CHAPTER 28

Numerical Solution of Equations

28.1 INTRODUCTION

The limitations of analytical methods have led the engineers and scientists to evolve graphical and numerical methods. As seen in § 1.8, the graphical methods, though simple, give results to a low degree of accuracy. Numerical methods can, however, be derived which are more accurate. With the advent of high speed digital computers and increasing demand for numerical answers to various problems, numerical techniques have become indispensible tool in the hands of engineers.

Numerical methods are often, of a repetitive nature. These consist in repeated execution of the same process where at each step the result of the preceding step is used. This is known as *iteration process* and is repeated till the result is obtained to a desired degree of accuracy.

In this chapter, we shall discuss some numerical methods for the solution of algebraic and transcendental equations and simultaneous linear and non-linear equations. We shall close the chapter by describing an iterative method for the solution of eigen-value problem. For a detailed study of these topics, the reader should refer to author's book *'Numerical Methods in Engineering & Science'*.

28.2 SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

To find the roots of an equation $f(x) = 0$, we start with a known approximate solution and apply any of the following methods :

(1) Bisection method. This method consists in locating the root of the equation $f(x) = 0$ between a and b. If $f(x)$ is continuous between a and b, and $f(a)$ and $f(b)$ are of opposite signs then there is a root between a and b. For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then the first approximation to the root is $x_1 = \frac{1}{2}(a + b)$.

If $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$. Otherwise, the root lies between a and x_1 or x_1 and b according as $f(x_1)$ is positive or negative. Then we bisect the interval as before and continue the process until the root is found to desired accuracy.

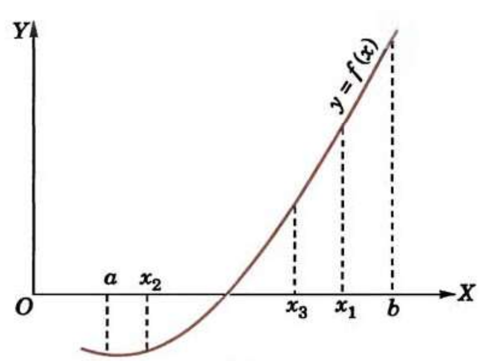

Fig. 28.1

In the Fig. 28.1, $f(x_1)$ is +ve, so that the root lies between a and x_1. Then the second approximation to the root is $x_2 = \frac{1}{2}(a + x_1)$. If $f(x_2)$ is – ve, the root lies between x_1 and x_2. Then the third approximation to the root is $x_3 = \frac{1}{2}(x_1 + x_2)$ and so on.

Example 28.1. *(a) Find a root of the equation $x^3 - 4x - 9 = 0$, using the bisection method correct to three decimal places.* (Mumbai, 2003)

(b) Using bisection method, find the negative root of the equation $x^2 - 4x + 9 = 0$. (J.N.T.U., 2009)

Solution. (*a*) Let $f(x) = x^3 - 4x - 9$

Since $f(2)$ is –ve and $f(3)$ is +ve, a root lies between 2 and 3

$\therefore$ first approximate to the root is

$$x_1 = \frac{1}{2}(2 + 3) = 2.5$$

Thus $\quad f(x_1) = (2.5)^3 - 4(2.5) - 9 = -3.375 \quad$ *i.e.*, – ve

$\therefore$ the root lies between x_1 and 3. Thus the second approximation to the root is

$$x_2 = \frac{1}{2}(x_1 + 3) = 2.75$$

Then $\quad f(x_2) = (2.75)^3 - 4(2.75) - 9 = 0.7969 \quad$ *i.e.*, + ve

$\therefore$ the root lies between x_1 and x_2. Thus the third approximation to the root is

$$x_3 = \frac{1}{2}(x_1 + x_2) = 2.625$$

Then $\quad f(x_3) = (2.625)^3 - 4(2.625) - 9 = -1.4121 \quad$ *i.e.*, – ve

$\therefore$ the root lies between x_2 and x_3. Thus the fourth approximation to the root is

$$x_4 = \frac{1}{2}(x_2 + x_3) = 2.6875$$

Repeating this process, the successive approximations are

$$x_5 = 2.71875, \quad x_6 = 2.70313, \quad x_7 = 2.71094$$
$$x_8 = 2.70703, \quad x_9 = 2.70508, \quad x_{10} = 2.70605$$
$$x_{11} = 2.70654, \quad x_{12} = 2.70642$$

Hence the root is 2.7064

(*b*) If α, β, γ are the roots of the given equation, then $-\alpha, -\beta, -\gamma$ are the roots of $(-x)^3 - 4(-x) + 9 = 0$

$\therefore$ the negative root of the given equation is the positive root of $x^3 - 4x - 9 = 0$ which we have found above to be 2.7064.

Hence the negative root for the given equation is – 2.7064.

Example 28.2. *By using the bisection method, find an approximate root of the equation $\sin x = 1/x$, that lies between $x = 1$ and $x = 1.5$ (measured in radians). Carry out computations upto the 7th stage.* (V.T.U., 2003 S)

Solution. Let $f(x) = x \sin x - 1$. We know that $1^r = 57.3°$.

Since $\quad f(1) = 1 \times \sin(1) - 1 = \sin(57.3°) - 1 = -0.15849$

and $\quad f(1.5) = 1.5 \times \sin(1.5)^r - 1 = 1.5 \times \sin(85.95)° - 1 = 0.49625$;

a root lies between 1 and 1.5.

$\therefore$ first approximation to the root is $x_1 = \frac{1}{2}(1 + 1.5) = 1.25$.

Then $f(x_1) = (1.25)\sin(1.25) - 1 = 1.25 \sin(71.625°) - 1 = 0.18627$ and $f(1) < 0$.

$\therefore$ a root lies between 1 and $x_1 = 1.25$.

Thus the second approximation to the root is $x_2 = \frac{1}{2}(1 + 1.25) = 1.125$.

Then $f(x_2) = 1.125 \sin(1.125) - 1 = 1.125 \sin(64.46)° - 1 = 0.01509$ and $f(1) < 0$.

$\therefore$ a root lies between 1 and $x_2 = 1.125$.

Thus the third approximation to the root is $x_3 = \frac{1}{2}(1 + 1.125) = 1.0625$

Then $f(x_3) = 1.0625 \sin(1.0625) - 1 = 1.0625 \sin(60.88) - 1 = -0.0718 < 0$

and $f(x_2) > 0$, *i.e.* now the root lies between $x_3 = 1.0625$ and $x_2 = 1.125$.

$\therefore$ fourth approximation to the root is $x_4 = \frac{1}{2}(1.0625 + 1.125) = 1.09375$

Then $f(x_4) = -0.02836 < 0$ and $f(x_2) > 0$,

i.e., the root lies between $x_4 = 1.09375$ and $x_2 = 1.125$.

$\therefore$ fifth approximation to the root is $x_5 = \frac{1}{2}(1.09375 + 1.125) = 1.10937$

Then $f(x_5) = -0.00664 < 0$ and $f(x_2) > 0$.

$\therefore$ the root lies between $x_5 = 1.10937$ and $x_2 = 1.125$.

Thus the sixth approximation to the root is

$$x_6 = \frac{1}{2}(1.10937 + 1.125) = 1.11719$$

Then $f(x_6) = 0.00421 > 0$. But $f(x_5) < 0$.

$\therefore$ the root lies between $x_5 = 1.10937$ and $x_6 = 1.11719$.

Thus the seventh approximation to the root is $x_7 = \frac{1}{2}(1.10937 + 1.11719) = 1.11328$

Hence the desired approximation to the root is 1.11328.

(2) Method of false position *or* **Regula-falsi method.** This is the oldest method of finding the real root of an equation $f(x) = 0$ and closely resembles the bisection method. Here we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs *i.e.*, the graph of $y = f(x)$ crosses the x-axis between these points (Fig. 28.2). This indicates that a root lies between x_0 and x_1 consequently $f(x_0) f(x_1) < 0$.

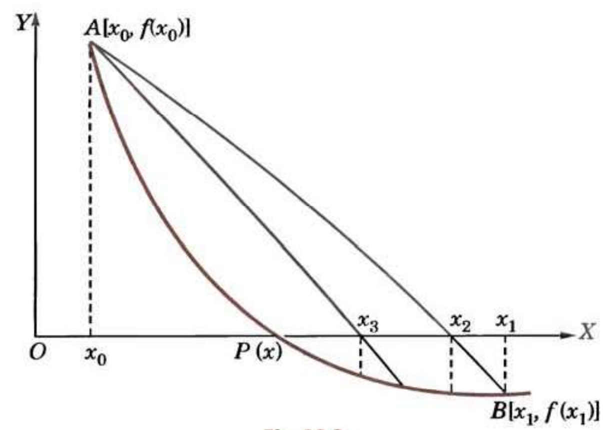

Fig. 28.2

Equation of the chord joining the points $A[x_0, f(x_0)]$ and $B[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \quad \text{...(1)}$$

The method consists in replacing the curve AB by means of the chord AB and taking the point of intersection of the chord with the x-axis as an approximation to the root. So the abscissa of the point where the chord cuts the x-axis ($y = 0$) is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \quad \text{...(2)}$$

which is an approximation to the root.

If now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2. So replacing x_1 by x_2 in (2), we obtain the next approximation x_3. (The root could as well lie between x_1 and x_2 and we would obtain x_3 accordingly). This procedure is repeated till the root is found to desired accuracy. The iteration process based on (1) is known as the *method of false position.*

Example 28.3. *Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of false position correct to three decimal places.* *(Manipal, 2005)*

Solution. Let $f(x) = x^3 - 2x - 5$

so that $f(2) = -1$ and $f(3) = 16$ *i.e.*, A root lies between 2 and 3.

$\therefore$ taking $x_0 = 2, x_1 = 3, f(x_0) = -1, f(x_1) = 16$, in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 + \frac{1}{17} = 2.0588 \quad \text{...}(i)$$

Now $f(x_2) = f(2.0588) = -0.3908$ *i.e.*, the root lies between 2.0588 and 3.

$\therefore$ taking $x_0 = 2.0588$, $x_1 = 3$, $f(x_0) = -0.3908$, $f(x_1) = 16$, in (*i*), we get

$$x_3 = 2.0588 - \frac{0.9412}{16.3908}(-0.3908) = 2.0813$$

Repeating this process, the successive approximations are

$$x_4 = 2.0862, x_5 = 2.0915, x_6 = 2.0934, x_7 = 2.0941, x_8 = 2.0943 \text{ etc.}$$

Hence the root is 2.094 correct to 3 decimal places.

Example 28.4. *Find the root of the equation* $\cos x = xe^x$ *using the regula-falsi method correct to four decimal places.* (*Bhopal, 2009*)

Solution. Let $f(x) = \cos x - xe^x = 0$

So that $f(0) = 1, f(1) = \cos 1 - e = -2.17798$

i.e., the root lies between 0 and 1.

$\therefore$ taking $x_0 = 0$, $x_1 = 1$, $f(x_0) = 1$ and $f(x_1) = -2.17798$ in the regula-falsi method, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 0 + \frac{1}{3.17798} \times 1 = 0.31467 \quad ...(i)$$

Now $f(0.31467) = 0.51987$

i.e., the root lies between 0.31467 and 1.

$\therefore$ taking $x_0 = 0.31467$, $x_1 = 1$, $f(x_0) = 0.51987$, $f(x_1) = -2.17798$ in (*i*), we get

$$x_3 = 0.31467 + \frac{0.68533}{2.69785} \times 0.51987 = 0.44673$$

Now $f(0.44673) = 0.20356$

i.e., the root lies between 0.44673 and 1.

$\therefore$ taking $x_0 = 0.44673$, $x_1 = 1$, $f(x_0) = 0.20356$, $f(x_1) = -2.17798$ in (*i*), we get

$$x_4 = 0.44673 + \frac{0.55327}{2.38154} \times 0.20356 = 0.49402$$

Repeating this process, the successive approximations are

$$x_5 = 0.50995, \quad x_6 = 0.51520, \quad x_7 = 0.51692$$
$$x_8 = 0.51748, \quad x_9 = 0.51767, \quad x_{10} = 0.51775 \text{ etc.}$$

Hence the root is 0.5177 correct to 4 decimal places.

Example 28.5. *Find a real root of the equation* $x \log_{10} x = 1.2$ *by regula-falsi method correct to four decimal places.* (*V.T.U., 2010 ; J.N.T.U., 2008 ; Kottayam, 2005*)

Solution. Let $f(x) = x \log_{10} x - 1.2$

so that $f(1) = -\text{ve}$, $f(2) = -\text{ve}$ and $f(3) = +\text{ve}$.

$\therefore$ a root lies between 2 and 3.

Taking $x_0 = 2$ and $x_1 = 3$, $f(x_0) = -0.59794$ and $f(x_1) = 0.23136$, in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2.72102 \quad ...(i)$$

Now $f(x_2) = f(2.72102) = -0.01709$

i.e., the root lies between 2.72102 and 3.

$\therefore$ taking $x_0 = 2.72102$, $x_1 = 3$, $f(x_0) = -0.01709$

and $f(x_1) = 0.23136$ in (*i*), we get

$$x_3 = 2.72102 + \frac{0.27898}{0.23136 + 0.01709} \times 0.01709 = 2.74021$$

Repeating this process, the successive approximations are

$$x_4 = 2.74024, x_5 = 2.74063 \text{ etc.}$$

Hence the root is 2.7406 correct to 4 decimal places.

Example 28.6. *Use the method of false position, to find the fourth root of 32 correct to three decimal places.*

Solution. Let $x = (32)^{1/4}$ so that $x^4 - 32 = 0$

Take $f(x) = x^4 - 32$. Then $f(2) = -16$ and $f(3) = 49$, *i.e.*, a root lies between 2 and 3.

$\therefore$ taking $x_0 = 2, x_1 = 3, f(x_0) = -16, f(x_1) = 49$ in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) = 2 + \frac{16}{65} = 2.2462 \qquad ...(i)$$

Now $f(x_2) = f(2.2462) = -6.5438$ *i.e.* the root lies between 2.2462 and 3.

$\therefore$ taking $x_0 = 2.2462, x_1 = 3, f(x_0) = -6.5438, f(x_1) = 49$

in (*i*), we get
$$x_3 = 2.2462 - \frac{3 - 2.2462}{49 + 6.5438}(-6.5438) = 2.335$$

Now $f(x_3) = f(2.335) = -2.2732$ *i.e.* the root lies between 2.335 and 3.

$\therefore$ taking $x_0 = 2.335$ and $x_1 = 3, f(x_0) = -2.2732$ and $f(x_1) = 49$ in (*i*), we obtain

$$x_4 = 2.335 - \frac{3 - 2.335}{49 + 2.2732}(-2.2732) = 2.3645$$

Repeating this process, the successive approximations are $x_5 = 2.3770, x_6 = 2.3779$ etc.

Since $x_5 = x_6$ upto 3 decimal places, we take $(32)^{1/4} = 2.378$.

(3) Newton-Raphson method*. Let x_0 be an approximate root of the equation $f(x) = 0$. If $x_1 = x_0 + h$ be the exact root, then $f(x_1) = 0$.

$\therefore$ expanding $f(x_0 + h)$ by Taylor's series

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots\ldots = 0$$

Since h is small, neglecting h^2 and higher powers of h, we get

$$f(x_0) + hf'(x_0) = 0 \quad \text{or} \quad h = -\frac{f(x_0)}{f'(x_0)} \qquad ...(1)$$

$\therefore$ a closer approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Similarly, starting with x_1, a still better approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general,
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad ...(2)$$

which is known as the *Newton-Raphson formula* or *Newton's iteration formula*.

Obs. 1. *Newton's method is useful in cases of large values of* $f'(x)$ *i.e.* when the graph of $f(x)$ while crossing the x-axis is nearly vertical.

Obs. 2. Newton's method has a second order of quadratic convergence. Suppose x_n differs from the root α by a small quantity ε_n so that $x_0 = \alpha + \varepsilon_n$ and $x_{n+1} = \alpha + \varepsilon_{n+1}$.

Then (2) becomes $\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \dfrac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$

i.e.,
$$\varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)} = \varepsilon_n - \frac{f(\alpha) + \varepsilon_n f'(\alpha) + \frac{1}{2!}\varepsilon_n^2 f''(\alpha) + \ldots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \ldots} \qquad \text{[By Taylor's expansion.]}$$

$$= \varepsilon_n - \frac{\varepsilon_n f'(\alpha) + \frac{1}{2}\varepsilon_n^2 f''(\alpha) + \ldots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \ldots} \qquad [\because \ f(\alpha) = 0]$$

$$= \frac{\varepsilon_n^2 f''(\alpha)}{2[f'(\alpha) + \varepsilon_n f''(\alpha)]} \approx \frac{\varepsilon_n^2}{2} \cdot \frac{f''(\alpha)}{f'(\alpha)}. \qquad \left[\begin{matrix}\text{neglecting third and} \\ \text{higher powers of } \varepsilon_n.\end{matrix}\right]$$

This shows that the subsequent error at each step, is proportional to the square of the previous error and as such the convergence is quadratic. (*P.T.U., 2005*)

Obs. 3. Geometrical interpretation. Let x_0 be a point near the root α of the equation $f(x) = 0$ (Fig. 28.3). Then the equation of the tangent at $A_0 [x_0, f(x_0)]$ is $y - f(x_0) = f'(x_0)(x - x_0)$.

*See footnote p. 466. Named after the English mathematician *Joseph Raphson* (1648–1715) who suggested a method similar to Newton's method.

It cuts the x-axis at $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$ which is a first approximation to the root α. If A_1 is the point corresponding to x_1 on the curve, then the tangent at A_1 will cut the x-axis of x_2 which is nearer to α and is, therefore, a second approximation to the root. Repeating this process, we approach to the root α quite rapidly. Hence *the method consists in replacing the part of the curve between the point A_0 and the x-axis by means of the tangent to the curve at A_0.*]

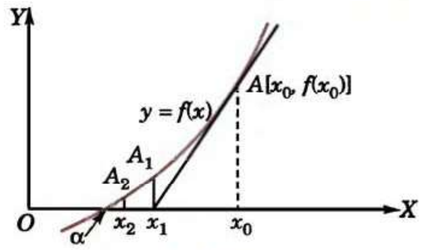

Fig. 28.3

Example 28.7. *Find the positive root of $x^4 - x = 10$ correct to three decimal places, using Newton-Raphson method.* *(J.N.T.U., 2008 ; Madras, 2006)*

Solution. Let $f(x) = x^4 - x - 10$

So that $f(1) = -10 = -\text{ve}, f(2) = 16 - 2 - 10 = 4 = +\text{ve}$

$\therefore$ a root of $f(x) = 0$ lies between 1 and 2. Let us take $x_0 = 2$

Also $f'(x) = 4x^3 - 1$

Newton-Raphson's formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \ldots(i)$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{4}{4 \times 2^3 - 1} = 2 - \frac{4}{31} = 1.871$$

Putting $n = 1$ in (*i*), the second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.871 - \frac{f(1.871)}{f'(1.871)}$$

$$= 1.871 - \frac{(1.871)^4 - (1.871) - 10}{4(1.871)^3 - 1} = 1.871 - \frac{0.3835}{25.199} = 1.856$$

Putting $n = 2$ in (*i*), the third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.856 - \frac{(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1}$$

$$= 1.856 - \frac{0.010}{24.574} - 1.856$$

Here $x_2 = x_3$. Hence the desired is 1.856 correct to three decimal places.

Example 28.8. *Find the Newton's method, the real root of the equation $3x = \cos x + 1$.* *(V.T.U., 2009 ; S.V.T.U., 2007)*

Solution. Let $f(x) = 3x - \cos x - 1$

$f(0) = -2 = -\text{ve}, f(1) = 3 - 0.5403 - 1 = 1.4597 = +\text{ve}.$

So a root of $f(x) = 0$ lies between 0 and 1. It is nearer to 1. Let us take $x_0 = 0.6$.

Also $f'(x) = 3 + \sin x$

$\therefore$ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \qquad \ldots(i)$$

Putting $n = 0$, the first approximation x_1 is given by

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0} = \frac{(0.6) \sin (0.6) + \cos (0.6) + 1}{3 \sin (0.6)}$$

$$= \frac{0.6 \times 0.5729 + 0.82533 + 1}{3 + 0.5729} = 0.6071$$

Putting $n = 1$ in (*i*), the second approximation is

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} = \frac{0.6071 \sin (0.6071) + \cos (0.6071) + 1}{3 + \sin (0.6071)}$$

$$= \frac{0.6071 \times 0.57049 + 0.8213 + 1}{3 + 0.57049} = 0.6071 \quad \text{Clearly, } x_1 = x_2.$$

Hence the desired root is 0.6071 correct to four decimal places.

Example 28.9. *Using Newton's iterative method, find the real root of $x \log_{10} x = 1.2$ correct to five decimal places.* *(V.T.U., 2005 ; Mumbai, 2004 ; Burdwan, 2003)*

Solution. Let $f(x) = x \log_{10} x - 1.2$

$f(1) = -1.2 = -\text{ve}, f(2) = 2 \log_{10} 2 - 1.2 = 0.59794 = -\text{ ve}$

and $f(3) = 3 \log_{10} 3 - 1.2 = 1.4314 - 1.2 = 0.23136 = +\text{ ve}$

So a root of $f(x) = 0$ lies between 2 and 3. Let us take $x_0 = 2$

Also $f'(x) = \log_{10} x + x \cdot \frac{1}{x} \log_{10} e = \log_{10} x + 0.43429$

$\therefore$ Newton's iteration formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{0.43429\, x_n + 1.2}{\log_{10} x_n + 0.43429} \qquad \ldots(i)$$

Putting $n = 0$, the first approximation is

$$x_1 = \frac{0.43429 \times x_0 + 1.2}{\log_{10} x_0 + 0.43429} = \frac{0.43429 \times 2 + 1.2}{\log_{10} 2 + 0.43429} = \frac{0.86858 + 1.2}{0.30103 + 0.43429} = 2.81$$

Similarly putting $n = 1, 2, 3, 4$ in (*i*), we get

$$x_2 = \frac{0.43429 \times 2.81 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.741$$

$$x_3 = \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.81 + 0.43429} = 2.74064$$

$$x_4 = \frac{0.43429 \times 2.74064 + 1.2}{\log_{10} 2.74064 + 0.43429} = 2.74065$$

$$x_5 = \frac{0.43429 \times 2.74065 + 1.2}{\log_{10} 2.74065 + 0.43429} = 2.74065$$

Clearly $x_4 = x_5$.

Hence the required root is 2.74065 correct to five decimal places.

28.3 USEFUL DEDUCTIONS FROM THE NEWTON-RAPHSON FORMULA

(1) *Iterative formula to find $1/N$ is* $\quad \boldsymbol{x_{n+1} = x_n (2 - Nx_n)}$

(2) *Iterative formula to find $\sqrt{N}$ is* $\quad \boldsymbol{x_{n+1} = \frac{1}{2}(x_n + N/x_n)}$

(3) *Iterative formula to find $1/\sqrt{N}$ is* $\quad \boldsymbol{x_{n+1} = \frac{1}{2}(x_n + 1/Nx_n)}$

(4) *Iterative formula to find $\sqrt[k]{N}$ is* $\quad \boldsymbol{x_{n+1} = \frac{1}{k}[(k-1)x_n + N/x_n^{k-1})]}$

Proofs. (1) Let $x = 1/N$ or $1/x - N = 0$

Taking $f(x) = 1/x - N$, we have $f'(x) = -x^{-2}$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(1/x_n - N)}{-x_n^{-2}} = x_n + \left(\frac{1}{x^n} - N\right) x_n^2 = x_n + x_n - Nx_n^2 = x_n(2 - Nx_n)$$

(2) Let $x = \sqrt{N}$ or $x^2 - N = 0$

Taking $f(x) = x^2 - N$, we have $f'(x) = 2x$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n} = \frac{1}{2}(x_n + N/x_n) \qquad \textit{(Madras, 2006)}$$

(3) Let $x = \frac{1}{\sqrt{N}}$ or $x^2 - \frac{1}{N} = 0$

Taking $f(x) = x^2 - 1/N$, we have $f'(x) = 2x$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 1/N}{2x_n} = \frac{1}{2}\left(x_n + \frac{1}{Nx_n}\right)$$

(4) Let $x = \sqrt[k]{N}$ or $x^k - N = 0$

Taking $f(x) = x^k - N$, we have $f'(x) = kx^{k-1}$

Then Newton's formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^k - N}{kx_n^{k-1}} = \frac{1}{k}\left[(k-1)x_n + \frac{N}{x_n^{k-1}}\right].$$

Example 28.10. *Evaluate the following (correct to four decimal places) by Newton's iteration method :*

(*i*) $1/31$ (*ii*) $\sqrt{5}$ *(Anna, 2007)*

(*iii*) $1/\sqrt{14}$ (*iv*) $\sqrt[3]{24}$ *(Madras, 2003)* (*v*) $(30)^{-1/5}$.

Solution. (*i*) Taking $N = 31$, the above formula (1) becomes

$$x_{n+1} = x_n(2 - 31x_n)$$

Since an approximate value of $1/31 = 0.03$, we take $x_0 = 0.03$

Then $x_1 = x_0(2 - 31x_0) = 0.03\,(2 - 31 \times 0.03) = 0.0321$

$x_2 = x_1(2 - 31x_1) = 0.0321(2 - 31 \times 0.0321) = 0.032257$

$x_3 = x_2\,(2 - 31x_2) = 0.032257\,(2 - 31 \times 0.032257) = 0.03226$

Since $x_2 = x_3$ upto 4 decimal places, we have $1/31 = 0.0323$.

(*ii*) Taking $N = 5$, the above formula (2), becomes $x_{n+1} = \frac{1}{2}(x_n + 5/x_n)$

Since an approximate value of $\sqrt{5} = 2$, we take $x_0 = 2$

Then $x_1 = \frac{1}{2}(x_0 + 5/x_0) = \frac{1}{2}(2 + 5/2) = 2.25$

$x_2 = \frac{1}{2}(x_1 + 5/x_1) = 2.2361$

$x_3 = \frac{1}{2}(x_2 + 5/x_2) = 2.2361$

Since $x_2 = x_3$ upto 4 decimal places, we have $\sqrt{5} = 2.2361$.

(*iii*) Taking $N = 14$, the above formula (3), becomes $x_{n+1} = \frac{1}{2}[x_n + 1/(14x_n)]$

Since an approximate value of $1/\sqrt{14} = 1/\sqrt{16} = \frac{1}{4} = 0.25$, we take $x_0 = 0.25$

Then $x_1 = \frac{1}{2}[x_0 + (14x_0)^{-1}] = \frac{1}{2}[0.25 + (14 \times 0.25)^{-1}] = 0.26785$

$x_2 = \frac{1}{2}[x_1 + (14x_1)^{-1}] = \frac{1}{2}[0.26785 + (14 \times 0.26785)^{-1}] = 0.2672618$

$x_3 = \frac{1}{2}[x_2 + (14x_2)^{-1}] = \frac{1}{2}[0.2672618 + (14 \times 0.2672618)^{-1}] = 0.2672612$

Since $x_2 = x_3$ upto 4 decimal places, we take $1/\sqrt{14} = 0.2673$.

(*iv*) Taking $N = 24$ and $k = 3$, the above formula (4) becomes $x_{n+1} = \frac{1}{3}[2x_n + 24/x_n^2]$

Since an approximate value of $(24)^{1/3} = (27)^{1/3} = 3$, we take $x_0 = 3$.

Then $$x_1 = \frac{1}{3}(2x_0 + 24/x_0^2) = \frac{1}{3}(6 + 24/9) = 2.88889$$

$$x_2 = \frac{1}{3}(2x_1 + 24/x_1^2) = \frac{1}{3}[(2 \times 2.88889) + 24/(2.88889)^2] = 2.88451$$

$$x_3 = \frac{1}{3}(2x_2 + 24/x_2^2) = \frac{1}{3}[2 \times 2.88451 + 24/(2.88451)^2] = 2.8845$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(24)^{1/3} = 2.8845$

(*v*) Taking $N = 30$ and $k = -5$, the above formula (4) becomes

$$x_{n+1} = \frac{1}{-5}(-6x_n + 30/x_n^{-6}) = \frac{x_n}{5}(6 - 30x_n^5)$$

Since an approximate value of $(30)^{-1/5} = (32)^{-1/5} = 1/2$, we take $x_0 = 1/2$

Then $$x_1 = \frac{x_0}{5}(6 - 30\,x_0^5) = \frac{1}{10}(6 - 30/2^5) = 0.50625$$

$$x_2 = \frac{x_1}{5}(6 - 30\,x_1^5) = \frac{0.50625}{5}[6 - 30(0.50625)^5] = 0.506495$$

$$x_3 = \frac{x_2}{5}(6 - 30\,x_2^5) = \frac{0.506495}{5}[6 - 30(0.506495)^5] = 0.506496.$$

Since $x_2 = x_3$ upto 4 decimal places, we take $(30)^{-1/5} = 0.5065$.

PROBLEMS 28.1

1. Find a root of the following equations, using the bisection method correct to three decimal places :
(*i*) $x^3 - 2x - 5 = 0$ (*P.T.U., 2005*) (*ii*) $x^3 - x^2 - 1 = 0$ (*J.N.T.U., 2009*)
(*iii*) $x^3 - x - 11 = 0$ which lies between 2 and 3 (*iv*) $2x^3 + x^2 - 20x + 12 = 0$.

2. Using the bisection method, find a real root of the following equations correct to three decimal places :
(*i*) $\cos x = xe^x$ (*Mumbai, 2004*) (*ii*) $x \log_{10} x = 1.2$ lying between 2 and 3
(*iii*) $e^x - x = 2$ lying between 1 and 1.4 (*iv*) $e^x = 4 \sin x$.

3. Find a real root of the following equations correct to three decimal places by the method of false position :
(*i*) $x^3 + x - 1 = 0$ (*ii*) $x^3 - 4x - 9 = 0$ (*V.T.U., 2007*)
(*iii*) $x^3 + x - 1 = 0$ near $x = 1$ (*iv*) $x^6 - x^4 - x^3 - 1 = 0$. (*Nagarjuna, 2001*)

4. Using regula-falsi method, compute the real root of the following equations correct to three decimal places :
(*i*) $xe^x = 2$ (*S.V.T.U., 2007*) (*ii*) $\cos x = 3x - 1$ (*iii*) $x \tan x - 1 = 0$
(*iv*) $2x - \log x = 7$ (*J.N.T.U., 2006*) (*v*) $xe^x = \sin x$. (*P.T.U., 2005*)

5. Find the fourth root of 12 correct to three decimal places using the method of false position.

6. Find by Newton's method, a root of the following equations correct to 3 decimal places :
(*i*) $x^3 - 3x + 1 = 0$ (*Bhopal, 2009*) (*ii*) $x^3 - 2x - 5 = 0$ (*P.T.U., 2005*)
(*iii*) $x^3 - 5x + 3 = 0$ (*Mumbai, 2004*)
(*iv*) $3x^3 - 9x^2 + 8 = 0$ lying between 1 and 2. (*Madras, 2003*)

7. Find a root of the following equations correct to three significant figures using Newton's iterative method :
(*i*) $x^4 + x^3 - 7x^2 - x + 5 = 0$ lying between 2 and 3 (*Madras, 2003*)
(*ii*) $x^5 - 5x^2 + 3 = 0$.

8. Find the negative root of the equation $x^3 - 21x + 3500 = 0$ correct to two decimal places by Newton's method.

9. Using Newton-Raphson method, find a root of the following equations correct to the three decimal places :
(*i*) $xe^x - 2 = 0$ (*V.T.U., 2005*) (*ii*) $x^2 + 4 \sin x = 0$ (*Hazaribagh, 2009*)
(*iii*) $x \tan x + 1 = 0$ which is near $x = \pi$ (*J.N.T.U., 2006 ; V.T.U., 2006*)
(*iv*) $e^x = x^2 + \cos 25x$ which is near $x = 4.5$. (*V.T.U., 2007*)

10. Find by Newton's method, the root of the equations :
(*i*) $\cos x = xe^x$ (*J.N.T.U., 2009 ; V.T.U., 2003*) (*ii*) $x \log_{10} x = 12.34$ (*Anna, 2004*)
(*iii*) $10^x + x - 4 = 0$ (*iv*) $x + \log_{10} x = 3.375$ (*Rohtak, 2003*)

11. Develop a recurrence formula for finding $\sqrt{N}$, using Newton-Raphson method and hence compute to three decimal places
(*i*) $\sqrt{13}$ (*U.P.T.U., 2008*) (*ii*) $\sqrt{10}$ (*J.N.T.U., 2008*)

12. Find the cube root of 41, using Newton-Raphson method. *(Madras, 2003)*
13. Develop an algorithm using N-R method, to find the fourth root of a positive number N and hence find $(32)^{1/4}$. *(W.B.T.U., 2005)*
14. Evaluate the following (correct to 3 decimal places) by using the Newton-Raphson method :
(*i*) 1/18 *J.N.T.U., 2004)* (*ii*) $1/\sqrt{15}$ (*iii*) $(28)^{-1/4}$.

28.4 APPROXIMATE SOLUTION OF EQUATIONS—HORNER'S METHOD

This is the best method of finding approximate values of both rational and irrational roots of a numerical equation. Horner's method consists in diminution of the root of an equation by successive digits occurring in the roots.

If the root of an equation lies between a and $a + 1$, then the value of this root will be $a\,.\,bcd$ ..., where b, c, d ... are digits in its decimal part. To obtain these, we proceed as follows :

(*i*) Diminish the roots of the given equation by a so that the root of the new equation is $0\,.\,bcd$ ...

(*ii*) Then multiply the roots of the transformed equation by 10 so that the root of the new equation is $b\,.\,cd$...

(*iii*) Now diminish the root by b and multiply the roots of the resulting equation by 10 so that the root is $c.\,d$...

(*iv*) Next diminish the root by c and so on. By continuing this process, the root may be evaluated to any desired degree of accuracy digit by digit. The method will be clear from the following example.

Example 28.11. *Find by Horner's method, the positive root of the equation $x^3 + x^2 + x - 100 = 0$ correct to three decimal places.*

Solution. *Step I.* Let $f(x) = x^3 + x^2 + x - 100$

By Descartes' rule of signs, there is only one positive root. Also $f(4) = -$ ve and $f(5) = +$ ve, therefore, the root lies between 4 and 5.

Step II. Diminish the roots of given equation by 4 so that the transformed equation is

$$x^3 + 13x^2 + 57x - 16 = 0 \qquad ...(i)$$

Its root lies between 0 and 1. (We draw a zig-zag line above the set of figures 13, 57, – 16 which are the coefficients of the terms in (*i*) as shown below. Now multiply the roots of (*i*) by 10 for which multiply the second term by 10, the third term by 100 and the fourth term by 1000 (*i.e.* attach one zero to the second term, two zeros to the third term and three zeros to the fourth term). Then we get the equation

$$f_1(x) = x^3 + 130x^2 + 5700x - 16000 = 0 \qquad ...(ii)$$

1	1	1	– 100	(4.264
	4	20	84	
	5	21	– 16000	
	4	36	11928	
	9	5700	– 4072000	
	4	264	3788376	
	130	5964	– 283624000	
	2	268		
	132	623200		
	2	8196		
	134	631396		
	2	8232		
	1360	63962800		
	6			
	1366			
	6			
	1372			
	6			
	13780			

Its root lies between 0 and 10.

Clearly $f_1(2) = -$ ve, $f_1(3) = +$ ve

$\therefore$ the root of (*ii*) lies between 2 and 3 *i.e.*, first figure after decimal is 2.

Step III. Diminish the roots of $f_1(x) = 0$ by 2 so that the next transformed equation is

$$x^3 + 136x^2 + 6232x - 4072 = 0 \qquad ...(iii)$$

Its root lies between 0 and 1. (We draw the second zig-zag line above the set of figures 136, 6232, – 4072). Multiply the roots of (*iii*), by 10, *i.e.* attach one zero to second term, two zeros to third term and three zeros to the fourth term. Then the new equation is

$$f_2(x) = x^3 + 1360x^2 + 623200x - 4072000 = 0$$

Its root lies between 0 and 10, which is nearly $= \dfrac{4072000}{623200} = 6$

Hence second figure after decimal place is 6.

Step IV. Diminish the roots of $f_2(x) = 0$ by 6, so that the transformed equation is

$$x^3 + 1378x^2 + 639628x - 283624 = 0.$$

Its root lies between 0 and 1. (We draw the third zig-zag line above the set of figures 1378, 639628, – 283624.) As before multiply its roots by 10, *i.e.* attach one zero to the second term, two zeros to the third term and three zeros to the fourth term. Then the equation becomes

$$f_3(x) = x^3 + 13780x^2 + 63962800x - 283624000 = 0$$

Its root lies between 0 and 10, which is nearly $= \dfrac{283624000}{63962800} = 4$. Thus the roots of $f_3(x) = 0$ are to be diminished by 4 *i.e.* the third figure after decimal place is 4. But there is no need to proceed further as the root is required correct to three decimal places only. Hence the root is 4.264.

Obs. 1. *After two steps of diminishing, we apply the* **principle of trial divisor** *in which we divide the last coefficient by last but one coefficient to get the next integer by which the roots are to be diminished. These last two coefficients should have opposite signs.*

Obs. 2. *At any stage if the trial divisor suggests the next integer to be zero, then we should again multiply the roots by 10 and write zero in decimal place of the root.*

Example 28.12. *Find the cube root of 30 correct to 3 decimal places, using Horner's method.*

Solution. *Step I.* Let $x = \sqrt[3]{30}$ *i.e.* $f(x) = x^3 - 30 = 0$

Now $f(3) = -3$ (– ve), $f(4) = 34$ (+ ve)

$\therefore$ the root lies between 3 and 4.

Step II. Diminish the roots of the given equation by 3 so that the transformed equation is

$$x^3 + 9x^2 + 27x - 3 = 0 \qquad ...(i)$$

Its roots lies between 0 and 1. (We draw a zig-zag line above the set of numbers 9, 27, – 3 which are the coefficients of the terms in (*i*)). Now multiply the roots of (*i*) by 10 for which attach one zero to the second term, two zeros to the third term and three zeros to the fourth term. Then we get the equation

$$f_1(x) = x^3 + 90x^2 + 2700x - 3000 = 0 \qquad ...(ii)$$

Its roots lies between 0 and 10.

Clearly $f_1(1) = -$ ve, $f_2(2) = +$ ve

$\therefore$ the root of (*ii*) lies between 1 and 2 *i.e.*, first figure after decimal place is 1.

Step III. Diminish the roots of $f_1(x) = 0$ by 1, so that the next transformed equation is

$$x^3 + 93x^2 + 2883x - 209 = 0 \qquad ...(iii)$$

Its root lies between 0 and 1. (We draw a second zig-zag line above the set of figures 93, 2883, – 209). Multiply the roots of (*iii*) by 10 *i.e.*, attach one zero to second term, two zeros to third term and three zeros to the fourth term. Then the new equation is

$$f_2(x) = x^3 + 930x^2 + 288300x - 209000 = 0$$

Its root lies between 0 and 10, which is nearly

$$= 209000/288300 = 0.724 > 0 \text{ and } < 1.$$

Hence second figure after decimal place is 0.

1	0	0	– 30 (3.107
	3	9	27
	3	9	– 30000
	3	18	2791
	6	2700	– 209000000
	3	91	
	90	2791	
	1	92	
	91	28830000	
	1		
	92		
	1		
	9300		

Step IV. Diminish the root of $f_2(x) = 0$ by 0 and then multiply its roots by 10 so that
$f_3(x) = x^3 + 9300x^2 + 28830000x - 209000000 = 0$.

Its root lies between 0 and 10, which is nearly $= 209000000/28830000 = 7.2 > 7$ and < 8. Thus the roots of $f_3(x) = 0$ are to be diminished by 7 *i.e.,* the third figure after decimal is 7. Hence the required root is 3.107.

PROBLEMS 28.2

1. Find by Horner's method, the root (correct to three decimal places) of the equation
   (*i*) $x^3 - 3x + 1 = 0$ which lies between 1 and 2
   (*ii*) $x^3 + x - 1 = 0$ (*Coimbatore, 1997*) (*iii*) $x^3 - 6x - 13 = 0$
   (*iv*) $x^3 - 3x^2 + 2.5 = 0$ which lies between 1 and 2. (*Madras, 2000 S*)
2. Using Horner's method, find the largest real root of $x^3 - 4x + 2 = 0$ correct to three decimal places.
3. Show that the root of the equation $x^4 + x^3 - 4x^2 - 16 = 0$ lies between 2 and 3. Find its value correct to two decimal places by Horner's method.
4. Find the negative root of the equation $x^3 - 9x^2 + 18 = 0$ correct to two decimal places by Horner's method.
5. Find the cube root of 25 by Horner's method correct to 3 decimal places.

28.5 SOLUTION OF LINEAR SIMULTANEOUS EQUATIONS

Simultaneous linear equations occur in various engineering problems. The student knows that a given system of linear equations can be solved by Cramer's rule or by Matrix method (§ 2.10). But these methods become tedious for large systems. However, there exist other numerical methods of solution which are well-suited for computing machines. We now explain some direct and iterative methods of solution.

28.6 DIRECT METHODS OF SOLUTION

(1) Gauss elimination method*. In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular system from which the unknowns are found by back substitution. The method is quite general and is well-adapted for computer operations. Here we shall explain it by considering a system of three equations for the sake of clarity.

Consider the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \qquad ...(1)$$

Step I. To eliminate x from second and third equations.

