indefinite integral of $\mathbf{f}(t)$. The constant vector $\mathbf{c}$ is called the *constant of integration.* It can be determined if we are given some initial conditions.

If $\dfrac{d}{dt}\mathbf{F}(t)=\mathbf{f}(t)$ for all t in the interval $[a, b]$, then the ***definite integral*** between the limits $t=a$ and $t=b$ can in such case be written

$$\int_a^b\mathbf{f}(t)\,dt=\int_a^b\left\{\frac{d}{dt}\mathbf{F}(t)\right\}dt$$

$$=\Big[\mathbf{F}(t)+\mathbf{c}\Big]_a^b=\mathbf{F}(b)-\mathbf{F}(a).$$

Theorem. *If* $\mathbf{f}(t)=f_1(t)\,\mathbf{i}+f_2(t)\,\mathbf{j}+f_3(t)\,\mathbf{k}$, *then*

$$\int \mathbf{f}(t)\,dt=\mathbf{i}\int f_1(t)\,dt+\mathbf{j}\int f_2(t)\,dt+\mathbf{k}\int f_3(t)\,dt.$$

Proof. Let $\frac{d}{dt}\,\mathbf{F}(t)=\mathbf{f}(t)$. ...(1)

Then $\int \mathbf{f}(t)\,dt=\mathbf{F}(t)$. ...(2)

Let $\mathbf{F}(t)=F_1(t)\,\mathbf{i}+F_2(t)\,\mathbf{j}+F_3(t)\,\mathbf{k}$.

Then from (1), we have

$$\frac{d}{dt}\,\{F_1(t)\,\mathbf{i}+F_2(t)\,\mathbf{j}+F_3(t)\,\mathbf{k}\}=\mathbf{f}(t)$$

or

$$\left\{\frac{d}{dt}\,F_1(t)\right\}\mathbf{i}+\left\{\frac{d}{dt}\,F_2(t)\right\}\mathbf{j}+\left\{\frac{d}{dt}\,F_3(t)\right\}\mathbf{k}=f_1(t)\,\mathbf{i}+f_2(t)\,\mathbf{j}+f_3(t)\,\mathbf{k}.$$

Equating the coefficients of **i, j, k**, we get

$$\frac{d}{dt}\,F_1(t)=f_1(t),\ \frac{d}{dt}\,F_2(t)=f_2(t),\ \frac{d}{dt}\,F_3(t)=f_3(t).$$

$$\therefore\quad F_1(t)=\int f_1(t)\,dt,\ F_2(t)=\int f_2(t)\,dt,\ F_3(t)=\int f_3(t)\,dt.$$

$$\therefore\quad \mathbf{F}(t)=\left\{\int f_1(t)\,dt\right\}\mathbf{i}+\left\{\int f_2(t)\,dt\right\}\mathbf{j}+\left\{\int f_3(t)\,dt\right\}\mathbf{k}.$$

So from (2), we get

$$\int \mathbf{f}(t)\,dt=\mathbf{i}\int f_1(t)\,dt+\mathbf{j}\int f_2(t)\,dt+\mathbf{k}\int f_3(t)\,dt.$$

Note. From this theorem we conclude that the definition of the integral of a vector function implies the definition of integrals of three scalar functions which are the components of that vector function. **Thus in order to integrate a vector function we should integrate its components.**

§ 13. Some Standard Results.

We have already obtained some standard results for differentiation. With the help of these results we can obtain some standard results for integration.

1. We have $\frac{d}{dt}\,(\mathbf{r}\cdot\mathbf{s})=\frac{d\mathbf{r}}{dt}\cdot\mathbf{s}+\mathbf{r}\cdot\frac{d\mathbf{s}}{dt}$.

Therefore $\int\left(\frac{d\mathbf{r}}{dt}\cdot\mathbf{s}+\mathbf{r}\cdot\frac{d\mathbf{s}}{dt}\right)dt=\mathbf{r}\cdot\mathbf{s}+c$,

where c is the constant of integration. It should be noted that c is here a scalar quantity since the integrand is also scalar.

2. We have $\frac{d}{dt}\,(\mathbf{r}^2)=2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}$.

Therefore $\int\left(2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right)dt=\mathbf{r}^2+c$.

Here the constant of integration c is a scalar quantity.

3. We have $\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)^2=2\,\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}$.

Therefore we have

$$\int\left(2\,\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}\right)dt=\left(\frac{d\mathbf{r}}{dt}\right)^2+c.$$

Here the constant of integration c is a scalar quantity.

Also $$\left(\frac{d\mathbf{r}}{dt}\right)^2=\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}.$$

4. We have $\frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)=\frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}+\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}=\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}$.

$$\therefore\quad \int\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\mathbf{r}\times\frac{d\mathbf{r}}{dt}+\mathbf{c}.$$

Here the constant of integration $\mathbf{c}$ is a vector quantity since the integrand $\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}$ is also a vector quantity.

5. If $\mathbf{a}$ is a constant vector, we have

$$\frac{d}{dt}(\mathbf{a}\times\mathbf{r})=\frac{d\mathbf{a}}{dt}\times\mathbf{r}+\mathbf{a}\times\frac{d\mathbf{r}}{dt}=\mathbf{a}\times\frac{d\mathbf{r}}{dt}.$$

Therefore $$\int\left(\mathbf{a}\times\frac{d\mathbf{r}}{dt}\right)dt=\mathbf{a}\times\mathbf{r}+\mathbf{c}.$$

Hence the constant of integration $\mathbf{c}$ is a vector quantity.