Assuming $a_1 \neq 0$, we eliminate x from the second equation by subtracting (a_2/a_1) times the first equation from the second equation. Similarly we eliminate x from the third equation by eliminating (a_3/a_1) times the first equation from the third equation. We thus, get the new system

*See footnote p. 37.

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ b_3'y + c_3'z &= d_3' \end{aligned}\right\} \qquad ...(2)$$

Here the first equation is called the *pivotal equation* and a_1 is called the *first pivot*.

Step II. To eliminate y from third equation in (2).

Assuming $b_2' \neq 0$, we eliminate y from the third equation of (2), by subtracting (b_3'/b_2') times the second equation from the third equation. We thus, get the new system

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ b_2'y + c_2'z &= d_2' \\ c_3''z &= d_3'' \end{aligned}\right\} \qquad ...(3)$$

Here the second equation is the *pivotal equation* and b_2' is the *new pivot*.

Step III. To evaluate the unknowns.

The values of x, y, z are found from the reduced system (3) by back substitution.

Obs. 1. On writing the given equations as $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ *i.e.*, $AX = D$, this method consists in ***transforming the coefficient matrix A to upper triangular matrix*** by elementary row transformations only.

Obs. 2. Clearly the method will fail if any one of the pivots a_1, b_2' or c_3'' becomes zero. In such cases, we rewrite the equations in a different order so that the pivots are non-zero.

Obs. 3. *Partial and complete pivoting.* In the first step, the numerically largest coefficient of x is chosen from all the equations and brought as the first pivot by interchanging the first equation with the equation having the largest coefficient of x. In the second step, the numerically largest coefficient of y is chosen from the remaining equations (leaving the first equation) and brought as the *second pivot* by interchanging the second equation with the equation having the largest coefficient of y'. This process is continued till we arrive at the equation with the single variable. This modified procedure is called *partial pivoting.*

If we are not taken about the elimination of x, y, z in a specified order, then we choose at each stage the numerically largest coefficient of the entire matrix of coefficients. This requires not only an interchange of equations but also an interchange of the position of the variables. This method of elimination is called *complete pivoting.* It is more complicated and does not appreciably improve the accuracy.

Example 28.13. *Apply Gauss elimination method to solve the equations $x + 4y - z = -5$; $x + y - 6z = -12$; $3x - y - z = 4$.* *(Mumbai, 2009)*

Solution. We have

	Check sum	
$x + 4y - z = -5$	-1	...(*i*)
$x + y - 6z = -12$	-16	...(*ii*)
$3x - y - z = 4$	5	...(*iii*)

Step I. Operate (*ii*) – (*i*) and (*iii*) – 3(*i*) to eliminate x :

	Check sum	
$-3y - 5z = -7$	-15	...(*iv*)
$-13y + 2z = 19$	8	...(*v*)

Step II. Operate (*v*) $- \dfrac{13}{3}$ (*iv*) to eliminate y :

	Check sum	
$\dfrac{71}{3}z = \dfrac{148}{3}$	73	...(*vi*)

Step III. By back-substitution, we get

From (*vi*) : $\quad z = \dfrac{148}{71} = 2.0845$

From (*iv*) : $\quad y = \dfrac{7}{3} - \dfrac{5}{3}\left(\dfrac{148}{71}\right) = -\dfrac{81}{71} = -1.1408$

From (*i*) : $x = -5 - 4\left(-\frac{81}{71}\right) + \frac{148}{71} = \frac{117}{71} = 1.6479$

Hence $x = 1.6479, y = -1.1408, z = 2.0845$

Note. A useful check is provided by noting the sum of the coefficients and terms on the right, operating on those numbers as on the equations and checking that the derived equations have the correct sum.

Otherwise : We have $\begin{bmatrix} 1 & 4 & -1 \\ 1 & 1 & -6 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -12 \\ 4 \end{bmatrix}$

Operate $R_2 - R_1$ and $R_3 - 3R_1$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & -13 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 19 \end{bmatrix}$

Operate $R_3 - \frac{13}{3}R_2$, $\begin{bmatrix} 1 & 4 & -1 \\ 0 & -3 & -5 \\ 0 & 0 & 71/3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ -7 \\ 148/3 \end{bmatrix}$

Thus, we have $z = 148/71 = 2.0845$,

$$3y = 7 - 5z = 7 - 10.4225 = -3.4225 \quad i.e., \quad y = -1.1408$$

and

$$x = -5 - 4y + z = -5 + 4(1.1408) + 2.0845 = 1.6479$$

Hence $x = 1.6479, y = -1.1408, z = 2.0845$.

Example 28.14. *Solve $10x - 7y + 3z + 5u = 6$, $-6x + 8y - z - 4u = 5$, $3x + y + 4z + 11u = 2$, $5x - 9y - 2z + 4u = 7$ by Gauss elimination method.* *(S.V.T.U., 2007)*

Solution. We have

	Check sum	
$10x - 7y + 3z + 5u = 6$	17	...(*i*)
$-6x + 8y - z - 4u = 5$	2	...(*ii*)
$3x + y + 4z + 11u = 2$	21	...(*iii*)
$5x - 9y - 2z + 4u = 7$	5	...(*iv*)

Step I. To eliminate x, operate $\left[(ii) - \left(\frac{-6}{10}\right)(i)\right], \left[(iii) - \frac{3}{10}(i)\right], \left[(iv) - \frac{5}{10}(i)\right]$:

	Check sum	
$3.8y + 0.8z - u = 8.6$	12.2	...(*v*)
$3.1y + 3.1z + 9.5u = 0.2$	15.9	...(*vi*)
$-5.5y - 3.5z + 1.5u = 4$	-3.5	...(*vii*)

Step II. To eliminate y, operate $\left[(vi) - \frac{3.1}{3.8}(v)\right], \left[(vii) - \left(\frac{-5.5}{3.8}\right)(v)\right]$:

$$2.4473684z + 10.315789u = -6.8157895 \quad ...(viii)$$

$$-2.3421053z + 0.0526315u = 16.447368 \quad ...(ix)$$

Step III. To eliminate z, operate $\left[(ix) - \left(\frac{-2.3421053}{2.4473684}\right)(viii)\right]$:

$$9.9249319u = 9.9245977$$

Step IV. By back-substitution, we get

$$u = 1, z = -7, y = 4 \text{ and } x = 5.$$

(2) Gauss-Jordan method*. This is a modification of the Gauss elimination method. In this method, elimination of unknowns is performed not in the equations below but in the equations above also, ultimately **reducing the system to a diagonal matrix form *i.e.,*** *each equation involving only one unknown. From these equations the unknowns x, y, z can be obtained readily.*

Thus in this method, the labour of back-substitution for finding the unknowns is saved at the cost of additional calculations.

*See footnote p. 37.

Example 28.15. *Apply Gauss-Jordan method to solve the equations*
$x + y + z = 9$; $2x - 3y + 4z = 13$; $3x + 4y + 5z = 40$. (*V.T.U., 2009 ; P.T.U., 2005*)

Solution. We have

$$x + y + z = 9 \quad ...(i)$$
$$2x - 3y + 4z = 13 \quad ...(ii)$$
$$3x + 4y + 5z = 40 \quad ...(iii)$$

Step I. Operate $(ii) - 2(i)$ and $(iii) - 3(i)$ to eliminate x from (ii) and (iii).

$$x + y + z = 9 \quad ...(iv)$$
$$-5y + 2z = -5 \quad ...(v)$$
$$y + 2z = 13 \quad ...(vi)$$

Step II. Operate $(iv) + \frac{1}{5}(v)$ and $(vi) + \frac{1}{5}(v)$ to eliminate y from (iv) and (vi) :

$$x + \frac{7}{5}z = 8 \quad ...(vii)$$
$$-5y + 2z = -5 \quad ...(viii)$$
$$\frac{12}{5}z = 12 \quad ...(ix)$$

Step III. Operate $(vii) - \frac{7}{12}(ix)$ and $(viii) - \frac{5}{6}(ix)$ to eliminate z from (vii) and $(viii)$:

$$x = 1$$
$$-5y = -15$$
$$\frac{12}{5}z = 12$$

Hence the solution is $x = 1, y = 3, z = 5$.

Otherwise : Rewriting the equations as $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$

Operate $R_2 - 2R_1, R_3 - 3R_1$, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 13 \end{bmatrix}$

Operate $R_3 + \frac{1}{5}R_2$, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -5 & 2 \\ 0 & 0 & 12/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ 12 \end{bmatrix}$

Operate $-R_2, 5R_3$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & -2 \\ 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 60 \end{bmatrix}$

Operate $R_2 + \frac{1}{6}R_3, \frac{1}{12}R_3$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 15 \\ 5 \end{bmatrix}$

Operate $\frac{1}{5}R_2$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 5 \end{bmatrix}$

Operate $R_1 - R_2 - R_3$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

Hence, $x = 1, y = 3, z = 5$.

Obs. Here the process of elimination of variables amounts to *reducing the given coefficient metric to a* **diagonal matrix** *by elementary row transformations only.*

Example 28.16. *Solve the equations of example 28.14, by Gauss-Jordan method.*

Solution. We have

$$10x - 7y + 3z + 5u = 6 \quad \text{...(i)}$$
$$-6x + 8y - z - 4u = 5 \quad \text{...(ii)}$$
$$3x + y + 4z + 11u = 2 \quad \text{...(iii)}$$
$$5x - 9y - 2z + 4u = 7 \quad \text{...(iv)}$$

Step I. To eliminate x, operate $\left[(ii) - \left(\frac{-6}{10}\right)(i)\right], \left[(iii) - \left(\frac{3}{10}\right)(i)\right], \left[(iv) - \left(\frac{5}{10}\right)(i)\right]$:

$$10x - 7y + 3z + 5u = 6 \quad \text{...(v)}$$
$$3.8y + 0.8z - u = 8.6 \quad \text{...(vi)}$$
$$3.1y + 3.1z + 9.5u = 0.2 \quad \text{...(vii)}$$
$$-5.5y - 3.5z + 1.5u = 4 \quad \text{...(viii)}$$

Step II. To eliminate y, operate $\left[(v) - \left(\frac{-7}{3.8}\right)(vi)\right], \left[(vii) - \left(\frac{3.1}{3.8}\right)(vi)\right], \left[(viii) - \left(\frac{-5.5}{3.8}\right)(vi)\right]$:

$$10x + 4.4736842z + 3.1578947u = 21.842105 \quad \text{...(ix)}$$
$$3.8y + 0.8z - u = 8.6 \quad \text{...(x)}$$
$$2.4473684z + 10.315789u = -6.8157895 \quad \text{...(xi)}$$
$$-2.3421053x + 0.0526315u = 16.447368 \quad \text{...(xii)}$$

Step III. To eliminate z, operate $\left[(ix) - \left(\frac{4.473684}{2.4473684}\right)(xi)\right]$,

$$\left[(x) - \left(\frac{0.8}{2.4473684}\right)(xi)\right], \left[(xii) - \left(\frac{-2.3421053}{2.4473684}\right)(xi)\right]:$$

$$10x - 15.698923u = 34.301075$$
$$3.8y - 4.3720429u = 10.827957$$
$$2.4473684z + 10.315789u = -6.8157895$$
$$9.9247309u = 9.9245975$$

Step IV. From the last equation $u = 1$ nearly.

Substitution of $u = 1$ in the above three equations gives $x = 5, y = 4, z = -7$.

(3) Factorization method*. This method is based on the fact that every matrix A can be expressed as the product of a lower triangular matrix and an upper triangular matrix, provided all the principal minors of A are non-singular, *i.e.*, if $A = [a_{ij}]$, then

$$a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, \text{ etc.}$$

Also such a factorization if it exists, is unique.

Now consider the equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

which can be written as $$AX = B \quad \text{...(1)}$$

where $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let $$A = LU, \quad \text{...(2)}$$

where $$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

*Another name given to this discomposition is **Doolittle's method.**

Then (1) becomes $\quad LUX = B$ ...(3)

Writing $\quad UX = V,$ ...(4)

(3) becomes $\quad LV = B$ which is equivalent to the equations

$$v_1 = b_1 \,;\, l_{21}v_1 + v_2 = b_2 \,;\, l_{31}v_1 + l_{31}v_2 + v_3 = b_3$$

Solving these for v_1, v_2, v_3, we know V. Then, (4) becomes

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = v_1 \,;\, u_{22}x_2 + u_{23}x_3 = v_2 \,;\, u_{33}x_3 = v_3,$$

from which x_3, x_2 and x_1 can be found by *back-substitution.*

To compute the matrices L and U, we write (2) as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 0 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiplying the matrices on the left and equating corresponding elements from both sides, we obtain

(*i*) $u_{11} = a_{11}, \quad u_{12} = a_{12}, \quad u_{13} = a_{13}$

(*ii*) $l_{21}u_{11} = a_{21}$ or $l_{21} = a_{21}/a_{11}$

$l_{31}u_{11} = a_{31}$ or $l_{31} = a_{31}/a_{11}$

(*iii*) $l_{21}u_{12} + u_{22} = a_{22}$ or $u_{22} = a_{22} - \dfrac{a_{21}}{a_{11}}\, a_{12}$

$l_{21}u_{13} + u_{23} = a_{23}$ or $u_{23} = a_{23} - \dfrac{a_{21}}{a_{11}}\, a_{13}$

(*iv*) $l_{31}u_{12} + l_{33}u_{22} = a_{32}$ or $l_{32} = \dfrac{1}{u_{22}}\left[a_{32} - \dfrac{a_{31}}{a_{11}}a_{12}\right]$

(*v*) $l_{31}u_{13} + l_{33}u_{23} + u_{33} = a_{33}$ which gives u_{33}.

Thus we compute the elements of L and U in the following set order :

(*i*) First row of U, (*ii*) First column of L,

(*iii*) Second row of U, (*iv*) Second column of L, (*v*) Third row of U.

This procedure can easily be generalised.

Obs. This method is superior to Gauss elimination method and is often used for the solution of linear systems and **for finding the inverse of a matrix.** Among the direct methods, Factorization method is also preferred as the software for computers.

Example 28.17. *Apply factorization method to solve the equations :*

$$3x + 2y + 7z = 4 \,;\, 2x + 3y + z = 5 \,;\, 3x + 4y + z = 7.$$ (*Madras, 2000 S*)

Solution. Let $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$ (*i.e.* A),

so that

(*i*) $u_{11} = 3, u_{12} = 2, \quad u_{13} = 7.$

(*ii*) $l_{21}u_{11} = 2, \quad \therefore\ l_{21} = 2/3$

$l_{31}u_{11} = 3, \quad \therefore\ l_{31} = 1.$

(*iii*) $l_{21}u_{12} + u_{22} = 3, \quad \therefore\ u_{22} = 5/3,$

$l_{21}u_{13} + u_{23} = 1, \quad \therefore\ u_{23} = -11/3.$

(*iv*) $l_{31}u_{12} + l_{32}u_{22} = 4, \quad \therefore\ l_{32} = 6/5.$

(*v*) $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$

$\therefore \quad u_{33} = -8/5$

Thus $\quad A = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}$

Writing $UX = V$, the given system becomes $\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$

Solving this system, we have $v_1 = 4$,

$$\frac{2}{3} v_1 + v_2 = 5 \qquad \text{or} \qquad v_2 = \frac{7}{3}$$

$$v_1 + \frac{6}{5} v_2 + v_3 = 7 \qquad \text{or} \qquad v_3 = \frac{1}{5}$$

Hence the original system becomes

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

i.e.,
$$3x + 2y + 7z = 4 \;;\; \frac{5}{3} y - \frac{11}{3} z = \frac{7}{3} \;;\; -\frac{8}{5} z = \frac{1}{5}$$

By back-substitution, we have $z = -1/8$, $y = 9/8$ and $x = 7/8$.

Example 28.18. *Solve the equations of Example 28.14 by factorization method.*

Solution. Let $\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{13} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} 10 & -7 & 3 & 5 \\ -6 & 8 & -1 & -4 \\ 3 & 1 & 4 & 11 \\ 5 & -9 & -2 & 4 \end{bmatrix}$ (*i.e.*, A)

so that

(*i*) R_1 of U : $u_{11} = 10$, $u_{12} = -7$, $u_{13} = 3$, $u_{14} = 5$

(*ii*) C_1 of L : $l_{21} = -0.6$, $l_{31} = 0.3$, $l_{41} = 0.5$

(*iii*) R_2 of U : $u_{22} = 3.8$, $u_{23} = 0.8$, $u_{24} = -1$

(*iv*) C_2 of L : $l_{32} = 0.81579$, $l_{42} = -1.44737$

(*v*) R_3 of U : $u_{33} = 2.44737$, $u_{34} = 10.31579$

(*vi*) C_3 of L : $l_{43} = -0.95699$

(*vii*) R_4 of U : $u_{44} = 9.92474$

Thus

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.6 & 1 & 0 & 0 \\ 0.3 & 0.81579 & 1 & 0 \\ 0.5 & -1.44737 & -0.95699 & 1 \end{bmatrix} \begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44737 & 10.31579 \\ 0 & 0 & 0 & 9.92474 \end{bmatrix}$$

Writing $UX = V$, the given system becomes

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.6 & 1 & 0 & 0 \\ 0.3 & 0.81579 & 1 & 0 \\ 0.5 & -1.44737 & -0.95699 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 2 \\ 7 \end{bmatrix}$$

Solving this system, we get

$$v_1 = 6, v_2 = 8.6, v_3 = -6.81579, v_4 = 9.92474.$$

Hence the original system becomes

$$\begin{bmatrix} 10 & -7 & 3 & 5 \\ 0 & 3.8 & 0.8 & -1 \\ 0 & 0 & 2.44737 & 10.31579 \\ 0 & 0 & 0 & 9.92474 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 6 \\ 8.6 \\ -6.81579 \\ 9.92474 \end{bmatrix}$$

i.e.,
$$10x - 7y + 3z + 5u = 6, \quad 3.8y + 0.8z - u = 8.6,$$
$$2.44737z + 10.31579u = -6.81579, \; u = 1.$$

By back-substitution, we get $u = 1$, $z = -7$, $y = 4$, $x = 5$.

PROBLEMS 28.3

Solve the following equations by Gauss elimination method :

1. $2x + y + z = 10$; $3x + 2y + 3z = 18$; $x + 4y + 9z = 16$. *(P.T.U., 2005)*
2. $2x + 2y + z = 12$; $3x + 2y + 2z = 8$; $5x + 10y - 8z = 10$. *(W.B.T.U., 2004)*
3. $2x - y + 3z = 9$; $x + y + z = 6$; $x - y + z = 2$. *(Bhopal, 2009)*
4. $2x_1 + 4x_2 + x_3 = 3$; $3x_1 + 2x_2 - 2x_3 = -2$; $x_1 - x_2 + x_3 = 6$. *(Marathwada, 2008)*
5. $5x_1 + x_2 + x_3 + x_4 = 4$; $x_1 + 7x_2 + x_3 + x_4 = 12$;
   $x_1 + x_2 + 6x_3 + x_4 = -5$; $x_1 + x_2 + x_3 + 4x_4 = -6$.

Solve the following equations by Gauss-Jordan method :

6. $2x + 5y + 7z = 52$; $2x + y - z = 0$; $x + y + z = 9$. *(V.T.U., 2010)*
7. $2x - 3y + z = -1$; $x + 4y + 5z = 25$; $3x - 4y + z = 2$. *(Kerala, 2003)*
8. $x + 3y + 3z = 16$; $z + 4y + 3z = 18$; $x + 3y + 4z = 19$. *(Anna, 2005)*
9. $2x + y + z = 10$; $3x + 2y + 3z = 18$; $x + 4y + 9z = 16$. *(V.T.U., 2008)*
10. $2x_1 + x_2 + 5x_3 + x_4 = 5$; $x_1 + x_2 - 3x_3 + 4x_4 = -1$;
    $3x_1 + 6x_2 - 2x_3 + x_4 = 8$; $2x_1 + 2x_2 + 2x_3 - 3x_4 = 2$.

Solve the following equations by factorization method :

11. $10x + y + z = 12$; $2x + 10y + z = 13$; $2x + 2y + 10z = 14$. *(Andhra, 2004 ; P.T.U., 2003)*
12. $x + 2y + 3z = 14$; $2x + 3y + 4z = 20$; $3x + 4y + z = 14$.
13. $2x + 3y + z = 9$; $x + 2y + 3z = 6$; $3x + y + 2z = 8$.
14. $2x_1 - x_2 + x_3 = -1$; $2x_2 - x_3 + x_4 = 1$; $x_1 + 2x_3 - x_4 = -1$; $x_1 + x_2 + 2x_4 = 5$.
15. Find the inverse of the matrix $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & 4 \\ 1 & 2 & 2 \end{bmatrix}$ by Crout's method.

28.7 ITERATIVE METHODS OF SOLUTION

The preceding methods of solving simultaneous linear equations are known as *direct methods* as they yield exact solutions. On the other hand, an iterative method is that in which we start from an approximation to the true solution and obtain better and better approximations from a computation cycle repeated as often as may be necessary for achieving a desired accuracy.

Simple iteration methods can be devised for systems in which the coefficients of the leading diagonal are large compared to others. We now explain three such methods :

(1) Jacobi's iteration method*. Consider the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \qquad \text{...(1)}$$

If a_1, b_2, c_3 are large as compared to other coefficients, then solving these for x, y, z respectively, the system can be written in the form

$$\left.\begin{aligned} x &= k_1 - l_1y - m_1z \\ y &= k_2 - l_2x - m_2z \\ z &= k_3 - l_3x - m_3y \end{aligned}\right\} \qquad \text{...(2)}$$

Let us start with the initial approximations x_0, y_0, z_0 (each = 0) for the values of x, y, z. Substituting these on the right, we get the first approximations $x_1 = k_1, y_1 = k_2, z_1 = k_3$.

Substituting these on the right-hand sides of (2), the second approximations are given by

$$\begin{aligned} x_2 &= k_1 - l_1y_1 - m_1z_1 \\ y_2 &= k_2 - l_2x_1 - m_2z_1 \\ z_2 &= k_3 - l_3x_1 - m_3y_1 \end{aligned}$$

This process is repeated till the difference between two consecutive approximations is negligible.

*See footnote p. 215.

Example 28.19. *Solve by Jacobi's iteration method, the equations $10x + y - z = 11.19$, $x + 10y + z = 28.08$, $-x + y + 10z = 35.61$, correct to two decimal places.* (Anna, 2007)

Solution. Rewriting the given equations as

$$x = \frac{1}{10}(11.19 - y + z),\ y = \frac{1}{10}(28.08 - x - z),\ z = \frac{1}{10}(35.61 + x - y)$$

We start from an approximation, $x_0 = y_0 = z_0 = 0$.

First iteration $\quad x_1 = \frac{11.19}{10} = 1.119,\ y_1 = \frac{28.08}{10} = 2.808,\ z_1 = \frac{35.61}{10} = 3.561$

Second iteration

$$x_2 = \frac{1}{10}(11.19 - y_1 + z_1) = 1.19$$

$$y_2 = \frac{1}{10}(28.08 - x_1 - z_1) = 2.24$$

$$z_2 = \frac{1}{10}(35.61 + x_1 - y_1) = 3.39$$

Third iteration

$$x_3 = \frac{1}{10}(11.19 - y_2 + z_2) = 1.22$$

$$y_3 = \frac{1}{10}(28.03 - x_2 - z_2) = 2.35$$

$$z_3 = \frac{1}{10}(35.61 + x_2 - y_2) = 3.45$$

Fourth iteration

$$x_4 = \frac{1}{10}(11.19 - y_3 + z_3) = 1.23$$

$$y_4 = \frac{1}{10}(28.03 - x_3 - z_3) = 2.34$$

$$z_4 = \frac{1}{10}(35.61 + x_3 - y_3) = 3.45$$

Fifth iteration

$$x_5 = \frac{1}{10}(11.19 - y_4 + z_4) = 1.23$$

$$y_5 = \frac{1}{10}(28.08 - x_4 - z_4) = 2.34$$

$$z_5 = \frac{1}{10}(35.61 + x_4 - y_4) = 3.45$$

Hence $x = 1.23$, $y = 2.34$, $z = 3.45$.

Example 28.20. *Solve, by Jacobi's iteration method, the equations*

$20x + y - 2z = 17$; $3x + 20y - z = -18$; $2x - 3y + 20z = 25$. (Bhopal, 2009)

Solution. We write the given equations in the form

$$\left.\begin{aligned} x &= \frac{1}{20}(17 - y + 2z) \\ y &= \frac{1}{20}(-18 - 3x + z) \\ z &= \frac{1}{20}(25 - 2x + 3y) \end{aligned}\right\} \quad \ldots(i)$$

We start from an approximation $x_0 = y_0 = z_0 = 0$.

Substituting these on the right sides of the equations (*i*), we get

$$x_1 = \frac{17}{20} = 0.85\ ;\ y_1 = -\frac{18}{20} = -0.9\ ;\ z_1 = \frac{25}{20} = 1.25$$

Putting these values on the right of the equations (*i*), we obtain

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.02$$

$$y_2 = \frac{1}{20}(-18 - 3x_1 + z_1) = -0.965$$

$$z_2 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.1515$$

Substituting these values in the right sides of the equations (*i*), we have

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0134$$

$$y_3 = \frac{1}{20}(-18 - 3x_2 + z_2) = -0.9954$$

$$z_3 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 1.0032$$

Substituting these values, we get

$$x_4 = \frac{1}{20}(17 - y_3 + 2z_3) = 1.0009$$

$$y_4 = \frac{1}{20}(-18 - 3x_3 + z_3) = -1.0018$$

$$z_4 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 0.9993$$

Putting these values, we have

$$x_5 = \frac{1}{20}(17 - y_4 + 2z_4) = 1.0000$$

$$y_5 = \frac{1}{20}(-18 - 3x_4 + z_4) = -1.0002$$

$$z_5 = \frac{1}{20}(25 - 2x_4 + 3y_4) = 0.9996$$

Again substituting these values, we get

$$x_6 = \frac{1}{20}(17 - y_5 + 2z_5) = 1.0000$$

$$y_6 = \frac{1}{20}(-18 - 3x_5 + z_5) = -1.0000$$

$$z_6 = \frac{1}{20}(25 - 2x_5 + 3y_5) = 1.0000$$

The values in the 5th and 6th iterations being practically the same, we can stop.

Hence the solution is $x = 1, y = -1, z = 1$.

(2) **Gauss-Seidel iteration method***. This is a modification of the Jacobi's iteration method. As before, we start with initial approximations x_0, y_0, z_0 (each = 0) for x, y, z respectively. Substituting $y = y_0, z = z_0$ in the first of the equations (2) on page 837, we get

$$x_1 = k_1$$

Then putting $x = x_1, z = z_0$ in the second of the equations (2) on page 837, we have

$$y_1 = k_2 - l_2 x_1 - m_2 z_0$$

Next substituting $x = x_1, y = y_1$ in the third of the equations (2) on page 837, we obtain

$$z_1 = k_3 - l_3 x_1 - m_3 y_1$$

and so on, *i.e.*, as soon as new approximation for an unknown is found, it is immediately used in the next step.

This process of iteration is continued till convergency to the desired degree of accuracy is obtained.

Obs 1. Since the most recent approximation of the unknowns are used while proceeding to the next step, the convergence in the Gauss-Seidel method is faster than in Jacobi's method.

Obs 2. Gauss-Sedial method converges if in each equation, the absolute value of the largest coefficient is greater than the sum of the absolute values of the remaining coefficients.

*See footnote p. 37. *After Philipp Ludwig Von Seidel* (1821–1896) who also suggested a similar method.

Example 28.21. *Apply Gauss-Seidel iteration method to solve the equations of Ex. 28.20.*
(V.T.U., 2011 ; Rohtak, 2005 ; Madras, 2003)

Solution. We write the given equation in the form

$$x = \frac{1}{20}(17 - y + 2z)\ ;\ y = \frac{1}{20}(-18 - 3x + z)\ ;\ z = \frac{1}{20}(25 - 2x + 3y) \qquad ...(i)$$

We start from the approximation $x_0 = y_0 = z_0 = 0$. Substituting $y = y_0$, $z = z_0$ in the right side of the first of equations (*i*), we get

$$x_1 = \frac{1}{20}(17 - y_0 + 2z_0) = 0.8500$$

Putting $x = x_1$, $z = z_0$ in the second of the equations (*i*), we have

$$y_1 = \frac{1}{20}(-18 - 3x_1 + z_0) = -1.0275$$

Putting $x = x_1$, $y = y_1$ in the last of the equations (*i*), we obtain

$$z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.0109$$

For the second iteration, we have

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.0025$$

$$y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9998$$

$$z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 0.9998$$

For the third iteration, we get

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0000$$

$$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -1.0000$$

$$z_3 = \frac{1}{20}(25 - 3x_3 + 2y_3) = 1.0000$$

The values in the 2nd and 3rd iterations being practically the same, we can stop.

Hence the solution is $x = 1, y = -1, z = 1$.

Example 28.22. *Solve the equations :*

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$
$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$
$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$
$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

by Gauss-Seidal iteration method. *(Bhopal, 2009 ; J.N.T.U., 2004)*

Solution. Rewriting the given equations as

$$x_1 = 0.3 + 0.2x_2 + 0.1x_3 + 0.1x_4 \qquad ...(i)$$
$$x_2 = 1.5 + 0.2x_1 + 0.1x_3 + 0.1x_4 \qquad ...(ii)$$
$$x_3 = 2.7 + 0.1x_1 + 0.1x_2 + 0.2x_4 \qquad ...(iii)$$
$$x_4 = -0.9 + 0.1x_1 + 0.1x_2 + 0.2x_3 \qquad ...(iv)$$

First iteration

Putting $x_2 = 0, x_3 = 0, x_4 = 0$ in (*i*), we get $x_1 = 0.3$

Putting $x_1 = 0.3, x_3 = 0, x_4 = 0$ in (*ii*), we obtain $x_2 = 1.56$

Putting $x_1 = 0.3, x_2 = 1.56, x_4 = 0$ in (*iii*), we obtain $x_3 = 2.886$

Putting $x_1 = 0.3, x_2 = 1.56, x_3 = 2.886$ in (*iv*), we get $x_4 = -0.1368$

Second iteration

Putting $x_2 = 1.56, x_3 = 2.886, x_4 = -0.1368$ in (*i*), we obtain

$x_1 = 0.8869$

Putting $x_1 = 0.8869, x_3 = 2.886, x_4 = -0.1368$ in (*ii*), we obtain

$x_2 = 1.9523$

Putting $x_1 = 0.8869, x_2 = 1.9523, x_4 = -0.1368$ in (*iii*), we have

$x_3 = 2.9566$

Putting $x_1 = 0.8869, x_2 = 1.9523, x_3 = 2.9566$ in (*iv*), we get

$x_4 = -0.0248.$

Third iteration

Putting $x_2 = 1.9523, x_3 = 2.9566, x_4 = -0.0248$ in (*i*), we obtain

$x_1 = 0.9836$

Putting $x_1 = 0.9836, x_3 = 2.9566, x_4 = -0.0248$ in (*ii*), we obtain

$x_2 = 1.9899$

Putting $x_1 = 0.9836, x_2 = 1.9899, x_4 = -0.0248$ in (*iii*), we get

$x_3 = 2.9924$

Putting $x_1 = 0.9836, x_1 = 1.9899, x_3 = 2.9924$ in (*iv*), we get

$x_4 = -0.0042.$

Fourth iteration. Proceeding as above

$x_1 = 0.9968, x_2 = 1.9982, x_3 = 2.9987, x_4 = -0.0008.$

Fifth iteration is

$x_1 = 0.9994, x_2 = 1.9997, x_3 = 2.9997, x_4 = -0.0001.$

Sixth iteration is

$x_1 = 0.9999, x_2 = 1.9999, x_3 = 2.9999, x_4 = -0.0001.$

Hence the solution is $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 0.$

(3) **Relaxation method*.** Consider the equations

$$a_1x + b_1y + c_1z = d_1 \,;\, a_2x + b_2y + c_2z = d_2 \,;\, a_3x + b_3y + c_3z = d_3$$

We define the residuals R_x, R_y, R_z by the relations

$$R_x = d_1 - a_1x - b_1y - c_1z \,;\, R_y = d_2 - a_2x - b_2y - c_2z \,;\, R_z = d_3 - a_3x - b_3y - c_3z \quad \text{...(1)}$$

To start with we assume $x = y = z = 0$ and calculate the initial residuals. Then the residuals are reduced step by step by giving increments to the variables. For this purpose, we construct the following *operation table :*

	δR_x	δR_y	δR_z
$\delta x = 1$	$-a_1$	$-a_2$	$-a_3$
$\delta y = 1$	$-b_1$	$-b_2$	$-b_3$
$\delta z = 1$	$-c_1$	$-c_2$	$-c_3$

We note from the equations (1) that if x is increased by 1 (keeping y and z constant), R_x, R_y and R_z decrease by a_1, a_2, a_3 respectively. This is shown in the above table alongwith the effects on the residuals when y and z are given unit increments. (The table is the transpose of the coefficient matrix).

At each step, the numerically largest residual is reduced to almost zero. To reduce a particular residual, the value of the corresponding variable is changed ; *e.g.*, to reduce R_x by p, x should be increased by p/a_1.

When all the residuals have been reduced to almost zero, the increments in x, y, z are added separately to give the desired solution.

Obs. As a check, the computed values of x, y, z are substituted in (1) and the residuals are calculated. If these residuals are not all negligible, then there is some mistake and the entire process should be rechecked.

Example 28.23. *Solve, by Relaxation method, the equations :*

$$9x - 2y + z = 50,\ x + 5y - 3z = 18,\ -2x + 2y + 7z = 19. \qquad \textit{(Madras, 2000 S)}$$

*This method was originally developed by *R.V. Southwell* in 1935, for application to structural engineering problems.

Solution. The residuals are given by

$R_x = 50 - 9x + 2y - z$; $R_y = 18 - x - 5y + 3z$; $R_z = 19 + 2x - 2y - 7z$

The operations table is

	δR_x	δR_y	δR_z
$\delta x = 1$	-9	-1	2
$\delta y = 1$	2	-5	-2
$\delta z = 1$	-1	3	-7

The relaxation table is

	R_x	R_y	R_z	
$x = y = z = 0$	50	18	19	...(*i*)
$\delta x = 5$	5	13	29	...(*ii*)
$\delta z = 4$	1	25	1	...(*iii*)
$\delta y = 5$	11	0	-9	...(*iv*)
$\delta x = 1$	2	-1	-7	...(*v*)
$\delta z = -1$	3	-4	0	...(*vi*)
$\delta y = -0.8$	1.4	0	1.6	...(*vii*)
$\delta z = 0.23$	1.17	0.69	$-.09$	...(*viii*)
$\delta x = 0.13$	0	0.56	0.17	...(*ix*)
$\delta y = 0.112$	0.224	0	-0.054	...(*x*)

$\Sigma\delta x = 6.13$, $\Sigma\delta y = 4.31$, $\Sigma\delta z = 3.23$

Thus $x = 6.13$, $y = 4.31$, $z = 3.23$.

[**Explanation.** In (*i*), the largest residual is 50. To reduce it, we give an increment $dx = 5$ and the resulting residuals are shown in (*ii*). Of these $R_x = 29$ is the largest and we given an increment $\delta z = 4$ to get the results in (*iii*). In (*vi*), $R_y = -4$ is the (numerically) largest and we give an increment $\delta y = -4/5 = -0.8$ to obtain the results in (*vii*). Similarly the other steps have been carried out.]

Example 28.24. *Solve by Relaxation method, the equations :*

$10x - 2y - 3z = 205$; $-2x + 10y - 2z = 154$; $-2x - y + 10z = 120$. *(V.T.U., 2011 S ; Rohtak, 2005)*

Solution. The residuals are given by

$R_x = 205 - 10x + 2y + 3z$; $R_y = 154 + 2x - 10y + 2z$; $R_z = 120 + 2x + y - 10z$.

The operations table is

	δR_x	δR_y	δR_z
$\delta x = 1$	-10	2	2
$\delta y = 1$	2	-10	-1
$\delta z = 1$	3	2	-10

The relaxation table is :

	R_x	R_y	R_z
$x = y = z = 0$	205	154	120
$\delta x = 20$	5	194	160
$\delta y = 19$	43	4	179
$\delta z = 18$	97	40	-1
$\delta x = 10$	-3	60	19
$\delta y = 6$	9	0	25
$\delta z = 2$	15	4	5
$\delta x = 2$	-5	8	9
$\delta z = 1$	-2	10	-1
$\delta y = 1$	0	0	0

$\Sigma\delta x = 32$, $\Sigma y = 26$, $\Sigma z = 21$.

Hence $x = 32$, $y = 26$, $z = 21$.

PROBLEMS 28.4

1. Solve by Jacobi's method, the equations : $5x - y + z = 10$; $2x + 4y = 12$; $x + y + 5z = -1$. Start with the solution (2, 3, 0).
2. Solve the equations $27x + 6y - z = 85$; $x + y + 54z = 110$; $6x + 15y + 2z = 72$.
   by (*a*) Jacobi's method (*b*) Gauss-Seidel method. (*Anna, 2006*)

Solve the following equations by Gauss-Seidel method :

3. $2x + y + 6z = 9$; $8x + 3y + 2z = 13$; $x + 5y + z = 7$.
4. $28x + 4y - z = 32$; $x + 3y + 10z = 24$; $2x + 17y + 4z = 35$. (*Mumbai, 2009*)
5. $10x + y + z = 12$; $2x + 10y + z = 13$; $2x + 2y + 10z = 104$. (*V.T.U., MCA, 2007*)
6. $83x + 11y - 4z = 95$; $7x + 52y + 13z = 104$; $3x + 8y + 29z = 71$. (*Hazaribagh, 2009*)
7. $3x_1 - 0.1x_2 - 0.2x_3 = 7.85$; $0.1x_1 + 7x_2 - 0.3x_3 = -19.3$; $0.3x_1 - 0.2x_2 + 10x_3 = 71.4$ (*Mumbai, 2004*)
8. $1.2x + 2.1y + 4.2z = 9.9$; $5.3x + 6.1y + 4.7z = 21.6$; $9.2x + 8.3y + z = 15.2$.
9. $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$

Solve by Relaxation method, the following sets of equations :

10. $3x + 9y - 2z = 11$; $4x + 2y + 13z = 24$; $4x - 4y + 3z = -8$. (*Bhopal, 2002*)
11. $10x - 2y - 2z = 6$; $-x + 10y - 2z = 7$; $-x - y + 10z = 8$.
12. $-9x + 3y + 4z + 100 = 0$; $x - 7y + 3z + 80 = 0$; $2x + 3y - 5z + 60 = 0$.
13. $54x + y + z = 110$; $2x + 15y + 6z = 72$; $-x + 6y + 27z = 85$. (*Bhopal, 2003*)

28.8 SOLUTION OF NON-LINEAR SIMULTANEOUS EQUATIONS—NEWTON-RAPHSON METHOD

Consider the equations

$$f(x, y) = 0, g(x, y) = 0 \quad \text{...(1)}$$

If an initial approximation (x_0, y_0) to a solution has been found by graphical method or otherwise, then a better approximation (x_1, y_1) can be obtained as follows :

Let $x_1 = x_0 + h, y_1 = y_0 + k$, so that $f(x_0 + h, y_0 + k) = 0, g(x_0 + h, y_0 + k) = 0$ ...(2)

Expanding each of the functions in (2) by Taylor's series to first degree terms, we get approximately

$$\left.\begin{aligned} f_0 + h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} = 0 \\ g_0 + h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} = 0 \end{aligned}\right\} \quad \text{...(3)}$$

where $\quad f_0 = f(x_0, y_0), \ \dfrac{\partial f}{\partial x_0} = \left(\dfrac{\partial f}{\partial x}\right)_{x_0, y_0}$ etc.

Solving the equations (3) for h and k, we get a new approximation to the root as

$$x_1 = x_0 + h, y_1 = y_0 + k$$

This process is repeated till we get the values to the desired accuracy.

Example 28.25. *Solve the system of non-linear equations :*

$$x^2 + y = 11, \ y^2 + x = 7. \quad \text{(Pune, 2000)}$$

Solution. An initial approximation to the solution is obtained from a rough graph of the given equations, as $x_0 = 3.5$ and $y_0 = -1.8$.

We have $\quad f = x^2 + y - 11$ and $g = y^2 + x - 7$ so that

$$\frac{\partial f}{\partial x} = 2x, \ \frac{\partial f}{\partial y} = 1 \quad \text{and} \quad \frac{\partial g}{\partial x} = 1, \ \frac{\partial g}{\partial y} = 2y.$$

Then Newton-Raphson's equations (3) above will be

$$7h + k = 0.55, \ h - 3.6k = 0.26$$

Solving these, we get $h = 0.0855, k = -0.0485$

∴ the better approximation to the root is

$$x_1 = x_0 + h = 3.5855, y_1 = y_0 + k = -1.8485$$

Repeating the above process, replacing (x_0, y_0) by (x_1, y_1), we obtain $x_2 = 3.5844$, $y_2 = -1.8482$.

PROBLEMS 28.5

1. Solve the equations $x^2 + y = 5$; $y^2 + x = 3$.
2. Solve the non-linear equations $x = 2(y + 1)$, $y^2 = 3xy - 7$ correct to three decimals.
3. Use Newton-Raphson method to solve the equations $x = x^2 + y^2$, $y = x^2 - y^2$ correct to two decimals, starting with the approximation (0.8, 0.4).
4. Solve the non-linear equations $x^2 - y^2 = 4$, $x^2 + y^2 = 16$ numerically with $x_0 = y_0 = 2.828$ using N.R. method. Carry out two iterations. *(V.T.U., MCA, 2007)*
5. Solve the equations $2x^2 + 3xy + y^2 = 3$; $4x^2 + 2xy + y^2 = 30$. Correct to three decimal places, using Newton-Raphson method, given that $x_0 = -3$, and $y_0 = 2$.