6. If $r=|\mathbf{r}|$ and $\hat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$ then

$$\frac{d}{dt}(\hat{\mathbf{r}})=\frac{d}{dt}\left(\frac{1}{r}\,\mathbf{r}\right)=\frac{1}{r}\frac{d\mathbf{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\,\mathbf{r}.$$

Therefore $$\int\left(\frac{1}{r}\frac{d\mathbf{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\,\mathbf{r}\right)dt=\hat{\mathbf{r}}+\mathbf{c}.$$

7. If c is a constant scalar and $\mathbf{r}$ a vector function of a scalar t, then obviously $\int c\mathbf{r}\,dt=c\int\mathbf{r}\,dt$.

8. If $\mathbf{r}$ and $\mathbf{s}$ are two vector functions of the scalar t, then obviously $\int(\mathbf{r}+\mathbf{s})\,dt=\int\mathbf{r}\,dt+\int\mathbf{s}\,dt$.

SOLVED EXAMPLES

Ex. 1. *If* $\mathbf{f}(t)=(t-t^2)\,\mathbf{i}+2t^3\,\mathbf{j}-3\mathbf{k}$, *find*

(*i*) $\int\mathbf{f}(t)\,dt$ *and* (*ii*) $\int_1^2\mathbf{f}(t)\,dt$.

Solution. (i) $\int\mathbf{f}(t)\,dt=\int\{(t-t^2)\,\mathbf{i}+2t^3\,\mathbf{j}-3\mathbf{k}\}\,dt$

$$=\mathbf{i}\int(t-t^2)\,dt+\mathbf{j}\int 2t^3\,dt+\mathbf{k}\int -3dt$$

$$=\mathbf{i}\left(\frac{t^2}{2}-\frac{t^3}{3}\right)+\mathbf{j}\left(2\frac{t^4}{4}\right)+\mathbf{k}\,(-3t)+\mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector

$$=\left(\frac{t^2}{2}-\frac{t^3}{3}\right)\mathbf{i}+\frac{t^4}{2}\mathbf{j}-3t\mathbf{k}+\mathbf{c}.$$

(ii) $$\int_1^2 \mathbf{f}(t)\,dt=\int_1^2 \{(t-t^2)\,\mathbf{i}+2t^3\,\mathbf{j}-3\mathbf{k}\}\,dt$$

$$=\mathbf{i}\int_1^2(t-t^2)\,dt+\mathbf{j}\int_1^2 2t^3\,dt-\mathbf{k}\int_1^2 3dt$$

$$=\mathbf{i}\left[\frac{t^2}{2}-\frac{t^3}{3}\right]_1^2+\mathbf{j}\left[2\frac{t^4}{4}\right]_1^2-3\mathbf{k}\left[t\right]_1^2=-\frac{5}{6}\mathbf{i}+\frac{15}{2}\mathbf{j}-3\mathbf{k}.$$

Ex. 2. *Find the value of* $\mathbf{r}$ *satisfying the equation* $\dfrac{d^2\mathbf{r}}{dt^2}=\mathbf{a}$, *where* $\mathbf{a}$ *is a constant vector. Also it is given that when* $t=0$, $\mathbf{r}=0$ *and* $\dfrac{d\mathbf{r}}{dt}=\mathbf{u}$. **[Agra 1978]**

Solution. Integrating the equation $\dfrac{d^2\mathbf{r}}{dt^2}=\mathbf{a}$, we get

$\dfrac{d\mathbf{r}}{dt}=t\mathbf{a}+\mathbf{b}$, where $\mathbf{b}$ is an arbitrary constant vector.

But it is given that when $t=0$, $\dfrac{d\mathbf{r}}{dt}=\mathbf{u}$.

$\therefore\quad \mathbf{u}=0\,\mathbf{a}+\mathbf{b}\quad$ or $\quad\mathbf{b}=\mathbf{u}$.

$\therefore\quad \dfrac{d\mathbf{r}}{dt}=t\mathbf{a}+\mathbf{u}$.

Integrating again with respect to t, we get

$\mathbf{r}=\frac{1}{2}t^2\,\mathbf{a}+t\mathbf{u}+\mathbf{c}$, where $\mathbf{c}$ is constant.

But when $t=0$, $\mathbf{r}=0$.

$\therefore\quad 0=0+0+\mathbf{c}\quad$ or $\quad\mathbf{c}=0$.

$\therefore\quad \mathbf{r}=\frac{1}{2}t^2\,\mathbf{a}+t\mathbf{u}$.

Ex. 3. *Find the value of* $\mathbf{r}$ *satisfying the equation* $\dfrac{d^2\mathbf{r}}{dt^2}=t\mathbf{a}+\mathbf{b}$, *where* $\mathbf{a}$ *and* $\mathbf{b}$ *are constant vectors.* **[Agra 1979]**

Solution. Integrating the equation $\dfrac{d^2\mathbf{r}}{dt^2}=t\mathbf{a}+\mathbf{b}$, we get

$\dfrac{d\mathbf{r}}{dt}=\frac{1}{2}t^2\,\mathbf{a}+t\mathbf{b}+\mathbf{c}$, where $\mathbf{c}$ is constant.

Again integrating, we get

$$\mathbf{r}=\tfrac{1}{6}t^3\,\mathbf{a}+\tfrac{1}{2}t^2\,\mathbf{b}+t\,\mathbf{c}+\mathbf{d},\ \text{where } \mathbf{d} \text{ is constant.}$$

Ex. 4. *Integrate* $\dfrac{d^2\mathbf{r}}{dt^2}=-n^2\mathbf{r}$

Solution. We have $\dfrac{d^2\mathbf{r}}{dt^2}=-n^2\mathbf{r}$. ...(1)

Forming the scalar product of each side of (1) with the vector $2\dfrac{d\mathbf{r}}{dt}$, we get $2\dfrac{d\mathbf{r}}{dt}\cdot\dfrac{d^2\mathbf{r}}{dt^2}=-2n^2\mathbf{r}\cdot\dfrac{d\mathbf{r}}{dt}$.