28.9 DETERMINATION OF EIGEN VALUES BY ITERATION

In § 2.14, we came across equations of the type

$$\left.\begin{aligned}(a_{11} - \lambda)x_1 + a_{12}x_2 + a_{13}x_3 = 0\\ a_{21}x_1 + (a_{22} - \lambda)x_2 + a_{23}x_3 = 0\\ a_{31}x_1 + a_{32}x_2 + (a_{33} - \lambda)x_3 = 0\end{aligned}\right\} \quad ...(1)$$

which in matrix form, may be written as $[A - \lambda I]\, X = 0$ or $AX = \lambda X$ ...(2)

where $A = [a_{ij}]$ and X is the column matrix $[x_i]$.

Equation (1) will have a non-trivial solution if the coefficient matrix vanishes *e.g.*,

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0$$

This gives a cubic in λ whose roots are *eigen values* of (2) and corresponding to each *eigen value*, we have a non-zero solution $X = [x_1, x_2, x_3]$ which is called an *eigen vector*. Such an equation can ordinarily be solved easily.

In some applications, it is required to compute the numerically largest *eigen value* and the corresponding *eigen vector*. In such cases, the following iterative method is more convenient which is also well-suited for computing machines.

If X_1, X_2, X_3 be the eigen vectors corresponding to the eigen values $\lambda_1, \lambda_2, \lambda_3$, then an arbitrary column vector can be written as $X = k_1X_1 + k_2X_2 + k_3X_3$

Then $\quad AX = k_1AX_1 + k_2AX_2 + k_3AX_3 = k_1\lambda_1X_1 + k_2\lambda_2X_2 + k_3\lambda_3X_3$

Similarly $\quad A^2X = k_1\lambda_1^2X_1 + k_2\lambda_2^2X_2 + k_3\lambda_3^2X_3$

and $\quad A^rX = k_1\lambda_1^r X_1 + k_2\lambda_2^rX_2 + k_3\lambda_3^rX_3$

If $|\lambda_1| > |\lambda_2| > |\lambda_3|$, then the contribution of the term $k_1\lambda_1^rX_1$ to the sum on the right increases with r and therefore, every time we multiply a column vector by A, it becomes nearer to the eigen vector X_1. Then we make the largest component of the resulting column vector unity to avoid the factor k_1.

Thus we start with a column vector X which is as near the solution as possible and evaluate AX which is written as $\lambda^{(1)} X^{(1)}$ after normalisation. This gives the first approximation $\lambda^{(1)}$ to the eigen value and $X^{(1)}$ to eigen vector. Similarly we evaluate $AX^{(1)} = \lambda^{(2)} X^{(2)}$ which gives the second approximation. We repeat this process till $[X^{(r)} - X^{(r-1)}]$ becomes negligible. Then $\lambda^{(r)}$ will be the largest eigen value of (1) and $X^{(r)}$, the corresponding eigen vector.

*This iterative procedure for finding the dominant eigen value of a matrix is known as Rayleigh's power method.**

*After the English mathematician and physicist John William Strut known as *Lord Rayleigh* (1842–1919) who made important contributions to the theory of waves, elasticity and hydrodynamics. He was professor at Cambridge and London.

Example 28.26. *Determine the largest eigen value and the corresponding eigen vector of the matrices using the power method :*

(i) $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ (V.T.U., 2007)

Solution. (*i*) Let the initial approximation to the eigen vector corresponding to the largest eigen value of A be $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Then $$AX = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \lambda^{(1)}X^{(1)}$$

So the first approximation to the eigen value is $\lambda^{(1)} = 5$ and the corresponding eigen vector is $X^{(1)} = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}$.

Now $$AX^{(1)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.8 \\ 1.4 \end{bmatrix} = 5.8\begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = \lambda^{(2)}X^{(2)}$$

Thus the second approximation to the eigen-value is $\lambda^{(2)} = 5.8$ and the corresponding eigen-vector is $X^{(2)} = \begin{bmatrix} 1 \\ 0.241 \end{bmatrix}$, repeating the above process, we get

Now $$AX^{(2)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.241 \end{bmatrix} = 5.966\begin{bmatrix} 1 \\ 0.248 \end{bmatrix} = \lambda^{(3)}X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.249 \end{bmatrix} = 5.994\begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = \lambda^{(4)}X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.250 \end{bmatrix} = 5.999\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{(5)}X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{(6)}X^{(6)}$$

Clearly $\lambda^{(5)} = \lambda^{(6)}$ and $X^{(5)} = X^{(6)}$ upto 3 decimal places. Hence the largest eigen-value is 6 and the corresponding eigen vector is $\begin{bmatrix} 1 \\ 0.25 \end{bmatrix}$.

(*ii*) Let the initial approximation to the required eigen vector be $X = [1, 0, 0]'$.

Then $$AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda^{(1)}X^{(1)}.$$

So the first approximation to the eigen value is $\lambda^{(1)} = 2$ and the corresponding eigen vector
$$X^{(1)} = [1, -0.5, 0]'.$$

Hence $$AX^{(1)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \lambda^{(2)}X^{(2)}.$$

Repeating the above process, we get

$$AX^{(2)} = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.43 \end{bmatrix} = \lambda^{(3)}X^{(3)};\ AX^{(3)} = 3.43\begin{bmatrix} 0.87 \\ -1 \\ 0.54 \end{bmatrix} = \lambda^{(4)}X^{(4)}$$

$$AX^{(4)} = 3.41\begin{bmatrix} 0.80 \\ -1 \\ 0.61 \end{bmatrix} = \lambda^{(5)}X^{(5)};\ AX^{(5)} = 3.41\begin{bmatrix} 0.76 \\ -1 \\ 0.65 \end{bmatrix} = \lambda^{(6)}X^{(6)};\ AX^{(6)} = 3.41\begin{bmatrix} 0.74 \\ -1 \\ 0.67 \end{bmatrix} = \lambda^{(7)}X^{(7)}$$

Clearly $\lambda^{(6)} = \lambda^{(7)}$ and $X^{(6)} = X^{(7)}$ approximately.

Hence the largest eigen value is 3.41 and the corresponding eigen vector is $[0.74, -1, 0.67]'$.

PROBLEMS 28.6

1. Find by power method, the larger eigen-value of the matrices :

   (a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (*Anna, 2005*) (b) $\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$.

2. Obtain the largest eigen-value and the corresponding eigen-vector for the equations
   $(2-\lambda)x_1 - x_2 = 0\ ;\ -x_1 + (2-\lambda)x_2 - x_3 = 0\ ;\ -x_2 + (2-\lambda)x_3 = 0$
   by Rayleigh Quotient method.

3. Find the dominant eigen value and the corresponding eigen vector of the following matrices using the power method :

   (a) $\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}$ (*V.T.U., 2011*) (b) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (*V.T.U., 2011 S*)

4. Find the largest eigen-value and the corresponding eigen-vector of the matrices :

   (a) $\begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ (*Anna, 2005*) (b) $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ (*V.T.U., 2008*)

   (c) $\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$ with initial approximation $[1, 1, 0]^T$. (*Madras, 2006*)

28.10 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 28.7

Fill up the blanks or select the correct answer to each of the following problems :

1. Out of Regula-falsi method and Newton-Rephson method, the rate of convergence is faster for ....
2. If x_n is the nth iterate, then the Newton-Raphson formula is ....
3. In the Regula-falsi method of finding the real root of an equation, the curve AB is replaced by ....
4. Newton's iterative formula to find the value of $\sqrt{N}$ is ....
5. Newton-Raphson formula converges when ....
6. In solving simultaneous equations by Gauss-Jordan method, the coefficient matrix is reduced to .... matrix.
7. In the case of bisection method, the convergence is
   (a) linear (b) quadratic (c) very slow.
8. The order of convergence in Newton-Raphson method is
   (a) 2 (b) 3 (c) 0 (d) none.
9. The Newton-Raphson algorithm for finding the cube root of N is ....
10. The bisection method for finding the root of an equation $f(x) = 0$ is ....
11. In Regula-falsi method, the first approximation is given by ....
12. The order of convergence in Newton-Raphson method is
    (a) 2 (b) 3 (c) 0 (d) none.
13. The iterative formula for finding the reciprocal of N is $x_{n+1} =$ ....
14. As soon as a new value of a variable is found by iteration, it is used immediately in the following equations, this method is called
    (a) Gauss-Jordan method (b) Gauss-Seidal method
    (c) Jacobi's method (d) Relaxation method.
15. Out of Regula-falsi method and Newton-Raphson method, the rate of convergence is faster for ....
16. The difference between direct and iterative methods of solving simultaneous linear equations is ....
17. To which form the coefficient matrix is transformed when $AX = B$ is solved by Gauss elimination method.
18. Jacobi's iteration method can be used to solve a system of non-linear equations. (True or False)
19. The convergence in the Gauss-Seidal method is thrice as fast as in Jacobi's method. (True or False)
20. By Gauss elimination method, solve $x + y = 2$ and $2x + 3y = 5$. (*Anna, 2007*)

CHAPTER 29

Finite Differences and Interpolation

1. Finite differences. **2.** Differences of a polynomial. **3.** Factorial notation **4.** Relations between the operators. **5.** To find one or more missing terms. **6.** Newton's interpolation formulae. **7.** Central difference interpolation formulae—Gauss's interpolation formulae ; Stirling's formula ; Bessel's formula ; Everett's formula. **8.** Choice of an interpolation formula. **9.** Interpolation with unequal intervals. **10.** Lagrange's formula. **11.** Divided differences. **12.** Newton's divided difference formula. **13.** Inverse interpolation. **14.** Objective Type of Questions.

29.1 FINITE DIFFERENCES

Suppose we are given the following values of $y = f(x)$ for a set of values of x :

x :	x_0	x_1	$x_2 \ldots x_n$.
y :	y_0	y_1	$y_2 \ldots y_n$.

Then the process of finding the values of y corresponding to any value of $x = x_i$ between x_0 and x_n is called *interpolation*. Thus *interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable* while the process of computing the value of the function outside the given range is called *extrapolation*. The study of the interpolation is based on the concept of differences of a function which we proceed to discuss. For a detailed study, the reader should refer to author's book *'Numerical Methods in Engineering and Science'*.

Suppose that the function $y = f(x)$ is tabulated for the equally spaced values $x = x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh$ giving $y = y_0, y_1, y_2, \ldots, y_n$. To determine the values of $f(x)$ or $f'(x)$ for some intermediate values of x, the following three types of differences are found useful :

(1) Forward differences. The differences $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, \ldots, \Delta y_{n-1}$ respectively are called the *first forward differences* where Δ is the *forward difference operator*. Thus the first forward differences are $\Delta y_r = y_{r+1} - y_r$.

Similarly, the second forward differences are defined by

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$$

In general,
$$\Delta^p y_r = \Delta^{p-1} y_{r+1} - \Delta^{p-1} y_r$$

defines the *pth forward differences*.

These differences are systematically set out as follows in what is called a *Forward Difference Table*.

In a difference table, x is called the *argument* and y the *function* or the *entry* y_0, the first entry is called the *leading term* and $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ etc. are called the *leading differences*.

Obs. Any higher order forward difference can be expressed in terms of the entries.

We have
$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$
$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0$$
$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0) = y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

The coefficients occurring on the right hand side being the binomial coefficient, we have in general,

$$\Delta^n y_0 = y_n - {}^n c_1 y_{n-1} + {}^n c_2 y_{n-2} - \ldots + (-1)^n y_0$$

Forward Difference Table

Value of x	*Value of y*	*1st. diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		Δy_0				
$x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_0 + 5h$	y_5					

(2) Backward differences. The differences $y_1 - y_0, y_2 - y_1, ..., y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, ..., \nabla y_n$ respectively, are called the *first backward differences* where ∇ is the *backward difference operator*. Similarly we define higher order backward differences. Thus we have

$$\nabla y_r = y_r - y_{r-1}, \nabla^2 y_r = \nabla y_r - \nabla y_{r-1},$$
$$\nabla^3 y_r = \nabla^2 y_r - \nabla^2 y_{r-1} \text{ etc.}$$

The differences are exhibited in the following :

Backward Difference Table

Value of x	*Value of y*	*1st. diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		∇y_1				
$x_0 + h$	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
$x_0 + 2h$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
$x_0 + 3h$	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
$x_0 + 4h$	y_4		$\nabla^2 y_5$			
		∇y_5				
$x_0 + 5h$	y_5					

(3) Central differences. Sometimes it is convenient to employ another system of differences known as *central differences.* In this system, the *central difference operator* δ is defined by the relations :

$$y_1 - y_0 = \delta y_{1/2}, y_2 - y_1 = \delta y_{3/2}, ..., y_n - y_{n-1} = \delta y_{n-1/2}$$

Similarly, higher order central differences are defined as

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1, \delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2, ...,$$
$$\delta^2 y_2 - \delta^2 y_1 = \delta^3 y_{3/2} \text{ and so on.}$$

These differences are shown in the following :

Central Difference Table

Value of x	*Value of y*	*1st. diff.*	*2nd diff.*	*3rd diff.*	*4th diff.*	*5th diff.*
x_0	y_0					
		$\delta y_{1/2}$				
$x_0 + h$	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
$x_0 + 2h$	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
$x_0 + 3h$	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
$x_0 + 4h$	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
$x_0 + 5h$	y_5					

We see from this table that the central differences on the same horizontal line have the same suffix. Also the differences of odd order are known only for half values of the suffix and those of even order for only integral values of the suffix.

It is often required to find the mean of adjacent values in the same column of differences. We denote this mean by μ. Thus

$$\mu\delta y_1 = \tfrac{1}{2}(\delta y_{1/2} + \delta y_{3/2}),\ \mu\delta^2 y_{3/2} = \tfrac{1}{2}(\delta^2 y_1 + \delta^2 y_2) \text{ etc.}$$

Obs. The reader should note that it is only the notation which changes and not the differences.

$$y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{1/2}.$$

Of all the interpolation formulae, those involving central differences are most useful in practice as the coefficients in such formulae decrease much more rapidly.

Example 29.1. *Evaluate (i) $\Delta \tan^{-1} x$ (ii) $\Delta(e^x \log 2x)$ (iii) $\Delta(x^2/\cos 2x)$ (iv) $\Delta^2 \cos 2x$.* (P.T.U., 2001)

Solution. (i) $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$

$$= \tan^{-1}\left\{\frac{x+h-x}{1+(x+h)x}\right\} = \tan^{-1}\left\{\frac{h}{1+hx+x^2}\right\}$$

(ii)
$$\begin{aligned}\Delta(e^x \log 2x) &= e^{x+h}\log 2(x+h) - e^x \log 2x\\ &= e^{x+h}\log 2(x+h) - e^{x+h}\log 2x + e^{x+h}\log 2x - e^x\log 2x\\ &= e^{x+h}\log\frac{x+h}{x} + (e^{x+h}-e^x)\log 2x\\ &= e^x\left[e^h \log\left(1+\frac{h}{x}\right) + (e^h-1)\log 2x\right]\end{aligned}$$

(iii)
$$\begin{aligned}\Delta\left(\frac{x^2}{\cos 2x}\right) &= \frac{(x+h)^2}{\cos 2(x+h)} - \frac{x^2}{\cos 2x} = \frac{(x+h)^2\cos 2x - x^2\cos 2(x+h)}{\cos 2(x+h)\cos 2x}\\ &= \frac{[(x+h)^2 - x^2]\cos 2x + x^2[\cos 2x - \cos 2(x+h)]}{\cos 2(x+h)\cos 2x}\\ &= \frac{(2hx+h^2)\cos 2x + 2x^2 \sin(h)\sin(2x+h)}{\cos 2(x+h)\cos 2x}\end{aligned}$$

(iv)
$$\begin{aligned}\Delta^2 \cos 2x &= \Delta\{\cos 2(x+h) - \cos 2x\}\\ &= \Delta\cos 2(x+h) - \Delta\cos 2x\\ &= [\cos 2(x+2h) - \cos 2(x+h)] - [\cos 2(x+h) - \cos 2x]\\ &= -2\sin(2x+3h)\sin h + 2\sin(2x+h)\sin h\\ &= -2\sin h\,[\sin(2x+3h) - \sin(2x+h)]\\ &= -2\sin h\,[2\cos(2x+2h)\sin h] = -4\sin^2 h\cos(2x+2h).\end{aligned}$$

Example 29.2. *Evaluate (i) $\Delta^2\left(\dfrac{5x+12}{x^2+5x+16}\right)$ (Mumbai, 2003) (ii) $\Delta^2(ab^x)$ (iii) $\Delta^n(e^x)$ interval of differencing being unity.* (Rohtak, 2003)

Solution. (i)
$$\begin{aligned}\Delta^2\left(\frac{5x+12}{x^2+5x+6}\right) &= \Delta^2\left\{\frac{5x+12}{(x+2)(x+3)}\right\} = \Delta^2\left\{\frac{2}{x+2} + \frac{3}{x+3}\right\}\\ &= \Delta\left\{\Delta\left(\frac{2}{x+2}\right) + \Delta\left(\frac{3}{x+3}\right)\right\} = \Delta\left\{2\left(\frac{1}{x+3} - \frac{1}{x+2}\right) + 3\left(\frac{1}{x+4} - \frac{1}{x+3}\right)\right\}\\ &= -2\Delta\left\{\frac{1}{(x+2)(x+3)}\right\} - 3\Delta\left\{\frac{1}{(x+3)(x+4)}\right\}\\ &= -2\left\{\frac{1}{(x+3)(x+4)} - \frac{1}{(x+2)(x+3)}\right\} - 3\left\{\frac{1}{(x+4)(x+5)} - \frac{1}{(x+3)(x+4)}\right\}\end{aligned}$$

$$= \frac{4}{(x+2)(x+3)(x+4)} + \frac{6}{(x+3)(x+4)(x+5)} = \frac{2(5x+16)}{(x+2)(x+3)(x+4)(x+5)}$$

(*ii*) $\Delta(ab^x) = a\,\Delta(b^x) = a(b^{x+1} - b^x) = ab^x(b-1)$

$\Delta^2(ab^x) = \Delta[\Delta ab^x] = a(b-1)\,\Delta(b^x)$

$= a(b-1)(b^{x+1} - b^x) = a(b-1)^2 - b^x.$

(*iii*) $\Delta e^x = e^{x+1} - e^x = (e-1)e^x$

$\Delta^2 e^x = \Delta(\Delta e^x) = \Delta[(e-1)\,e^x]$

$= (e-1)\,\Delta e^x = (e-1)(e-1)\,e^x = (e-1)^2\,e^x$

Similarly $\Delta^3 e^x = (e-1)^3 e^x,\ \Delta^4 e^x = (e-1)^4 e^x, \ldots$ and $\Delta^n e^x = (e-1)^n e^x.$

29.2 DIFFERENCES OF A POLYNOMIAL

The nth differences of a polynomial of the nth degree are constant and all higher order differences are zero.

Let the polynomial of the nth degree in x, be

$$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \ldots + k(x+h) + l$$

$\therefore$ $\Delta f(x) = f(x+h) - f(x)$

$= a[(x+h)^n - x^n] + b[(x+h)^{n-1} - x^{n-1}] + \ldots + kh$

$= anhx^{n-1} + b'x^{n-2} + c'x^{n-3} + \ldots + k'x + l'$ ...(1)

where $b', c', \ldots, l'$ are new constant coefficients.

Thus the first differences of a polynomial of the nth degree is a polynomial of degree $(n-1)$.

Similarly $\Delta^2 f(x) = \Delta[f(x+h) - f(x)] = \Delta f(x+h) - \Delta f(x)$

$= anh[(x+h)^{n-1} - x^{n-1}] + b'[(x+h)^{n-2} - x^{n-2}] + \ldots + k'h$

$= an(n-1)\,h^2 x^{n-2} + b'' x^{n-3} + c'' x^{n-4} + \ldots + k''$, [by (1)]

$\therefore$ the second differences represent a polynomial of degree $(n-2)$.

Continuing this process, for the nth differences we get a polynomial of degree zero *i.e.*

$\Delta^n f(x) = an(n-1)(n-2)\ldots 1 \,.\, h^n = an!\,h^n$ ...(2)

which is a constant. Hence the $(n+1)$th and higher differences of a polynomial of nth degree will be zero.

Obs. The converse of this theorem is also true *i.e. if the nth differences of a function tabulated at equally spaced intervels are constant, the function is a polynomial of degree n.* This fact is important in numerical analysis as it enables us to approximate a function by a polynomial of nth degree, if its nth order differences become nearly constant.

Example 29.3. *Evaluate* $\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)]$.

Solution. $\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)] = \Delta^{10}[abcd\,x^{10} + (\ )x^9 + (\ )x^8 + \ldots + 1]$

$= abcd\,\Delta^{10}(x^{10})$ $[\because\ \Delta^{10}(x^n) = 0 \text{ for } n < 10]$

$= abcd\,(10!)$. [by (2) above]

29.3 (1) FACTORIAL NOTATION

A product of the form $x(x-1)(x-2)\ldots(x-r+1)$ is denoted by $[x]^r$ and is called a **factorial.**

In particular $[x] = x,\ [x]^2 = x(x-1)$

$[x]^3 = x(x-1)(x-2)$, etc.

In general $[x]^n = x(x-1)(x-2)\ldots(x-n+1)$

In case, the interval of differencing is h, then

$$[x]^n = x(x-h)(x-2h)\ldots(x-\overline{n-1}h)$$

which is called a *Factorial polynomial or function.*

The factorial notation is of special utility in the theory of finite differences. It helps in finding the successive differences of a polynomial directly by simple rule of differentiation.

The result of differencing $[x]^r$ is analogous to that of differentiating x^r.

(2) To express a polynomial in the factorial notation

(*i*) *arrange the coefficients of the powers of x in descending order, replacing missing powers by zeros ;*

(*ii*) *using detached coefficients divide by x, x – 1, x – 2, etc. successively.*

Obs. Every polynomial of degree n can be expressed as a factorial polynomial of the same degree and *vice versa*.

Example 29.4. *Express* $y = 2x^3 - 3x^2 + 3x - 10$ *in a factorial notation and hence show that* $\Delta^3 y = 12$.

(*Bhopal, 2007 ; P.T.U., 2005*)

Solution. First method : Let $y = A[x]^3 + B[x]^2 + C[x] + D$.

Then

	x^3	x^2	x	
1	2	– 3	3	– 10 = D
	—	2	– 1	
2	2	– 1	2 = C	
	—	4		
3	2	3 = B		
	—			
	2 = A			

Hence $\qquad y = 2[x]^3 + 3[x]^2 + 2[x] - 10$

$\therefore \qquad \Delta y = 2 \times 3[x]^2 + 3 \times 2[x] + 2$

$\Delta^2 y = 6 \times 2[x] + 6$

$\Delta^3 y = 12$, which shows that the third differences of y are constant, as they should be.

Obs. *The coefficient of the highest power of x remains unchanged while transforming a polynomial to factorial notation.*

Second method (*Direct method*) :

Let $\qquad y = 2x^3 - 3x^2 + 3x - 10$

$= 2x(x-1)(x-2) + Bx(x-1) + Cx + D$

Putting $x = 0, -10 = D$

Putting $x = 1, 2 - 3 + 3 - 10 = C + D$

$\therefore \quad C = -8 - D = -8 + 10 = 2$

Putting $x = 2, 16 - 12 + 6 - 10 = 2B + 2C + D$

$$\therefore \qquad B = \frac{1}{2}(-2C - D) = \frac{1}{2}(-4 + 10) = 3.$$

Hence $\qquad y = 2x(x-1)(x-2) + 3x(x-1) + 2x - 10 = 2[x]^3 + 3[x]^2 + 2[x] - 10$

$\therefore \qquad \Delta y = 2 \times 3[x]^2 + 3 \times 2[x] + 2, \Delta^2 y = 6 \times 2[x] + 6, \Delta^3 y = 12.$

Example 29.5. *Find the missing values in the following table :*

x :	45	50	55	60	65
y :	3.0	—	2.0	—	– 2.4

(*Bhopal, 2007 ; V.T.U., 2001*)

Solution. The difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	$y_0 = 3$			
		$y_1 - 3$		
50	y_1		$5 - 2y_1$	
		$2 - y_1$		$3y_1 + y_3 - 9$
55	$y_2 = 2$		$y_1 + y_3 - 4$	
		$y_3 - 2$		$3.6 - y_1 - 3y_3$
60	y_3	$-2.4 - y_3$	$-0.4 - 2y_3$	
65	$y_4 = -2.4$			

As only three entries y_0, y_2, y_4 are given, the function y can be represented by a second degree polynomial.

$\therefore$ $\qquad \Delta^3 y_0 = 0 \quad$ and $\quad \Delta^3 y_1 = 0$

i.e., $\qquad 3y_1 + y_3 = 9\,; \qquad y_1 + 3y_3 = 3.6$

Solving these, we get $y_1 = 2.925$, $y_3 = 0.225$.

Otherwise : As only three entries $y_0 = 3$, $y_2 = 2$, $y_4 = -2.4$ are given, the function y can be represented by a second degree polynomial.

$\therefore$ $\qquad \Delta^3 y_0 = 0 \quad$ and $\quad \Delta^3 y_1 = 0$

i.e., $\qquad (E-1)^3 y_0 = 0 \quad$ and $\quad (E-1)^3 y_1 = 0$

i.e., $\qquad (E^3 - 3E^2 + 3E - 1)\, y_0 = 0 \quad$ and $\quad (E^3 - 3E^2 + 3E - 1) y_1 = 0$

i.e., $\qquad y_3 - 3y_2 + 3y_1 - y_0 = 0$

$\qquad y_4 - 3y_3 + 3y_2 - y_1 = 0$

i.e., $\qquad y_3 + 3y_1 = 9\,; \; 3y_3 + y_1 = 3.6$

Solving these, we get $y_1 = 2.925$, $y_3 = 0.225$.

Example 29.6. *Assuming that the following values of y belong to a polynomial of degree 4, compute the next three values :*

x :	0	1	2	3	4	5	6	7
y :	1	– 1	1	– 1	1	—	—	—

Solution. We construct the following difference table from the given data :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	$y_0 = 1$				
		-2			
1	$y_1 = -1$		4		
		2		-8	
2	$y_2 = 1$		-4		16
		-2		8	
3	$y_3 = -1$		4		16
		2		$\Delta^3 y_2$,	
4	$y_4 = 1$		$\Delta^2 y_3$		16
		Δy_4		$\Delta^2 y_3$,	
5	y_5		$\Delta^2 y_4$		16
		Δy_5		$\Delta^3 y_4$,	
6	y_6		$\Delta^2 y_5$		
		Δy_6			
7	y_7				

Since the values of y belong to a polynomial of degree 4, the fourth differences must be constant. But $\Delta^4 y = 16$.

$\therefore$ The other fourth order differences must also be 16. Thus

$$\Delta^4 y_1 = 16 = \Delta^3 y_2 - \Delta^3 y_1$$

i.e.,
$$\Delta^3 y_2 = \Delta^3 y_1 + \Delta^4 y_1 = 8 + 16 = 24$$
$$\Delta^2 y_3 = \Delta^2 y_2 + \Delta^3 y_2 = 4 + 24 = 28$$
$$\Delta y_4 = \Delta y_3 + \Delta^2 y_3 = 2 + 28 = 30$$

and
$$y_5 = y_4 + \Delta y_4 = 1 + 30 = 31$$

Similarly starting with $\Delta^4 y_2 = 16$, we get

$$\Delta^3 y_3 = 40,\ \Delta^2 y_4 = 68,\ \Delta y_5 = 98,\ y_6 = 129.$$

Starting with $\Delta^4 y_3 = 16$, we obtain

$$\Delta^3 y_4 = 56,\ \Delta^2 y_5 = 124,\ \Delta y_6 = 222,\ y_7 = 351.$$

PROBLEMS 29.1

1. Construct the table of differences for the data below :

x :	0	1	2	3	4
$f(x)$:	1.0	1.5	2.2	3.1	4.6

Evaluate $\Delta^3 f(2)$.

2. If $u_0 = 3, u_1 = 12, u_2 = 18, u_3 = 2000, u_4 = 100$, calculate Δu_0.
3. Show that $\Delta^3 y_i = y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i$.
4. Form the table of backward differences of the function
$$f(x) = x^3 - 3x^2 - 5x - 7 \text{ for } x = -1, 0, 1, 2, 3, 4, 5.$$
5. Form a table of differences for the function
$$f(x) = x^3 + 5x - 7 \text{ for } x = -1, 0, 1, 2, 3, 4, 5$$
Continue the table to obtain $f(6)$.
6. Extend the following table to two more terms on either side by constructing the difference table :

x :	$-.2$	0.0	0.2	0.4	0.6	0.8	1.0
y :	2.6	3.0	3.4	4.28	7.08	14.2	29.0

7. Show that

(i) $\Delta\left[\dfrac{1}{f(x)}\right] = \dfrac{-\Delta f(x)}{f(x)\, f(x+1)}$; *(Raipur, 2005)* (ii) $\Delta \log f(x) = \log\left\{1 + \dfrac{\Delta f(x)}{f(x)}\right\}$.

8. Evaluate :

(i) $\Delta (x + \cos x)$ (ii) $\Delta \tan^{-1}\left(\dfrac{n-1}{n}\right)$ (iii) $\Delta\left\{\dfrac{1}{x(x+4)(x+6)}\right\}$ *(Madras, 2001)*

(iv) $\Delta^2\left(\dfrac{1}{x^2 + 5x + 6}\right)$ *(P.T.U., 2001)*

9. Evaluate :

(i) $\Delta(e^{3x} \log 2x)$ (ii) $\Delta(2^x/x\,!)$ (iii) $\Delta^n(a^x)$ *(Burdwan, 2003)* (iv) $\Delta^n\left(\dfrac{1}{x}\right)$.

10. If $f(x) = e^{ax+b}$, show that its leading differences form a geometric progression. *(Mumbai, 2003)*
11. Prove that

(i) $y_3 = y_2 + \Delta y_1 + \Delta^2 y_0 + \Delta^3 y_0$ (ii) $\nabla^2 y_8 = y_8 - 2y_7 + y_6$; $\delta^2 y_5 = y_6 - 2y_5 + y_4$.

12. Evaluate :

(i) $\Delta^3 [(1-x)(1-2x)(1-3x)]$

(ii) $\Delta^{10}[(1-x)(1-2x^2)(1-3x^3)(1-4x^4)]$, if the interval of differencing is 2.

13. Express $x^3 - 2x^2 + x - 1$ into factorial polynomial. Hence show that $\Delta^4 f(x) = 0$ *(P.T.U., 2001)*
14. Express $u = x^4 - 12x^3 + 24x^2 - 30x + 9$ and its successive differences in factorial notation. Hence show that $\Delta^5 u = 0$.
15. Find the first and second differences of $x^4 - 6x^3 + 11x^2 - 5x + 8$ with $h = 1$. Show that the fourth difference is constant.
16. Obtain the function whose first difference is $2x^3 + 3x^2 - 5x + 4$.
17. Find the first term of the series whose second and subsequent terms are 8, 3, 0, −1, 0.
18. If $u(x)$ and $v(x)$ be two functions of x, prove that

(i) $\Delta[u(x)\, v(x)] = u(x)\, \Delta v(x) + v(x+1)\, \Delta u(x)$. (ii) $\Delta\left\{\dfrac{u(x)}{v(x)}\right\} = \dfrac{v(x)\, \Delta u(x) - u(x)\, \Delta v(x)}{v(x)\, v(x+1)}$.

29.4 (1) OTHER DIFFERENCE OPERATORS

We have already introduced the operators Δ, ∇ and δ. Besides these, there are the operators E and μ, which we define below :

(i) **Shift operator E** *is the operation of increasing the argument x by h* so that
$$Ef(x) = f(x+h),\ E^2 f(x) = f(x+2h),\ E^3 f(x) = f(x+3h) \text{ etc.}$$
The inverse operator E^{-1} is defined by $E^{-1} f(x) = f(x-h)$

If y_x is the function $f(x)$, then $Ey_x = y_{x+h}$, $E^{-1}y_x = y_{x-h}$, $E^n y_x = y_{x+nh}$,

where n may be any real number.

(*ii*) **Averaging operator** μ is defined by the equation $\mu y_x = \frac{1}{2}(y_{x+h/2} + y_{x-h/2})$

Obs. *In the difference calculus, Δ and E are regarded as the fundamental operators and ∇, δ, μ can be expressed in terms of these.*

(2) Relations between the operators. We shall now establish the following identities :

(*i*) $\Delta = \mathbf{E} - \mathbf{1}$ (*ii*) $\nabla = \mathbf{1} - \mathbf{E}^{-1}$

(*iii*) $\delta = \mathbf{E}^{1/2} - \mathbf{E}^{-1/2}$ (*iv*) $\mu = \frac{1}{2}(\mathbf{E}^{1/2} + \mathbf{E}^{-1/2})$

(*v*) $\Delta = \mathbf{E}\nabla = \nabla\mathbf{E} = \delta\mathbf{E}^{1/2}$ (*vi*) $\mathbf{E} = e^{\mathbf{hD}}$.

Proofs. (*i*) $\Delta y_x = y_{x+h} - y_x = Ey_x - y_x = (E-1)\,y_x$.

This shows that the operators Δ and E are connected by the symbolic relation

$$\Delta = E - 1 \quad \text{or} \quad E = 1 + \Delta$$

(*ii*) $$\nabla y_x = y_x - y_{x-h} = y_x - E^{-1} y_x = (1 - E^{-1}) y_x$$

$\therefore$ $$\nabla = 1 - E^{-1} \quad \text{or} \quad E = (1 - \nabla)^{-1}$$

(*iii*) $$\delta y_x = y_{x+h/2} - y_{x-h/2} = E^{1/2} y_x - E^{-1/2} y_x = (E^{1/2} - E^{-1/2}) y_x$$

$\therefore$ $$\delta = E^{1/2} - E^{-1/2}.$$

(*iv*) $$\mu y_x = \tfrac{1}{2}(y_{x+h/2} + y_{x-y/2}) = \tfrac{1}{2}(E^{1/2} y_x + E^{-1/2} y_x) = \tfrac{1}{2}(E^{1/2} + E^{-1/2})\, y_x$$

$\therefore$ $$\mu = \tfrac{1}{2}(E^{1/2} + E^{-1/2}).$$

(*v*) $$E\nabla y_x = E(y_x - y_{x-h}) = Ey_x - Ey_{x-h} = y_{x+h} - y_x = \Delta y_x \quad \therefore \quad E\nabla = \Delta$$

Also $$\nabla E y_x = \nabla y_{x+h} = y_{x+h} - y_x = \Delta y_x \quad \therefore \quad \nabla E = \Delta$$

$$\delta E^{1/2} y_x = \delta y_{x+h/2} = y_{x+h/2+h/2} - y_{x+h/2-h/2} = y_{x+h} - y_x = \Delta y_x$$

$\therefore$ $$\delta E^{1/2} = \Delta$$

Hence $$\Delta = E\nabla = \nabla E = \delta E^{1/2}.$$

(*vi*) $$Ef(x) = f(x+h)$$

$$= f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots \qquad \text{[By Taylor's series]}$$

$$= f(x) + hD\,f(x) + \frac{h^2}{2!} D^2 f(x) + \ldots = \left(1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \ldots\ldots\right) f(x) = e^{hD} f(x)$$

$\therefore$ $$E = e^{hD}$$

Cor. 1. $$E = 1 + \Delta = e^{hD}.$$

2. $$D = \frac{1}{h}\log(1+\Delta) = \frac{1}{h}\left(\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \ldots\right) \qquad \textit{(Burdwan, 2003)}$$

Note. *A table showing the symbolic relations between the various operators is given below for ready reference. To prove such relations between the operators, always express each operator in terms of the fundamental operator E.*

(3) Relations between the various operators

In terms of	E	Δ	∇	δ	hD
E		$\Delta + 1$	$(1+\nabla)^{-1}$	$1 + \frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$	e^{hD}
Δ	$E - 1$	—	$(1-\nabla)^{-1} - 1$	$\frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$	$e^{hD} - 1$
∇	$1 - E^{-1}$	$1 - (1+)^{-1} - 1$	—	$-\frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$	$1 - e^{-hD}$
δ	$E^{1/2} - E^{-1/2}$	$\Delta(1+\Delta)^{-1/2}$	$\nabla(1-\nabla)^{-1/2}$	—	$2\sinh(hD/2)$
μ	$\frac{1}{2}(E^{1/2} + E^{-1/2})$	$(1+\Delta/2)(1+\Delta)^{-1/2}$	$(1+\nabla/2)(1+\nabla)^{-1/2}$	$\sqrt{(1+\delta^2/4)}$	$\cosh(hD/2)$
hD	$\log E$	$\log(1+\Delta)$	$\log(1-)^{-1}$	$2\sinh^{-1}(\delta/2)$	

Example 29.7. *Prove that*

$$e^x = \left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{Ee^x}{\Delta^2 e^x}, \text{ the interval of differencing being } h. \qquad \textit{(Bhopal, 2009)}$$

Solution. Since $\left(\frac{\Delta^2}{E}\right)e^x = \Delta^2 . E^{-1}e^x = \Delta^2 e^{x-h} = \Delta^2 e^x . e^{-h} = e^{-h}\Delta^2 e^x$

$\therefore$ $$\text{R.H.S.} = e^{-h}\Delta^2 e^x . \frac{Ee^x}{\Delta^2 e^x} = e^{-h} Ee^x = e^{-h} . e^{x+h} = e^x.$$

Example 29.8. *Prove with the usual notations, that*

(*i*) $hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$ *(Rohtak, 2005)*

(*ii*) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2 + \Delta$ *(Bhopal, 2009 ; U.P.T.U., 2009)*

(*iii*) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$ (*iv*) $\Delta^3 y_2 = \nabla^3 y_5$.

Solution. (*i*) We know that $e^{hD} = E = 1 + \Delta \quad \therefore \quad hD = \log(1+\Delta)$

Also $hD = \log E = -\log(E^{-1}) = -\log(1-\nabla)$ $[\because \; E^{-1} = 1 - \nabla]$

We have proved that $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$ and $\delta = E^{1/2} - E^{-1/2}$

$\therefore$ $\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2}) = \frac{1}{2}(E - E^{-1}) = \frac{1}{2}(e^{hD} - e^{-hD}) = \sinh(hD)$

i.e. $hD = \sinh^{-1}(\mu\delta)$.

Hence $hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$

(*ii*) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = (E^{1/2} + E^{-1/2})E^{1/2} = E + 1 = 1 + \Delta + 1 = 2 + \Delta$.

(*iii*) $\frac{1}{2}\delta^2 + \delta\sqrt{(1+\delta^2/4)}$

$$= \tfrac{1}{2}(E^{1/2} - E^{-1/2})^2 + (E^{1/2} - E^{-1/2})\sqrt{[1 + (E^{1/2} - E^{-1/2})^2/4]}$$

$$= \tfrac{1}{2}(E + E^{-1} - 2) + (E^{1/2} - E^{-1/2})\sqrt{[(E + E^{-1} + 2)/4]}$$

$$= \tfrac{1}{2}(E + E^{-1} - 2) + \tfrac{1}{2}(E^{1/2} - E^{-1/2})(E^{1/2} + E^{-1/2})$$

$$= \tfrac{1}{2}[(E + E^{-1} - 2) + (E - E^{-1})] = \tfrac{1}{2}(2E - 2) = E - 1 = \Delta.$$

(*iv*) $\Delta^3 y_2 = (E-1)^3 y_2$ $[\because \; \Delta = E - 1]$

$= (E^3 - 3E^2 + 3E - 1)\,y_2 = y_5 - 3y_4 + 3y_3 - y_2$ ...(1)

$\nabla^3 y_5 = (1 - E^{-1})^3 y_5$ $[\because \; \Delta = 1 - E^{-1}]$

$= (1 - 3E^{-1} + 3E^{-2} - E^{-3})\,y_5 = y_5 - 3y_4 + 3y_3 - y_2$ ...(2)

From (1) and (2), $\Delta^3 y_2 = \nabla^3 y_5$.

29.5 TO FIND ONE OR MORE MISSING TERMS

When one or more values of $y = f(x)$ corresponding to the equidistant values of x are missing, we can find these using any of the following two methods :

First method : We assume the missing term or terms as a, b etc. and form the difference table. Assuming the last difference as zero, we solve these equations for a, b. These give the missing term/terms.

Second method : If n entries of y are given, $f(x)$ can be represented by a$(n-1)$th degree polynomial *i.e.*, $\Delta^n y = 0$. Since $\Delta = E - 1$, therefore $(E-1)^n y = 0$. Now expanding $(E-1)^n$ and substituting the given values, we obtain the missing term/terms.

Example 29.9. *Find the missing term in the table :* *(U.P.T.U., 2008)*

x :	2	3	4	5	6
y :	45.0	49.2	54.1	...	67.4

Solution. Let the missing term be a. Then the difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	45.0 $(= y_0)$				
		4.2			
3	49.2 $(= y_1)$		0.7		
		49.9		$a - 59.7$	
4	54.1 $(= y_2)$		$a - 59.0$		$240.2 - 4a$
		$a - 54.1$		$180.5 - 3a$	
5	$a(= y_3)$		$121.5 - a$		
		$67.4 - a$			
6	67.4 $(= y_4)$				

We know that $\Delta^4 y = 0$ *i.e.*, $240.2 - 4a = 0$.

Hence $a = 60.05$.

Otherwise: As only four entries y_0, y_1, y_2, y_3 are given, therefore $y = f(x)$ can be represented by a third degree polynomial.

$\therefore$ $\Delta^3 y =$ constant or $\Delta^4 y = 0$ *i.e.*, $(E - 1)^4 = 0$

i.e., $(E^4 - 4E^3 + 6E^2 - 4E + 1) = 0$ or $y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

Let the missing entry y_3 be a so that

$$67.4 - 4a + 6(54.1) - 4(49.2) + 45 = 0 \text{ or } - 4a = - 240.2$$

Hence $a = 60.05$.

Example 29.10. *Find the missing values in the following data :*

x :	45	50	55	60	65
y :	3.0	...	2.0	...	– 2.4

(Bhopal, 2007)

Solution. Let the missing value be a, b. Then the difference table is as follows :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	$3(= y_0)$			
		$a - 3$		
50	$a(= y_1)$		$5 - 2a$	
		$2 - a$		$3a + b - 9$
55	$2(= y_2)$		$b + a - 4$	
		$b - 2$		$3.6 - a - 36$
60	$b(= y_3)$		$- 0.4 - 2b$	
		$- 2.4 - b$		
65	$- 2.4\ (= y_4)$			

As only three entries y_0, y_2, y_4 are given, y can be represented by a second degree polynomial having third differences as zero.

$\therefore$ $$\Delta^3 y_0 = 0 \text{ and } \Delta^3 y_1 = 0$$

i.e., $$3a + b = 9, a + 3b = 3.6$$

Solving these, we get $a = 2.925, b + 0.0225$.

Otherwise. As only three entries $y_0 = 3, y_2 = 2, y_4 = - 2.4$ are given, y can be represented by a second degree polynomial having third differences as zero.

$\therefore$ $$\Delta^3 y_0 = 0 \quad \text{and} \quad \Delta^3 y_1 = 0$$

i.e., $$(E - 1)^3 y_0 = 0 \text{ and } (E - 1)^3 y_1 = 0$$

i.e., $$(E^3 - 3E^3 + 3E - 1) y_0 = 0 ; (E^3 - 3E^2 + 3E - 1) . y_1 = 0$$

or $$y_3 - 3y_2 + 3y_1 - y_0 = 0 ; y_4 - 3y_3 + 3y_2 - y_1 = 0$$

or $$y_3 + 3y_1 = 9 ; 3y_3 + y_1 = 3.6$$

Solving three, we get $y_1 = 2.925, y_2 = 0.225$.

Example 29.11. *If $y_{10} = 3$, $y_{11} = 6$, $y_{12} = 11$, $y_{13} = 18$, $y_{14} = 27$, find y_4.* (Mumbai, 2005)

Solution. Taking y_{14} as u_0, we are required to find y_4 *i.e.*, u_{-10}. Then the difference table is

x	u	Δu	$\Delta^2 u$	
x_{-4}	$y_{10} = u_{-4} = 3$			
		3		
x_{-3}	$y_{11} = u_{-3} = 6$		2	
		5		0
x_{-2}	$y_{12} = u_{-2} = 11$		2	
		7		0
x_{-1}	$y_{13} = u_{-1} = 18$		2	
		9		
x_0	$y_{14} = u_0 = 27$			

Then
$$y_4 = u_{-10} = (E^{-1})^{10} u_0 = (1 - \nabla)^{10} u_8$$
$$= \left(1 - 10\nabla + \frac{10.9}{2}\nabla^2 - \frac{10.9.8}{1.2.3}\nabla^3 + \ldots\right) u_0$$
$$= u_0 - 10\nabla u_0 + 45\nabla^2 u_0 - 120\nabla^3 u_0$$
$$= 27 - 10 \times 9 + 45 \times 2 - 120 \times 0 = 27.$$

Example 29.12. *If y_x is a polynomial for which fifth difference is constant and $y_1 + y_7 = -7845$, $y_2 + y_6 = 686$, $y_3 + y_5 = 1088$, find y_4.* (Mumbai, 2004)

Solution. Starting with y_1 instead of y_0, we note that $\Delta^6 y_1 = 0$ [$\because$ $\Delta^5 y_1$ is constant.

i.e., $$(E - 1)^6 y_1 = (E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1) y_1 = 0$$

$\therefore$ $$y_7 - 6y_6 + 15y_5 - 20y_4 + 15y_3 - 6y_2 + y_1 = 0$$

or $$(y_7 + y_1) - 6(y_6 + y_2) + 15(y_5 + y_3) - 20y_4 = 0$$

i.e. $$y_4 = \frac{1}{20}[(y_1 + y_7) - 6(y_2 + y_6) + 15(y_3 + y_5)]$$
$$= \frac{1}{20}[-784 - 6(686) + 15(1088)] = 571.$$

Example 29.13. *Prove the following identities :*

(*i*) $u_1 x + u_2 x^2 + u_3 x^3 + \ldots = \dfrac{x}{1-x} u_1 + \left(\dfrac{x}{1-x}\right)^2 \Delta u_1 + \left(\dfrac{x}{1-x}\right)^3 \Delta^2 u_1 + \ldots$

(*ii*) $u_0 + \dfrac{u_1 x}{1!} + \dfrac{u_2 x^2}{2!} + \dfrac{u_3 x^3}{3!} + \ldots = e^x \left(u_0 + x\Delta u_0 + \dfrac{x^2}{2!}\Delta^2 u_0 + \dfrac{x^3}{3!}\Delta^3 u_0 + \ldots\right).$

Solution. (*i*) L.H.S. $= xu_1 + x^2 E u_1 + x^3 E^2 u_1 + \ldots = x(1 + xE + x^2E^2 + \ldots) u_1$ [$\because$ $u_{x+h} = E^h u_x$]

$$= x \cdot \frac{1}{1 - xE} u_1, \text{ taking sum of infinite G.P.}$$

$$= x\left[\frac{1}{1 - x(1 + \Delta)}\right] u_1 \qquad [\because \; E = 1 + \Delta]$$

$$= x\left(\frac{1}{1 - x - x\Delta}\right) u_1 = \frac{x}{1-x}\left(1 - \frac{x\Delta}{1-x}\right)^{-1} u_1 = \frac{x}{1-x}\left(1 + \frac{x\Delta}{1-x} + \frac{x^2\Delta^2}{(1-x^2)} + \ldots\ldots\right) u_1$$

$$= \frac{x}{1-x} u_1 + \frac{x^2}{(1-x)^2}\Delta u_1 + \frac{x^2}{(1-x)^3}\Delta^2 u_1 + \ldots = \text{R.H.S.}$$

(ii) $$\text{L.H.S.} = u_0 + \frac{x}{1!}Eu_0 + \frac{x^2}{2!}E^2u_0 + \frac{x^3}{3!}E^3u_0 + \dots$$

$$= \left(1 + \frac{xE}{1!} + \frac{x^2E^2}{2!} + \frac{x^3E^3}{3!} + \dots\right)u_0 = e^{xE}\,u_0 = e^{x(1+\Delta)}\,u_0$$

$$= e^x \,.\, e^{x\Delta}\,u_0 = e^x\left(1 + \frac{x\Delta}{1!} + \frac{x^2\Delta^2}{2!} + \frac{x^3\Delta^3}{3!} + \dots\right)u_0$$

$$= e^x\left(u_0 + \frac{x}{1!}\Delta u_0 + \frac{x^2}{2!}\Delta^2 u_0 + \frac{x^3}{3!}\Delta^3 u_0 + \dots\right) = \text{R.H.S.}$$

PROBLEMS 29.2

1. Explain the difference between $\left(\frac{\Delta^2}{E}\right)u_x$ and $\frac{\Delta^2 u_x}{Eu_x}$. (*Madras, 2003*)

2. Evaluate taking h as the interval of differencing :

   (i) $\frac{\Delta^2}{E}\sin x$ (ii) $\left(\frac{\Delta^2}{E}\right)x^4$, $(h = 1)$ (*W.B.T.U., 2005*)

   (iii) $\left(\frac{\Delta^2}{E}\right)\sin(x+h) + \frac{\Delta^2\sin(x+h)}{E\sin(x+h)}$ (iv) $(\Delta + \nabla)^2(x^2 + x)$, $(h = 1)$.

3. With the usual notations, show that

   (i) $\nabla = 1 - e^{-hD}$ (ii) $D = \frac{2}{h}\sinh^{-1}\left(\frac{\delta}{2}\right)$

   (iii) $(1 + \Delta)(1 - \nabla) = 1$. (iv) $\Delta - \nabla = \nabla\Delta = \delta^2$. (*Mumbai, 2005*)

4. Prove that

   (i) $\delta = \Delta(1 + \Delta)^{-1/2} = \nabla(1 - \nabla)^{-1/2}$ (ii) $\mu^2 = 1 + \frac{\delta^2}{4}$ (*U.P.T.U, 2009*)

   (iii) $\delta(E^{1/2} + E^{-1/2}) = \Delta E^{-1} + \Delta$ (iv) $\nabla = \Delta E^{-1} = E^{-1}\Delta = 1 - E^{-1}$

5. Show that (i) $\mu\delta = \frac{1}{2}(\Delta + \nabla)$ (ii) $1 + \delta^2/2 = \sqrt{(1 + \delta^2\mu^2)}$ (*U.P.T.U., MCA, 2008*)

   (iii) $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$ (*U.P.T.U., 2009*) (iv) $\nabla^2 = h^2D^2 - h^3D^3 + \frac{7}{12}h^4D^4 - \dots\dots$

6. Prove that

   (i) $\nabla^r f_k = \Delta^r f_{k-r}$ (ii) $\Delta f_k^{\,2} = (f_k + f_{k+1})\Delta f_k$ (*J.N.T.U., MCA, 2006*)

   (iii) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{\left(1 + \frac{\delta^2}{4}\right)}$ (iv) $E^{1/2} = (1 + \delta^2/4)^{1/2} + \delta/2$.

7. Prove that $\nabla y_{n+1} = h\left(1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \dots\right)y'_n$.

8. The following table gives the values of y which is a polynomial of degree five. It is known that $f(3)$ is in error. Correct the error.

x :	0	1	2	3	4	5	6
y :	1	2	33	254	1025	3126	7777

(*Mumbai, 2004*)

9. Estimate the missing term in the following table :

x :	0	1	2	3	4
$f(x)$:	1	3	9	—	81

(*S.V.T.U., 2007*)

10. Find the missing terms of the following data :

x :	1	1.5	2	2.5	3	3.5	4
$f(x)$:	6	?	10	20	?	1.5	5

(*U.P.T.U., 2010*)

11. Find the missing values in the following table :

x:	0	1	2	3	4	5	6
y:	5	11	22	40	...	140	...

(V.T.U., 2006)

12. If $u_{13} = 1, u_{14} = -3, u_{15} = -1, u_{16} = 13$ find u_8. (Mumbai, 2004)
13. Evaluate y_4 from the following data (stating the assumptions you make) :
$$y_0 + y_8 = 1.9243, y_1 + y_7 = 1.9590, y_2 + y_6 = 1.9823, y_3 + y_5 = 1.9956.$$ (Mumbai, 2003)
14. Using the method of separation of symbols, prove that
(i) $u_0 + u_1 + u_2 + \ldots\ldots + u_n = {}^{n+1}C_1 u_0 + {}^{n+1}C_3 \Delta^2 u_0 + \ldots\ldots + {}^{n+1}C_{n+1} \Delta^n u_0$.
(ii) $y_x = y_n - {}^{n-x}C_1 \Delta y_{n-1} + {}^{n-x}C_2 \Delta^2 y_{n-2} - \ldots\ldots + (-1)^{n-x} \Delta^{n-x} y_{n-(n-x)}$.
15. Using the method of finite differences, sum the following series :
(i) 2.5 + 5.8 + 8.11 + 11.14 + ... to n terms.
(ii) 1.2.3 + 2.3.4 + 3.4.5 + ... to n terms.
16. Prove that $u_0 + u_1 x + u_2 x^2 + \ldots\ldots \infty = \dfrac{u_0}{1-x} + \dfrac{x\Delta u_0}{(1-x)^2} + \dfrac{x^2\Delta^2 u_0}{(1-x)^3} + \ldots\ldots \infty$
Hence sum the series $1.2 + 2.3x + 3.4x^2 + \ldots\ldots \infty$.

29.6 NEWTON'S INTERPOLATION FORMULAE*

We now derive two important interpolation formulae by means of the forward and backward differences of a function. These formulae are often employed in engineering and scientific problems.

(1) Newton's forward interpolation formula. Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number.

For any real number p, we have defined E such that

$$E^p f(x) = f(x + ph)$$

$$\therefore \quad y_p = f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^{-p} y_0 \qquad [\because \; E = 1 + \Delta]$$

$$= \left\{1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \ldots\right\} y_0 \qquad \text{[Using Binomial theorem]}$$

i.e.,
$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots \qquad \ldots(1)$$

It is called **Newton's forward interpolation formula** as (1) contains y_0 and the forward differences of y_0.

Obs. *This formula is used for interpolating the values of y near the beginning of a set of tabulated values and extrapolating values of y a little backward (i.e. to the left) of* y_0.

(2) Newton's backward interpolation formula. Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \ldots$ of x. Suppose it is required to evaluate $f(x)$ for $x = x_0 + ph$, where p is any real number. Then we have

$$y_p = f(x_n + ph) = E^p f(x_n) = (1 - \nabla)^{-p} y_n \qquad [\because \; E^{-1} = 1 - \nabla]$$

$$= \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^2 + \ldots\right] y_n \qquad \text{[Using Binomial theorem]}$$

i.e.,
$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots \qquad \ldots(2)$$

It is called *Newton's backward interpolation formula as* (2) contains y_n and backward differences of y_n.

Obs. *This formula is used for interpolating the values of y near the end of a set of tabulated values and also for extrapolating values of y a little ahead* (to the right) *of* y_n.

Example 29.14. *The table gives the distances in nautical miles of the visible horizon for the given heights in feet above the earth's surface :*

x = height :	100	150	200	250	300	350	400
y = distance :	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when (i) x = 218 ft (Madras, 2003 S) *(ii) 410 ft.* (V.T.U., 2002)

*See foot note p.466.

Solution. The difference table is as under :

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	13.03		– 0.39		
		2.01		0.15	
200	**15.04**		– 0.24		– 0.07
		1.77		0.08	
250	16.81		**– 0.16**		– 0.05
		1.61		**0.03**	
300	18.42		– 0.13		**– 0.01**
		1.48		**0.02**	
350	19.90		**– 0.11**		
		1.37			
400	**21.27**				

(*i*) If we take $x_0 = 200$, then $y_0 = 15.04$, $\Delta y_0 = 1.77$, $\Delta^2 y_0 = -0.16$, $\Delta^3 y_0 = 0.03$ etc.

Since $x = 218$ and $h = 50$, $\therefore\ p = \dfrac{x - x_0}{h} = \dfrac{18}{50} = 0.36$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{218} = y_0 + p\Delta y_0 + \frac{p(p-1)}{1\,.\,2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1\,.\,2\,.\,3}\Delta^3 y_0 + \ldots$$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36(-0.64)}{2}(-0.16) + \frac{0.36(-0.64)(-1.64)}{6}(0.03) + \ldots$$

$$= 15.04 + 0.637 + 0.018 + 0.001 + \ldots = 15.696 \quad i.e., \text{ 15.7 nautical miles}$$

(*ii*) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

$\therefore$ taking $x_n = 400$, $p = \dfrac{x - x_n}{h} = \dfrac{10}{50} = 0.2$

Using the line of backward differences

$$y_n = 21.27,\ \nabla y_n = 1.37,\ \nabla^2 y_n = -0.11,\ \nabla^3 y_n = 0.02 \text{ etc.}$$

$\therefore$ newton's backward formula gives

$$y_{410} = y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2}\nabla^2 y_{400} + \frac{p(p+1)(p+2)}{1\,.\,2\,.\,3\,.}\nabla^3 y_{400} + \ldots$$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2}(-0.11) + \ldots = 21.53 \text{ nautical miles.}$$

Example 29.15. *From the following table, estimate the number of students who obtained marks between 40 and 45 :*

Marks :	*30—40*	*40—50*	*50—60*	*60—70*	*70—80*
No. of Students :	*31*	*42*	*51*	*35*	*31*

(V.T.U., 2011 S ; S.V.T.U., 2007 ; Madras, 2006)

Solution. First we prepare the cumulative frequency table, as follows :

Marks less than (x) :	40	50	60	70	80
No. of Students (y_x) :	31	73	124	159	190

Now the difference table is

x	y	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	**31**				
		42			
50	73		**9**		
		51		**– 25**	
60	124		– 16		37
		35		12	
70	159		– 4		
		31			
80	190				

We shall find y_{45} *i.e.* number of students with marks less than 45.

Taking $x_0 = 40, x = 45$, we have $p = \dfrac{x - x_0}{h} = \dfrac{5}{10} = 0.5$ $\quad [\because \ h = 10]$

$\therefore$ using Newton's forward interpolation formula, we get

$$y_{45} = y_{40} + p\Delta y_{40} + \frac{p(p-1)}{2} \Delta^2 y_{40} + \frac{p(p-1)(p-2)}{1.2.3} \Delta^3 y_{40} + \ldots$$

$$= 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2} \times 9 + \frac{0.5(0.5)(-1.5)}{6} \times (-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24} \times 37$$

$= 47.87$, on simplification.

$\therefore$ the number of students with marks less than 45 is 47.87 *i.e.*, 48.

But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and 45 = 48 – 31 = 17.

Example 29.16. *Find the cubic polynomial which takes the following values :*

x :	0	1	2	3
$f(x)$:	1	2	1	10

Hence or otherwise evaluate f (4). (Bhopal, 2009 ; Rohtak, 2005 ; W.B.T.U., 2005)

Solution. The difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	**1**			
		1		
1	2		**–2**	
		–1		**12**
2	1		10	
		9		
3	10			

We take $x_0 = 0$ and $p = \dfrac{x - 0}{h} = x$ $\quad [\because \ h = 1]$

$\therefore$ using Newton's forward interpolation formula, we get

$$f(x) = f(0) + \frac{x}{1} \Delta f(0) + \frac{x(x-1)}{1.2} \Delta^2 f(0) + \frac{x(x-1)(x-2)}{1.2.3} \Delta^3 f(0)$$

$$= 1 + x(1) + \frac{x(x-1)}{2} (-2) + \frac{x(x-1)(x-2)}{6} (12)$$

$= 2x^3 - 7x^2 + 6x + 1$, which is the required polynomial.

To compute $f(4)$, we take $x_n = 3, x = 4$ so that $p = \dfrac{x - x_n}{h} = 1$ $\quad [\because \ h = 1]$

Using Newton's backward interpolation formula, we get

$$f(4) = f(3) + p\nabla f(3) + \frac{p(p+1)}{1.2} \nabla^2 f(3) + \frac{p(p+1)(p+2)}{1.2.3} \nabla^3 f(3)$$

$= 10 + 9 + 10 + 12 + 41.$

which is the same value as that obtained by substituting $x = 4$ in the cubic polynomial above.

Obs. The above example shows that *if a tabulated function is a polynomial, then interpolation and extrapolation give the same values.*

Example 29.17. *In the table below, the values of y are consecutive terms of a series of which 23.6 is the 6th term. Find the first and tenth terms of the series :*

x :	3	4	5	6	7	8	9
y :	4.8	8.4	14.5	23.6	36.2	52.8	73.9

(Anna, 2007)

Solution. The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3	4.8				
		3.6			
4	8.4		2.5		
		6.1		0.5	
5	14.5		3.0		0
		9.1		0.5	
6	23.6		3.5		0
		12.6		0.5	
7	36.2		4.0		0
		16.6		0.5	
8	52.8		4.5		
		21.1			
9	73.9				

To find the first term, use Newton's forward interpolation formula with $x_0 = 3, x = 1, h = 1$ and $p = -2$. We have

$$y(1) = 4.8 + \frac{(-2)}{1} \times 3.6 + \frac{(-2)(-3)}{1.2} \times 2.5 + \frac{(-2)(-3)(-4)}{1.2.3} \times 0.5 = 3.1$$

To obtain the tenth term, use Newton's backward interpolation formula with $x_n = 9, x = 10, h = 1$ and $p = 1$. This gives

$$y(10) = 73.9 + \frac{1}{1} \times 21.1 + \frac{1(2)}{1.2} \times 4.5 + \frac{1(2)(3)}{1.2.3} \times 0.5 = 100.$$

PROBLEMS 29.3

1. Using Newton's forward formula, find the value of $f(1.6)$, if

x :	1	1.4	1.8	2.2
$f(x)$:	3.49	4.82	5.96	6.5

(*J.N.T.U., 2006*)

2. State Newton's interpolation formula and use it to calculate the value of exp (1.85), given the following table :

x :	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$f(x)$:	5.474	6.050	6.686	7.389	8.166	9.025	9.974

(*Kottayam, 2005*)

3. If $f(1.15) = 1.0723$, $f(1.20) = 1.0954$, $f(1.25) = 1.1180$ and $f(1.30) = 1.1401$, find $f(1.28)$.
4. Given sin 45° = 0.7071, sin 50° = 0.7660, sin 55° = 0.8192, sin 60° = 0.8660, find sin 52°, using Newton's forward formula.
5. From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age of 46 :

Age :	45	50	55	60	65
Premium (in rupees) :	114.84	96.16	83.32	74.48	68.48

(*U.P.T.U., 2010*)

6. The area A of a circle of diameter d is given for the following values :

d :	80	85	90	95	100
A :	5026	5674	6362	7088	7854

(*V.T.U., 2010*)

Calculate the area of a circle of diameter 105.

7. Estimate the value of $f(22)$ and $f(42)$ from the following available data :

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

(*J.N.T.U., 2007*)

8. From the following table :

$x°$:	10	20	30	40	50	60	70	80
cos x :	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420	0.1737

Calculate cos 25° and cos 73° using Gregory Newton formulae. (*U.P.T.U., 2006*)

9. Find the number of men getting wages below Rs. 15 from the following data :

Wages in Rs. :	0—10	10—20	20—30	30—40
Frequency :	9	30	35	42

(Nagarjuna, 2001)

10. Find the polynomial interpolating the data :

x :	0	1	2
$f(x)$:	0	5	2

(U.P.T.U., 2008)

11. Construct Newton's forward interpolation polynomial for the following data :

x :	4	6	8	10
y :	1	3	8	16

(Madras, 2006)

Hence evaluate y for $x = 5$.

12. Construct the difference table for the following data :

x :	0.1	0.3	0.5	0.7	0.9	1.1	1.3
$f(x)$:	0.003	0.067	0.148	0.248	0.370	0.518	0.697

Evaluate $f(0.6)$ *(J.N.T.U., 2007)*

13. Estimate from following table $f(3.8)$ to three significant figures using Gregory Newton backward interpolation formula:

x :	0	1	2	3	4
$f(x)$:	1	1.5	2.2	3.1	4.6

(U.P.T.U., 2009)

14. The following table gives the population of a town during the last six censuses. Estimate the increase in the population during the period from 1976 to 1978 :

Year :	1941	1951	1961	1971	1981	1991
Population (in thousands) :	12	15	20	27	39	52

(U.P.T.U., 2009)

15. In the following table, the values of y are consecutive terms of a series of which 12.5 is the 5th term. Find the first and tenth terms of the series.

x :	3	4	5	6	7	8	9
y :	2.7	6.4	12.5	21.6	34.3	51.2	72.9

(P.T.U., 2001)

16. Given $u_1 = 40$, $u_3 = 45$, $u_5 = 54$, find u_2 and u_4. *(Nagarjuna, 2003 S)*

17. If $u_{-1} = 10$, $u_1 = 8$, $u_2 = 10$, $u_4 = 50$, find u_0 and u_3.

18. Given $y_0 = 3, y_1 = 12, y_2 = 81, y_3 = 200, y_4 = 100, y_5 = 8$, without forming the difference table, find $\Delta^5 y_0$.

29.7 CENTRAL DIFFERENCE INTERPOLATION FORMULAE

In the preceding section, we derived Newton's forward and backward interpolation formulae which are applicable for interpolation near the beginning and end of tabulated values. Now we shall develop central difference formulae which are best suited for interpolation near the middle of the table.

If x takes the values $x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h$ and the corresponding values of $y = f(x)$ are $y_{-2}, y_{-1}, y_0, y_1, y_2$, then we can write the difference table in the two notations as follows :

x	y	1st diff.	2nd diff.	3rd diff.	4th diff.
$x_0 - 2h$	y_{-2}				
		$\Delta y_{-2} (= \delta y_{-3/2})$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2} (= \delta^2 y_{-1})$		
		$\Delta y_{-1} (= \delta y_{-1/2})$		$\Delta^3 y_{-2} (= \delta^3 y_{-1/2})$	
x_0	y_0		$\Delta^2 y_{-1} (\delta^2 y_0)$		$\Delta^4 y_{-2} (= \delta^4 y_0)$
		$\Delta y_0 (= \delta y_{1/2})$		$\Delta^3 y_{-1} (= \delta^3 y_{1/2})$	
$x_0 + h$	y_1		$\Delta^2 y_0 (= \delta^2 y_1)$		
		$\Delta y_1 (= \delta y_{3/2})$			
$x_0 + 2h$	y_2				

(1) Gauss's forward interpolation formula. The Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1 \cdot 2} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{1 \cdot 2 \cdot 3} \Delta^3 y_0 + \dots \qquad \dots(1)$$

We have $\quad \Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

i.e., $$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1} \qquad \text{...(2)}$$

Similarly $$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \qquad \text{...(3)}$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \text{ etc.} \qquad \text{...(4)}$$

Also $\quad \Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

i.e., $$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

Similarly $$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \text{ etc.} \qquad \text{...(5)}$$

Substituting for $\Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0$ ... from (2), (3), (4) ... in (1), we get

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{1.2.3}(\Delta^3 y_{-1} + \Delta^4 y_{-1})$$
$$+ \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots.$$

Hence $$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots \text{ [Using (5)]}$$

which is called *Gauss's forward interpolation formula.*

Cor. *In the central differences notation, this formula will be*

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{1/2} + \frac{(p+1)p(p-1)(p-2)}{4!}\delta^4 y_0 + \ldots$$

Obs. 1. *It employs odd differences just below the central line and even difference on the central line as shown below:*

$$y_0 \cdots\cdots \Delta^2 y_{-1} \cdots\cdots \Delta^4 y_{-2} \cdots\cdots \Delta^6 y_{-3} \cdots\cdots \text{ Central line}$$
$$\Delta y_0 \qquad \Delta^3 y_{-1} \qquad \Delta^5 y_{-2} \qquad \Delta^7 y_{-3}$$

Obs. 2. This formula is used to interpolate the values of y for p $(0 < p < 1)$ measured forwardly from the origin.

(2) Gauss's backward interpolation formula. The Newton's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{1.2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{1.2.3}\Delta^3 y_0 + \ldots \qquad \text{...(1)}$$

We have $\quad \Delta y_0 - \Delta y_{-1} = \Delta^2 y_{-1}$

i.e., $$\Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1} \qquad \text{...(2)}$$

Similarly $$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1} \qquad \text{...(3)}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \text{ etc.} \qquad \text{...(4)}$$

Also $\quad \Delta^3 y_{-1} - \Delta^3 y_{-2} = \Delta^4 y_{-2}$

i.e., $$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2} \qquad \text{...(5)}$$

Similarly $$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \text{ etc.} \qquad \text{...(6)}$$

Substituting for $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$, ... from (2), (3), (4) in (1), we get

$$y_p = y_0 + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{1.2}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{1.2.3}(\Delta^3 y_{-1} + \Delta^4 y_{-1})$$
$$+ \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \ldots$$
$$= y_0 + p\Delta y_{-1} + \frac{(p+1)p}{1.2}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{1.2.3}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{1.2.3.4}\Delta^4 y_{-1}$$
$$+ \frac{p(p-1)(p-2)(p-3)}{1.2.3.4}\Delta^5 y_{-1} + \ldots$$

$$= y_0 + p\Delta y_{-1} + \frac{(p+1)p}{1.2}\Delta^2 y_{-1} + \frac{(p+1)\,p\,(p-1)}{1.2.3}(\Delta^3 y_{-2} + \Delta^4 y_{-2})$$

$$+ \frac{(p+1)\,p\,(p-1)(p-2)}{1.2.3.4}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \ldots \quad \text{[Using (5) and (6)]}$$

Hence $$y_p + y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p\,(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)\,p\,(p-1)}{4!}\Delta^4 y_{-2} + \ldots$$

which is called *Gauss's backward interpolation formula.*

Cor. *In the central differences notation, this formula will be*

$$y_p = y_0 + p\delta y_{-1/2} + \frac{(p+1)p}{2!}\delta^2 y_0 + \frac{(p+1)\,p(p-1)}{3!}\delta^3 y_{-1/2} + \frac{(p+2)(p+1)\,p\,(p-1)}{4!}\delta^4 y_0 + \ldots$$

Obs. 1. This formula contains odd differences above the central line and even differences on the central line as shown below :

$$\begin{array}{ccccccc} & \Delta y_{0-1} & & \Delta^3 y_{-2} & & \Delta^5 y_{-3} & \\ \nearrow & & \searrow \quad \nearrow & & \searrow \quad \nearrow & & \searrow \\ y_0 \ldots\ldots & & \Delta^2 y_{-1} \ldots\ldots & & \Delta^4 y_{-2} \ldots\ldots & & \Delta^6 y_{-3} \ldots\ldots \text{Central line}\end{array}$$

Obs. 2. It is used to interpolate the values of y for a negative value of p lying between -1 and 0.

Obs. 3. Gauss's forward and backward formulae are not of much practical use. However, these serve as intermediate steps for obtaining the important formulae of the following sections.

(3) Stirling's formula.* Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(1)$$

Gauss's backward interpolation formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)\,p\,(p-1)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(2)$$

Taking the mean of (1) and (2), we obtain

$$y_p = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p\,(p^2-1)}{3!} \times \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2\,(p^2-1)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(3)$$

which is called *Stirling's formula.*

Cor. *In the central differences notation, (3) takes the form*

$$y_p = y_0 + p\,\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \frac{p\,(p^2+1^2)}{3!}\mu\delta^3 y_0 + \frac{p^2\,(p^2-1^2)}{4!}\delta^4 y_0 + \ldots \quad \ldots(4)$$

for $$\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) = \frac{1}{2}(\delta y_{1/2} + \delta y_{-1/2}) = \mu\delta y_0$$

$$\frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) = \frac{1}{2}(\delta^3 y_{1/2} + \delta^3 y_{-1/2}) = \mu\delta^3 y_0 \text{ etc.}$$

Obs. *This formula involves means of the odd differences just above and below the central line and even differences on this line* as shown below :

$$\ldots y_0 \ldots \begin{pmatrix}\Delta y_{-1} \\ \Delta y_0\end{pmatrix} \ldots \Delta^2 y_{-1} \ldots \begin{pmatrix}\Delta^3 y_{-2} \\ \Delta^3 y_{-1}\end{pmatrix} \ldots \Delta^4 y_{-2} \ldots \begin{pmatrix}\Delta^5 y_{-3} \\ \Delta^5 y_{-2}\end{pmatrix} \ldots \Delta^6 y_{-3} \ldots \text{ Central line}$$

(4) Bessel's formula.** Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p\,(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} \ldots \quad \ldots(1)$$

*Named after the Scotish mathematicians *James Stirling* (1692-1770).

**See footnote p. 550.

We have $\Delta^2 y_0 - \Delta^2 y_{-1} = \Delta^3 y_{-1}$

i.e., $$\Delta^2 y_{-1} = \Delta^2 y_0 - \Delta^3 y_{-1} \quad ...(2)$$

Similarly $$\Delta^4 y_{-2} = \Delta^4 y_{-1} - \Delta^5 y_{-2} \text{ etc.} \quad ...(3)$$

Now (1) can be written as

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{1}{2}\Delta^2 y_{-1} + \frac{1}{2}\Delta^2 y_{-1}\right) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1} + \frac{p(p^2-1)(p-2)}{4!}\left(\frac{1}{2}\Delta^4 y_{-2} + \frac{1}{2}\Delta^4 y_{-2}\right) + \ldots$$

$$= y_0 + p\Delta y_0 + \frac{1}{2}\frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{1}{2}\frac{p(p-1)}{2!}(\Delta^2 y_0 - \Delta^3 y_{-1}) + \frac{p(p^2-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{1}{2}\frac{p(p^2-1)(p-2)}{4!} \times (\Delta^4 y_{-1} - \Delta^5 y_{-2}) + \ldots \quad \text{[Using (2), (3) etc.]}$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \cdot \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{p(p-1)}{2!} \times \left(\frac{p+1}{3} - \frac{1}{2}\right)\Delta^3 y_{-1} + \frac{p(p^2-1)(p-2)}{4!} \cdot \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots$$

Hence $$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \cdot \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p-1/2)\,p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!} \cdot \frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \ldots \quad ...(4)$$

which is known as the *Bessel's formula.*

Cor. *In the central differences notation, (4) becomes*

$$y_p = y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\mu\delta^2 y_{1/2} + \frac{(p-1/2)\,p(p-1)}{3!}\delta^3 y_{1/2} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\mu\delta^4 y_{1/2} + \ldots \quad ...(5)$$

for $$\frac{1}{2}(\Delta^2 y_{-1} + \Delta^2 y_0) = y\delta^2 y_{1/2},\ \frac{1}{2}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) = \mu\delta^4 y_{1/2} \text{ etc.}$$

Obs. *This is a very useful formula for practical purposes. It involves odd differences below the central line and means of even differences of and below his line as shown below :*

$$\begin{array}{ccccccccl} y_0 & & \left.\begin{array}{c}\Delta^2 y_{-1}\\ \\ \end{array}\right\} & & \left.\begin{array}{c}\Delta^4 y_{-2}\\ \\ \end{array}\right\} & & \left.\begin{array}{c}\Delta^6 y_{-3}\\ \\ \end{array}\right\} & & \text{Central line} \\ & \Delta y_0 & & \Delta^3 y_{-1} & & \Delta^5 y_{-2} & & & \\ & & \Delta^2 y_0 & & \Delta^4 y_{-1} & & \Delta^6 y_{-2} & & \end{array}$$

(5) Everett's formula. Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)\,p(p-1)(p-2)}{5!}\Delta^5 y_{-2} + \ldots \quad ...(1)$$

We eliminate the odd difference in (1) by using the relations

$$\Delta y_0 = y_1 - y_0,\ \Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1},\ \Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2} \text{ etc.}$$

Then (1) becomes

$$y_p = y_0 + p(y_1 - y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}(\Delta^2 y_0 - \Delta^2 y_{-1})$$

$$+ \frac{(p+1)\,p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \frac{(p+2)(p+1)\,p(p-1)(p-2)}{5!} \times (\Delta^4 y_{-1} - \Delta^4 y_{-2}) + \ldots$$

$$= (1-p)\,y_0 + py_1 - \frac{p(p-1)(p-2)}{3!}\Delta^2 y_{-1} + \frac{(p+1)\,p(p-1)}{3!}\Delta^2 y_0$$

$$-\frac{(p+1)\,p\,(p-1)(p-2)(p-3)}{5\,!}\Delta^4 y_{-2}+\frac{(p+2)(p+1)\,p\,(p-1)(p-2)}{5\,!}\Delta^4 y_{-1}-\ldots$$

To change the terms with negative sign, putting $p = 1 - q$, we obtain

$$y_p = qy_0 + \frac{q(q^2-1^2)}{3\,!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5\,!}\Delta^4 y_{-2} + \ldots\ldots + py_1 + \frac{p(p^2-1^2)}{3\,!}\Delta^2 y_0 + \frac{p\,(p^2-1^2)(p^2-2^2)}{5\,!}\Delta^4 y_{-1} + \ldots$$

This is known as *Everett's formula*.

Obs. *This formula is extensively used and involves only even differences on and below the central line* as shown below :

$$\begin{array}{ccccl} y_0 \ldots\ldots & \Delta^2 y_{-1}\ldots\ldots & \Delta^4 y_{-2}\ldots\ldots & \Delta^6 y_{-3}\ldots\ldots & \text{Central line} \\ - & - & - & - & \\ y_1 & \Delta^2 y_0 & \Delta^4 y_{-1} & \Delta^6 y_{-2} & \end{array}$$

29.8 CHOICE OF AN INTERPOLATION FORMULA

The coefficients in the central difference formulae are smaller and converge faster than those in Newton's formulae. After a few terms, the coefficients in the Stirling's formula decrease more rapidly than those of the Bessel's formula and the coefficients of Bessel's formula decrease more rapidly than those of Newton's formula. As much, whenever possible, *central difference formulae should be used in preference to Newton's formulae.*

The right choice of an interpolation formula however, depends on the position of the interpolated value in the given data.

The following rules will be found useful :

1. *To find a tabulated value near the beginning of the table, use Newton's forward formula.*
2. *To find a value near the end of the table, use Newton's backward formula.*
3. *To find an interpolated value near the centre of the table, use either Stirling's or Bessel's or Everett's formula.*

If interpolation is required for p lying between –1/4 and 1/4, prefer Stirling's formula.

If interpolation is desired for p lying between 1/4 and 3/4, use Bessel's or Everett's formula.

Example 29.18. *Find f(22) from the Gauss forward formula :*

x :	20	25	30	35	40	45
$f(x)$:	354	332	291	260	231	204

(J.N.T.U., 2007)

Solution. Taking $x_0 = 25$, $h = 5$, we have to find the value of $f(x)$ for $x = 22$.

i.e., for $p = \dfrac{x - x_0}{h} = \dfrac{22-25}{5} = -0.6$

The difference table is as follows :

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
20	– 1	354 $(= y_{-1})$					
			– 22				
25	0	332 $(= y_0)$		$\underline{-19}$			
			– 41		29		
30	1	291 $(= y_1)$		10		$\underline{-37}$	
			– 31		$\underline{-8}$		45
35	2	260 $(= y_2)$		2		8	
			– 29		0		
40	3	231 $(= y_3)$		2			
			– 27				
45	4	204 $(= y_4)$					

Gauss forward formula is

$$y^p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta y_{-1} + \frac{(p+1)\,p\,(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{(p+1)\,p\,(p-1)\,(p-2)}{4!}\Delta^4 y_{-2} + (p+1)\,(p-1)\,(p-2)\,(p+2)\,\Delta^5 y_{-2}$$

$$\therefore \quad f(22) = 332 + (0.6)\,(-41) + \frac{(-0.6)\,(-0.6-1)}{2!}\,(-19) + \frac{(-0.6+1)\,(-0.6)\,(-0.6-1)}{3!}\,(-8)$$

$$+ \frac{(-0.6-1)\,(-0.6)\,(-0.6-1)\,(-0.6-2)}{4!}\,(-37)$$

$$+ \frac{(-0.6+1)\,(-0.6)\,(-0.6-1)\,(-0.6-2)\,(-0.6+2)}{5!}\,(45)$$

$$= 332 + 24.6 - 9.12 + 1.5392 - 0.5241$$

Hence $\quad f(22) = 347.983.$

Example 29.19. *Interpolate by means of Gauss's backward formula, the population of a town for the year 1974, given that :*

Year :	*1939*	*1949*	*1959*	*1969*	*1979*	*1989*
Population (in thousands) :	*12*	*15*	*20*	*27*	*39*	*52*

(Kottayam, 2005 ; Madras, 2003)

Solution. Taking $x_0 = 1969$, $h = 10$, the population of the town is to be found for $p = \frac{1974 - 1969}{10} = 0.5$.

The central difference table is

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$	$\Delta^5 y_p$
1939	−3	12					
			3				
1949	−2	15		2			
			5		0		
1959	−1	20		2		3	
			7		3		−10
1969	0	27		5		−7	
			12		−4		
1979	1	39		1			
			13				
1989	2	52					

Gauss's backward formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)\,p}{2!}\Delta^2 y_{-1} + \frac{(p+1)\,p\,(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+1)\,(p+1)\,p\,(p-1)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{(p+2)\,(p+1)\,p\,(p-1)\,(p-2)}{5!}\Delta^5 y_3 + \ldots$$

i.e.,

$$y_5 = 27 + (0.5)(7) + \frac{(1.5)(.5)}{2}\,(5) + \frac{(1.5)\,(.5)\,(-.5)}{6}\,(3) + \frac{(2.5)\,(1.5)\,(-.5)}{24}\,(-7)$$

$$+ \frac{(2.5)\,(1.5)\,(.5)\,(-.5)\,(-1.5)}{120}\,(-10)$$

$$= 27 + 3.5 + 1.875 - 0.1875 + 0.2743 - 0.1172 = 32.345 \text{ thousands approx.}$$

Example 29.20. *Given*

$\theta°$:	0	5	10	15	20	25	30
$\tan\theta$:	0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Using Stirling's formula, estimate the value of tan 16°. (Anna, 2005)

Solution. Taking the origin at $\theta° = 15°$, $h = 5°$ and $p = \dfrac{\theta - 15}{5}$, we have the following central difference table :

p						
−3	0.0000					
		0.08575				
−2	0.0875		0.0013			
		0.0888				
−1	0.1763		0.0028			
		0.0916		0.0015		
0	0.2679		0.0045		0.0002	
		0.0961		0.0017		− 0.0002
1	0.3640		0.0062		0.0000	
		0.1023		0.0017		0.0009
2	0.4663		0.0088		0.0009	
		0.1111		0.0026		
3	0.5774					

At $\theta = 16°$, $\qquad p = \dfrac{16-15}{5} = 0.2$

Stirling's formula is

$$y_p = y_0 + \frac{p}{1}\cdot\frac{\Delta y_{-1} + \Delta y_0}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\cdot\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \dots$$

$$\therefore \qquad y_{0.2} = 0.2679 + (0.2)\left(\frac{0.0916 + 0.0961}{2}\right) + \frac{(0.2)^2}{2!}(0.0045) + \dots$$

$$= 0.2679 + 0.01877 + 0.00009 + \dots = 0.28676$$

Hence $\qquad \tan 16° = 0.28676.$

Example 29.21. *Employ Stirling's formula to compute $y_{12.2}$ from the following table ($y_x = 1 + \log_{10}\sin x$) :*

$x°$:	10	11	12	13	14
$10^5 y_x$:	23,967	28,060	31,788	35,209	38,368

(V.T.U., 2004)

Solution. Taking the origin at $x_0 = 12°$, $h = 1$ and $p = x - 12$, we have the following central table :

p	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
−2	0.23967				
		0.04093			
−1	0.28060		− 0.00365		
		0.03728		0.00058	
0	0.31788		− 0.00307		− 0.00013
		0.034121		− 0.00045	
1	0.35209		− 0.00062		
		0.03159			
2	0.38368				

At $x = 12.2$, $p = 0.2$. (As p lies between − 1/4 and 1/4, the use of Stirling's formula will be quite suitable.)