Now integrating we get

$$\left(\frac{d\mathbf{r}}{dt}\right)^2=-n^2\mathbf{r}^2+c,\ \text{where } c \text{ is constant.}$$

Ex. 5. *Integrate* $\mathbf{a}\times\dfrac{d^2\mathbf{r}}{dt^2}=\mathbf{b}$, *where* **a** *and* **b** *are constant vectors.*

Solution. We have $\dfrac{d}{dt}\left\{\mathbf{a}\times\dfrac{d\mathbf{r}}{dt}\right\}=\mathbf{a}\times\dfrac{d^2\mathbf{r}}{dt^2}$.

Therefore integrating $\mathbf{a}\times\dfrac{d^2\mathbf{r}}{dt^2}=\mathbf{b}$, we get

$$\mathbf{a}\times\frac{d\mathbf{r}}{dt}=t\mathbf{b}+\mathbf{c},\ \text{where } \mathbf{c} \text{ is constant.}$$

Again integrating, we get

$$\mathbf{a}\times\mathbf{r}=\tfrac{1}{2}t^2\mathbf{b}+t\mathbf{c}+\mathbf{d},\ \text{where } \mathbf{d} \text{ is constant.}$$

Ex. 6. *If* $\mathbf{r}(t)=5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\,\mathbf{k}$, *prove that*

$$\int_1^2\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=-14\mathbf{i}+75\mathbf{j}-15\mathbf{k}.$$

[Kanpur 1976, 78; Agra 80]

Solution. We have $\displaystyle\int\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\mathbf{r}\times\frac{d\mathbf{r}}{dt}+\mathbf{c}$.

$$\therefore\quad \int_1^2\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\left[\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right]_1^2.$$

Let us now find $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}$. We have $\dfrac{d\mathbf{r}}{dt}=10t\,\mathbf{i}+\mathbf{j}-3t^2\,\mathbf{k}$.

$$\therefore\quad \mathbf{r}\times\frac{d\mathbf{r}}{dt}=(5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\,\mathbf{k})\times(10t\,\mathbf{i}+\mathbf{j}-3t^2\,\mathbf{k})$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5t^2 & t & -t^3\\ 10t & 1 & -3t^2\end{vmatrix}=-2t^3\,\mathbf{i}+5t^4\,\mathbf{j}-5t^2\,\mathbf{k}.$$

$$\therefore\quad \int_1^2\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\Big[-2t^3\,\mathbf{i}+5t^4\,\mathbf{j}-5t^2\,\mathbf{k}\Big]_1^2$$

$$=\Big[-2t^3\Big]_1^2\,\mathbf{i}+\Big[5t^4\Big]_1^2\,\mathbf{j}-\Big[5t^2\Big]_1^2\,\mathbf{k}=-14\mathbf{i}+75\mathbf{j}-15\mathbf{k}.$$

Ex. 7. *Given that*

$$\mathbf{r}(t)=2\mathbf{i}-\mathbf{j}+2\mathbf{k}, \text{ when } t=2$$
$$=4\mathbf{i}-2\mathbf{j}+3\mathbf{k}, \text{ when } t=3,$$

show that $\int_2^3 \left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right) dt=10.$

[Kanpur 1980; Rohilkhand 80; Agra 76]

Solution. We have $\int\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right) dt=\frac{1}{2}\mathbf{r}^2+c.$

$\therefore \quad \int_2^3 \left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right) dt=\left[\frac{1}{2}\mathbf{r}^2\right]_2^3.$

When $t=3$, $\mathbf{r}=4\mathbf{i}-2\mathbf{j}+3\mathbf{k}$.

$\therefore$ when $t=3$, $\mathbf{r}^2=(4\mathbf{i}-2\mathbf{j}+3\mathbf{k})\cdot(4\mathbf{i}-2\mathbf{j}+3\mathbf{k})=16+4+9=29.$

When $t=2$, $\mathbf{r}=2\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

$\therefore$ When $t=2$, $\mathbf{r}^2=4+1+4=9.$

$\therefore \quad \int_2^3 \left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right) dt=\frac{1}{2}[29-9]=10.$

Ex. 8. *The acceleration of a particle at any time $t \geqslant 0$ is given by*

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=12\cos 2t\,\mathbf{i}-8\sin 2t\,\mathbf{j}+16t\,\mathbf{k}.$$

If the velocity $\mathbf{v}$ *and displacement* $\mathbf{r}$ *are zero at $t=0$, find* $\mathbf{v}$ *and* $\mathbf{r}$ *at any time.* **[Kerala 1974]**

Solution. We have $\frac{d\mathbf{v}}{dt}=12\cos 2t\,\mathbf{i}-8\sin 2t\,\mathbf{j}+16t\,\mathbf{k}.$

Integrating, we get

$\mathbf{v}=\mathbf{i}\int 12\cos 2t\,dt+\mathbf{j}\int -8\sin 2t\,dt+\mathbf{k}\int 16t\,dt$

or $\quad \mathbf{v}=6\sin 2t\,\mathbf{i}+4\cos 2t\,\mathbf{j}+8t^2\,\mathbf{k}+\mathbf{c}.$

When $t=0$, $\mathbf{v}=\mathbf{0}$.