Stirling's formula is

$$y_p = y_0 + \frac{p}{1} \cdot \frac{\Delta y_{-1} + \Delta y_{-0}}{2} + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} \cdot \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{p^2(p^2-1)}{4!} \Delta^4 y_{-2} + \ldots$$

When $p = 0.2$, we have

$$\therefore \quad y_{0.2} = 0.31788 + 0.2 \left(\frac{0.03728 + 0.03421}{2}\right) + \frac{(0.2)^2}{2}(-0.00307)$$

$$+ \frac{(0.2)[(0.2)^2 - 1]}{6}\left(\frac{0.00058 - 0.00045}{2}\right) + \frac{(0.2)^2 [(0.2)^2 - 1]}{24}(-0.00013)$$

$$= 0.31788 + 0.00715 - 0.00006 - 0.000002 + 0.0000002 = 0.32497.$$

Example 29.22. *Apply Bessel's formula to obtain y_{25}, given $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$.* (S.V.T.U., 2007 ; V.T.U., 2000 S)

Solution. Taking the origin at $x_0 = 24$, $h = 4$, we have $p = \frac{1}{4}(x - 24)$.

$\therefore$ The central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-1	2854			
		308		
0	*3162*		*74*	
		382		*−8*
1	3544		*66*	
		448		
2	3992			

At $x = 25$, $p = (25 - 24)/4 = 1/4$. (As p lies between 1/4 and 3/4, the use of Bessel's formula will yield accurate esult.)

Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{(p - 1/2)\, p\,(p-1)}{3!} \Delta^3 y_{-1} + \ldots \qquad \ldots(1)$$

When $p = 0.25$, we have

$$y_p = 3162 + 0.25 \times 382 + \frac{0.25(-0.75)}{2}\left(\frac{74+66}{2}\right) + \frac{(-0.25)\,0.25\,(-0.75)}{6}(-8)$$

$$= 3162 + 95.5 - 6 - 5625 - 0.0625 = 3250.875 \text{ approx.}$$

Example 29.23. *Apply Bessel's formula to find the value of f(27.5) from the table :*

x :	25	26	27	28	29	30
$f(x)$:	4.000	3.846	3.704	3.571	3.448	3.333

(U.P.T.U., 2009)

Solution. Taking the origin at $x_0 = 27$, $h = 1$, we have $p = x - 27$

The central difference table is

x	p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
25	-2	4.000				
			-0.154			
26	-1	3.846		0.012		
			-0.142		-0.003	
27	0	3.704		0.009		0.004
			-0.133		-0.001	
28	1	3.571		0.010		-0.001
			-0.123		-0.002	
29	2	3.448		0.008		
			-0.115			
30	3	3.333				

At $x = 27.5$, $p = 0.5$ (As p lies between 1/4 and 3/4, the use of Bessel's formula will yield accurate result)

Bessel's formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{\left(p - \frac{1}{2}\right) p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)\,p(p-1)(p-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \ldots$$

When $p = 0.5$, we have

$$y_p = 3.704 - \frac{(0.5)(0.5-1)}{2}\left(\frac{0.009 + 0.010}{2}\right) + 0$$
$$+ \frac{(0.5+1)(0.5)(0.5-1)(0.5-2)}{24}\left(\frac{-0.001 - 0.004}{2}\right)$$
$$= 3.704 - 0.11875 - 0.00006 = 3.585$$

Hence $f(27.5) = 3.585$.

Example 29.24. *Given the table*

x :	310	320	330	340	350	360
$\log x$:	2.49136	2.50515	2.51851	2.53148	2.54407	2.55630

find the value of log 337.5 by Everett's formula.

Solution. Taking the origin at $x_0 = 330$ and $h = 10$, we have $p = \dfrac{x - 330}{10}$

∴ The central difference table is

p	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
−2	2.49136					
		0.01379				
−1	2.50515		−0.00043			
		0.01336		0.00004		
0	2.51881		−0.00039		−0.00003	
		0.01297		0.00001		0.00004
1	2.53148		−0.00038		0.00001	
		0.01259		0.00002		
2	2.54407		−0.00036			
		0.01223				
3	2.55630					

To evaluate log 337.5 *i.e.* for $x = 337.5$, $p = \dfrac{337.5 - 330}{10} = 0.75$

(As $p > 0.5$ and $= 0.75$, Everett's formula will be quite suitable)

Everett's formula is

$$y_p = qy_0 + \frac{q(q^2 - 1^2)}{3!}\Delta^2 y_{-1} + \frac{q(q^2-1^2)(q^2-2^2)}{5!}\Delta^4 y_{-2} + \ldots + py_1 + \frac{p(p^2-1^2)}{3!}\Delta^2 y_0$$
$$+ \frac{p(p^2-1^2)(p^2-2^2)}{5!}\Delta^4 y_{-1} + \ldots$$
$$= 0.25 \times 2.51851 + \frac{0.25\,(0.0625 - 1)}{6} \times (-0.00039) + \frac{0.25\,(0.0625-1)(0.0625-4)}{120}$$
$$\times (-0.00003) + 0.75 \times 2.53148 + \frac{0.75\,(0.5625-1)}{6} \times (-0.00038)$$
$$+ \frac{0.75\,(0.5625-1)(0.5625-4)}{120} \times (0.00001)$$
$$= 0.62963 + 0.00002 - 0.0000002 + 1.89861 + 0.00002 + 0.0000001 = 2.52828 \text{ nearly.}$$

PROBLEMS 29.4

1. Using Gauss's forward formula, evaluate $f(3.75)$ from the table :

x :	2.5	3.0	3.5	4.0	4.5	5.0	
y :	24.145	22.043	20.225	18.644	17.262	16.047	*(Bhopal, 2002 ; Madras, 2000)*

2. Using Gauss's backward difference formula, find $y(8)$ from the following table :

x :	0	5	10	15	20	25	
y :	7	11	14	18	24	32	*(J.N.T.U., 2007)*

3. Using Gauss's backward formula, estimate the number of persons earning wages between Rs. 60 and Rs. 70 from the following data :

Wages (₹) :	Below 40	40–60	60–80	80–100	100–120	
No. of persons : (in thousands)	250	120	100	70	50	*(Madras, 2000)*

4. From the following table :

x :	1.00	1.05	1.10	1.15	1.20	1.25	1.30	
e^x :	2.7183	2.8577	3.0042	3.1582	3.3201	3.4903	3.6693	*(U.P.T.U., 2006)*

Find $e^{1.17}$, using Gauss forward formula.

5. The pressure p of wind corresponding to velocity v is given by the following data. Estimate p when $y = 25$.

v :	10	20	30	40
p :	1.1	2	4.4	7.9

6. Using Stirling's formula find y_{35}, given $y_{20} = 512$, $y_{30} = 439$, $y_{40} = 346$, $y_{50} = 243$, where y_x represents the number of persons at age x years in a life table. *(Nagarjuna, 2003 S)*

7. Employ Bessel's formula to find the value of F at $x = 1.95$, given that

x :	1.7	1.8	1.9	2.0	2.1	2.2	2.3
F :	2.979	3.144	3.283	3.391	3.463	3.997	4.491

Which other interpolation formula can be used here ? Which is more appropriate ? Give reasons.

8. Calculate the value of $f(1.5)$ using Bessel's interpolation formula, from the following table :

x :	0	1	2	3	
$f(x)$:	3	6	12	15	*(U.P.T.U., 2008)*

9. Apply Everett's formula to obtain u_{25}, given $u_{20} = 854$, $u_{24} = 3162$, $u_{28} = 3544$, $u_{32} = 3992$. *(S.V.T.U., 2007)*

10. Using Everett's formula, evaluate $f(30)$, if $f(20) = 2854$, $f(28) = 3162$, $f(36) = 7088$, $f(44) = 7984$ *(U.P.T.U., 2006)*

11. Given the table :

x :	310	320	330	340	350	360
$\log x$:	2.4914	2.5052	2.5185	2.5315	2.5441	2.5563

Find the value of log 337.5 by Gauss's, Stirling's and Bessel's formulae.

29.9 INTERPOLATION WITH UNEQUAL INTERVALS

The various interpolation formulae derived so far possess the disadvantages of being applicable only to equally spaced values of the argument. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x. Now we shall study two such formulae :

(*i*) Lagrange's interpolation formula

(*ii*) Newton's general interpolation formula with divided differences.

29.10 LAGRANGE'S INTERPOLATION FORMULA

If $y = f(x)$ takes the value $y_0, y_1, \ldots, y_n$ corresponding to $x = x_0, x_1, \ldots, x_n$, then

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}\,y_0 + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}\,y_1 + \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}\,y_n \quad \ldots(1)$$

This is known as *Lagrange's interpolation formula for unequal intervals.*

Proof. Let $y = f(x)$ be a function which takes the values $(x_0, y_0), (x_1, y_1), ..., (x_n, y_n)$. Since there are $n + 1$ pairs of values of x and y, we can represent $f(x)$ by a polynomial in x of degree n. Let this polynomial be of the form

$$y = f(x) = a_0(x - x_1)(x - x_2) \dots (x - x_n) + a_1(x - x_0)(x - x_2) \dots (x - x_n)$$
$$+ a_2(x - x_0)(x - x_1)(x - x_3) \dots (x - x_n) + \dots + a_n(x - x_0)(x - x_1) \dots (x - x_{n-1}) \quad ...(2)$$

Putting $x = x_0, y = y_0$, in (2), we get

$$y_0 = a_0(x_0 - x_1)(x_0 - x_2)...(x_0 - x_n)$$
$$a_0 = y_0/[(x_0 - x_1)(x_0 - x_2)...(x_0 - x_n)]$$

Similarly putting $x = x_1, y = y_1$ in (2), we have $a_1 = y_1/[(x_1 - x_0)(x_1 - x_2)...(x_1 - x_n)]$

Proceeding the same way, we find $a_2, a_3...a_n$

Substituting the values of $a_0, a_1,..., a_n$ in (2), we get (1).

Obs. Lagranges interpolation formula (1) for n points is a polynomial of degree $(n - 1)$ which is known as *Lagrangian polynomial* and is very simple to implement on a computer.

This formula can also be used to split the given function into partial fractions.

For on dividing both sides of (1) by $(x - x_0)(x - x_1) \dots (x - x_n)$, we get

$$\frac{f(x)}{(x_0 - x_0)(x_0 - x_1)...(x_0 - x_n)} = \frac{y_0}{(x_0 - x_1)(x_0 - x_2)...(x_0 - x_n)} \cdot \frac{1}{x - x_0}$$
$$+ \frac{y_0}{(x_1 - x_0)(x_1 - x_2)...(x_1 - x_n)} \cdot \frac{1}{x - x_1} + \dots + \frac{y_n}{(x_n - x_0)(x_n - x_1)...(x_n - x_{n-1})} \cdot \frac{1}{x - x_n}.$$

Example 29.25. *Given the values*

x :	5	7	11	13	17
$f(x)$:	150	392	1492	2366	5202

evaluate f (9), using (i) Lagrange's formula. (Anna, 2006)

Solution. (*i*) Here $x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13, x_4 = 17$

and $y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202.$

Putting $x = 9$ and substituting the above values in Lagrange's formula, we get

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$
$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 + \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$
$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202 = -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = 810.$$

Example 29.26. *Find the polynomial f(x) by using Lagrange's formula and hence find f (3) for*

x :	0	1	2	5
$f(x)$:	2	3	12	147

(Anna, 2005)

Solution. Here $x_0 = 0, x_1, x_2 = 2, x_3 = 5$

and $y_0 = 2, y_1 = 3, y_2 = 12, y_3 = 147$

Lagrange's formula is

$$y = \frac{(x - x_1)(x - x_2)...(x - x_3)}{(x_0 - x_1)(x_0 - x_2)...(x_0 - x_3)} y_0 + \frac{(x - x_0)(x - x_2)...(x - x_3)}{(x_1 - x_0)(x_1 - x_2)...(x_1 - x_3)} y_1$$
$$+ \frac{(x - x_0)(x - x_1)...(x - x_3)}{(x_2 - x_0)(x_2 - x_1)...(x_2 - x_3)} y_2 + \frac{(x - x_0)(x - x_2)...(x - x_5)}{(x_3 - x_0)(x_3 - x_2)...(x_3 - x_2)} y_3$$
$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$
$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

Hence $f(x) = x^3 + x^2 - x + 2$

$\therefore$ $f(3) = 27 + 9 - 3 + 2 = 35.$

Example 29.27. *A curve passes through the point (0, 18), (1, 10), (3, – 18) and (6, 90). Find the slope of the curve at x = 2.* *(J.N.T.U., 2009)*

Solution. Here $x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 6$ and $y_0 = 18, y_1 = 10, y_2 = -18, y_3 = 90$

Since the values of x are unequally spaced, we use the Lagrange's formula :

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-1)(x-3)(x-6)}{(0-1)(0-2)(0-6)}(18) + \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}(10) + \frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18) + \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)}(90)$$

$$= (-x^3 + 10x^2 - 27x + 18) + (x^3 - 9x^2 + 18x) + (x^3 - 7x^2 + 6x) + (x^3 - 4x^2 + 3x)$$

i.e., $y = 2x^3 - 10x^2 + 18$

Thus the slope of the curve at $(x = 2) = \left(\frac{dy}{dx}\right)_{x=2}$

$$= (6x^2 - 20x)_{x=2} = -16.$$

Example 29.28. *Using Lagrange's formula, express the function* $\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ *as a sum of partial fractions.*

Solution. Let us evaluate $y = 3x^2 + x + 1$ for $x = 1, x = 2$ and $x = 3$

These values are

x :	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$
y :	$y_0 = 5$	$y_1 = 15$	$y_2 = 31$

Lagrange's formula is

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$= \frac{(x-1)(x-2)}{(0-1)(0-2)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-1)}(31)$$

Substituting the above values, we get

$$= \frac{(x-2)(x-3)}{(1-2)(1-3)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(31)$$

$$= 2.5(x-2)(x-3) - 15(x-1)(x-3) + 15.5(x-1)(x-2)$$

Thus $$\frac{3x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{2.5(x-2)(x-3) - 15(x-1)(x-3) + 15.5(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$= \frac{2.5}{x-1} - \frac{15}{x-2} + \frac{15.5}{x-3}.$$

Example 29.29. *Find the distance moved by a particle and its acceleration at the end of 4 seconds, if the time verses velocity data is as follows :*

t :	*0*	*1*	*3*	*4*
v :	*21*	*15*	*12*	*10*

Solution. Since the values of t are not equispaced, we use Lagrange's formula :

$$v = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)}v_0 + \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}v_1$$
$$+ \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)}v_2 + \frac{(t-t_0)(t-t_2)(t-t_5)}{(t_3-t_0)(t_3-t_2)(t_3-t_2)}v_3$$

$$v = \frac{(t-1)(t-3)(t-4)}{(-1)(-2)(-4)}(21) + \frac{t(t-3)(t-4)}{(1)(-2)(-3)}(15) + \frac{t(t-1)(t-4)}{(3)(2)(-1)}(12) + \frac{t(t-1)(t-3)}{(4)(3)(1)}(10)$$

i.e., $$v = \frac{1}{12}(-5t^3 + 38t^2 - 105t + 252)$$

$\therefore$ Distance moved $s = \int_0^4 v\,dt = \frac{1}{12}\int_0^4 (-5t^3 + 38t^2 - 105t + 252)\,dt$ $\left[\because\ v = \frac{ds}{dt}\right]$

$$= \frac{1}{12}\left(-\frac{5t^4}{4} + \frac{38t^3}{3} - \frac{105t^2}{2} + 252t\right)_0^4$$

$$= \frac{1}{12}\left(-320 + \frac{2432}{3} - 840 + 1008\right) = 54.9$$

Also acceleration $= \frac{dv}{dt} = \frac{1}{2}(-15t^2 + 76t - 105 + 0)$

Hence acceleration at $(t = 4) = \frac{1}{12}(-15(16) + 76(4) - 105) = -3.4.$

PROBLEMS 29.5

1. Use Lagrange's interpolation formula to find the value of y when $x = 10$, if the following values of x and y are given :

x :	5	6	9	11
y :	12	13	14	16

(*U.P.T.U., 2009 ; J.N.T.U., 2008*)

2. Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$, find by using Lagrange's formula, the value of $\log_{10} 656$. (*Hazaribagh, 2009*)

3. The following are the measurements T made on a curve recorded by oscilograph representing a change of current I due to a change in the conditions of an electric current.

T :	1.2	2.0	2.5	3.0
I :	1.36	0.58	0.34	0.20

Using Lagrange's formula, find I at $T = 16$. (*J.N.T.U., 2009*)

4. Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data :

Year :	1997	1999	2001	2002
Profit in Lakhs of ₹ :	43	65	159	248

(*Anna, 2004*)

5. Use Lagrange's formula to find the form of $f(x)$, given

x :	0	2	3	6
$f(x)$:	648	704	729	792

(*Madras, 2003 S*)

6. If $y(1) = -3$, $y(3) = 9$, $y(4) = 30$, $y(6) = 132$, find the Lagrange's interpolation polynomial that takes the same values as y at the given points. (*V.T.U., 2006*)

7. Given $f(0) = -18$, $f(1) = 0$, $f(3) = 0$, $f(5) = -248$, $f(6) = 0$, $f(9) = 13104$, find $f(x)$. (*Nagarjuna, 2003*)

8. Find the missing term in the following table using interpolation

x :	1	2	4	5	6
y :	14	15	5	...	9

9. Using Lagrange's formula, express the function $\frac{x^2 + x - 3}{x^3 - 2x^2 - x + 2}$ as sum of partial fractions.

29.11 DIVIDED DIFFERENCES

The Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated. The labour of recomputing the interpolation coefficients is saved by using Newton's general interpolation formula which employs what are called 'divided differences'. Before deriving this formula, we shall first define these differences.

If (x_0, y_0), (x_1, y_1), (x_2, y_2), ... be given points, then the *first divided difference* for the arguments, x_0, x_1 is defined by the relation $[x_0, x_1] = \dfrac{y_1 - y_0}{x_1 - x_0}$.

Similarly $[x_1, x_2] = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $[x_2, x_3] = \dfrac{y_3 - y_2}{x_3 - x_2}$ etc.

The *second divided difference* for x_0, x_1, x_2 is defined as $[x_0, x_1, x_2] = \dfrac{[x_1 - x_2] - [x_0, x_1]}{x_2 - x_0}$

The *third divided difference* for x_0, x_1, x_2, x_3 is defined as

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0} \text{ and so on.}$$

Obs. 1. *The divided differences are symmetrical in their arguments i.e. independent of the order of the arguments.*

For it is easy to write $[x_0, x_1] = \dfrac{y_0}{x_0 - x_1} + \dfrac{y_1}{x_1 - x_0} = [x_1, x_0]$ $[x_0, x_1, x_2]$

$$= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} \cdot \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

$= [x_1, x_2, x_0]$ or $[x_2, x_0, x_1]$ and so on.

Obs. 2. *The nth divided differences of a polynomial of the nth degree are constant.*

Let the arguments be equally spaced so that, $x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h$. Then

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} = \frac{1}{2h}\left\{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right\}$$

$$= \frac{1}{2!h^2}\Delta^2 y_0 \text{ and in general, } [x_0, x_1, x_2, \ldots x_n] = \frac{1}{n!h^n}\Delta^n y_0.$$

If the tabulated function is a nth degree polynomial, then $\Delta^n y_0$ will be constant. Hence the nth divided differences will also be constant.

29.12 NEWTON'S DIVIDED DIFFERENCE FORMULA

Let $y_0, y_1, \ldots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$. Then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

so that
$$y = y_0 + (x - x_0)[x, x_0] \quad \ldots(1)$$

Again
$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

which gives
$$[x, x_0] = [x_0, x_1] + (x - x_1) + (x - x_1)[x, x_0, x_1]$$

Substituting this value of $[x, x_0]$ in (1), we get

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \quad \ldots(2)$$

Also
$$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

which gives $[x, x_0, x_1,] = [x_0, x_1, x_2] + (x, x_2)[x, x_0, x_1, x_2]$

Substituting this value of $[x, x_0, x_1]$ in (2), we obtain

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2]$$

Proceeding in this manner, we get

$$y = f(x) = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \dots + (x - x_0)(x - x_1)\dots(x - x_n)[x, x_0, x_1, \dots, x_n] \quad \dots(3)$$

which is called *Newton's general interpolation formula with divided differences.*

Example 29.30. *Given the values*

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202,

evaluate $f(9)$, using Newton's divided difference formula. (V.T.U., 2010 ; P.T.U., 2005)

Solution. The divided difference table is

x	y	*1st divided differences*	*2nd divided differences*	*3rd divided differences*
5	150			
		$\frac{392-150}{7-5} = 121$		
7	392		$\frac{265-121}{11-5} = 24$	
		$\frac{1452-392}{11-7} = 265$		$\frac{32-24}{13-5} = 1$
11	1452		$\frac{457-265}{13-7} = 32$	
		$\frac{2366-1452}{13-11} = 457$		$\frac{42-32}{17-7} = 1$
13	2366		$\frac{709-457}{17-11} = 42$	
		$\frac{5202-2366}{17-13} = 709$		
17	5202			

Taking $x = 9$ in the Newton's divided difference formula, we obtain

$$f(9) = 150 + (9-5) \times 121 + (9-5)(9-7) \times 24 + (9-5)(9-7)(9-11) \times 1$$
$$= 150 + 484 + 192 - 16 = 810.$$

Example 29.31. *Determine $f(x)$ as a polynomial in x for the following data :*

x :	-4	-1	0	2	5
$f(x)$:	1245	33	5	9	1335

(V.T.U., 2007)

Solution. The divided differences table is

x	$f(x)$	*1st divided differences*	*2nd divided differences*	*3rd divided differences*	*4th divided differences*
-4	1245				
		-404			
-1	33		94		
		-28		-14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

Applying Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \ldots$$
$$= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94)$$
$$+ (x + 4)(x + 1)(x - 0)(-14) + (x + 4)(x + 1)x(x - 2)(3)$$
$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5.$$

PROBLEMS 29.6

1. Find the third divided difference with arguments 2, 4, 9, 10 of the function $f(x) = x^3 - 2x$. *(U.P.T.U., 2005)*

2. Use Newton's divided difference method to compute $f(5.5)$ from the following data :

x :	0	1	4	5	6
$f(x)$:	1	14	15	6	3

(U.P.T.U., 2010)

3. Using Newton's divided difference formula, evaluate $f(8)$ and $f(15)$ given :

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

(U.P.T.U., MCA, 2009, V.T.U., 2008)

4. Obtain the Newton's divided difference interpolation polynomial and hence find $f(6)$:

x :	3	7	9	10
$f(x)$:	168	120	72	63

(U.P.T.U., 2007)

5. Using Newton's divided difference interpolation, find the polynomial of the given data :

x :	-1	0	1	3
$f(x)$:	2	1	0	-1

(Anna, 2007)

6. For the following table, find $f(x)$ as a polynomial in x using Newton's divided difference formula:

x :	5	6	9	11
$f(x)$:	12	13	14	16

7. Using the following table, find $f(x)$ as a polynomial in

x :	-1	0	3	6	7
$f(x)$:	3	-6	39	822	1611

(U.P.T.U., 2009)

8. Find the missing term in the following table using Newton's divided difference formula

x :	0	1	2	3	4
y :	1	3	9	...	81

29.13 INVERSE INTERPOLATION

So far, given a set of values of x and y, we have been finding the values of y corresponding to a certain value of x. On the other hand, the process of estimating the value of x for a value of y (which is not in the table) is called the *inverse interpolation*.

Lagrange's formula is merely a relation between two variables either of which may be taken as the independent variable. Therefore, on inter-changing x and y in the Lagrange's formula, we obtain

$$x = \frac{(y - y_1)(y - y_2)\ldots(y - y_n)}{(y_0 - y_1)(y_0 - y_2)\ldots(y_0 - y_n)} x_0 + \frac{(y - y_0)(y - y_2)\ldots(y - y_n)}{(y_1 - y_0)(y_1 - y_2)\ldots(y_1 - y_n)} x_1$$
$$+ \ldots + \frac{(y - y_0)(y - y_1)\ldots(y - y_{n-1})}{(y_n - y_0)(y_n - y_1)\ldots(y_n - y_{n-1})} x_n \quad \ldots(1)$$

which is used for inverse interpolation.

Example 29.32. *The following table gives the values of x and y :*

x :	1.2	2.1	2.8	4.1	4.9	6.2
y :	4.2	6.8	9.8	13.4	15.5	19.6

Find the value of x corresponding to y = 12, using Lagrange's technique. *(V.T.U., 2009)*

Solution. Here $x_0 = 1.2, x_1 = 2.1, x_2 = 2.8, x_3 = 4.1, x_4 = 4.9, x_5 = 6.2$

and $y_0 = 4.2, y_1 = 6.8, y_2 = 9.8, y_3 = 13.4, y_4 = 15.5, y_5 = 19.6$

Taking $y = 12$, the above formula (1) gives

$$x = \frac{(12-6.8)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(4.2-6.8)(4.2-9.8)(4.2-13.4)(4.2-15.5)(4.2-19.6)} \times 1.2$$

$$+ \frac{(12-4.2)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(6.8-4.2)(6.8-9.8)(6.8-13.4)(6.8-15.5)(6.8-19.6)} \times 2.1$$

$$+ \frac{(12-4.2)(12-6.8)(12-13.4)(12-15.5)(12-19.6)}{(9.8-4.2)(9.8-6.8)(9.8-13.4)(9.8-15.5)(9.8-19.6)} \times 2.8$$

$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-15.5)(12-19.6)}{(13.4-4.2)(13.4-6.8)(13.4-9.8)(13.4-15.5)(13.4-19.6)} \times 4.1$$

$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)(12-19.6)}{(15.5-4.2)(15.5-6.8)(15.5-9.8)(15.5-13.4)(15.5-19.6)} \times 4.9$$

$$+ \frac{(12-4.2)(12-6.8)(12-9.8)(12-13.4)(12-15.5)}{(19.6-4.2)(19.6-6.8)(19.6-9.8)(19.6-13.4)(19.6-15.5)} \times 6.2$$

$$= 0.022 - 0.234 + 1.252 + 3.419 - 0.964 + 0.055 = 3.55.$$

Example 29.33. *Apply Lagrange's formula inversely to obtain a root of the equation $f(x) = 0$, given that $f(30) = -30$, $f(34) = -13$, $f(38) = 3$, and $f(42) = 18$.* *(V.T.U., 2009 S)*

Solution. Here $x_0 = 30, x_1 = 34, x_2 = 38, x_3 = 42$

and $y_0 = -30, y_1 = -13, y_2 = 3, y_3 = 18$

It is required to find x corresponding to $y = f(x) = 0$.

Taking $y = 0$, the Lagrange's formula gives,

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)} x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y-y_2)(y_1-y_3)} x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)} x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)} x_3$$

$$= \frac{13(-3)(-18)}{(-17)(-33)(-48)} \times 30 + \frac{30(-3)(-18)}{17(-16)(-31)} \times 34 + \frac{30(13)(-18)}{33(16)(-15)} \times 38 + \frac{30(13)(-3)}{48(31)(15)} \times 42$$

$$= -0.782 + 6.532 + 33.682 - 2.202 = 37.23$$

Hence the desired root of $f(x) = 0$ is 37.23.

PROBLEMS 29.7

1. Apply Lagrange's method to find the value of x when $f(x) = 15$ from the given data :

x :	5	6	9	11
$f(x)$:	12	13	14	16

(Madras, 2000)

2. Obtain the value of t when $A = 85$ from the following table, using Lagrange's method :

t :	2	5	8	14
A :	94.8	87.9	81.3	68.7

29.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 29.8

Select the correct answer or fill up the blanks in the following problems :

1. Newton's backward interpolation formula is ......
2. Bessel's formula is most appropriate when p lies between

(a) -0.25 and 0.25 (b) 0.25 and 0.75 (c) 0.75 and 1.00.

3. From the divided difference table for the following data :

x :	5	15	22
y :	7	36	160

4. Interpolation is the technique of estimating the value of a function for any ....
5. Bessel's formula for interpolation is ....
6. The 4th divided differences for x_0, x_1, x_2, x_3, x_4 = ...
7. Stirling's formula is best suited for p lying between ....
8. Newton's divided differences formula is ....
9. Given $(x_0, y_0), (x_1, y_1), (x_2, y_2)$, Lagrange's interpolation formula is ....
10. If $f(0) = 1$, $f(2) = 5$, $f(3) = 10$ and $f(x) = 14$, then $x = 0$ ....
11. Gauss forward interpolation formula involves
    (*a*) even differences above the central line and odd differences on the central line
    (*b*) even differences below the central line and odd differences on the central line
    (*c*) odd differences below the central line and even differences on the central line
    (*d*) odd differences above the central line and even differences on the central line.
12. If $y(1) = 4$, $y(3) = 12$, $y(4) = 19$ and $y(x) = 7$ find x using Lagrange's formula.
13. Extrapolation is defined as ....
14. The second divided difference of $f(x) = 1/x$, with arguments, a, b, c, is....... . (*Anna, 2007*)
15. Gauss-forward interpolation formula is used to interpolate values of y for
    (*a*) $0 < p < 1$ (*b*) $-1 < p < 0$
    (*c*) $0 < p < -\alpha$ (*d*) $-\alpha < p < 0$.
16. Given

x :	0	1	3	4
y :	– 12	0	6	12

Using Lagrange's formula, a polynomial that can be fitted to the data is ....

17. The nth divided difference of a polynomial of degree n is
    (*a*) zero (*b*) a constant
    (*c*) a variable (*d*) none of these.
18. If h is the interval of differencing, $\Delta^2 x^3$ = ...

CHAPTER 30

Numerical Differentiation & Integration

1. Numerical differentiation. 2. Formulae for derivatives. 3. Maxima and minima of a tabulated function. 4. Numerical integration. 5. Newton-Cotes quadrature formula. 6. Trapezoidal rule. 7. Simpson's 1/3rd rule. 8. Simpson's 3/8th rule. 9. Boole's rule. 10. Weddle's rule. 11. Objective Type of Questions.

30.1 NUMERICAL DIFFERENTIATION

It is the process of calculating the value of the derivative of a function at some assigned value of x from the given set of values (x_i, y_i). To compute dy/dx, we first replace the latter as many times as we desire. The choice of the interpolation formula to be used, will depend on the assigned value of x at which dy/dx, is desired.

If the values of x are equi-spaced and dy/dx, is required near the beginning of the table, we employ Newton's forward formula. If it is required near the end of the table, we use Newton's backward formula. For values near the middle of the table, dy/dx, is calculated by means of Stirling's or Bessel's formula. If the values of x are not equi-spaced, we use Newton's divided difference formula to represent the function.

30.2 FORMULAE FOR DERIVATIVES

Consider the function $y = f(x)$ which is tabulated for the values $x_i\ (= x_0 + ih)$, $i = 0, 1, 2, \ldots n$.

(1) Derivatives using forward difference formula. Newton's forward interpolation formula (p. 958) is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

Differentiating both sides w.r.t. p, we have

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \ldots$$

Since $p = \dfrac{(x-x_0)}{h}$, therefore $\dfrac{dp}{dx} = \dfrac{1}{h}\cdot$

Now
$$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4i}\Delta^4 y_0 + \cdots\right] \quad \ldots(1)$$

At $x = x_0, p = 0$. Hence putting $p = 0$,

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \cdots\right] \quad \ldots(2)$$

Again differentiating (1) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dp}\right)\frac{dp}{dx}$$

$$= \frac{1}{h}\left[\frac{2}{2!}\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \frac{12p^2-36p+22}{4!}\Delta^4 y_0 + \cdots\right]\frac{1}{h}$$

Putting $p = 0$, we obtain

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 + \cdots\right] \quad ...(3)$$

Similarly $$\left(\frac{d^3y}{dx^3}\right)_{x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right] \quad ...(4)$$

Otherwise : We know that $1 + \Delta = E = e^{hD}$

$\therefore$ $$hD = \log(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + ...$$

or $$D = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots\right]$$

and $$D^2 = \frac{1}{h^2}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots\right]^2 = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 + \cdots\right]$$

and $$D^3 = \frac{1}{h^2}\left[\Delta^3 - \frac{3}{2}\Delta^4 + \cdots\right]$$

Now applying the above identities to y_0, we get

Dy_0, *i.e.*, $$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0 + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0 - \cdots\right]$$

and $$\left(\frac{d^3y}{dx^3}\right)_{x_0} = \frac{1}{h^3}\left[\Delta^2 y_0 - \frac{3}{2}\Delta^4 y_0 + \cdots\right]$$

which are the same as (2), (3) and (4) respectively.

(2) Derivatives using backward difference formula. Newton's backward interpolation formula (p. 958) is

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

Differentiating both sides w.r.t. p, we get

$$\frac{dy}{dp} = \nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \cdots$$

Since $p = \dfrac{x-x_n}{h}$, therefore $\dfrac{dp}{dx} = \dfrac{1}{h}$.

Now $$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\nabla y_n \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^3+6p+2}{3!}\nabla^3 y_n + \cdots\right] \quad ...(5)$$

At $x = x_n$, $p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \cdots\right] \quad ...(6)$$

Again differentiating (5) w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx} = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6p+6}{3!}\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \cdots\right]$$

Putting $p = 0$, we obtain

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right] \qquad ...(7)$$

Similarly, $$\left(\frac{d^3y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right] \qquad ...(8)$$

Otherwise : We know that $1 - \nabla = E^{-1} = e^{-hD}$

$$\therefore \quad -hD = \log(1-\nabla) = -\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \cdots\right]$$

or $$D = \frac{1}{h}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{1}{4}\nabla^4 + \cdots\right]$$

$$\therefore \quad D^2 = \frac{1}{h^2}\left[\nabla + \frac{1}{2}\nabla^2 + \frac{1}{3}\nabla^3 + \cdots\right]^2 = \frac{1}{h^2}\left[\nabla^2 + \nabla^3 + \frac{11}{12}\nabla^4 + \cdots\right]$$

Similarly, $$D^3 = \frac{1}{h^3}\left[\nabla^3 + \frac{3}{2}\nabla^4 + \cdots\right]$$

Applying these identities to y_n, we get

Dy_n, *i.e.*, $$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y^n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + \frac{1}{6}\nabla^6 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left(\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right)$$

and $$\left(\frac{d^3y}{dx^3}\right)_{x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \cdots\right]$$

which are the same as (6), (7) and (8).

(3) Derivatives using central difference formulae. Stirling's formula (p. 964) is

$$y_p = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1^2)}{4!}\Delta^4 y_{-2} + \ldots$$

Differentiating both sides w.r.t. p., we get

$$\frac{dy}{dp} = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{2p}{2!}\Delta^2 y_{-1} + \frac{3p^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{4p^3-2p}{4!}\Delta^4 y_{-2} + \ldots$$

Since $p = \dfrac{x - x_0}{h}$, $\quad \therefore \quad \dfrac{dp}{dx} = \dfrac{1}{h}$.

Now $$\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + p\Delta^2 y_{-1} + \frac{3p^2-1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{2p^3-p}{12}\Delta^4 y_{-2} + \cdots\right]$$

At $x = x_0$, $p = 0$. Hence putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\,\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \frac{1}{30}\,\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} + \cdots\right] \qquad ...(9)$$

Similarly $$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \cdots\right] \qquad ...(10)$$

Obs. We can similarly use any other interpolation formula for computing the derivatives.

Example 30.1. *Given that*

x :	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y :	7.989	8.403	8.781	9.129	9.451	9.750	10.031

find $\dfrac{dy}{dx}$ *and* $\dfrac{d^2y}{dx^2}$ *at (a)* $x = 1.1$ *(V.T.V., 2006; Madras, 2003 S)*

(b) $x = 1.6$. *(Rohtak, 2006; J.N.T.U., 2004 S)*

Solution. (*a*) The difference table is :

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	7.989						
		0.414					
1.1	8.403		– 0.036				
		0.378		0.006			
1.2	8.781		– 0.030		– 0.002		
		0.348		0.004		0.001	
1.3	9.129		– 0.026		– 0.001		0.002
		0.322		0.003		0.003	
1.4	9.451		– 0.023		0.002		
		0.299		0.005			
1.5	9.750		– 0.018				
		0.281					
1.6	10.031						

We have

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^2 y_0 - \frac{1}{6}\Delta^6 y_0 + \cdots\right] \quad ...(i)$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 - \cdots\right] \quad ...(ii)$$

Here $h = 0.1, x_0 = 1.1, \Delta y_0 = 0.378, \Delta^2 y_0 = -0.03$ etc.

Substituting these values in (*i*) and (*ii*), we get

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004) - \frac{1}{4}(-0.001) + \frac{1}{5}(0.003)\right] = 3.952$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.1} = \frac{1}{(0.1)^2}\left[-0.03 - (0.004) + \frac{11}{12}(-0.001) - \frac{5}{6}(0.003)\right] = -3.74$$

(*b*) We use the above difference table and the backward difference operator ∇ instead of Δ.

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^2 y_n + \frac{1}{6}\nabla^6 y_n + \cdots\right] \quad ...(i)$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \cdots\right] \quad ...(ii)$$

Here $h = 0.1, x_n = 1.6, \nabla y_n = 0.281, \nabla^2 y_n = -0.018$ etc.

Putting these values in (*i*) and (*ii*), we get

$$\left[\frac{dy}{dx}\right]_{1.6} = \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(0.002) + \frac{1}{5}(0.003) + \frac{1}{6}(0.002)\right] = 2.75$$

$$\left(\frac{d^2y}{dx^2}\right)_{1.6} = \frac{1}{(0.1)^2}\left[-0.018 + 0.005 + \frac{11}{12}(0.002) + \frac{5}{6}(0.003) + \frac{137}{180}(0.002)\right]$$

$$= -0.715.$$

Example 30.2. *The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds. Find the initial acceleration using the entire data :*

Time t (sec) :	*0*	*5*	*10*	*15*	*20*
Velocity v (m/sec) :	*0*	*3*	*14*	*69*	*228*

(*Anna, 2004*)

Solution. The difference table is :

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
0	0				
		3			
5	3		8		
		11		36	
10	14		44		24
		55		60	
15	69		104		
		159			
20	228				

An initial acceleration $\left(i.e.\ \frac{dv}{dt}\right)$ at $t = 0$ is required, we use Newton's forward formula :

$$\left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{h}\left(\Delta v_0 - \frac{1}{2}\Delta^2 v_0 + \frac{1}{3}\Delta^3 v_0 - \frac{1}{4}\Delta^4 v_0 + \cdots\right)$$

$$\therefore \quad \left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{5}\left[3 - \frac{1}{2}(8) + \frac{1}{3}(36) - \frac{1}{4}(24)\right] = \frac{1}{5}\ (3 - 4 + 12 - 6) = 1$$

Hence the initial acceleration is 1 m/sec^2.

Example 30.3. *A slider in a machine moves along a fixed straight rod. Its distance x cm. along the rod is given below for various values of the time t seconds. Find the velocity of the slider and its acceleration when t = 0.3 seconds.*

t =	*0*	*0.1*	*0.2*	*0.3*	*0.4*	*0.5*	*0.6*
x =	*30.13*	*31.62*	*32.87*	*33.64*	*33.95*	*33.81*	*33.24*

(*V.T.U, 2009*)

Solution. The difference table is :

t	x	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
0	30.13						
		1.49					
0.1	31.62		– 0.24				
		1.25		– 0.24			
0.2	32.87		– 0.48		0.26		
		0.77		0.02		– 0.27	
0.3	33.64		– 0.46		– 0.01		0.29
		0.31		0.01		0.02	
0.4	33.95		– 0.45		0.01		
		– 0.14		0.02			
0.5	33.81		– 0.43				
		– 0.57					
0.6	33.24						

As the derivatives are required near the middle of the table, we use Stirling's formulae :

$$\left(\frac{dx}{dt}\right)_{t_0} = \frac{1}{h}\left(\frac{\Delta x_0 + \Delta x_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 x_{-2} + \Delta^5 x_{-3}}{2}\right) + \ldots \quad \ldots(i)$$

$$\left(\frac{d^2x}{dt^2}\right)_{t_0} = \frac{1}{h^2}\left[\Delta^2 x_{-1} - \frac{1}{12}\Delta^4 x_{-2} + \frac{1}{90}\Delta^6 x_{-3} \cdots\right] \quad ...(ii)$$

Here $h = 0.1$, $t_0 = 0.3$, $\Delta x_0 = 0.31$, $\Delta x_{-1} = 0.77$, $\Delta^2 x_{-1} = -0.46$ etc.