$\therefore \quad \mathbf{0}=0\mathbf{i}+4\mathbf{j}+0\mathbf{k}+\mathbf{c}$

or $\quad \mathbf{c}=-4\mathbf{j}.$

$\therefore \quad \mathbf{v}=\frac{d\mathbf{r}}{dt}=6\sin 2t\,\mathbf{i}+(4\cos 2t-4)\,\mathbf{j}+8t^2\mathbf{k}.$

Integrating, we get

$\mathbf{r}=\mathbf{i}\int 6\sin 2t\,dt+\mathbf{j}\int(4\cos 2t-4)\,dt+\mathbf{k}\int 8t^2\,dt$

$=-3\cos 2t\,\mathbf{i}+(2\sin 2t-4t)\,\mathbf{j}+\frac{8}{3}t^3\,\mathbf{k}+\mathbf{d}$, where $\mathbf{d}$ is constant.

When $t=0$, $\mathbf{r}=\mathbf{0}$.

$\therefore \quad \mathbf{0}=-3\mathbf{i}+0\mathbf{j}+0\mathbf{k}+\mathbf{d}. \qquad \therefore \quad \mathbf{d}=3\mathbf{i}.$

$\therefore \quad \mathbf{r}=-3 \cos 2t\, \mathbf{i}+(2 \sin 2t-4t)\, \mathbf{j}+\frac{8}{3}t^3\, \mathbf{k}+3\mathbf{i}$
$=(3-3 \cos 2t)\, \mathbf{i}+(2 \sin 2t-4t)\, \mathbf{j}+\frac{8}{3}t^3\mathbf{k}.$

Exercises

1. Evaluate $\int_0^1 (e^t\, \mathbf{i}+e^{-2t}\, \mathbf{j}+t\mathbf{k})\, dt.$

Ans. $(e-1)\, \mathbf{i}-\frac{1}{2}(e^{-2}-1)\, \mathbf{j}+\frac{1}{2}\mathbf{k}.$

2. If $\mathbf{f}(t)=t\, \mathbf{i}+(t^2-2t)\, \mathbf{j}+(3t^2+3t^3)\, \mathbf{k}$, find
$\int_0^1 \mathbf{f}(t)\, dt.$ [Agra 1977]

Ans. $\frac{1}{2}\mathbf{i}-\frac{2}{3}\mathbf{j}+\frac{7}{4}\mathbf{k}.$

3. If $\mathbf{r}=t\mathbf{i}-t^2\mathbf{j}+(t-1)\, \mathbf{k}$ and $\mathbf{s}=2t^2\, \mathbf{i}+6t\mathbf{k}$, evaluate
(i) $\int_0^2 \mathbf{r}\cdot\mathbf{s}\, dt$, (ii) $\int_0^2 \mathbf{r}\times\mathbf{s}\, dt.$

Ans. (i) 12, (ii) $-24\mathbf{i}-\frac{40}{3}\mathbf{j}+\frac{64}{5}\mathbf{k}.$

4. Solve the equation $\frac{d^2\mathbf{r}}{dt^2}=\mathbf{a}$ where $\mathbf{a}$ is a constant vector; given that $\mathbf{r}=0$ and $\frac{d\mathbf{r}}{dt}=0$ when $t=0$. **Ans.** $\mathbf{r}=\frac{1}{2}t^2\mathbf{a}.$

5. Find the value of $\mathbf{r}$ satisfying the equation
$$\frac{d^2\mathbf{r}}{dt^2}=6t\mathbf{i}-24t^2\mathbf{j}+4 \sin t\mathbf{k},$$
given that $\mathbf{r}=2\mathbf{i}+\mathbf{j}$ and $\frac{d\mathbf{r}}{dt}=-\mathbf{i}-3\mathbf{k}$ at $t=0$.

Ans. $\mathbf{r}=(t^3-t-2)\, \mathbf{i}+(1-2t^4)\, \mathbf{j}+(t-4 \sin t)\, \mathbf{k}.$

6. The acceleration of a particale at any time t is $e^t\, \mathbf{i}+e^{2t}\, \mathbf{j}+\mathbf{k}$. Find $\mathbf{v}$, given that $\mathbf{v}=\mathbf{i}+\mathbf{j}$ at $t=0$. [Agra 1973]

2

Gradient, Divergence and Curl

§ 1. **Partial Derivatives of Vectors.** Suppose **r** is a vector depending on more than one scalar variable. Let $\mathbf{r}=\mathbf{f}(x, y, z)$ *i.e.* let **r** be a function of three scalar variables x, y and z. The partial derivative of **r** with respect to x is defined as

$$\frac{\partial \mathbf{r}}{\partial x}=\lim_{\delta x \to 0} \frac{\mathbf{f}(x+\delta x, y, z)-\mathbf{f}(x, y, z)}{\delta x}$$

if this limit exists. Thus $\partial \mathbf{r}/\partial x$ is nothing but the ordinary derivative of **r** with respect to x provided the other variables y and z are regarded as constants. Similarly we may define the partial derivatives $\frac{\partial \mathbf{r}}{\partial y}$ and $\frac{\partial \mathbf{r}}{\partial z}$.

Higher partial derivatives can also be defined as in Scalar Calculus. Thus, for example,

$$\frac{\partial^2 \mathbf{r}}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{r}}{\partial x}\right), \frac{\partial^2 \mathbf{r}}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial \mathbf{r}}{\partial y}\right), \frac{\partial^2 \mathbf{r}}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial \mathbf{r}}{\partial z}\right),$$

$$\frac{\partial^2 \mathbf{r}}{\partial x\, \partial y}=\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{r}}{\partial y}\right), \frac{\partial^2 \mathbf{r}}{\partial y\, \partial x}=\frac{\partial}{\partial y}\left(\frac{\partial \mathbf{r}}{\partial x}\right).$$