Putting these values in (*i*) and (*ii*), we get

$$\left(\frac{dx}{dt}\right)_{0.3} = \frac{1}{0.1}\left[\frac{0.31+0.77}{2} - \frac{1}{6}\left(\frac{0.01+0.02}{2}\right) + \frac{1}{30}\left(\frac{0.02-0.27}{2}\right) - \cdots\right] = 5.33$$

$$\left(\frac{d^2x}{dt^2}\right)_{0.3} = \frac{1}{(0.1)^2}\left[-0.46 - \frac{1}{12}(-0.01) + \frac{1}{90}(0.29) - \cdots\right] = -45.6$$

Hence the required velocity is 5.33 cm/sec and acceleration is – 45.6 cm/sec^2.

Example 30.4. *Using Bessel's formula, find f′ (7.5) from the following table :*

x :	7.47	7.48	7.49	7.50	7.51	7.52	7.53
$f(x)$:	0.193	0.195	0.198	0.201	0.203	0.206	0.208

(J.N.T.U., 2006)

Solution. Taking $x_0 = 7.50$, $h = 0.1$, we have $p = \dfrac{x - x_0}{h} = \dfrac{x - 7.50}{0.01}$

The difference table is :

x	p	y_p	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
7.47	– 3	0.193						
			0.002					
7.48	– 2	0.195		0.001				
			0.003		– 0.001			
7.49	– 1	0.198		0.000		0.000		
			0.003		– 0.001		0.003	
7.50	0	0.201		– 0.001		0.003		– 0.01
			0.002		0.002		– 0.007	
7.51	1	0.203		0.001		– 0.004		
			0.003		– 0.002			
7.52	2	0.206		– 0.001				
			0.002					
7.53	3	0.208						

Bessel's formula (p. 550) is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\cdot\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(p-\frac{1}{2}\right)p(p-1)}{3!}\cdot\Delta^3 y_{-1}$$
$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\cdot\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2} + \frac{\left(p-\frac{1}{2}\right)(p+1)p(p-1)(p-2)}{5!}\cdot\Delta^5 y_{-2}$$
$$+ \frac{(p+2)p(p+1)p(p-1)(p-2)(p-3)}{6!}\cdot\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2} + \cdots \quad ...(i)$$

Since $\quad p = \dfrac{x - x_0}{h}$, $\quad \therefore \quad \dfrac{dp}{dx} = \dfrac{1}{h}$ and $\dfrac{dy}{dx} = \dfrac{dy}{dp}\cdot\dfrac{dp}{dx} = \dfrac{1}{h}\dfrac{dy}{dp}$

Differentiating (*i*) w.r.t. p and putting $p = 0$, we get

$$\left(\frac{dy}{dx}\right)_{7.5} = \frac{1}{h}\left(\frac{dy}{dp}\right)_{p=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{h}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{24}(\Delta^4 y_{-2} + \Delta^4 y_{-1})\right.$$
$$\left. - \frac{1}{120}\Delta^5 y_{-2} - \frac{1}{240}(\Delta^6 y_{-3} + \Delta^6 y_{-2})\right]$$

$$\left(\frac{dy}{dx}\right)_{7.5} = \frac{1}{0.01}\left[0.002 - \frac{1}{4}(-0.001+0.001) + \frac{1}{12}(0.002)^1 + \frac{1}{24}(-0.004+0.003) - \frac{1}{120}(-0.007)\frac{-1}{240}(0.010+0)\right]$$

$[\because \ \Delta^6 y_{-2} = 0]$

$$= 0.2 + 0 + 0.01666 - 0.00583 + 0.00416 = 0.223.$$

Example 30.5. *Find $f'(0)$ from the following data :*

x :	3	5	11	27	34
$f(x)$:	– 13	23	899	17315	35606

Solution. As the values of x are not equi-spaced, we shall use Newton's divided difference formula. The divided difference table is

x	$f(x)$	*1st div. diff.*	*2nd div. diff.*	*3rd div. diff.*	*4th div. diff.*
3	– 13				
		18			
5	23		16		
		146		0.998	
11	899		39.96		0.0002
		1025		1.003	
27	17315		69.04		
		2613			
34	35606				

Fifth difference being zero, Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)\, f(x_0 - x_1) + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\, f(x_0, x_1, x_2, x_3) + (x - x_0)(x - x_1)$$
$$\times (x - x_2)(x - x_3)\, f(x_0, x_1, x_2\, x_3, x_4)$$

Differentiating it w.r.t. x, we get

$$f'(x) = f(x_0, x_1) + (2x - x_0 - x_1)\, f(x_0, x_1, x_2)$$
$$+ [3x^2 - 2x(x_0 + x_1 + x_2) + (x_0x_1 + x_1x_2 + x_2x_0)] \times f(x_0, x_1, x_2, x_3)$$
$$+ [4x^3 - 3x^2(x_0 + x_1 + x_2 + x_3) + 2x\,(x_0x_1 + x_1x_2 + x_2x_3 + x_3x_0 + x_1x_3 + x_0x_2)$$
$$- x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_0 + x_0x_1x_3]\, f(x_0, x_1, x_2, x_3, x_4)$$

Putting $x_0 = 3, x_1 = 5, x_2 = 11, x_3 = 27$ and $x = 10$, we obtain

$$f'(x) = 18 + 12 \times 16 + 23 \times 0.998 - 426 \times 0.0002 = 232.869.$$

30.3 MAXIMA AND MINIMA OF A TABULATED FUNCTION

Newton's forward interpolation formula is

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

Differentiating it w.r.t. p, we get

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{6}\Delta^3 y_0 + \ldots \quad \ldots(1)$$

For maxima or minima, $dy/dp = 0$. Hence equating the right hand side of (1) to zero and retaining only upto third differences, we obtain

$$\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{6}\Delta^3 y_0 = 0$$

i.e.,

$$\left(\frac{1}{2}\Delta^3 y_0\right)p^2 + (\Delta^2 y_0 - \Delta^3 y_0)\,p + \left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0\right) = 0$$

Substituting the values of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ from the difference table, we solve this quadratic for p. Then the corresponding values of $x = x_0 + ph$ at which y is maximum or minimum.

Example 30.6. *Find the maximum and minimum value of y from the following data :*

x :	-2	-1	0	1	2	3	4
y :	2	-0.25	0	-0.25	2	15.75	56

(Anna, 2004)

Solution. The difference table is :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
-2	2					
		-2.25				
-1	-0.25		2.5			
		0.25		-3		
0	0		-0.5		6	
		-0.25		3		0
1	-0.25		2.5		6	
		0.25		9		0
2	2		11.5		6	
		13.75		15		
3	15.75		26.5			
		40.25				
4	56					

Taking $x_0 = 0$, we have $y_0 = 0$, $\Delta y_0 = -0.25$, $\Delta^2 y_0 = 2.5$, $\Delta^3 y_0 = 9$, $\Delta^4 y_0 = 6$.
Newton's forward difference formula for the first derivative gives

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 - \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 - \cdots\right]$$

$$= \frac{1}{1}\left[-0.25 + \frac{2x-1}{2}(2.5) + \frac{1}{6}(3x^2-6x+2)(9) + \frac{1}{24}(4x^2-18x^2+22x-6)(6)\right]$$

$$= \frac{1}{1}[-0.25 + 2.5x - 1.25 + 4.5x^2 - 9x + 3 + x^3 - 4.5x^2 + 5.5x - 1.5] = x^3 - x$$

For y to be maximum or minimum, $\dfrac{dy}{dx} = 0$ *i.e.*, $x^3 - x = 0$

i.e., $$x = 0, 1, -1$$

Now $$\frac{d^2y}{dx^2} = 3x^2 - 1 = -\text{ve for } x = 0$$
$$= +\text{ve for } x = 1$$
$$= +\text{ve for } x = -1$$

Since $$y = y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + \ldots, y(0) = 0$$

Thus y is maximum for $x = 0$, and maximum value $= y(0) = 0$.
Also y is minimum for $x = 1$ and minimum value $= y(0) = -0.25$.

PROBLEMS 30.1

1. Find $y'(0)$ and y'' from the following table :

x :	0	1	2	3	4	5
y :	4	8	15	7	6	2

2. Find the first and second derivatives of $f(x)$ at $x = 1.5$ if

x :	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$:	3.375	7.000	13.625	24.000	38.875	59.000

(S.V.T.U., 2007)

3. Find the first and second derivatives of the function tabulated below, at the point $x = 1.1$:

x :	1.0	1.2	1.4	1.6	1.8	2.0
y :	0	0.128	0.544	1.296	2.432	4.000

(*U.P.T.U., 2010; Bhopal, 2009*)

4. Given the following table of values of x and y

x :	1.00	1.05	1.10	1.15	1.20	1.25	1.30
y :	1.000	1.025	1.049	1.072	1.095	1.118	1.140

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (*a*) $x = 1.05$ (*b*) $x = 1.25$ (*c*) $x = 1.15$. (*V.T.U., 2008*)

5. For the following values of x and y, find the first derivative at $x = 4$.

x :	1	2	4	8	10
y :	0	1	5	21	27

(*J.N.T.U., 2009*)

6. From the following table, find the values of dy/dx and d^2y/dx^2 at $x = 2.03$.

x :	1.96	1.98	2.00	2.02	2.04
y :	0.7825	0.7739	0.7651	0.7563	0.7473

(*Anna, 2005*)

7. Find the value of cos 1.74 from the following table :

x :	1.7	1.74	1.78	1.82	1.86
$\sin x$:	0.9916	0.9857	0.9781	0.9691	0.9584

(*J.N.T.U., 2009*)

8. The distance covered by an athlete for the 50 metre is given in the following table :

Time (sec) :	0	1	2	3	4	5	6
Distance (metre) :	0	2.5	8.5	15.5	24.5	36.5	50

Determine the speed of the athlete at $t = 5$ sec. correct to two decimals. (*U.P.T.U., 2009*)

9. The following data gives corresponding values of pressure and specific volume of a superheated stream.

v :	2	4	6	8	10
p :	105	42.7	25.3	16.7	13

Find the rate of change of

(*i*) pressure with respect to volume when $v = 2$,

(*ii*) volume with respect to pressure when $p = 105$.

10. The table below reveals the velocity v of a body during the specific time t, find its acceleration at $t = 1.1$?

t :	1.0	1.1	1.2	1.3	1.4
v :	43.1	47.7	52.1	56.4	60.8

(*J.N.T.U., 2009*)

11. The elevation above a datum line of 7 points of a road is given below :

x :	0	300	600	900	1200	1500	1800
y :	135	149	157	183	201	205	193

Find the gradient of the road at the middle point.

12. A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has turned for various values of the time t second.

t :	0	0.2	0.4	0.6	0.8	1.0	1.2
θ :	0	0.12	0.49	1.12	2.02	3.20	4.67

Calculate the angular velocity and the angular acceleration of the rod, when $t = 0.6$ second. (*V.T.U., 2004*)

13. Find the value of $f'(x)$ at $x = 0.4$ from the following table using Bessel's formula

x :	0.01	0.02	0.03	0.04	0.05	0.06
$f(x)$:	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

14. If $y = f(x)$ and y_n denotes $f(x_0 + nh)$, prove that, if powers of h above h^6 be neglected.

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{3}{4h}\left[(y_1 - y_{-1}) - \frac{1}{5}(y_2 - y_{-2}) + \frac{1}{45}(y_3 - y_{-3})\right]$$ (*U.P.T.U., 2006*)

[**Hint:** Differentiate Stirling's formula w.r.t. x, and put $x = 0$]

15. Find the value of $f'(8)$ from the table given below :

x :	6	7	9	12
$f(x)$:	1.556	1.690	1.908	2.158

(*Anna, 2007*)

16. Find the $f'(6)$ from the following data :

x :	0	2	3	4	7	8
$f(x)$:	4	26	58	112	466	922

(*J.N.T.U., 2009 ; U.P.T.U., 2008*)

17. Find the maximum and minimum values of y from the following table :

x :	0	1	2	3	4	5
$f(x)$:	0	0.25	0	2.25	16	56.25

18. Find the value of x for which $f(x)$ is minimum, using the table

x :	9	10	11	12	13	14
$f(x)$:	1330	1340	1320	1250	1120	930

Also find the maximum value of $f(x)$?

30.4 NUMERICAL INTEGRATION

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$ is called *numerical integration*. This process when applied to a function of a single variable, is known as *quadrature*.

The problem of numerical integration, like that of numerical differentiation, is solved by representing $f(x)$ by an interpolation formula and then integrating it between the given limits. In this way, we can derive quadrature formula for approximate integration of a function defined by a set of numerical values only.

30.5 NEWTON-COTES QUADRATURE FORMULA

Let $$I = \int_a^b f(x)\,dx$$

where $f(x)$ takes the values $y_0, y_1, y_2, \ldots y_n$ for $x = x_0, x_1, x_2, \ldots x_n$. (Fig. 30.1)

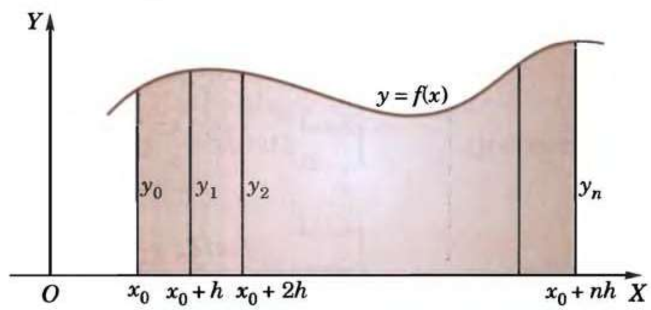

Fig. 30.1

Let us divide the interval (a, b) into n sub-intervals of width h so that $x_0 = a, x_1 = x_0 + h, x_2 = x_0 + 2h, \ldots x_n = x_0 + nh = b$. Then

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx = h\int_0^n f(x_0 + rh)\,dr,$$

putting $x = x_0 + rh$, $dx = hdr$

$$= h\int_0^n \Big[y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{6!}\Delta^6 y_0 + \cdots \Big] dr$$

[By Newton's forward interpolation formula]

Integrating term by, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = nh\Big[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \left(\frac{n^4}{5} - \frac{3n^3}{2} + \frac{11n^2}{3} - 3n\right)\frac{\Delta^4 y_0}{4!} + \left(\frac{n^5}{6} - 2n^4 + \frac{35n^3}{4} - \frac{50n^2}{3} + 12n\right)\frac{\Delta^5 y_0}{5!} + \left(\frac{n^6}{7} - \frac{15n^5}{6} + 17n^4 - \frac{225n^3}{4} + \frac{274n^2}{3} - 60n\right)\frac{\Delta^6 y_0}{6!} + \cdots \Big] \quad \ldots(A)$$

This is known as *Newton-Cotes quadrature formula*. From this general formula, we deduce the following important quadrature rules by taking $n = 1, 2, 3 \ldots$

30.6 TRAPEZOIDAL RULE

Putting $n = 1$ in (A) § 30.5 and taking the curve through (x_0, y_0) and (x_1, y_1) as a straight line *i.e.* a polynomial of first order so that differences of order higher than first become zero, we get

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}(y_0 + y_1)$$

Similarly $$\int_{x_0}^{x_0+2h} f(x)\,dx = h\left(y_1 + \frac{1}{2}\Delta y_1\right) = \frac{h}{2}(y_1 + y_2)$$

....

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{(n-1)} + y_n)$$

Adding these n integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})]$$

This is known as the ***trapezium rule.***

Obs. The area of each strip (trapezium) is found separately. Then the area under the curve and the *ordinates* at x_0 and $x_0 + nh$ is approximately equal to the areas of the trapeziums.

30.7 SIMPSON'S ONE-THIRD RULE

Putting $n = 2$ in (A) above and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola *i.e.*, a polynomial of second order so that differences of order higher than second vanish, we get

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right)\frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly, $$\int_{x_0+2h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4) \qquad \text{when}$$

....

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n),\ n \text{ being even.}$$

Adding all these integrals, we have (when n is even)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3) + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})]$$

This is known as the *Simpson's one-third rule* or simply *Simpson's rule* and is most commonly used.

Obs. While applying *Simpson's 1/3rd rule, the given interval must be divided into even number of equal subintervals*, since we find the area of two strips at a time.

30.8 SIMPSON'S THREE-EIGHTH RULE

Putting $n = 3$ in (A) above and taking the curve through (x_i, y_i): $i = 0, 1, 2, 3$ as a polynomial of third order so that differences above the third order vanish, we get

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}(y_0 + 3y_1 + 3y_2 + y_3)$$

Similarly,

$$\int_{x_0+3h}^{x_0+nh} f(x)\,dx = \frac{3h}{8}(y_3 + 3y_4 + 3y_5 + y_6) \text{ and so on.}$$

Adding all such expressions from x_0 to $x_0 + nh$, where n is a multiple of 3, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1}) + 2(y_3 + y_6 + \dots + y_{n-3})]$$

which is known as *Simpson's three-eighth rule*.

Obs. While applying *Simpson's 3/8th rule, the number of sub-intervals should be taken as multiple of 3.*

30.9 BOOLE'S RULE

Putting $n = 4$ in (A) above and neglecting all differences above the fourth, we obtain

$$\int_{x_0}^{x_0+4h} f(x)\,dx = 4h\left(y_0 + 2\Delta y_0\,\frac{5}{3}\Delta^2 y_0 + \frac{2}{3}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0\right)$$

$$= \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_8)$$

Similarly

$$\int_{x_0+4h}^{x_0+8h} f(x)\,dx = \frac{2h}{45}(7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 4, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8 + \ldots$$

This is known as *Boole's rule*.

Obs. While applying *Boole's rule*, the number of sub-intervals should be taken as a multiple of 4.

30.10 WEDDLE'S RULE

Putting $n = 6$ in (A) above and neglecting all differences above the sixth, we obtain

$$\int_{x_0}^{x_0+6h} f(x)\,dx = \left(y_0 + 3\Delta y_0 + \frac{9}{2}\Delta^2 y_0 + 4\Delta^3 y_0 + \frac{123}{60}\Delta^4 y_0 + \frac{11}{20}\Delta^5 x_0 + \frac{1}{6}\cdot\frac{41}{140}\Delta^6 y_0\right)$$

If we replace $\frac{41}{140}\Delta^6 y_0$ by $\frac{3}{10}\Delta^6 y_0$, the error made will be negligible.

$$\therefore \quad \int_{x_0}^{x_0+6h} f(x)\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

Similarly

$$\int_{x_0+6h}^{x_0+12h} f(x)\,dx = \frac{3h}{10}(y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}) \text{ and so on.}$$

Adding all these integrals from x_0 to $x_0 + nh$, where n is a multiple of 6, we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + \ldots)$$

This is known as *Weddle's rule*.

Obs. While applying *Weddle's rule the number of sub-intervals should be taken as a multiple of 6*. Weddle's rule is generally more accurate than any of the others. Of the two Simpson rules, the 1/3 rule is better.

Example 30.7. *Evaluate* $\int_0^6 \frac{dx}{1+x^2}$ *by using (i) Trapezoidal rule,*

(i) Simpson's 1/3 rule, *(Mumbai, 2005)*
(ii) Simpson's 3/8 rule, *(J.N.T.U., 2008)*
(iii) Weddle's rule and compare the results with its actual value. *(V.T.U., 2008)*

Solution. Divide the interval (0, 6) into six parts each of width $h = 1$. The values of $f(x) = \frac{1}{1+x^2}$ are given below :

x	0	1	2	3	4	5	6
$f(x)$ $= y$	1 y_0	0.5 y_1	0.2 y_2	0.1 y_3	0.05884 y_4	0.0385 y_5	0.027 y_6

(*i*) By Trapezoidal rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{h}{2}[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)]$$

$$= \frac{1}{2}[(1+\ 0.027)+2(0.5+0.2+0.1+0.0588+0.0385)] = 1.4108.$$

(*ii*) By Simpson's 1/3 rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{h}{3}[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)]$$

$$= \frac{1}{3}[(1+0.027)+4(0.5+0.1+0.0385)+2(0.2+0.0588)] = 1.3662.$$

(*iii*) By Simpson's 3/8 rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{3h}{8}[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3]$$

$$= \frac{3}{8}[(1+0.027)+3(0.5+0.2+0.0588+0.0385)+2(0.1)] = 1.3571.$$

(*iv*) By Weddle's rule,

$$\int_0^6 \frac{1}{1+x^2} = \frac{3h}{10}[y_0+5y_1+y_2+6y_3+y_4\ 5y_5+y_6]$$

$$= 0.3[1+5(0.5)+0.2+6(0.1)+0.0588+5(0.0385)+0.027] = 1.3735.$$

Also, $\int_0^6 \frac{dx}{1+x^2} = \left|\tan^{-1} x\right|_0^6 = 1.4056$

Obs. This shows that the value of the integral found by Weddle's rule is the nearest to the actual value followed by its value given by Simpson's 1/3rd.

Example 30.8. *Use the Trapezoidal rule to estimate the integral* $\int_0^2 e^{x^2}\,dx$ *taking 10 intervals.*

(U.P.T.U., 2008)

Solution. Let $y = e^{x^2}$, $h = 0.2$ and $n = 10$.

The values of x and y are as follows :

x :	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y :	1	1.0408	1.1735	1.4333	1.8964	2.1782	4.2206	7.0993	12.9358	25.5337	54.5981
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

By Trapezoidal rule, we have

$$\int_0^1 e^{x^2}dx = \frac{h}{2}[(y_0+y_{10})+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8+y_9)]$$

$$= \frac{0.2}{2}[(1+54.5981)+2(1.0408+1.1735+1.4333+1.8964$$

$$+2.178+4.2206+7.0993+12.9358+25.5337)]$$

Hence $\int_0^2 e^{x^2}\,dx = 17.0621.$

Example 30.9. *Use Simpson's 1/3rd rule to find* $\int_0^{0.6} e^{-x^2}\,dx$ *by taking seven ordinates.*

(V.T.U., 2011 ; Bhopal, 2009)

Solution. Divide the interval (0, 0.6) into six parts each of width $h = 0.1$. The values of $y = f(x) = e^{-x^2}$ are given below :

x	0	0.1	0.2	0.3	0.4	0.5	0.6
x^2	0	0.01	0.04	0.09	0.16	0.25	0.36
y	1 y_0	0.9900 y_1	0.9608 y_2	0.9139 y_3	0.8521 y_4	0.7788 y_5	0.6977 y_6

By Simpson's 1/3rd rule, we have

$$\int_0^{0.6} e^{-x^2} dx = \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5 + 2(y_2 + y_4)]$$

$$= \frac{0.1}{3} [(1 + 0.6977) + 4(0.99 + 0.9139 + 0.7788 + 2(0.9608 + 0.8521)]$$

$$= \frac{0.1}{3} [1.6977 + 10.7308 + 3.6258] = \frac{0.1}{3} (16.0543) = 0.5351.$$

Example 30.10. *Compute the value of* $\int_{0.2}^{1.4} (\sin x - \log x + e^x)\, dx$ *using Simpson's* $\frac{3}{8}$*th rule.* *(Mumbai, 2005)*

Solution. Let $y = \sin x - \log_e x + e^x$ and $h = 0.2, n = 6$.
The values of y are as given below :

x :	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y :	3.0295 y_0	2.7975 y_1	2.8976 y_2	3.1660 y_3	3.5597 y_4	4.0698 y_5	4.4042 y_6

By Simpson's $\frac{3}{8}$th rule, we have

$$\int_{0.2}^{1.4} y\, dx = \frac{3h}{8} [(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3}{8} (0.2)\ [7.7336 + 2(3.1660) + 3\ (13.3247)] = 4.053$$

Hence $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\, dx = 4.053.$

Obs. Applications of Simpson's rule. If the various ordinates in §30.5 represent equispaced cross-sectional areas, then Simpson's rule gives the volume of the solid. As such, Simpson's rule is very useful to civil engineers for calculating the amount of earth that must be moved to fill a depression or make a dam. Similar if the ordinates denote velocities at equal intervals of time, the Simpson's rule gives the distance travelled. The following examples illustrate these applications.

Example 30.11. *The velocity v (km / min) of a moped which starts from rests, is given at fixed intervals of time t (min) as follows :*

t :	2	4	6	8	10	12	14	16	18	20
y :	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in 20 minutes.

Solution. If s km be the distance covered in t (min), then $\frac{ds}{dt} = v$

$$\therefore \qquad \Big| s \Big|_{t=0}^{20} = \int_0^{20} v\, dt = \frac{h}{3} [X + 4.O + 2E], \text{ by Simpson's rule}$$

Hence $h = 2, v_0 = 0, v_1 = 10, v_2 = 18, v_3 = 25$ etc.

$$\therefore \qquad X = v_0 + v_{10} = 0 + 0 = 0$$

$$O = v_1 + v_3 + v_5 + v_7 + v_9 = 10 + 25 + 32 + 11 + 2 = 80$$
$$E = v_2 + v_4 + v_6 + v_8 = 18 + 29 + 20 + 5 = 72$$

Hence the required distance $= \left| s \right|_{t=0}^{20} = \frac{2}{3}(0 + 4 \times 80 + 2 \times 72)$

$= 309.33$ km.

Example 30.12. *The velocity v of a particle at distance s from a point on its linear path is given by the following table :*

s (m) :	0	2.5	5.0	7.5	10.0	12.5	15.0	17.5	20.0
v (m/sec) :	16	19	21	22	20	17	13	11	9

Estimate the time taken by the particle to traverse the distance of 20 metres, using Boole's rule.)
(U.P.T.U. 2007)

Solution. If t sec be the time taken to traverse a distance s (m) then $\frac{ds}{dt} = v$

or $$\frac{dt}{ds} = \frac{1}{v} = y \text{ (say)},$$

$\therefore$ then $$\left| t \right|_{s=0}^{s=20} = \int_0^{20} y\,ds$$

Here $h = 2.5$ and $n = 8$

Also $$y_0 = \frac{1}{16}, y_1 = \frac{1}{19}, y_2 = \frac{1}{4}, y_3 = \frac{1}{22}, y_4 = \frac{1}{20}, y_5 = \frac{1}{17}, y_6 = \frac{1}{13}, y_7 = \frac{1}{11}, y_8 = \frac{1}{9}$$

$\therefore$ by Boole's Rules, we have

$$\left| t \right|_{s=0}^{s=20} = \int_0^{20} y\,ds = \frac{2h}{45}\,[7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8]$$

$$= \frac{2(2.5)}{45}\left[7\left(\frac{1}{16}\right) + 32\left(\frac{1}{19}\right) + 12\left(\frac{1}{21}\right) + 32\left(\frac{1}{22}\right) + 14\left(\frac{1}{20}\right) + 32\left(\frac{1}{17}\right) + 12\left(\frac{1}{13}\right) + 32\left(\frac{1}{11}\right) + 14\left(\frac{1}{9}\right)\right]$$

$$= \frac{1}{9}(12.11776) = 1.35$$

Hence the required time = 1.35 sec.

Example 30.13. *A solid of revolution is formed by rotating about the x-axis, the area between the x-axi the lines x = 0 and x = 1 and a curve through the points with the following co-ordinates*

x :	0.00	0.25	0.50	0.75	1.00
y :	1.0000	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed using Simpson's rule. *(Raipur, 200*

Solution. Here $h = 0.25$, $y_0 = 1$, $y_1 = 0.9896$, $y_2 = 0.9589$, etc.

$\therefore$ Required volume of the solid generated

$$= \int_0^1 \pi y^2\, dx = \pi \cdot \frac{h}{3}[(y_0^2 + y_4^2) + 4(y_1^2 + y_3^2) + 2y_2^2]$$

$$= 0.25\,\frac{\pi}{3}\,[\{1 + (0.8415)^2\} + 4\{(0.9896)^2 + (0.9089)^2\} + 2(0.0589)^2]$$

$$= \frac{0.25 \times 3.1416}{3}\,[1.7081 + 7.2216 + 1.839] = 0.2618\,(10.7687) = 2.8192.$$

PROBLEMS 30.2

1. Evaluate $\int_0^1 \frac{dx}{1+x}$ applying

(*i*) Trapezoidal rule (*J.N.T.U., 2009*) (*ii*) Simpson's 1/3rd rule

(*iii*) Simpson's 3/8th rule. (*Mumbai, 2004*)

2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using (*i*) Trapezoidal rule taking $h = 1/4$

(*ii*) Simpson's 1/3rd rule taking $h = 1/4$. (*J.N.T.U., 2008*)

(*iii*) Simpson's 3/8th rule taking $h = 1/6$. (*U.P.T.U, 2010 ; V.T.U., 2007*)

(*iv*) Weddle's rule taking $h = 1/6$. (*Bhopal, 2009*)

Hence compute an approximate value of π in each case.

3. Find an approximate value of $\log_e 5$ by calculating to 4 decimal places, by Simpson's 1/3 rule, $\int_0^5 \frac{dx}{4x+5}$, dividing the range into 10 equal parts. (*Anna., 2005*)

4. Evaluate $\int_0^6 x \sec x \, dx$ using eight intervals by Trapezoidal rule. (*U.P.T.U., 2009*)

5. Evaluate using Simpson's $\frac{1}{3}$rd rule (*i*) $\int_0^6 \frac{e^x}{1+x} dx$ (*U.P.T.U., 2006*)

(*ii*) $\int_0^2 e^{-x^2} dx$ (Take $h = 0.25$). (*J.N.T.U., 2007*)

6. Evaluate using Simpson's 1/3rd rule $\int_0^1 \frac{dx}{x^3+x+1}$, choose step length 0.25. (*U.P.T.U., 2009*)

7. Evaluate using Simpson's 1/3rd rule, (*i*) $\int_0^\pi \sin x \, dx$ using 11 ordinates.

(*ii*) $\int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta$ taking 9 ordinates. (*V.T.U., 2009*)

8. Evaluate correct to 4 decimal places, by Simpson's $\frac{3}{8}$th rule

(*i*) $\int_0^9 \frac{dx}{1+x^3}$ (*U.P.T.U., M. Tech., 2010*) (*ii*) $\int_0^{\pi/2} e^{\sin x} dx$ (*U.P.T.U., 2007*)

9. Given that

x :	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$\log x$:	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

evaluate $\int^{5.2} \log x$ [illegible] by

(*a*) Trapezoidal r[illegible] (*b*) Simpson's 1/3rd rule, (*Kerala, 2003*)

(*c*) Simp[illegible] 3/8 [illegible] T.U., 2006) (*d*) Weddle's rule (*V.T.U., 2008*)

10. Use Boo[illegible] compute $\int_0^{\pi/2} \sqrt{\sin x}\, dx$. (*U.P.T.U., 2008*)

11. The [illegible] (t) as a function of time :

t	5	6	7
$f(t)$	78	70	60

Using Sim[illegible] (*J.N.T.U., 2007*)

12. A curve i[illegible] the following table :

x :	[illegible]	3.5	4
y :	[illegible]	2.6	[illegible] 1

Estimate the area bounded by the cu[illegible] $= 4$. (*Bhopal, 2007*)

13. A river is 80 ft wide. The depth d in feet at a distance x ft. from one bank is given by the following table :

x :	0	10	20	30	40	50	60	70	80
y :	0	4	7	9	12	15	14	8	3

Find approximately the area of the cross-section. (*Rohtak, 2005*)

14. A curve is drawn to pass through the points given by following table :

x :	1	1.5	2	2.5	3	3.5	4
y :	2	2.4	2.7	2.8	3	2.6	2.1

Using Weddle's rule, estimate the area bounded by the curve, the x-axis and the lines $x = 1, x = 4$.(*V.T.U., 2011 S*)

15. A body is in the form of a solid of revolution. The diameter D in cms of its sections at distances x cm. from the one end are given below. Estimate the volume of the solid.

x :	0	2.5	5.0	7.5	10.0	12.5	15.0
D :	5	5.5	6.0	6.75	6.25	5.5	4.0

16. The velocity v of a particle at distances s from a point on its path is given by the table :

s ft :	0	10	20	30	40	50	60
v ft/sec :	47	58	64	65	61	52	38

Estimate the time taken to travel 60 ft. by using Simpson's 1/3 rule. (*U.P.T.U., 2007*)
Compare the result with Simpson's 3/8 rule. (*Madras, 2003*)

17. The following table gives the velocity v of a particle at time t :

t (second) :	0	2	4	6	8	10	12
v (m/sec) :	4	6	16	34	60	94	136

Find the distance moved by the particle in 12 seconds and also the acceleration at $t = 2$ sec. (*S.V.T.U., 2007*)

18. A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's $\frac{1}{3}$rd rule, find the velocity of the rocket at $t = 80$ seconds.

t sec :	0	10	20	30	40	50	60	70	80
f (cm/sec^2) :	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

(*Mumbai, 2004*)

19. A reservoir discharging water through sluices at a depth h below the water surface has a surface area A for various values of h as given below :

h (ft.) :	10	11	12	13	14
A (sq.ft.) :	950	1070	1200	1350	1530

If t denotes time in minutes, the rate of fall of the surface is given by $dh/dt = -48\sqrt{h}/A$.
Estimate the time taken for the water level to fall from 14 to 10 ft. above the sluices.

30.12 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 30.3

Select the correct answer or fill up the blanks in the following questions :

1. The value of $\int_0^1 \frac{dx}{1+x}$ by Simpson's rule is
   (*a*) 0.96315 (*b*) 0.63915 (*c*) 0.69315 (*d*) 0.69...
2. Using forward differences, the formula for $f'(a) = \ldots$
3. In application of Simpson's 1/3 rule, the interval h for closer approximation should be ...
4. $f(x)$ is given by

x :	0	0.5	1
$f(x)$:	1	0.8	0.5,

then using Trapezoidal rule, the value of $\int_0^1 f(x)\,dx$ is ...

5. If

x :	0	0.5	1	1.5	2
$f(x)$:	0	0.25	1	2.25	4

then the value of $\int_0^2 f(x)\, dx$ by Simpson's 1/3rd rule is ...

6. Simpson's 3/8th rule states that ...

7. For the data :

t :	3	6	9	12
$y(t)$:	–1	1	2	3,

the value of $\int_3^{12} y(t)\, dt$ when computed by Simpson's $\frac{1}{3}$rd rule is

(*a*) 15 (*b*) 10 (*c*) 0 (*d*) 5.

8. While evaluating a definite integral by Trapezoidal rule, the accuracy can be increased by taking ...

9. The value of $\int_0^1 \frac{dx}{1+x^2}$ by Simpson's 1/3rd rule (taking $n = 1/4$) is ...

10. For the data:

x :	2	4	6	8
$f(x)$:	3	5	6	7,

$\int_2^8 f(x)\, dx$ when found by the Trapezoidal rule is

(*a*) 18 (*b*) 25 (*c*) 16 (*d*) 32.

11. The expression for $\left(\frac{dy}{dx}\right)_{x=x_0}$ using backward differences is ...

12. The number of strips required in Weddle's rule is ...

13. The number of strips required in Simpson's 3/8th rule is a multiple of

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 6.

14. If $y_0 = 1, y_1 = \frac{16}{17}, y_2 = \frac{4}{5}, y_3 = \frac{16}{25}, y_4 = \frac{1}{2}$ and $h = \frac{1}{4}$, then using Trapezoidal rule, $\int_0^4 ydx$ = ...

15. Using Simpson's $\frac{1}{3}$rd rule, $\int_0^1 \frac{dx}{x}$ = ... (taking $n = 4$).

16. If $y_0 = 1, y_1 = 0.5, y_2 = 0.2, y_3 = 0.1, y_4 = 0.06, y_5 = 0.04$ and $y_6 = 0.03$, then $\int_0^4 ydx$ by Simpson's $\frac{3}{8}$th, rule is = ...

17. If $f(0) = 1, f(1) = 2.7, f(2) = 7.4, f(3) = 20.1, f(4) = 54.6$ and $h = 1$, then $\int_0^4 f(x)\, dx$ by Simpson's $\frac{1}{3}$rd rule = ...

18. Simpson's 1/3rd rule and direct integration give the same result if ...

19. To evaluate $\int_{x_0}^{x_n} ydx$ by Simpson's 1/3rd rule as well as Simpson's 3/8th rule, the number of intervals should be .... and .... respectively.

20. Whenever Trapezoidal rule is applicable, Simpson's 1/3rd rule can also be applied. (True or False)

CHAPTER 31

Difference Equations

31.1 INTRODUCTION

Difference calculus also forms the basis of Difference equations. These equations arise in all situations in which sequential relation exists at various discrete values of the independent variable. The need to work with discrete functions arises because there are physical phenomena which are inherently of a discrete nature. In control engineering, it often happens that the input is in the form of discrete pulses of short duration. The radar tracking devices receive such discrete pulses from the target which is being tracked. As such differences equations arise in the study of electrical networks, in the theory of probability, in statistical problems and many other fields.

Just as the subject of Differential equations grew out Differential calculus to become one of the most powerful instruments in the hands of a practical mathematician when dealing with continuous processes in nature, so the subject of Difference equations is forcing its way to the fore for the treatment of discrete processes. Thus the difference equations may be thought of as the discrete counterparts of the differential equations.

31.2 DEFINITION

(1) A difference equation *is a relation between the differences of an unknown function at one or more general values of the argument.*

Thus $\Delta y_{(n+1)} + y_{(n)} = 2$ ...(1) and $\Delta y_{(n+1)} + \Delta^2 y_{(n-1)} = 1$ ...(2)

are difference equations.

An alternative way of writing a difference equation is as under :

Since $\Delta y_{(n+1)} = y_{(n+2)} - y_{(n+1)}$, therefore (1) may be written as

$$y_{(n+2)} - y_{(n+1)} + y_{(n)} = 2 \quad ...(3)$$

Also since, $\Delta^2 y_{(n-1)} = y_{(n+1)} - 2y_{(n)} + y_{(n-1)}$, therefore (2) takes the form :

$$y_{(n+2)} - 2y_{(n)} + y_{(n-1)} = 1 \quad ...(4)$$

Quite often, difference equations are met under the name of *recurrence relations.*

(2) Order of a difference equation *is the difference between the largest and the smallest arguments occuring in the difference equation divided by the unit of increment.*

Thus (3) above is the *second order*, for

$$\frac{\text{largest argument} - \text{smallest argument}}{\text{unit of increment}} = \frac{(n+2)-n}{1} = 2,$$

and (4) is of the *third order*, for $\dfrac{(n+2)-(n-1)}{1} = 3.$

Obs. *While finding the order of a difference equation, it must always be expressed in a form free of Δs,* for the highest power of Δ does not give order of the difference equation.

(3) Solution *of a difference equation is an expression for $y_{(n)}$ which satisfies the given difference equation.*

The **general solution** *of a difference equation is that in which the number of arbitrary constants is equal to the order of the difference equation.*

A **particular solution** or **particular integral** *is that solution which is obtained from the general solution by giving particular values to the constants.*

31.3 FORMATION OF DIFFERENCE EQUATIONS

The following examples illustrate the way in which difference equations arise and are formed.

Example 31.1. *Form the difference equation corresponding to the family of curves*

$$y = ax + bx^2 \qquad \text{...(i)}$$

Solution. We have $\Delta y = a\Delta(x) + b\Delta(x^2) = a\,(x + 1 - x) + b[(x + 1)^2 - x^2]$

$$= a + b(2x + 1) \qquad \text{...(ii)}$$

and $$\Delta^2 y = 2b[(x + 1) - x] = 2b \qquad \text{...(iii)}$$

To eliminate a and b, we have from (iii), $b = \frac{1}{2}\Delta^2 y$

and form (ii), $a = \Delta y - b\,(2x + 1) = \Delta y - \frac{1}{2}\Delta^2 y\,(2x + 1)$

Substituting these values of a and b in (i), we get

$$y = \left[\Delta y - \frac{1}{2}\Delta^2 y\,(2x+1)\right]x + \frac{1}{2}\Delta^2 y \,.\, x^2$$

or $$(x^2 + x)\,\Delta^2 y - 2x\,\Delta y + 2y = 0$$

This is the desired difference equation which may equally well be written in terms of E as

or $$(x^2 + x)\,y_{x+2} - (2x^2 + 4x)y_{x+1} + (x^2 + 3x + 2)y_x = 0.$$

Example 31.2. *From $y_n = A2^n + B(-3)^n$, derive a difference equation by eliminating the constants.*

Solution. We have $$y_n = A.2^n + B(-3)^n,\; y_{n+1} = 2A.2^n - 3B(-3)^n$$

and $$y_{n+2} = 4A.2^n + 9B(-3)^n.$$

Eliminating A and B, we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 2 & -3 \\ y_{n+2} & 4 & 9 \end{vmatrix} = 0 \quad \text{or} \quad y_{n+2} + y_{n+1} - 6y_n = 0$$

which is the desired difference equation.