If **r** has continuous partial derivatives of the second order at least, then, $\frac{\partial^2 \mathbf{r}}{\partial x\, \partial y}=\frac{\partial^2 \mathbf{r}}{\partial y\, \partial x}$ *i.e.* the order of differentiation is immaterial. If $\mathbf{r}=\mathbf{f}(x, y, z)$, the total differential $d\mathbf{r}$ of **r** is given by

$$d\mathbf{r}=\frac{\partial \mathbf{r}}{\partial x}\, dx+\frac{\partial \mathbf{r}}{\partial y}\, dy+\frac{\partial \mathbf{r}}{\partial z}\, dz.$$

§ 2. **The Vector Differential Operator Del. (∇).** The vector differential operator ∇ (read as *del* or *nabla*) is defined as

$$\nabla\equiv\frac{\partial}{\partial x}\,\mathbf{i}+\frac{\partial}{\partial y}\,\mathbf{j}+\frac{\partial}{\partial z}\,\mathbf{k}\equiv\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}$$

and operates distributively.

The vector operator ∇ can generally be treated to behave as an ordinary vector. It possesses properties like ordinary vectors. The symbols $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ can be treated as its components along **i, j, k.**

§ 3. **Gradient of a scalar Field. Definition.** *Let $f(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space (i.e., defines a differentiable scalar field). Then the gradient of f, written as ∇f or grad f, is defined as*

$$\nabla f = \left(\frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} \right) f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}.$$

[Kerala 1975; Allahabad 79]

It should be noted that ∇f is a vector whose three successive components are $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$. Thus the gradient of a scalar field defines a vector field. *If f is a scalar point function, then ∇f is a vector point function.*

§ 4. **Formulas involving gradient.**

Theorem 1. Gradient of the sum of two scalar point functions. *If f and g are two scalar point functions, then*

$$\text{grad } (f+g) = \text{grad } f + \text{grad } g$$

or

$$\nabla (f+g) = \nabla f + \nabla g.$$

Proof. We have $\nabla (f+g) = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) (f+g)$

$$= \mathbf{i} \frac{\partial}{\partial x} (f+g) + \mathbf{j} \frac{\partial}{\partial y} (f+g) + \mathbf{k} \frac{\partial}{\partial z} (f+g)$$

$$= \mathbf{i} \frac{\partial f}{\partial x} + \mathbf{i} \frac{\partial g}{\partial x} + \mathbf{j} \frac{\partial f}{\partial y} + \mathbf{j} \frac{\partial g}{\partial y} + \mathbf{k} \frac{\partial f}{\partial z} + \mathbf{k} \frac{\partial g}{\partial z}$$

$$= \left(\mathbf{i} \frac{\partial f}{\partial x} + \mathbf{j} \frac{\partial f}{\partial y} + \mathbf{k} \frac{\partial f}{\partial z} \right) + \left(\mathbf{i} \frac{\partial g}{\partial x} + \mathbf{j} \frac{\partial g}{\partial y} + \mathbf{k} \frac{\partial g}{\partial z} \right)$$

$$= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) f + \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) g$$

$$= \nabla f + \nabla g = \text{grad } f + \text{grad } g.$$

Similarly, we can prove that $\nabla (f-g) = \nabla f - \nabla g$.

Theorem 2. Gradient of a constant. *The necessary and sufficient condition for a scalar point function to be constant is that*

$$\nabla f = \mathbf{0}.$$

$$=-12\mathbf{i}-9\mathbf{j}-16\mathbf{k}.$$

Ex. 3. *If* $r=|\mathbf{r}|$ *where* $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, *prove that*

(*i*) $\nabla f(r)=f'(r)\nabla r$, (*ii*) $\nabla r=\frac{1}{r}\mathbf{r}$, **[Rohilkhand 1981]**

(*iii*) $\nabla f(r)\times\mathbf{r}=\mathbf{0}$, (*iv*) $\nabla\left(\frac{1}{r}\right)=-\frac{\mathbf{r}}{r^3}$, **[Kanpur 1976]**

(*v*) $\nabla \log|\mathbf{r}|=\frac{\mathbf{r}}{r^2}$,

(*vi*) $\nabla r^n=nr^{n-2}\mathbf{r}$.

[Kanpur 1970; Rohilkhand 76; B.H.U. 70]

Solution. If $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then $r=|\mathbf{r}|=\sqrt{(x^2+y^2+z^2)}$.

$$\therefore \quad r^2=x^2+y^2+z^2.$$

(i) $\nabla f(r)=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)f(r)$

$=\mathbf{i}\frac{\partial}{\partial x}f(r)+\mathbf{j}\frac{\partial}{\partial y}f(r)+\mathbf{k}\frac{\partial}{\partial z}f(r)$

$=\mathbf{i}f'(r)\frac{\partial r}{\partial x}+\mathbf{j}f'(r)\frac{\partial r}{\partial y}+\mathbf{k}f'(r)\frac{\partial r}{\partial z}$

$=f'(r)\left(\mathbf{i}\frac{\partial r}{\partial x}+\mathbf{j}\frac{\partial r}{\partial y}+\mathbf{k}\frac{\partial r}{\partial z}\right)=f'(r)\nabla r.$

(ii) We have $\nabla r=\mathbf{i}\frac{\partial r}{\partial x}+\mathbf{j}\frac{\partial r}{\partial y}+\mathbf{k}\frac{\partial r}{\partial z}$.

Now $r^2=x^2+y^2+z^2$; $\therefore$ $2r\frac{\partial r}{\partial x}=2x$ *i.e.* $\frac{\partial r}{\partial x}=\frac{x}{r}$.

Similarly $\frac{\partial r}{\partial y}=\frac{y}{r}$ and $\frac{\partial r}{\partial z}=\frac{z}{r}$.

$\therefore$ $\nabla r=\frac{x}{r}\mathbf{i}+\frac{y}{r}\mathbf{j}+\frac{z}{r}\mathbf{k}=\frac{1}{r}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=\frac{1}{r}\mathbf{r}=\hat{\mathbf{r}}.$

(iii) We have as in part (i), $\nabla f(r)=f'(r)\nabla r$.

But as in part (ii) $\nabla r=\frac{1}{r}\mathbf{r}$.

$$\therefore \quad \nabla f(r)=f'(r)\frac{1}{r}\mathbf{r}.$$

$\therefore$ $\nabla f(r)\times\mathbf{r}=\left\{f'(r)\frac{1}{r}\mathbf{r}\right\}\times\mathbf{r}=\left\{\frac{1}{r}f'(r)\right\}(\mathbf{r}\times\mathbf{r})$

$=\mathbf{0}$, since $\mathbf{r}\times\mathbf{r}=\mathbf{0}$.

(iv) We have $\nabla\left(\frac{1}{r}\right)=\mathbf{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\mathbf{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+\mathbf{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

be a neighbouring point on this surface. Then the position vector of $Q=\mathbf{r}+\delta\mathbf{r}=(x+\delta x)\ \mathbf{i}+(y+\delta y)\ \mathbf{j}+(z+\delta z)\ \mathbf{k}$.

$\therefore \quad \overrightarrow{PQ}=(\mathbf{r}+\delta\mathbf{r})-\mathbf{r}=\delta\mathbf{r}=\delta x\ \mathbf{i}+\delta y\ \mathbf{j}+\delta z\ \mathbf{k}$.

As $Q\to P$, the line PQ tends to tangent at P to the level surface. Therefore $d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k}$ lies in the tangent plane to the surface at P.

From the differential calculus, we have

$$df=\frac{\partial f}{\partial x}\,dx+\frac{\partial f}{\partial y}\,dy+\frac{\partial f}{\partial z}\,dz$$

$$=\left(\mathbf{i}\,\frac{\partial f}{\partial x}+\mathbf{j}\,\frac{\partial f}{\partial y}+\mathbf{k}\,\frac{\partial f}{\partial z}\right)\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})=\nabla f\cdot d\mathbf{r}.$$

Since $f(x, y, z)=\text{constant}$, therefore $df=0$.

$\therefore \quad \nabla f\cdot d\mathbf{r}=0$ so that ∇f is a vector perpendicular to $d\mathbf{r}$ and therefore to the tangent plane at P to the surface

$$f(x, y, z)=c.$$

Hence ∇f is a vector normal to the surface $f(x, y, z)=c$.

Thus if $f(x, y, z)$ is a scalar field defined over a region R, then ∇f at any point (x, y, z) is a vector in the direction of normal at that point to the level surface $f(x, y, z)=c$ passing through that point

§ 6. Directional Derivative of a scalar point function.

[Agra 1972; Kolhapur 73; Bombay 70]

Definition. *Let $f(x, y, z)$ define a scalar field in a region R and let P be any point in this region. Suppose Q is a point in this region in the neighbourhood of P in the direction of a given unit vector $\hat{\mathbf{a}}$.*

Then $\lim\limits_{Q\to P}\dfrac{f(Q)-f(P)}{PQ}$, *if it exists, is called the directional derivative of f at P in the direction of $\hat{\mathbf{a}}$.*

Interpretation of directional derivative. Let P be the point (x, y, z) and let Q be the point $(x+\delta x, y+\delta y, z+\delta z)$. Suppose $PQ=\delta s$. Then δs is a small element at P in the direction of $\hat{\mathbf{a}}$. If $\delta f=f(x+\delta x, y+\delta y, z+\delta z)-f(x, y, z)=f(Q)-f(P)$, then $\dfrac{\delta f}{\delta s}$ represents the average rate of change of f per unit distance in the direction of $\hat{\mathbf{a}}$. Now the directional derivative of f at P in the

Normal at P. Let $\mathbf{R}=X\,\mathbf{i}+Y\,\mathbf{j}+Z\mathbf{k}$ be the position vector of any current point $Q\,(X, Y, Z)$ on the normal at P to the surface. The vector $\overrightarrow{PQ}=\mathbf{R}-\mathbf{r}=(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k}$ lies along the normal at P to the surface. Therefore it is parallel to the vector ∇f.

$$\therefore \quad (\mathbf{R}-\mathbf{r})\times \nabla f=\mathbf{0} \qquad ...(2)$$

is the vector equation of the normal at P to the given surface.

Cartesian form. The vectors

$$(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k} \text{ and } \nabla f=\frac{\partial f}{\partial x}\,\mathbf{i}+\frac{\partial f}{\partial y}\,\mathbf{j}+\frac{\partial f}{\partial z}\,\mathbf{k}$$

will be parallel if

$$(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k}=p\left(\frac{\partial f}{\partial x}\,\mathbf{i}+\frac{\partial f}{\partial y}\,\mathbf{j}+\frac{\partial f}{\partial z}\,\mathbf{k}\right),$$

where p is some scalar.

Equating the coefficients of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, we get

$$X-x=p\,\frac{\partial f}{\partial x},\ Y-y=p\,\frac{\partial f}{\partial y},\ Z-z=p\,\frac{\partial f}{\partial z}$$

or
$$\frac{X-x}{\frac{\partial f}{\partial x}}=\frac{Y-y}{\frac{\partial f}{\partial y}}=\frac{Z-z}{\frac{\partial f}{\partial z}}$$

are the equations of the normal at P.

SOLVED EXAMPLES

Ex. 1. *Find a unit normal vector to the level surface $x^2y+2xz=4$ at the point $(2, -2, 3)$.*

Solution. The equation of the level surface is

$$f\,(x, y, z)\equiv x^2y+2xz=4.$$

The vector grad f is along the normal to the surface at the point (x, y, z).