PROBLEMS 31.1

1. Write the difference equation $\Delta^3 y_x + \Delta^2 y_x + \Delta y_x + y_x = 0$ in the subscript notation.
2. Assuming $\dfrac{\log(1-z)}{1+z} = y_0 + y_1 z + y_2 z^2 + \ldots\ldots + y_n z^n, \ldots\ldots$, find the difference equations satisfied by y_n.
3. Form a difference equation by eliminating arbitrary constant from $u_n = a2^{n+1}$. *(Anna, 2008)*
4. Find the difference equation satisfied by
   (i) $y = a/x + b$ *(Tiruchirapalli, 2001)* $\quad (ii)$ $y = ax^2 - bx$.
5. Derive the difference equations in each of the following cases :
   (i) $y_n = A.3^n + B.5^n$ $\quad (ii)$ $y_n = (A + Bx)\,2^x$. *(Madras, 2001)*
6. Form the difference equations generated by
   (i) $y_n = ax + b2^x$ $\quad (ii)$ $y_n = a2^n + b(-2)^n$ $\quad (iii)$ $y_x = a2^x + b3^x + c$.

31.4 LINEAR DIFFERENCE EQUATIONS

(1) Def. *A* ***linear difference equation*** *is that in which y_{n+1}, y_{n+2}, etc. occur to the first degree only and are not multiplied together.*

A ***linear difference equation with constant coefficient*** *is of the form*

$$y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \dots + a_r y_n = f(n) \qquad \text{...(1)}$$

where $a_1, a_2, \dots a_r$ are constants.

Now we shall deal with linear difference equations with constant coefficients only. Their properties are analogous to those of linear differential equations with constant co-efficients.

(2) Elementary properties. If $u_1(n), u_2(n), \dots, u_r(n)$ be r independent solution of the equation

$$y_{n+r} + a_1 y_{n+r-1} + \dots + a_r y_n = 0 \qquad \text{...(2)}$$

then its complete solution is $U_n = c_1 u_1(n) + \dots + c_r u_r(n)$

where $c_1, c_2, \dots, c_r$ are arbitrary constants.

If V_n is a particular solution of (1), then the complete solution of (1) is $y_n = U_n + V_n$. *The part U_n is called the* ***complementary function (C.F.)*** *and the part V_n is called the* ***particular integral (P.I.)*** *of (1).*

Thus the ***complete solution (C.S.)*** *of (1) is* $\boldsymbol{y_n = \text{C.F.} + \text{P.I.}}$

31.5 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

(*i.e., rules to solve a linear difference equation with constant coefficients having right hand side zero*).

(1) To begin with, consider the first order linear equation $y_{n+1} - \lambda y_n = 0$, where λ is a constant.

Rewriting it as $\dfrac{y_{n+1}}{\lambda^{n+1}} - \dfrac{y_n}{\lambda^n} = 0$, we have $\Delta\left(\dfrac{y_n}{\lambda^n}\right) = 0$, which gives $y_n/\lambda^n = c$, a constant.

Thus the solution of $(E - \lambda)\, y_n = 0$ is $\boldsymbol{y_n = c.\lambda^n}$.

(2) Now consider the *second order linear equation* $y_{n+2} + ay_{n+1} + by_n = 0$ which in *symbolic form* is

$$(E^2 + aE + b)y_n = 0 \qquad \text{...(1)}$$

Its symbolic co-efficient equated to zero *i.e.*, $E^2 + aE + b = 0$

is called the *auxiliary equation*. Let its roots be λ_1, λ_2.

Case I. *If these roots are real and distinct,* then (1) is to equivalent to

$$(E - \lambda_1)(E - \lambda_2)\, y_n = 0 \qquad \text{...(2)}$$

$$(E - \lambda_2)(E - \lambda_1)\, y_n = 0 \qquad \text{...(3)}$$

If y_n satisfies the subsidiary equation $(E - \lambda_1)y_n = 0$, then it will also satisfy (3).

Similarly, if y_n satisfies the subsidiary equation $(E - \lambda_2)y_n = 0$, then it will also satisfy (2).

$\therefore$ it follows that we can derive two independent solutions of (1), by solving the two subsidiary equations

$$(E - \lambda_1)\, y_n = 0 \quad \text{and} \quad (E - \lambda_2)\, y_n = 0$$

Their solutions are respectively, $y_n = c_1(\lambda_1)^n$ and $y_n = c_2(\lambda_2)^n$

where c_1 and c_2 are arbitrary constants.

Thus the general solution of (1) is $y_n = c_1(\lambda_1)^n + c_2(\lambda_2)^n$

Case II. *If the roots are real and equal* (*i.e.*, $\lambda_1 = \lambda_2$), then (2) becomes

$$(E - \lambda_1)^2 y_n = 0 \qquad \text{...(4)}$$

Let $\qquad y_n = (\lambda_1)^n z_n$

where z_n is a new dependent variable. Then (4) takes the form

$$(\lambda_1)^{n+2} z_{n+2} - 2\lambda_1(\lambda_1)^{n+1} z_{n+1} + \lambda_1^2 \cdot (\lambda_1)^n z_n = 0$$

or $\qquad z_{n+2} - 2z_{n+1} + z_n = 0 \quad i.e., \quad \Delta^2 z_n = 0$

$\therefore \qquad z_n = c_1 + c_2 n$, where c_1, c_2 are arbitrary constants.

Thus the solution of (1) becomes $y_n = (c_1 + c_2 n)(\lambda_1)^n$.

Case III. *If the roots are imaginary,* (*i.e.* $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$) then the solution of (1) is

$$y_n = c_1(\alpha + i\beta)^n + c_2(\alpha - i\beta)^n \qquad [\text{Put } \alpha = r\cos\theta \text{ and } \beta = r\sin\theta]$$

$$= r^n\, [c_1(\cos n\theta + i \sin n\theta) + c_2(\cos n\theta - i \sin n\theta)]$$

$$= r^n [A_1 \cos n\theta + A_2 \sin n\theta]$$

where A_1, A_2 are arbitrary constants are $r = \sqrt{(\alpha^2 + \beta^2)}$, $\theta = \tan^{-1}(\beta/\alpha)$.

(3) *In general, to solve the equation* $y_{n+r} + a_1 y_{n+r-2} + \ldots + a_r y_n = 0$ where a's are constants :

(*i*) Write the equation in the *symbolic form* $(E^r + a_1 E^{r-1} + \ldots + a_r)y_n = 0$.

(*ii*) Write down the *auxiliary equation i.e.*, $E^r + a_1 E^{r-1} \ldots + a_r = 0$ and solve it for E.

(*iii*) Write the solution as follows :

Roots of A.E.	*Solution, i.e. C.F.*
1. $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots$ (real and distinct roots)	$c_1(\lambda_1)^n + c_2(\lambda_2)^n + c_3(\lambda_3)^n + \ldots\ldots$
2. $\lambda_1, \lambda_1, \lambda_3, \ldots\ldots$ (2 real and equal roots)	$(c_1 + c_2 n)(\lambda_1)^n + c_3(\lambda_3)^n + \ldots\ldots$
3. $\lambda_1, \lambda_1, \lambda_1, \ldots\ldots$ (3 real and equal roots)	$(c_1 + c_2 n + c_3 n^2)(y_1)^n + \ldots\ldots$
4. $\alpha + i\beta, \alpha - i\beta, \ldots\ldots$ (a pair of imaginary roots)	$r^n (c_1 \cos\theta + c_2 \sin n\theta)$ where $r = \sqrt{(\alpha^2 + \beta^2)}$ and $\theta = \tan^{-1}(\beta/\alpha)$

Example 31.3. *Solve the difference equation* $u_{n+3} - 2u_{n+2} - 5u_{n+1} + 6u_n = 0$.

Solution. Given equation in *symbolic form* is $(E^3 - 2E^2 - 5E + 6)u_n = 0$

$\therefore$ its auxiliary equation is $E^3 - 2E^2 - 5E + 6 = 0$

or $$(E-1)(E+2)(E-3) = 0. \qquad \therefore \quad E = 1, -2, 3$$

Thus the complete solution is $u_n = c_1(1)^n + c_2(-2)^n + c_3(3)^n$.

Example 31.4. *Solve* $u_{n+2} - 2u_{n+1} + u_n = 0$.

Solution. Given difference equation in *symbolic form* is $(E^2 - 2E + 1)\, u_n = 0$.

$\therefore$ its auxiliary equation is $E^2 - 2E + 1 = 0$

or $$(E-1)^2 = 0. \qquad \therefore \quad E = 1, 1$$

Thus the required solution is $u_n = (c_1 + c_2 n)(1)^n$, *i.e.*, $u_n = c_1 + c_2 n$.

Example 31.5. *Solve* $y_{n+1} - 2y_n \cos\alpha + y_{n-1} = 0$.

Solution. This is a second order difference equation in y_{n-1}; which in *symbolic form* is

$$(E^2 - 2E\cos\alpha + 1)\, y_n = 0$$

The *auxiliary equation* is $E^2 - 2E\cos\alpha + 1 = 0$

$$E = \frac{2\cos\alpha \pm \sqrt{(4\cos^2\alpha - 4)}}{4} = \cos\alpha \pm i\sin\alpha$$

Thus the solution is $y_{n-1} = (1)^{n-1} [c_1 \cos(n-1)\alpha + c_2 \sin(n-1)\alpha]$

or $$y_n = c_1 \cos n\alpha + c_2 \sin n\alpha.$$

Example 31.6. *The integers 0, 1, 1, 2, 3, 5, 8, 13, 21, ... are said to form a Fibonacci sequence. Form the Fibonacci difference equation and solve it.*

Solution. In this sequence, each number beyond the second, is the sum of its two previous number. If y_n be the nth number then $y_n = y_{n-1} + y_{n-2}$ for $n > 2$.

or $$y_{n+2} - y_{n+1} - y_n = 0 \text{ (for } n > 0)$$

or $$(E^2 - E - 1)y_n = 0 \text{ is the difference equation.}$$

Its A.E. is $E^2 - E - 1 = 0$ which gives $E = \frac{1}{2}(1 \pm \sqrt{5})$.

Thus the solution is $$y_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^2 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n, \text{ for } n > 0$$

When $n = 1, y = 0$

$$\therefore \qquad c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 0 \qquad ...(i)$$

When $n = 2, y_2 = 0$

$$\therefore \qquad c_1\left(\frac{1+\sqrt{5}}{2}\right)^2 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^2 = 0 \qquad ...(ii)$$

Solving (*i*) and (*ii*), we get

$$c_1 = \frac{5-\sqrt{5}}{10} \text{ and } c_2 = \frac{5+\sqrt{5}}{10}$$

Hence the complete solution is

$$y_n = \frac{5-\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

PROBLEMS 31.2

Solve the following difference equations :

1. $u_{x+2} - 6u_{x+1} + 9u_x = 0.$
2. $y_{n+2} + y_{n+1} + 2y_n = 0.$
3. $\Delta^2 u_n + 2\Delta u_n + u_n = 0.$
4. $(\Delta^2 - 3\Delta + 2)y_n = 0.$
5. $4y_n - y_{n+2} = 0$ given that $y_0 = 0, y_1 = 2.$
6. $u_{k+3} - 3u_{k+2} + 4u_k = 0.$
7. $f(x+3) - 3f(x+1) - 2f(x) = 0.$
8. $u_{n+3} - 3u_{n+1} + 2u_n = 0$, given $u_1 = 0, u_2 = 8$ and $u_3 = -2.$
9. $(E^3 - 5E^2 + 8E - 4)y_n = 0$, given that $y_0 = 3, y_1 = 2, y_3 = 22.$
10. $u_{n+1} - 2u_n + 2u_{n-1} = 0.$
11. $y_{m+3} + 16y_{m-1} = 0.$
    [**Hint.** $E^4 = -16 = 16\,[\cos(2n+1)\pi + i\sin(2n+1)\pi]$; use De Moivre's theorem.]
12. Show that the difference equation $I_{m+1} - (2 + r_o/r)\,I_m + I_{m-1} = 0$ has the solution.
    $I_m = I_o \sinh(n-m)\,\alpha/\sinh(n-1)\,\alpha$, if $I = I_0$ and $I_n = 0$, α being $= 2\sinh^{-1}\frac{1}{2}(r_0/r)^{1/2}.$
13. A series of values of y_n satisfy the relation, $y_{n+2} + ay_{n+1} + by_n.$
    Given that $y_0 = 0, y_1 = 1, y_2 = y_3 = 2$. Show that $y_n = 2^{n/2}\sin n\pi/4.$
14. A plant is such that each of its seeds when one year old produces 8-fold and produces 18-fold when two years old or more. A seed is planted and as soon as a new seed is produced it is planted. Taking y_n to be the number of seeds produced at the end of the nth year, show that $y_{n+1} = 8y_n + 18(y_1 + y_2 + ... + y_{n-1}).$
    Hence show that $y_{n+2} - 9y_{n+1} - 10y_n = 0$ and find y_n.

31.6 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $y_{n+r} + a_1 y_{n+r-1} + ... + a_r y_n = f(n)$

which in *symbolic form* is $\phi(E)y_n = f(n)$ ...(1)

where $$\phi(E) = E^r + a_1 E^{r-1} + ... + a_r$$

Then the particular integral is given by P.I. $= \dfrac{1}{\phi(E)} f(n).$

Case I. When $f(n) = a^n$

$$\text{P.I.} = \frac{1}{\phi(E)}\, a_n, \text{ put } E = a$$

$$= \frac{1}{\phi(a)}\, a^n, \text{ provided } \phi(a) \neq 0$$

If $\phi(a) = 0$, *then for the equation*

(*i*) $(E - a)y_n = a^n$, $\qquad$ P.I. $= \dfrac{1}{E-a}a^n = na^{n-1}$

(ii) $(E-a)^2 y_n = a^n$, $\quad$ P.I. $= \dfrac{1}{(E-a)^2} a^n = \dfrac{n(n-1)}{2!} a^{n-2}$

(iii) $(E-a)^3 y^n = a^n$, $\quad$ P.I. $= \dfrac{1}{(E-a)^3} a^n = \dfrac{n(n-1)(n-2)}{3!} a^{n-3}$

and so on.

Example 31.7. *Solve* $y_{n+2} - 4y_{n+1} + 3y_n = 5^n$.

Solution. Given equation in *symbolic form* is $(E^2 - 4E + 3)y_n = 5^n$

$\therefore$ The auxiliary equation is $E^2 - 4E + 3 = 0$

or $\quad (E-1)(E-3) = 0. \quad \therefore \quad E = 1, 3$

$\therefore \quad$ C.F. $= c_1(1)^n + c_2(3)^n = c_1 + c_2 \cdot 3^n$

and $\quad$ P.I. $= \dfrac{1}{E^2 - 4E + 3} 5^n$ $\quad$ [Put $E = 5$]

$$= \frac{1}{25 - 4.5 + 3} 5^n = \frac{1}{8} \cdot 5^n$$

Thus the complete solution is $\quad y_n = c_1 + c_2 \cdot 3^n + 5^n/8$.

Example 31.8. *Solve* $u_{n+2} - 4u_{n+1} + 4u_n = 2^n$.

Solution. Given equation in *symbolic form* is $(E^2 - 4E + 4)u_n = 2^n$.

The auxiliary equation is $E^2 - 4E + 4 = 0. \quad \therefore \quad E = 2, 2.$

C.F. $= (c_1 + c_2 n)2^n$

P.I. $= \dfrac{1}{(E-2)^2} \cdot 2^n = \dfrac{n(n-1)}{2!} \cdot 2^{n-2} = n(n-1)\, 2^{n-3}$

Hence the complete solution is $u_n = (c_1 + c_2 n)\, 2^n + n(n-1)\, 2^{n-3}$.

Case II. *When* $f(n) = \sin kn$.

$$\text{P.I.} = \frac{1}{\phi(E)} \sin kn = \frac{1}{\phi(E)} \left(\frac{e^{ikn} - e^{-ikn}}{2i} \right) = \frac{1}{2i} \left[\frac{1}{\phi(E)} a^n - \frac{1}{\phi(E)} b^n \right]$$

where $a = e^{ik}$ and $b = e^{-ik}$.

Now proceed as in case I.

(2) *When* $f(n) = \cos kn$ $\quad$ P.I. $= \dfrac{1}{\phi(E)} \cos kn = \dfrac{1}{\phi(E)} \left(\dfrac{e^{ikn} + e^{-ikn}}{2} \right)$

$$= \frac{1}{2} \left[\frac{1}{\phi(E)} a^n + \frac{1}{\phi(E)} b^n \right] \text{ as before}$$

Now proceed as in case I.

Example 31.9. *Solve* $y_{n+2} - 2 \cos \alpha \cdot y_{n+1} + y_n = \cos \alpha n$. *(Nagpur, 2008)*

Solution. Given equation in *symbolic form* is $(E^2 - 2 \cos \alpha \cdot E + 1)y_n = \cos \alpha n$.

The *auxiliary equation is* $E^2 - 2 \cos \alpha \cdot E + 1 = 0$.

$$\therefore \quad E = \frac{2 \cos \alpha \pm \sqrt{(4 \cos^2 \alpha - 4)}}{2} = \cos \alpha \pm i \sin \alpha$$

$\therefore \quad$ C.F. $= (1)^n [c_1 \cos \alpha n + c_2 \sin \alpha n]$ *i.e.*, $c_1 \cos \alpha n + c_2 \sin \alpha n$

$$\text{P.I.} = \frac{1}{E^2 - 2E \cos \alpha + 1} \cos \alpha n$$

$$= \frac{1}{E^2 - E(e^{i\alpha} + e^{-i\alpha}) + 1} \left(\frac{e^{i\alpha n} + e^{-i\alpha n}}{2} \right)$$

$$= \frac{1}{2}\left[\underset{[\text{Put } E = e^{i\alpha}]}{\frac{1}{(E - e^{i\alpha})(E - e^{-i\alpha})} e^{i\alpha n}} + \underset{[\text{Put } E = e^{-i\alpha}]}{\frac{1}{(E - e^{i\alpha})(E - e^{-i\alpha})} e^{-i\alpha n}}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(E - e^{i\alpha})} \cdot \frac{1}{e^{i\alpha} - e^{-i\alpha}} e^{i\alpha n} + \frac{1}{E - e^{-i\alpha}} \cdot \frac{1}{e^{-i\alpha} - e^{i\alpha}} e^{-i\alpha n}\right]$$

$$= \frac{1}{4i \sin \alpha}\left[\frac{1}{E - e^{i\alpha}} e^{i\alpha n} - \frac{1}{E - e^{-i\alpha}} e^{-i\alpha n}\right] = \frac{1}{4i \sin \alpha} [n \, . \, e^{i\alpha(n-1)} - n \, e^{-i\alpha(n-1)}]$$

$$= \frac{n}{2 \sin \alpha}\left[\frac{e^{i\alpha(n-1)} - e^{-i\alpha(n-1)}}{2i}\right] = \frac{n \sin (n-1)\alpha}{2 \sin \alpha}$$

Hence the complete solution is

$$y_n = c_1 \cos \alpha n + c_2 \sin \alpha n + \frac{n \sin (n-1)\alpha}{2 \sin \alpha}.$$

Case III. *When* $f(n) = n^p$. P.I. $= \frac{1}{\phi(E)} n^p = \frac{1}{\phi(1+\Delta)} n^p$

(1) Expand $[\phi(1+\Delta)]^{-1}$ in ascending powers of Δ by the Binomial theorem as far as the term in Δ^p.

(2) Expand n^p in the factorial form (p. 950) and operate on it with each term of the expansion.

Example 31.10. *Solve* $y_{n+2} - 4y_n = n^2 + n - 1.$ *(Madras, 1999)*

Solution. Given equation is $(E^2 - 4)y_n = n^2 + n - 1$.

The *auxiliary equation is* $E^2 - 4 = 0$, $\therefore$ $E = \pm 2$.

$\therefore$ C.F. $= c_1 (2)^n + c_2 (-2)^n$.

$$\therefore \quad \text{P.I.} = \frac{1}{E^2 - 4}(n^2 + n - 1) = \frac{1}{(1+\Delta)^2 - 4}[n(n-1) + 2n - 1].$$

$$= \frac{1}{\Delta^2 + 2\Delta - 3}([n]^2 + 2[n] - 1) = -\frac{1}{3}\left[1 - \left(\frac{2}{3}\Delta + \frac{\Delta^2}{3}\right)\right]^{-1} \{[n]^2 + 2[n] - 1\}$$

$$= -\frac{1}{3}\left[1 + \left(\frac{2}{3}\Delta + \frac{\Delta^2}{3}\right) + \left(\frac{2}{3}\Delta + \frac{\Delta^2}{3}\right) + \ldots\right] \{[n]^2 + 2[n] - 1\}$$

$$= -\frac{1}{3}\left\{1 + \frac{2}{3}\Delta + \frac{7}{9}\Delta^2 + \ldots\right\}\{[n]^2 + 2[n] - 1\} = -\frac{1}{3}\left\{[n]^2 + 2[n] - 1 + \frac{2}{3}(2[n] + 2) + \frac{7}{9} \times 2\right\}$$

$$= -\frac{1}{3}\left\{[n]^2 + \frac{10}{3}[n] + \frac{17}{9}\right\} = -\frac{n^2}{3} - \frac{7}{9}n - \frac{17}{27}.$$

Hence the *complete solution* is $y_n = c_1 2^n + c_2(-2)^n - \frac{n^2}{3} - \frac{7}{9}n - \frac{17}{27}$.

Case IV. *When* $f(n) = a^n F(n)$, $F(n)$, being a polynomial of finite degree in n.

$$\text{P.I.} = \frac{1}{\phi(E)} a^n F(n) = a^n \frac{1}{\phi(aE)} F(a)$$

Now $F(n)$ being a polynomial in n, proceed as in case III.

Example 31.11. *Solve* $y_{n+2} - 2y_{n+1} + y_n = n^2 \, . \, 2^n.$ *(Nagpur, 2008)*

Solution. Given equation is $(E^2 - 2E + 1)y_n = n^2 \, . \, 2^n$.

Its C.F. $= c_1 + c_2 n$

and
$$\text{P.I.} = \frac{1}{(E-1)^2} 2^n . \, n^2 = 2^n \frac{1}{(2E-1)^2} n^2 = 2^n \frac{1}{(1+2\Delta)^2} n^2$$

$$= 2^n (1 + 2\Delta)^{-2} n(n-1) + n = 2^n (1 - 4\Delta + 12\Delta^2 - ...) ([n]^2 + [n])$$
$$= 2^n \{[n]^2 + [n] - 4(2[n] + 1) + 12 \times 2\}$$
$$= 2^n ([n]^2 - 7[n] + 20) = 2^n (n^2 - 8n + 20)$$

Hence the *complete solution* is $y_n = c_1 + c_2 n + 2^n (n^2 - 8n + 20)$.

PROBLEMS 31.3

Solve the following difference equations :

1. $y_{n+2} - 5y_{n+1} - 6y_n = 4^n, y_0 = 0, y_1 = 1.$ *(Madras, 2003)*
2. $y_{n+2} + 6y_{n+1} + 9y_n = 2^n, y_0 = y_1 = 0.$ *(V.T.U., 2009)*
3. $y_{p+3} - 3y_{p+2} 3y_{p+1} - y_p = 1.$ *(Kottayam, 2005)*
4. $y_{n+2} - 2y_{n+1} + 4y_n = 6$, given that $y_0 = 0$ and $y_1 = 2$.
5. $(E^2 - 4E + 3)y = 3^x.$
6. $y_{x+2} - 4y_{x+1} + 4y_x = 3.2^x + 5.4^x.$
7. $u_{n+2} - u_n = \cos n/2.$ *(Madras, 2001 S)*
8. $y_{p+2} - \left(2 \cos \frac{1}{2}\right) y_{p+1} + y_p = \sin p/2.$
9. $(E^2 - 4)y_x = x^2 - 1.$
10. $y_{n+3} + y_n = n^2 + 1 y_0 = y_1 = y_2 = 0.$ *(Tirchirapalli, 2001)*
11. $y_{n+3} - 5y_{n+2} + 3y_{n+1} + 9y_n = 2^n + 3n.$ *(Nagpur, 2009)*
12. $(4E^2 - 4E + 1)y = 2^n + 2^{-n}.$ *(Madras, 2001)*
13. $y_{n+2} + 5y_{n+1} + 6y_n = n + 2^n.$ *(Nagpur, 2006)*
14. $u_{x+2} + 6u_{x+1} + 9u_x = x2^x + 3^x + 7.$
15. $y_{n+3} + 8y_n = (2n + 3)\, 2^n$ *(Nagpur, 2005)*
16. $u_{n+2} - 4u_{n+1} 4u_n = n^2 2^n.$
17. $(E^2 - 5E + 6)y_k = 4^k (k^2 - k + 5).$
18. $(E^2 - 2E + 4)y_n = -2^n \left\{6 \cos \frac{n\pi}{3} + 2\sqrt{3} \sin \frac{n\pi}{3}\right\}.$
19. A beam of length l, supported at n points carries a uniform load w per unit length. The bending moments M_1, M_2 ..... M_n at the supports satisfy the Clapeyron's equation :

$$M_{r+2} + 4M_{r+1} + M_r = -\frac{1}{2} wl^2$$

If a beam weighing 30 kg is supported at its ends and at two other supports dividing the beam into three equal parts of 1 metre length, show that the bending moment at each of the two middle supports is 1 kg metre.

31.7 SIMULTANEOUS DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

The method used for solving simultaneous differential equations with constant coefficients also applies to simultaneous difference equations with constants coefficients. The following example illustrates the technique.

Example 31.12. *Solve the simultaneous difference equations*

$$u_{x+1} + v_x - 3u_x = x,\ 3u_x + v_{x+1} - 5v_x = 4^x$$

subject to the conditions $u_1 = 2, v_1 = 0.$

Solution. Given equation in symbolic form, are

$$(E - 3)\, u_x + v_x = x \quad ...(i)$$
$$3u_x + (E - 5)v_x = 4^x \quad ...(ii)$$

Operating the first equation with $E - 5$ and subtracting the second from it, we get

$$[(E - 5)(E - 3) - 3]u_x = (E - 5)x - 4^x$$

or
$$(E^2 - 8E + 12)u_x = 1 - 4x - 4^x$$

Its solution is
$$u_x = c_1 2^x + c_2 6^x - \frac{4}{5} x - \frac{19}{25} + \frac{4^x}{4} \quad ...(iii)$$

Substituting the value of u_x from (iii) in (i), we get

$$v_x = c_1 2^x - 3c_2 6^x - \frac{3x}{5} - \frac{34}{25} - \frac{4^x}{4} \quad ...(iv)$$

Taking $u_1 = 2, v_1 = 0$, in (iii) and (iv), we obtain

$$2c_1 + 6c_2 = \frac{64}{25},\ 2c_1 - 18c_2 = \frac{74}{25}$$

when
$$c_1 = 1.33,\ c_2 = -0.0167$$

Hence $u_x = 1.33.2^x - 0.0167.6^x - 0.8x - 0.76 + 4^{x-1}$

$u_x = 1.33.2^x - 0.05.6^x - 0.6x - 1.36 - 4^{x-1}$.

PROBLEMS 31.4

Solve the following simultaneous difference equations :

1. $y_{x+1} - z_x = 2(x+1),\ z_{x+1} - y_x = -2(x+1)$.
2. $y_{n+1} - y_n + 2z_{n+1} = 0,\ z_{n+1} - z_n - 2y_n = 2^n$.
3. $u_{n+1} + n = 3u_n + 2v_n,\ v_{n+1} - n = u_n + 2v_n$, given $u_0 = 0, v_0 = 3$.
4. $u_{x+1} + v_x + w_x = 1,\ u_x + v_{x+1} + w_x = x,\ u_x + v_x + w_{x+1} = 2x$.

31.8 APPLICATION TO DEFLECTION OF A LOADED STRING

Consider a light string of length l stretched tightly between A and B. Let the forces P_i be acting at its equispaced points x_i $(i = 1, 2, ..., n-1)$ and perpendicular to AB resulting in small transverse displacements y_i at these points (Fig. 31.1). Assuming the angle θ_i made by the portion between x_i and x_{i+1} with the horizontal, to be small, we have

$$\sin \theta_i = \tan \theta_i = \theta_i \text{ and } \cos \theta_i = 1$$

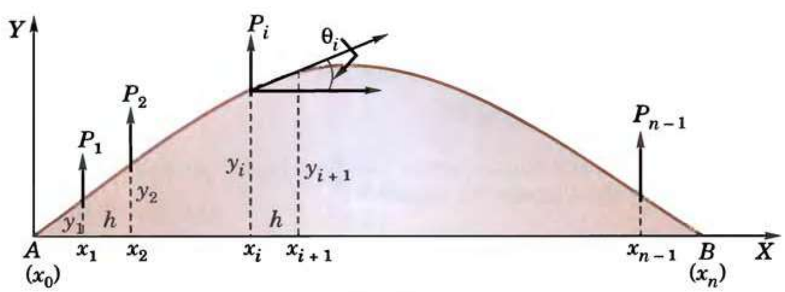

Fig. 31.1

If T be the tension of the string at x_i, then $T \cos \theta_i = T$

i.e., the tension may be taken as uniform.

Taking $x_{i+1} - x_i = h$, we have

$$y_{i+1} - y_i = h \tan \theta_i = h\theta_i \qquad ...(1)$$

$$y_i - y_{i-1} = h \tan \theta_{i-1} = h\theta_{i-1} \qquad ...(2)$$

Also resolving the forces in equilibrium at $(x_i, y_i) \perp$ to AB, we get

$$T \sin \theta_i - T \sin \theta_{i-1} + P_i = 0 \text{ i.e. } T(\theta_i - \theta_{i-1}) + P_i = 0 \qquad ...(3)$$

Eliminating θ_i and θ_{i-1} from (1), (2) and (3), we obtain

$$y_{i+1} - 2y_i + y_{i-1} = -\frac{hP_i}{T} \qquad ...(4)$$

which is a difference equation and its solution gives the displacements y_i. To obtain the arbitrary constants in the solution, we take $y_0 = y_n = 0$ as the boundary conditions, since the ends A and B of the string are fixed.

Example 31.13. *A light string stretched between two fixed nails 120 cm apart, carries 11 loads of weight 5 gm each at equal intervals and the resulting tension is 500 gm weight. Show that the sag at the mid-point is 1.8 cm.*

Solution. Taking $h = 10$ cm, $P_i = 5$ gm and $T = 500$ gm wt.,

the above equation (4) becomes $y_{i+1} - 2y_i + y_{i-1} = -1/10$

i.e., $$y_{i+2} - 2y_{i+1} + y_i = -\frac{1}{10}$$

Its A.E. is $(E-1)^2 = 0$ *i.e.* $E = 1, 1$. $\therefore$ C.F. $= c_1 + c_2 i$

and $$\text{P.I.} = \frac{1}{(E-1)^2}\left(-\frac{1}{10}\right) = -\frac{1}{10}\frac{1}{(E-1)^2}(1)^i = -\frac{1}{10}\frac{i(i-1)}{2} = \frac{1}{20}(i-i^2)$$

Thus the C.S. is $y_i = c_1 + c_2 i + \frac{1}{20}(i - i^2)$

Since $y_0 = 0, \quad \therefore \quad c_1 = 0$

and $y_{12} = 0, \quad \therefore \quad c_2 = \frac{11}{20}$.

Hence $y_i = \frac{11}{20} i + \frac{1}{20}(i - i^2)$

At the mid-point $i = 6$, we get $y_6 = 1.8$ cm.

PROBLEMS 31.5

1. A light string of length $(n + 1)l$ is stretched between two fixed points with a force P. It is loaded with n equal masses m at distance l. If the system starts rotating with angular velocity ω, find the displacement y_i of the ith mass.

31.9 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 31.6

Select the correct answer or fill up the blanks in the following questions :

1. $y_n = A\,2^n + B\,3^n$, is the solution of the difference equation ....
2. The solution of $(E-1)^3 u_n = 0$ is ....
3. The solution of the difference equation $u_{n+3} - 2u_{n+2} - 5u_{n+1} + 6u_n = 0$ is ....
4. The solution of $y_{n+1} - y_n = 2^n$ is ....
5. The difference equation $y_{n+1} - 2y_n = n$ has $y_n = ...$ as its solution.
6. The difference equation corresponding to the family of curves $y = ax^2 + bx$ is ....
7. The particular integral of the equation $(E-2)\,y_n = 1$.
8. The solution of $4y_n = y_{n+2}$ such that $y_0 = 0, y_1 = 2$, is ....
9. The equation $\Delta^2 u_{n+1} + \frac{1}{2}\Delta^2 u_n = 0$ is of order ....
10. The difference equation satisfied by $y = a + b/x$ is ....
11. The order of the difference equation $y_{n+2} - 2y_{n+1} + y_n = 0$ is ....
12. The solution of $y_{n+2} - 4y_{n+1} + 4y_n = 0$ is ....
13. The particular integral of $u_{x+2} - 6u_{x+1} + 9u_x = 3$ is ....
14. The difference equation generated by $u_n = (a + bn)\,3^n$ is ....
15. Solution of $6y_{n+2} + 5y_{n+1} - 6y_n = 2^n$ is $y_n = A(2/3)^n + B(-3/2)^n + 2^n/28$. (True or False)

CHAPTER 32

Numerical Solution of Ordinary Differential Equations

32.1 INTRODUCTION

The methods of solution so far presented are applicable to a limited class of differential equations. Frequently differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods. These methods are of even greater importance when we realise that computing machines are now available which reduce numerical work considerably.

A number of numerical methods are available for the solution of first order differential equations of the form :

$$\frac{dy}{dx} = f(x, y), \text{ given } y(x_0) = y_0 \qquad ...(1)$$

These methods yield solutions either as a power series in x from which the values of y can be found by direct substitution, or as a set of values of x and y. The methods of Picard and Taylor series belong to the former class of solutions whereas those of Euler, Runge-Kutta, Milne, Adams-Bashforth etc. belong to the latter class. In these later methods, the values of y are calculated in short steps for equal intervals of x and are therefore, termed as *step-by-step methods.*

Euler and Runge-Kutta methods are used for computing y over a limited range of x-values whereas Milne and Adams-Bashforth methods may be applied for finding y over a wider range of x-values. These later methods require starting values which are found by Picard's or Taylor series or Runge-Kutta methods.

The initial condition in (1) is specified at the point x_0. *Such problems in which all the initial conditions are given at the initial point only are called* ***initial value problems.*** *But there are problems involving second and higher order differential equations in which the conditions may be given at two or more points. These are known as* ***boundary value problems.*** In this chapter, we shall first explain methods for solving initial value problems and then give a method of solving boundary value problems.

32.2 PICARD'S METHOD*

Consider the first order equation $dy/dx = f(x, y)$ ...(1)

* Called after the French mathematician *Emile Picard* (1856—1941) who was professor in Paris since 1881 and is famous for his researches in the theory of functions.

It is required to find that particular solution of (1) which assumes the value y_0 when $x = x_0$. Integrating (1) between limits, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx \quad \text{or} \quad y = y_0 + \int_{x_0}^{x} f(x, y)\, dx \qquad ...(2)$$

This is an integral equation equivalent to (1), for it contains the unknown y under the integral sign.

As a first approximation y_1 to the solution, we put $y = y_0$ in $f(x, y)$ and integrate (2), giving

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

For a second approximation y_2, we put $y = y_1$ in $f(x, y)$ and integrate (2), giving

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx.$$

Similarly, *a third approximation is* $y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx.$

Continuing this process, a sequence of functions of x, *i.e.*, y_1, y_2, y_3 ... is obtained each giving a better approximation of the desired solution than the preceding one.

Obs. Picard's method is of considerable theoretical value, but can be applied only to a limited class of equations in which the successive integrations can be performed easily. The method can be extended to simultaneous equations and equations of higher order (See § 32.11 and 32.12).

Example 32.1. *Using Picard's process of successive approximation, obtain a solution upto the fifth approximation of the equation $dy/dx = y + x$, such that $y = 1$ when $x = 0$. Check your answer by finding the exact particular solution.*

Solution. (*a*) We have $y = 1 + \int_0^x (y + x)\, dx.$

First approximation. Put $y = 1$, in $y + x$, giving

$$y_1 = 1 + \int_0^x (1 + x)\, dx = 1 + x + x^2/2.$$

Second approximation. Put $y = 1 + x + x^2/2$ in $y + x$, giving

$$y_2 = 1 + \int_0^x (1 + 2x + x^2/2)\, dx = 1 + x + x^2 + x^3/6.$$

Third approximation. Put $y = 1 + x + x^2 + x^3/6$ in $y + x$, giving

$$y_3 = 1 + \int_0^x (1 + 2x + x^2 + x^3/6)\, dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}.$$

Fourth approximation. Put $y = y_3$ in $y + x$, giving

$$y_4 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}.$$

Fifth approximation. Put $y = y_4$ in $y + x$, giving

$$y_5 = 1 + \int_0^x \left(1 + 2x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{720} \qquad ...(i)$$

(*b*) Given equation :

$$\frac{dy}{dx} - y = x \text{ is a Leibnitz's linear in } x.$$

Its I.F. being e^{-x}, the solution is

$$ye^{-x} = \int xe^{-x}\, dx + c = -xe^{-x} - \int (-e^{-x})\, dx + c = -xe^{-x} - e^{-x} + c \qquad \text{[Integrate by parts]}$$

$\therefore \quad y = ce^x - x - 1.$

Since $y = 1$, when $x = 0$, $\quad \therefore \quad c = 2.$

Thus the desired particular solution is $y = 2e^x - x - 1 \qquad ...(ii)$

Or using the series : $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots \infty,$

we get
$$y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{360} + \ldots \infty \qquad \ldots(iii)$$

Comparing (i) and (iii), it is clear that (i) approximates to the exact particular solution (ii) upto the term in x^5.

Obs. At $x = 1$, the fourth approximation $y_4 = 3.433$ and the fifth approximation $y_5 = 3.434$ whereas exact value is 3.44.

Example 32.2. *Find the value of y for x = 0.1 by Picard's method, given that*
$$\frac{dy}{dx} = \frac{y - x}{y + x}, \; y(0) = 1. \qquad \text{(P.T.U., 2002)}$$

Solution. We have $y = 1 + \displaystyle\int_0^x \frac{y - x}{y + x}\, dx$

First approximation. Put $y = 1$ in the integrand, giving
$$y_1 = 1 + \int_0^x \frac{1 - x}{1 + x}\, dx = 1 + \int_0^x \left(-1 + \frac{2}{1 + x}\right) dx$$
$$= 1 + \left[-x + 2 \log(1 + x)\right]_0^x = 1 - x + 2 \log(1 + x) \qquad \ldots(i)$$

Second approximation. Put $y = 1 - x + 2 \log(1 + x)$ in the integrand, giving
$$y_2 = 1 + \int_0^x \frac{1 - x + 2\log(1 + x) - x}{1 - x + 2\log(1 + x) + x}\, dx = 1 + \int_0^x \left[1 - \frac{2x}{1 + 2\log(1 + x)}\right] dx$$

which is very difficult to integrate.

Hence we use the first approximation and taking $x = 0.1$ in (i) we obtain
$$y(0.1) = 1 - (.1) + 2 \log 1.1 = 0.9828.$$

32.3 TAYLOR'S SERIES METHOD*

Consider the first order equation $dy/dx = f(x, y)$ ...(1)

Differentiating (1), we have
$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \quad i.e., \quad y'' = f_x + f_y f' \qquad \ldots(2)$$

Differentiating this successively, we can get y''', y^{iv} etc. Putting $x = x_0$ and $y = 0$, the values of $(y')_0$, $(y'')_0$, $(y''')_0$ can be obtained. Hence the Taylor's series
$$y(x) = y_0 + (x - x_0)(y')_0 + \frac{(x - x_0)^2}{2!}(y'')_0 + \frac{(x - x_0)^3}{3!}(y''')_0 + \ldots \qquad \ldots(3)$$

gives the values of y for every value of x for which (3) converges.

On finding the value y_1 for $x = x_1$ from (3), y', y'' can be evaluated at $x = x_1$ by means of (1), (2) etc. Then y can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series (3).

Example 32.3. *Find by Taylor's series method the value of y at x = 0.1 and x = [illegible] to five places of decimals from $dy/dx = x^2y - 1$, $y(0) = 1$.* (V.T.U., 2009; [illegible]htak, 2005)

Solution. Here $(y)_0 = 1$, $y' = x^2y - 1$, $(y')_0 = -1$

$\therefore$ Differentiating successively and substituting, we get
$$y'' = 2xy + x^2y', \qquad (y'')_0 = 0$$
$$y''' = 2y + 4xy' + x^2y'', \qquad (y''')_0 = 2$$
$$y^{iv} = 6y' + 6xy'' + x^2y''', \qquad (y^{iv})_0 = -6 \text{ etc.}$$

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*See footnote p. 145.

Putting these values in the Taylor's series,

$$y(x) = y_0 + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \ldots,$$

we have $$y(x) = 1 + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \ldots = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Hence $y(0.1) = 0.90033$ and $y(0.2) = 0.80227$.

Example 32.4. *Employ Taylor's method to obtain approximate value of y at x = 0.2 for the differential equation $dy/dx = 2y + 3e^x$, y(0) = 0. Compare the numerical solution obtained with the exact solution.* (V.T.U., 2009 ; P.T.U., 2003)

Solution. (*a*) We have $y' = 2y + 3e^x$ $\qquad y'(0) = 2y(0) + 3e^0 = 3.$

Differentiating successively and substituting $x = 0, y = 0$, we get

$$y'' = 2y' + 3e^x, \qquad y''(0) = 2y'(0) + 3 = 9$$
$$y''' = 2y'' + 3e^x, \qquad y'''(0) = 2y''(0) + 3 = 21$$
$$y^{iv} = 2y''' + 3e^x, \qquad y^{iv}(0) = 2y'''(0) + 3 = 45 \text{ etc.}$$

Putting these values in the Taylor's series, we have

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \ldots$$

$$= 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \ldots = 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \ldots$$

Hence $\quad y(0.2) = 3(0.2) + 4.5(0.2)^2 + 3.5(0.2)^3 + 1.875(0.4)^4 + \ldots = 0.8110$ ...(*i*)

(*b*) Now $\dfrac{dy}{dx} - 2y = 3e^x$ is a Leibnitz's linear in x.