We have grad $f=\nabla\,(x^2y+2xz)=(2xy+2z)\,\mathbf{i}+x^2\,\mathbf{j}+2x\,\mathbf{k}$.

$\therefore$ at the point $(2, -2, 3)$, grad $f=-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$.

$\therefore$ $-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$ is a vector along the normal to the given surface at the point $(2, -2, 3)$.

Hence a unit normal vector to the surface at this point

$$=\frac{-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}}{|-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}|}=\frac{-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}}{\sqrt{(4+16+16)}}=-\tfrac{1}{3}\mathbf{i}+\tfrac{2}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k}.$$

The vector $-\left(-\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}\right)$ *i.e.*, $\frac{1}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$ is also a unit normal vector to the given surface at the point $(2, -2, 3)$.

Ex. 2. *Find the directional derivatives of a scalar point function f in the direction of coordinate axes.*

Solution. The grad f at any point (x, y, z) is the vector

$$\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}.$$

The directional derivative of f in the direction of **i**

$$=\text{grad } f\cdot\mathbf{i}=\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)\cdot\mathbf{i}=\frac{\partial f}{\partial x}.$$

Similarly the directional derivatives of f in the directions of **j** and **k** are $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$.

Ex. 3. *Find the directional derivative of* $f(x, y, z)=x^2yz+4xz^2$ *at the point* $(1, -2, -1)$ *in the direction of the vector* $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$.

[Allahabad 1978]

Solution. We have $f(x, y, z)=x^2yz+4xz^2$.

$\therefore$ grad $f=(2xyz+4z^2)\,\mathbf{i}+x^2z\,\mathbf{j}+(x^2y+8xz)\,\mathbf{k}$

$=8\mathbf{i}-\mathbf{j}-10\mathbf{k}$ at the point $(1, -2, -1)$.

If $\hat{\mathbf{a}}$ be the unit vector in the direction of the vector $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$,

then $$\hat{\mathbf{a}}=\frac{2\mathbf{i}-\mathbf{j}-2\mathbf{k}}{\sqrt{(4+1+4)}}=\frac{2}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}.$$

Therefore the required directional derivative is

$$\frac{df}{ds}=\text{grad } f\cdot\hat{\mathbf{a}}=(8\mathbf{i}-\mathbf{j}-10\mathbf{k})\cdot(\tfrac{2}{3}\mathbf{i}-\tfrac{1}{3}\mathbf{j}-\tfrac{2}{3}\mathbf{k})=\frac{16}{3}+\frac{1}{3}+\frac{20}{3}=\frac{37}{3}.$$

Since this is positive, f is increasing in this direction.

Ex. 4. *Find the directional derivative of*

$$f(x, y, z)=x^2-2y^2+4z^2$$

at the point $(1, 1, -1)$ *in the direction of* $2\mathbf{i}+\mathbf{j}-\mathbf{k}$ **[Agra 1979]**

Ans. $8/\sqrt{6}$.

Ex. 5. *Find the directional derivative of the function* $f=x^2-y^2+2z^2$ *at the point* $P(1, 2, 3)$ *in the direction of the line* PQ *where* Q *is the point* $(5, 0, 4)$. **[Agra 1980]**

Solution. Here grad $f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}$

$=2x\,\mathbf{i}-2y\,\mathbf{j}+4z\,\mathbf{k}=2\mathbf{i}-4\mathbf{j}+12\mathbf{k}$ at the point $(1, 2, 3)$.

Also $\overrightarrow{PQ}$= position vector of Q − position vector of P

$=(5\mathbf{i}+0\mathbf{j}+4\mathbf{k})-(\mathbf{i}+2\mathbf{j}+3\mathbf{k})=4\mathbf{i}-2\mathbf{j}+\mathbf{k}$.

If $\hat{\mathbf{a}}$ be the unit vector in the direction of the vector $\overrightarrow{PQ}$,

then $$\hat{\mathbf{a}}=\frac{4\mathbf{i}-2\mathbf{j}+\mathbf{k}}{\sqrt{(16+4+1)}}=\frac{4\mathbf{i}-2\mathbf{j}+\mathbf{k}}{\sqrt{(21)}}.$$

$\therefore$ the required directional derivative

$$=(\text{grad } f)\cdot\hat{\mathbf{a}}=(2\mathbf{i}-4\mathbf{j}+12\mathbf{k})\cdot\left\{\frac{4\mathbf{i}-2\mathbf{j}+\mathbf{k}}{\sqrt{(21)}}\right\}$$

$$=\frac{28}{\sqrt{(21)}}=\frac{28}{21}\sqrt{(21)}=\frac{4}{3}\sqrt{(21)}.$$

Ex. 6. *In what direction from the point* (1, 1, −1) *is the directional derivative of* $f=x^2-2y^2+4z^2$ *a maximum ? Also find the value of this maximum directional derivative.*

Solution. We have grad $f=2x\mathbf{i}-4y\mathbf{j}+8z\mathbf{k}$

$=2\mathbf{i}-4\mathbf{j}-8\mathbf{k}$ at the point (1, 1, −1).

The directional derivative of f is a maximum in the direction of grad $f=2\mathbf{i}-4\mathbf{j}-8\mathbf{k}$.

The maximum value of this directional derivative

$=|\text{grad } f|=|2\mathbf{i}-4\mathbf{j}-8\mathbf{k}|=\sqrt{(4+16+64)}=\sqrt{(84)}=2\sqrt{(21)}.$

Ex. 7. *For the function* $f=y/(x^2+y^2)$, *find the value of the directional derivative making an angle* 30° *with the positive x-axis at the point* (0, 1).

Solution. We have grad $f=\dfrac{\partial f}{\partial x}\mathbf{i}+\dfrac{\partial f}{\partial y}\mathbf{j}$

$$=\frac{-2xy}{(x^2+y^2)^2}\mathbf{i}+\frac{x^2-y^2}{(x^2+y^2)^2}\mathbf{j}=-\mathbf{j} \text{ at the point } (0, 1).$$

If $\hat{\mathbf{a}}$ is a unit vector along the line which makes an angle 30° with the positive x-axis, then

$$\hat{\mathbf{a}}=\cos 30°\ \mathbf{i}+\sin 30°\ \mathbf{j}=\frac{\sqrt{3}}{2}\mathbf{i}+\tfrac{1}{2}\mathbf{j}.$$

$\therefore$ the required directional derivative is

$$=\text{grad } f\cdot\hat{\mathbf{a}}=(-\mathbf{j})\cdot\left(\frac{\sqrt{3}}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}\right)=-\tfrac{1}{2}.$$

Ex. 8. *What is the greatest rate of increase of* $u=xyz^2$ *at the point* (1, 0, 3) ? **[Agra 1968]**

Solution. We have $\nabla u=yz^2\,\mathbf{i}+xz^2\,\mathbf{j}+2xyz\,\mathbf{k}$.

$\therefore$ at the point (1, 0, 3), we have

$\nabla u=0\,\mathbf{i}+9\,\mathbf{j}+0\,\mathbf{k}=9\,\mathbf{j}.$

The greatest rate of increase of u at the point (1, 0, 3)

$=$the maximum value of $\dfrac{du}{ds}$ at the point (1, 0, 3)

$=|\nabla u|$, at the point (1, 0, 3)

$=|9\mathbf{j}|=9.$

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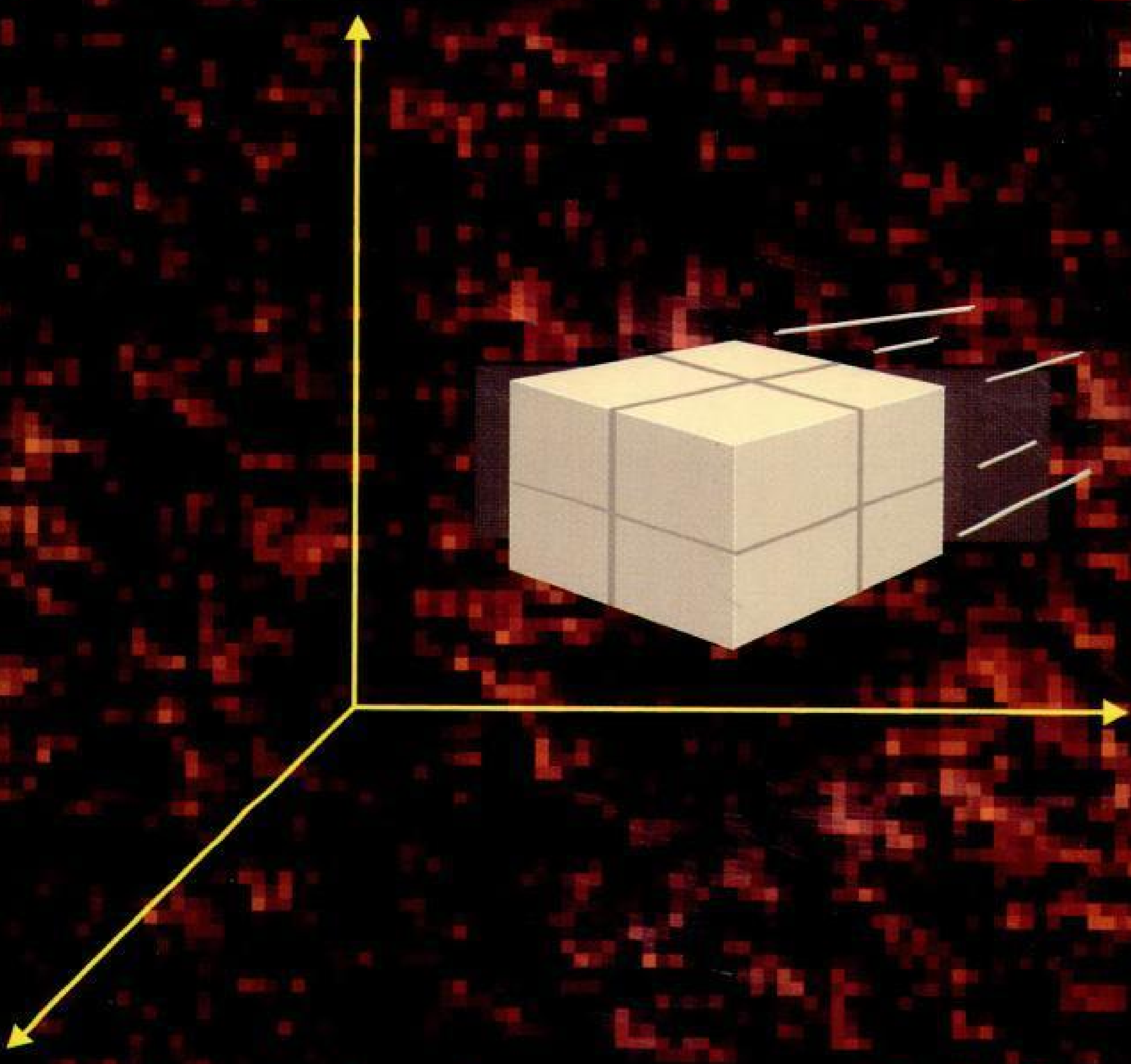

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