Its I.F. being e^{-2x}, the solution is

$$ye^{-2x} = \int 3e^x \,.\, e^{-2x}\,dx + c = -3e^{-x} + c \quad \text{or} \quad y = -3e^x + ce^{2x}$$

Since $y = 0$ when $x = 0$, $\quad \therefore \quad c = 3$.

Thus the exact solution is $y = 3(e^{2x} - e^x)$

When $x = 0.2$, $y = 3(e^{0.4} - e^{0.2}) = 0.8112$ ...(*ii*)

Comparing (*i*) and (*ii*), it is clear that (*i*) approximates to the exact value upto 3 decimal places.

Example 32.5. *Solve by Taylor's series method the equation $\dfrac{dy}{dx} = \log(xy)$ for y(1.1) and y(1.2), given y(1) = 2.* (Hazaribagh, 2009)

Solution. We have $y' = \log x + \log y$; $\quad y'(1) = \log 2$

Differentiating w.r.t. x and substituting $x = 1, y = 2$, we get

$$y'' = \frac{1}{x} + \frac{1}{y}y' \,;\, y''(1) = 1 + \frac{1}{2}\log 2$$

$$y''' = -\frac{1}{x^2} + \frac{1}{y} + y'' + y'\left(-\frac{1}{y1}\right)y' \,;\, y'''(1) = -1 + \frac{1}{2}\left(1 + \frac{1}{2}\log 2\right) - \frac{1}{4}(\log 2)^2$$

Substituting these values in the Taylor's series about $x = 1$, we have

$$y(x) = y(1) + (x-1)\,y'(1) + \frac{(x-1)^2}{2!}\,y''(1) + \frac{(x-1)^3}{3!}\,y'''(1) + \ldots$$

$$= 2 + (x-1)\log 2 + \frac{1}{2}(x-1)^2\left(1 + \frac{1}{2}\log 2\right) + \frac{1}{6}(x-1)^3\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right]$$

$$\therefore \quad y(1.1) = 2 + (0.1)\log 2 + \frac{(0.1)^2}{2}\left(1 + \frac{1}{2}\log 2\right) + \frac{(0.1)^3}{6}\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right] = 2.036$$

$$y(1.2) = 2 + (0.2)\log 2 + \frac{(0.2)^2}{2}\left(1 + \frac{1}{2}\log 2\right) + \frac{(0.2)^3}{6}\left[-\frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right] = 2.081.$$

PROBLEMS 32.1

1. Using Picard's method, solve $dy/dx = -xy$ with $x_0 = 0, y_0 = 1$ upto third approximation. *(Mumbai, 2005)*
2. Employ Picard's method to obtain, correct to four places of decimal, solution of the differential equation $dy/dx = x^2 + y^2$ for $x = 0.4$, given that $y = 0$ when $x = 0$. *(J.N.T.U., 2009)*
3. Obtain Picard's second approximate solution of the initial value problem : $y' = x^2/(y^2 + 1), y(0) = 0$. *(Marathwada, 2008)*
4. Find an approximate value of y when $x = 0.1$, if $dy/dx = x - y^2$ and y = 1 at $x = 0$, using
   (*a*) Picard's method (*b*) Taylor's series. *(V.T.U., 2010 ; Madras, 2006)*
5. Solve $y' = x + y$ given $y(1) = 0$. Find $y(1.1)$ and $y(1.2)$ by Taylor's method. Compare the result with its exact value. *(J.N.T.U., 2008 ; Anna, 2005)*
6. Evaluate $y(0.1)$ correct to six places of decimals by Taylor's series method if $y(x)$ satisfies
   $$y' = xy + 1, y(0) = 1.$$
7. Solve $y' = 3x + y^2, y(0) = 1$ using Taylor's series method and computer $y(0.1)$. *(Mumbai, 2007)*
8. Using Taylor series method, find $y(0.1)$ correct to 3-decimal places given that
   $$dy/dx = e^x - y^2, y(0) = 1.$$

32.4 EULER'S METHOD*

Consider the equation $\dfrac{dy}{dx} = f(x, y)$ ...(1)

given that $y(x_0) = y_0$. Its curve of solution through $P(x_0, y_0)$ is shown dotted in Fig. 32.1. Now we have to find the ordinate of any other point Q on this curve.

Let us divide LM into n sub-intervals each of width h at $L_1, L_2, \ldots$ so that h is quite small. In the interval LL_1, we approximate the curve by the tangent at P. If the ordinate through L_1 meets this tangent in $P_1(x_0 + h, y_1)$, then

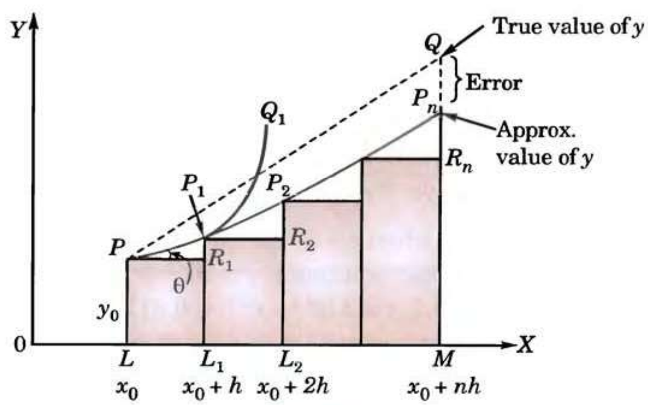

Fig. 32.1

$$y_1 = L_1P_1 = LP + R_1P_1$$
$$= y_0 + PR_1 \tan\theta = y_0 + h\left(\frac{dy}{dx}\right)_P$$
$$= y_0 + h\,f(x_0, y_0)$$

Let P_1Q_1 be the curve of solution of (1) through P_1 and let its tangent at P_1 meet the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$. Then

$$y_2 = y_1 + h\,f(x_0 + h, y_1) \quad \ldots(2)$$

Repeating this process n times, we finally reach an approximation MP_n of MQ given by

$$y_n = y_{n-1} + h\,f(x_0 + \overline{n-1}\,h, y_{n-1})$$

This is *Euler's method* of finding an approximate solution of (1).

Obs. In Euler's method, we approximate the curve of solution by the tangent in each interval, *i.e.* by a sequence of short lines. Unless h is small, the error is bound to be quite significant. This sequence of lines may also deviate considerably from the curve of solution. Hence there is a modification of this method which is given in the next section.

Example 32.6. *Using Euler's method, find an approximate value of y corresponding to x = 1, given that dy/dx = x + y and y = 1 when x = 0.* *(Mumbai, 2005 ; Rohtak, 2003)*

*See footnote p. 302.

Solution. We take $n = 10$ and $h = 0.1$ which is sufficiently small. The various calculations are arranged as follows :

x	y	$x + y = dy/dx$	Old y + 0.1 (dy/dx) = new y
0.0	1.00	1.00	1.00 + 0.1(1.00) = 1.10
0.1	1.10	1.20	1.10 + 0.1(1.20) = 1.22
0.2	1.22	1.42	1.22 + 0.1(1.42) = 1.36
0.3	1.36	1.66	1.36 + 0.1(1.66) = 1.53
0.4	1.53	1.93	1.53 + 0.1(1.93) = 1.72
0.5	1.72	2.22	1.72 + 0.1(2.22) = 1.94
0.6	1.94	2.54	1.94 + 0.1(2.54) = 2.19
0.7	2.19	2.89	2.19 + 0.1(2.89) = 2.48
0.8	2.48	3.89	2.48 + 0.1(3.89) = 2.81
0.9	2.81	3.71	2.81 + 0.1(3.71) = 3.18
1.0	3.18		

Thus the required approximate value of $y = 3.18$.

Obs. In example 32.1, the true value of y from its exact solution at $x = 1$ is 3.44 whereas by Euler's method $y = 3.18$ and by Picard's method $y = 3.434$. In the above solution, had we chosen $n = 20$, the accuracy would have been considerably increased but at the expense of double the labour of computation. Euler's method is no doubt very simple but cannot be considered as one of the best.

Example 32.7. *Given* $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$ *with initial condition* $y = 1$ *at* $x = 0$; *find* y *for* $x = 0.1$ *by Euler's method.*

(P.T.U., 2001)

Solution. We divide the interval (0, 0.1) into five steps *i.e.* we take $n = 5$ and $h = 0.02$. The various calculations are arranged as follows :

x	y	$(y - x)/(y + x) = dy/dx$	Old y + 0.02 (dy/dx) = new y
0.00	1.0000	1.0000	1.0000 + 0.02 (1.0000) = 1.0200
0.02	1.0200	0.9615	1.0200 + 0.02 (.9615) = 1.0392
0.04	1.0392	0.926	1.0392 + 0.02 (.926) = 1.0577
0.06	1.0577	0.893	1.0577 + 0.02 (.893) = 1.0756
0.08	1.0756	0.862	1.0756 + 0.02 (.862) = 1.0928
0.10	1.0928		

Hence the required approximate value of $y = 1.0928$.

32.5 MODIFIED EULER'S METHOD

In the Euler's method, the curve of solution in the interval LL_1 is approximated by the tangent at P (Fig. 32.1) such that at P_1, we have

$$y_1 = y_0 + h\,f(x_0, y_0) \qquad ...(1)$$

Then the slope of the curve of solution through P_1 [*i.e.* $(dy/dx)_{P_1} = f(x_0 + h, y_1)$] is computed and the tangent at P_1 to P_1Q_1 is drawn meeting the ordinate through L_2 in $P_2(x_0 + 2h, y_2)$.

Now we find a better approximation $y_1^{(1)}$ of $y(x_0 + h)$ by taking the slope of the curve as the mean of the slopes of the tangents at P and P_1, *i.e.*

$$y_1^{(1)} = y_0 + \frac{h}{2}\,[f(x_0, y_0) + f(x_0 + h, y_1)] \qquad ...(2)$$

As the slope of the tangent at P_1 is not known, we take y_1 as found in (1) by Euler's method and insert it on R.H.S. of (2) to obtain the first modified value $y_1^{(1)}$. The equation (1) is therefore, called the *predictor* while (2) serves as the *corrector of* y_1.

Again the corrector is applied and we find a still better value $y_1^{(2)}$ corresponding to L_1 as

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_0 + h, y_1^{(1)})]$$

We repeat this step, till two consecutive values of y agree. This is then taken as the starting point for the next interval L_1L_2.

Once y_1 is obtained to desired degree of accuracy, y corresponding to L_2 is found from the *predictor*

$$y_2 = y_1 + hf(x_0 + h, y_1)$$

and a better approximation $y_2^{(1)}$ is obtained from the *corrector*

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_0 + h, y_1) + f(x_0 + 2h, y_2)].$$

We repeat this step until y_2 becomes stationary. Then we proceed to calculate y_3 as above and so on.

This is the *modified Euler's method* which is a predictor-corrector method.

Example 32.8. *Using modified Euler's method, find an approximate value of y when $x = 0.3$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.* *(Rohtak, 2005 ; Bhopal, 2002 S ; Delhi, 2002)*

Solution. Taking $h = 0.1$, the various calculations are arranged as follows :

x	$x + y = y'$	*Mean slope*	*Old y + 0.1 (mean slope) = new y*
0.0	0 + 1	—	1.00 + 0.1 (1.00) = 1.10
0.1	.1 + 1.1	$\frac{1}{2}(1 + 1.2)$	1.00 + 0.1 (1.1) = 1.11
0.1	.1 + 1.11	$\frac{1}{2}(1 + 1.21)$	1.00 + 0.1 (1.105) = 1.1105
0.1	.1 + 1.1105	$\frac{1}{2}(1 + 1.2105)$	1.00 + 0.1 (1.1052) = 1.1105
0.1	1.2105	—	1.1105 + 0.1 (1.2105) = 1.2316
0.2	.2 + 1.2316	$\frac{1}{2}(1.2105 + 1.4316)$	1.1105 + 0.1 (1.3211) = 1.2426
0.2	.2 + 1.2426	$\frac{1}{2}(1.2105 + 1.4426)$	1.1105 + 0.1 (1.3266) = 1.2432
0.2	.2 + 1.2432	$\frac{1}{2}(1.2105 + 1.4432)$	1.1105 + 0.1 (1.3268) = 1.2432
0.2	1.4432	—	1.2432 + 0.1 (1.4432) = 1.3875
0.3	.3 + 1.3875	$\frac{1}{2}(1.4432 + 1.6875)$	1.2432 + 0.1 (1.5654) = 1.3997
0.3	.3 + 1.3997	$\frac{1}{2}(1.4432 + 1.6997)$	1.2432 + 0.1 (1.5715) = 1.4003
0.3	.3 + 1.4003	$\frac{1}{2}(1.4432 + 1.7003)$	1.2432 + 0.1 (1.5718) = 1.4004
0.3	.3 + 1.4004	$\frac{1}{2}(1.4432 + 1.7004)$	1.2432 + 0.1 (1.5718) = 1.4004

Hence $y(0.3) = 1.4004$ approximately.

Obs. In example 32.6, the approximate value of y for $x = 0.3$ would be 1.53 whereas by modified Euler's method the corresponding value is 1.4004 which is nearer its true value 1.3997, obtained from its exact solution $y = 2e^x - x - 1$ by putting $x = 0.3$.

Example 32.9. *Using modified Euler's method, find y(0.2) and y(0.4) given*

$$y' = y + e^x, \; y(0) = 0.$$

(J.N.T.U., 2009)

Solution. We have $y' = y + e^x = f(x, y)$; $x = 0, y = 0$ and $h = 0.2$

The various calculations are arranged as under :

To calculate $y(0.2)$:

x	$y + e^x = y'$	*Mean slope*	*Old y + h (mean slope) = new y*
0.0	1	—	$0 + 0.2\,(1) = 0.2$
0.2	$0.2 + e^{0.2} = 1.4214$	$\frac{1}{2}(1 + 1.4214) = 1.2107$	$0 + 0.2\,(1.2107) = 0.2421$
0.2	$0.2421 + e^{0.2} = 1.4635$	$\frac{1}{2}(1 + 1.4635) = 1.2317$	$0 + 0.2\,(1.2317) = 0.2463$
0.2	$0.2463 + e^{0.2} = 1.4677$	$\frac{1}{2}(1 + 1.4677) = 1.2338$	$0 + 0.2\,(1.2338) = 0.2468$
0.2	$0.2468 + e^{0.2} = 1.4682$	$\frac{1}{2}(1 + 1.4682) = 1.2341$	$0 + 0.2\,(1.2341) = 0.2468$

Since the last two values of y are equal, we take $y(0.2) = 0.2468$.

To calculate $y(0.4)$.

x	$y + e^x = y'$	*Mean slope*	*Old y + h (Mean slope) = new y*
0.2	$0.2468 + e^{0.2} = 1.4682$	—	$0.2468 + 0.2\,(1.4682) = 0.5404$
0.4	$0.5404 + e^{0.4} = 2.0322$	$\frac{1}{2}(1.4682 + 2.0322) = 1.7502$	$0.2468 + 0.2\,(1.7502) = 0.5968$
0.4	$0.5968 + e^{0.4} = 2.0887$	$\frac{1}{2}(1.4682 + 2.0887) = 1.7784$	$0.2468 + 0.2\,(1.7784) = 0.6025$
0.4	$0.6025 + e^{0.4} = 2.0943$	$\frac{1}{2}(1.4682 + 2.0943) = 1.78125$	$0.2468 + 0.2\,(1.78125) = 0.6030$
0.4	$0.6030 + e^{0.4} = 2.0949$	$\frac{1}{2}(1.4682 + 2.0949) = 1.7815$	$0.2468 + 0.2\,(1.7815) = 0.6031$
0.4	$0.6031 + e^{0.4} = 2.0949$	$\frac{1}{2}(1.4682 + 2.0949) = 1.7816$	$0.2468 + 0.2\,(1.7815) = 0.6031$

Since the last two value of y are equal, we take $y(0.4) = 0.6031$.

Hence $y(0.2) = 0.2468$ and $y(0.4) = 0.6031$ approximately.

Example 32.10. *Solve the following by Euler's modified method :*

$$\frac{dy}{dx} = \log(x + y),\ y(0) = 2.$$

at $x = 1.2$ and 1.4 with $h = 0.2$. *(Bhopal, 2009 ; U.P.T.U., 2007)*

Solution. The various calculations are arranged as follows :

x	$\log(x + y) = y'$	*Mean slope*	*Old y + 0.2 (mean slope) = new y*
0.0	$\log(0 + 2)$	—	$2 + 0.2\,(0.301) = 2.0602$
0.2	$\log(0.2 + 2.0602)$	$\frac{1}{2}(0.301 + 0.3541)$	$2 + 0.2\,(0.3276) = 2.0655$
0.2	$\log(0.2 + 2.0655)$	$\frac{1}{2}(0.301 + 0.3552)$	$2 + 0.2\,(0.3281) = 2.0656$
0.2	0.3552	—	$2.0656 + 0.2\,(0.3552) = 2.1366$
0.4	$\log(0.4 + 2.1366)$	$\frac{1}{2}(0.3552 + 0.4042)$	$2.0656 + 0.2\,(0.3797) = 2.1415$
0.4	$\log(0.4 + 2.1415)$	$\frac{1}{2}(0.3552 + 0.4051)$	$2.0656 + 0.2\,(0.3801) = 2.1416$

x	$\log(x+y) = y'$	*Mean slope*	*Old y + 0.2 (mean slope) = new y*
0.4	0.4051	—	2.1416 + 0.2 (0.4051) = 2.2226
0.6	log (0.6 + 2.2226)	$\frac{1}{2}$(0.4051 + 0.4506)	2.1416 + 0.2 (0.4279) = 2.2272
0.6	log (0.6 + 2.2272)	$\frac{1}{2}$(0.4051 + 0.4514)	2.1416 + 0.2 (0.4282) = 2.2272
0.6	0.4514	—	2.2272 + 0.2 (0.4514) = 2.3175
0.8	log (0.8 + 2.3175)	$\frac{1}{2}$(0.4514 + 0.4938)	2.2272 + 0.2 (0.4726) = 2.3217
0.8	log (0.8 + 2.3217)	$\frac{1}{2}$(0.4514 + 0.4943)	2.2272 + 0.2 (0.4727) = 2.3217
0.8	0.4943	—	2.3217 + 0.2 (0.4943) = 2.4206
1.0	log (1 + 2.4206)	$\frac{1}{2}$(0.4943 + 0.5341)	2.3217 + 0.2 (0.5142) = 2.4245
1.0	log (1 + 2.4245)	$\frac{1}{2}$(0.4943 + 0.5346)	2.3217 + 0.2 (0.5144) = 2.4245
1.0	0.5346	—	2.4245 + 0.2 (0.5346) = 2.5314
1.2	log (1.2 + 2.5314)	$\frac{1}{2}$(0.5346 + 0.5719)	2.4245 + 0.2 (0.5532) = 2.5351
1.2	log (1.2 + 2.5351)	$\frac{1}{2}$(0.5346 + 0.5723)	2.4245 + 0.2 (0.5534) = 2.5351
1.2	0.5723	—	2.5351 + 0.2 (0.5723) = 2.6496
1.4	log (1.4 + 2.6496)	$\frac{1}{2}$(0.5723 + 0.6074)	2.5351 + 0.2 (0.5898) = 2.6531
1.4	log (1.4 + 2.6531)	$\frac{1}{2}$(0.5723 + 0.6078)	2.5351 + 0.2 (0.5900) = 2.6531

Hence $y(1.2) = 2.5351$ and $y(1.4) = 2.6531$ approximately.

Example 32.11. *Using Euler's modified method, obtain a solution of the equation $dy/dx = x + |\sqrt{y}|$, with initial conditions $y = 1$ at $x = 0$, for the range $0 \le x \le 0.6$ in steps of 0.2.* (V.T.U., 2007)

Solution. The various calculations are arranged as follows :

x	$x + \|\sqrt{y}\| = y'$	*Mean slope*	*Old y + .2 (mean slope) = new y*
0.0	0 + 1 = 1	—	1 + 0.2 (1) = 1.2
0.2	$0.2 + \|\sqrt{(1.2)}\| = 1.2954$	$\frac{1}{2}$(1 + 1.2954) = 1.1477	1 + 0.2 (1.1477) = 1.2295
0.2	$0.2 + \|\sqrt{(1.2295)}\| = 1.3088$	$\frac{1}{2}$(1 + 1.3088) = 1.1544	1 + 0.2 (1.1544) = 1.2309
0.2	$0.2 + \|\sqrt{(1.2309)}\| = 1.3094$	$\frac{1}{2}$(1 + 1.3094) = 1.1547	1 + 0.2 (1.1547) = 1.2309
0.2	1.3094	—	1.2309 + 0.2 (1.3094) = 1.4927
0.4	$0.4 + \|\sqrt{(1.4927)}\| = 1.6218$	$\frac{1}{2}$(1.3094 + 1.6218) = 1.4654	1.2309 + 0.2 (1.4654) = 1.5240
0.4	$0.2 + \|\sqrt{(1.524)}\| = 1.6345$	$\frac{1}{2}$(1.3094 + 1.6345) = 1.4718	1.2309 + 0.2 (1.4718) = 1.5253
0.4	$0.4 + \|\sqrt{(1.5253)}\| = 1.6350$	$\frac{1}{2}$(1.3094 + 1.6350) = 1.4721	1.2309 + 0.2 (1.4721) = 1.5253

x	$x + \lvert\sqrt{y}\rvert = y'$	Mean slope	Old y + .2 (mean slope) = new y
0.4	1.6350	—	1.5253 + 0.2 (1.635) = 1.8523
0.6	$0.6 + \lvert\sqrt{(1.8523)}\rvert = 1.9610$	$\frac{1}{2}(1.635 + 1.961) = 1.798$	1.5253 + 0.2 (1.798) = 1.8849
0.6	$0.6 + \lvert\sqrt{(1.8849)}\rvert = 1.9729$	$\frac{1}{2}(1.635 + 1.9729) = 1.8040$	1.5253 + 0.2 (1.804) = 1.8861
0.6	$0.6 + \lvert\sqrt{(1.8861)}\rvert = 1.9734$	$\frac{1}{2}(1.635 + 1.9734) = 1.8042$	1.5253 + 0.2 (1.8042) = 1.8861

Hence $y(0.6) = 1.8861$ approximately.

PROBLEMS 32.2

1. Apply Euler's method to solve $y' = x + y$, $y(0) = 0$, choosing the step length = 0.2. (Carry out 6 steps). *(Kottayam, 2005)*
2. Using simple Euler's method solve for y at $x = 0.1$ from $dy/dx = x + y + xy$, $y(0) = 1$, taking step size $h = 0.025$.
3. Using Euler's method, find the approximate value of y when $dy/dx = x^2 + y^2$ and $y(0) = 1$ in five steps (*i.e.* $h = 0.2$). *(Mumbai, 2006)*
4. Solve $y' = 1 - y$, $y(0) = 0$ by modified Euler's method and obtain y at $x = 0.1, 0.2, 0.3$. *(Anna, 2005)*
5. Given $y' = x + \sin y$, $y(0) = 1$. Compute $y(0.2)$ and $y(0.4)$ with $h = 0.2$ using Euler's modified method. *(J.N.T.U., 2007)*
6. Given that $dy/dx = x^2 + y$ and $y(0) = 1$. Find an approximate value of $y(0.1)$ taking $h = 0.05$ by modified Euler's method. *(V.T.U., 2010)*
7. Given $\dfrac{dy}{dx} = \dfrac{y - x}{y + x}$ with boundary conditions $y = 1$ when $x = 0$, find approximately y for $x = 0.1$, by Euler's modified method (5 steps). *(V.T.U., 2007)*
8. Given that $dy/dx = 2 + \sqrt{(xy)}$ and $y = 1$ when $x = 1$. Find approximate value of y at $x = 2$ in steps of 0.2, using Euler's modified method. *(Anna, 2004)*

32.6 RUNGE'S METHOD*

Consider the differential equation,

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \qquad ...(1)$$

Clearly the slope of the curve through $P(x_0, y_0)$ is $f(x_0, y_0)$ (Fig. 32.2).

Integrate both sides of (1) from (x_0, y_0) to $(x_0 + h, y_0 + k)$, we have

$$\int_{y_0}^{y_0 + k} dy = \int_{x_0}^{x_0 + h} f(x, y)\, dx \qquad ...(2)$$

To evaluate the integral on the right, we take N as the mid-point of LM and find the values of $f(x, y)$ (*i.e.* dy/dx) at the points $x_0, x_0 + h/2, x_0 + h$. For this purpose, we first determine the values of y at these points.

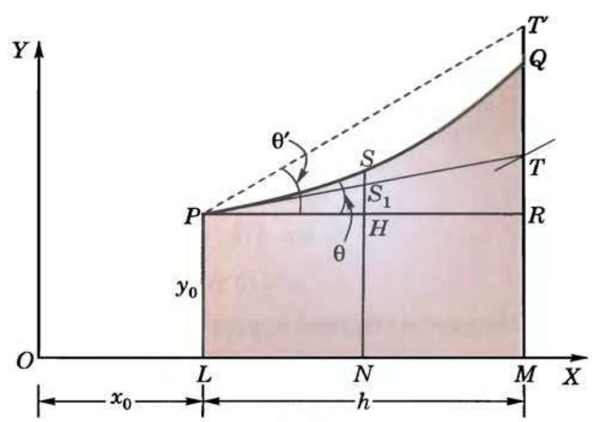

Fig. 32.2

Let the ordinate through N cut the curve PQ in S and the tangent PT in S_1. The value of y_s is given by the point S_1.

$$\therefore \qquad y_s = NS = LP + HS_1 = y_0 + PH \tan\theta$$

* Called after the German mathematician *Carl Runge* (1856–1927) who was professor at Gottingen.

$$= y_0 + \frac{h}{2}(dy/dx)_P = y_0 + \frac{h}{2} f(x_0, y_0) \quad ...(3)$$

Also $\quad y_T = MT = LP + RT = y_0 + PR \tan\theta = y_0 + hf(x_0, y_0)$.

Now the value of y_Q at $x_0 + h$ is given by the point T' where the line through P drawn with slope at $T(x_0 + h, y_T)$ meets MQ.

$\because$ Slope at $T = \tan\theta' = f(x_0 + h, y_T) = f[x_0 + h, y_0 + hf(x_0, y_0)]$

$$\therefore \quad y_Q = MR + RT' = y_0 + PT \tan\theta' = y_0 + hf[x_0 + h, y_0 + hf(x_0, y_0)] \quad ...(4)$$

Thus the value of $f(x, y)$ at $P = f(x_0, y_0)$,
the value of $f(x, y)$ at $S = f(x_0 + h/2, y_s)$
and the value of $f(x, y)$ at $Q = f(x_0 + h, y_Q)$
where y_S and y_Q are given by (3) and (4).

Hence from (2), we obtain

$$k = \int_{x_0}^{x_0+h} f(x, y)\,dx = \frac{h}{6}[f_P + 4f_S + f_Q] \quad \text{[By Simpsons' rule (p. 1106)]}$$

$$= \frac{h}{6}[f(x_0, y_0) + 4f(x_0 + h/2, y_S) + f(x_0 + h, y_Q)] \quad ...(5)$$

which gives a sufficiently accurate value of k and also of $y = y_0 + k$.

The repeated application of (5) gives the values of y for equispaced points.

Working rule *to solve (1) by Runge's method :*

Calculate successively

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1\right)$$

$$k' = hf(x_0 + h, y_0 + k_1)$$

and $$k_3 = hf(x_0 + h, y_0 + k')$$

Finally compute, $$k = \frac{1}{6}(k_1 + 4k_2 + k_3).$$

(Note that k is the weighted mean of k_1, k_2 and k_3)

Example 32.12. *Apply Runge's method to find an approximate value of y when $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.*

Solution. Here we have $x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$

$\therefore \quad k_1 = hf(x_0, y_0) = 0.2\,(1) \quad = 0.200$

$k_2 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1\right) = 0.2\,f(0.1, 1.1) \quad = 0.240$

$k' = hf(x_0 + h, y_0 + k_1) = 0.2\,f(0.2, 1.2) \quad = 0.280$

and $\quad k_3 = hf(x_0 + h, y_0 + k') = 0.2\,f(0.1, 1.28) \quad = 0.296$

$\therefore \quad k = \tfrac{1}{6}(k_1 + 4k_2 + k_3)$

$= \tfrac{1}{6}(0.200 + 0.960 + 0.296) \quad = 0.2426$

Hence the required approximate value of y is 1.2426.

32.7 RUNGE-KUTTA METHOD*

The Taylor's series method of solving differential equations numerically is restricted by the labour involved in finding the higher order derivatives. However there is a class of methods known as Runge-Kutta methods which do not require the calculations of higher order derivatives. These methods agree with Taylor's series solution upto the terms in h^r, where r differs from method to method and is called the *order of that method. Euler's method, Modified Euler's method and Runge's method are the Runge-Kutta methods of the first, second and third order respectively.*

* See footnote p. 1017. Named after *Wilhelm Kutta* (1867—1944).

The fourth-order Runge-Kutta method is most commonly used and is often referred to as 'Runge-Kutta method' only.

Working rule *for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method from*

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \text{ is as follows :}$$

Calculate successively

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1\right)$$

$$k_3 = hf\left(x_0 + \tfrac{1}{2}h,\ y + \tfrac{1}{2}k_2\right)$$

and $$k_4 = hf(x_0 + h, y_0 + k_3)$$

Finally compute $$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

which gives the required approximate value $y_1 = y_0 + k$.

(Note that k is the weighted mean of k_1, k_2, k_3 and k_4)

Obs. One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.

Example 32.13. *Apply Runge-Kutta fourth order method, to find an approximate value of y when $x = 0.2$, given that $dy/dx = x + y$ and $y = 1$ when $x = 0$.* *(V.T.U., 2009 ; P.T.U., 2007 ; S.V.T.U., 2007)*

Solution. Here $x_0 = 0, y_0 = 1, h = 0.2, f(x_0, y_0) = 1$

$\therefore$ $$k_1 = hf(x_0, y_0) = 0.2 \times 1 = 0.2000$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_1\right) = 0.2 \times f(0.1, 1.1) = 0.2400$$

$$k_3 = hf\left(x_0 + \tfrac{1}{2}h,\ y_0 + \tfrac{1}{2}k_2\right) = 0.2 \times f(0.1, 1.12) = 0.2440$$

and $$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.244) = 0.2888$$

$\therefore$ $$k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \tfrac{1}{6}(0.2000 + 0.4800 + 0.4880 + 0.2888) = \tfrac{1}{6} \times (1.4568) = 0.2468.$$

Hence the required approximate value of y is 1.2428.

Example 32.14. *Using Runge-Kutta method of fourth order, solve $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2, 0.4$.* *(U.P.T.U., 2010 ; J.N.T.U., 2009 ; V.T.U., 2008)*

Solution. We have $f(x, y) = \dfrac{y^2 - x^2}{y^2 + x^2}$

To find y(0.2) :

Here $x_0 = 0, y_0 = 1, h = 0.2$

$$k_1 = h\,f(x_0, y_0) = 0.2\,f(0, 1) = 0.2000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_1\right) = 0.2\,f(0.1, 1.1) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right) = 0.2\,f(0.1, 1.09836) = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2\,f(0.2, 1.1967) = 0.1891$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891]$$

$$= 0.19599$$

Hence $$y(0.2) = y_0 + k = 1.196.$$

To find y(0.4) :

Here $x_1 = 0.2, y_1 = 1.196, h = 0.2$

$$k_1 = h\,f(x_1, y_1) \qquad = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = 0.2\,f(0.3, 1.2906) \qquad = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = 0.2\,f(0.3, 1.2858) \qquad = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2\,f(0.4, 1.3753) \qquad = 0.1688$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688] \qquad = 0.1792$$

Hence $y(0.4) = y_1 + k = 1.196 + 0.1792 = 1.3752.$

Example 32.15. *Apply Runge-Kutta method to find an approximate value of y for x = 0.2 in steps of 0.1, if $dy/dx = x + y^2$, given that y = 1, where x = 0.* *(V.T.U., 2009 ; Osmania, 2007 ; Madras, 2000)*

Solution. Here we take $h = 0.1$ and carry out the calculations in two steps.

Step I. $x_0 = 0, y_0 = 1, h = 0.1$

$$\therefore \quad k_1 = hf(x_0, y_0) = 0.1\,f(0, 1) \qquad = 0.1000$$

$$k_2 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1\right) = 0.1\,f(0.05, 1.1) \qquad = 0.1152$$

$$k_3 = hf\left(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2\right) = 0.1\,f(0.05, 1.1152) \qquad = 0.1168$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1\,f(0.1, 1.1168) \qquad = 0.1347$$

$$\therefore \quad k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \tfrac{1}{6}(0.1000 + 0.2304 + 0.2336 + 0.1347) \qquad = 0.1165$$

giving $y(0.1) = y_0 + k = 1.1165.$

Step II. $x_1 = x_0 + h = 0.1, y_1 = 1.1165, h = 0.1$

$$\therefore \quad k_1 = hf(x_1, y_1) = 0.1\,f(0.1, 1.1165) \qquad = 0.1347$$

$$k_2 = hf\left(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_1\right) = 0.1\,f(0.15, 1.1838) \qquad = 0.1551$$

$$k_3 = hf\left(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_2\right) = 0.1\,f(0.15, 1.194) \qquad = 0.1576$$

$$k_4 = hf(x_1 + h, y_2 + k_3) = 0.1\,f(0.2, 1.1576) \qquad = 0.1823$$

$$\therefore \quad k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = 0.1571$$

Hence $y(0.2) = y_1 + k = 1.2736.$

Example 32.16. *Using Runge-Kutta method of fourth order, solve for y at x = 1.2, 1.4 from $\dfrac{dy}{dx} = \dfrac{2xy + e^x}{x^2 + xe^x}$ given $x_0 = 1, y_0 = 0$.* *(Mumbai, 2008)*

Solution. We have $f(x, y) = \dfrac{2xy + e^x}{x^2 + xe^2}$

To find y(1.2) :

Here $x_0 = 1, y_0 = 0, h = 0.2$

$$\therefore \quad k_1 = h\,f(x_0, y_0) = 0.2\,\frac{0 + e}{1 + e} = 0.1462$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\left\{\frac{2(1 + 0.1)(0 + 0.073) + e^{1+0.1}}{(1 + 0.1)^2 + (1 + 0.1)\,e^{1+0.1}}\right\} = 0.1402$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 \left\{\frac{2(1+0.1)(0+0.07)+e^{1.1}}{(1+0.1)^2+(1+0.1)\,e^{1.1}}\right\} = 0.1399$$

$$k_4 = hf(x_0 + k, y_0 + k_3) = 0.2 \left\{\frac{2(1.2)(0.1399)+e^{1.2}}{(1.2)^2+(1.2)\,e^{1.2}}\right\} = 0.1348$$

and $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}[0.1462 + 0.2804 + 0.2798 + 0.1348]$$

$$= 0.1402.$$

Hence $y(1.2) = y_0 + k = 0 + 0.1402 = 0.1402.$

To find y(1.4) :

Here $x_1 = 1.2, y_1 = 0.1402, h = 0.2$

$$k_1 = hf(x_1, y_1) = 0.2\,f(1.2, 0) = 0.1348$$
$$k_2 = hf(x_1 + h/2, y_1 + k_1/2) = 0.2\,f(1.3, 0.2076) = 0.1303$$
$$k_3 = hf(x_1 + h/2, y_1 + k_1/2) = 0.2\,f(1.3, 0.2053) = 0.1301$$
$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2\,f(1.3, 0.2703) = 0.1260$$

$\therefore$ $$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}[0.1348 + 0.2606 + 0.2602 + 0.1260] = 0.1303$$

Hence $y(1.4) = y_1 + k = 0.1402 + 0.1303 = 0.2705.$

PROBLEMS 32.3

1. Use Runge's method to approximate y when $x = 1.1$, given that $y = 1.2$ when $x = 1$ and $dy/dx = 3x + y^2$.
2. Using Runge-Kutta method of order 4, find $y(0.2)$ given that $dy/dx = 3x + \frac{1}{2}y$, $y(0) = 1$, taking $h = 0.1$. *(V.T.U., 2004)*
3. Using Runge-Kutta method of order 4, compute $y(.2)$ and $(.4)$ from $10\dfrac{dy}{dx} = x^2 + y^2$, $y(0) = 1$, taking $h = 0.1$. *(Rohtak, 2003 ; Bhopal, 2002)*
4. Use Runge-Kutta method to find y when $x = 1.2$ in steps of 0.1, given that :
   $dy/dx = x^2 + y^2$ and $y(1) = 1.5$. *(Mumbai, 2007)*
5. Find $y(0.1)$ and $y(0.2)$ using Runge-Kutta 4th order formula, given that $y' = x^2 - y$ and $y(0) = 1$. *(J.N.T.U., 2006)*
6. Using 4th order Runge-Kutta method, solve the following equation, taking each step of $h = 0.1$, given $y(0) = 3$, $dy/dx = (4x/y - xy)$. Calculate y for $x = 0.1$ and 0.2. *(Anna, 2007)*
7. Use fourth order Runge-Kutta method to find y at $x = 0.1$, given that $\dfrac{dy}{dx} = 3e^x + 2y$, $y(0) = 0$ and $h = 0.1$. *(V.T.U., 2006)*
8. Find by Runge-Kutta method an approximate value of y for $x = 0.8$, given that $y = 0.41$ when $x = 0.4$ and $dy/dx = \sqrt{(x+y)}$. *(S.V.T.U., 2007 S)*
9. Using Runge-Kutta method of order 4, find $y(0.2)$ for the equation $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$, $y(0) = 1$. Take $h = 0.2$. *(V.T.U., 2011 S)*
10. Given that $dy/dx = (y^2 - 2x)/(y^2 + x)$ and $y = 1$ at $x = 0$; find y for $x = 0.1, 0.2, 0.3, 0.4$ and 0.5. *(Delhi, 2002)*

32.8 PREDICTOR-CORRECTOR METHODS

If x_{i-1} and x_i be two consecutive mesh points, we have $x_i = x_{i-1} + h$. In the Euler's method (§ 32.4), we have

$$y_i = y_{i-1} + hf(x_0 + \overline{i-1}\,h, y_{i-1})\ ;\ i = 1, 2, 3, \ldots \qquad \ldots(1)$$

The modified Euler's method (§ 32.5), gives

$$y_i = y_{i-1} + \frac{h}{2}[f(x_{i-1}, y_{i-1}) + f(x_i, y_i)] \quad ...(2)$$

The value of y_i is first estimated by using (1), then this value is inserted on the right side of (2), giving a better approximation of y_i. This value of y_i is again substituted in (2) to find a still better approximation of y_i. This step is repeated till two consecutive values of y_i agree. *This technique of refining an initially crude estimate of y_i by means of a more accurate formula is known as* **predictor-corrector method.** The equation (1) is therefore called the *predictor* while (2) serves as a *corrector* of y_i.

In the methods so far explained, to solve a differential equation over an interval (x_i, x_{i+1}) only the value of y at the beginning of the interval was required. In the *predictor-corrector* methods, four prior values are required for finding the value of y at x_{i+1}. A predictor formula is used to predict the value of y at x_{i+1} and then a corrector formula is applied to improve this value.

We now describe two such methods, namely : Milne's method and Adams-Bashforth method.

32.9 MILNE'S METHOD

Given $dy/dx = f(x, y)$ and $y = y_0, x = x_0$; to find an approximate value of y for $x = x_0 + nh$ by Milne's method, we proceed as fo... :

The value $y_0 = y(x_0)$ being given, we compute

$$y_1 = y(x_0 + h),\ y_2 = y(x_0 + 2h),\ y_3 = y(x_0 + 3h),$$

by Picard's or Taylor's series method.

Next we calculate,

$$f_0 = f(x_0, y_0),\ f_1 = f(x_0 + h, y_1),\ f_2 = f(x_0 + 2h, y_2),\ f_3 = f(x_0 + 3h, y_3)$$

Then to find $y_4 = y(x_0 + 4h)$, we substitute Newton's forward interpolation formula

$$f(x, y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{6}\Delta^3 f_0 + ...$$

in the relation $y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x, y)\, dx$

$$\therefore \quad y_4 = y_0 + \int_{x_0}^{x_0+4h}\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + ...\right)dx \qquad [\text{Put } x = x_0 + nh,\ dx = h\,dn]$$

$$= y_0 + h\int_0^4\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + ...\right)dn$$

$$= y_0 + h\left(4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + ...\right)$$

Neglecting fourth and higher order differences and expressing Δf_0, $\Delta^2 f_0$ and $\Delta^3 f_0$ in terms of the function values, we get

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) \text{ which is called a } \textit{predictor.}$$

Having found y_4, we obtain a first approximation to $f_4 = f(x_0 + 4h, y_4)$.

Then a better value of y_4 is found by Simpson's rule (p. 1106) as

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \text{ which is called a } \textit{corrector.}$$

Then an improved value of f_4 is computed and again the corrector is applied to find a still better value of y_4. We repeat this step until y_4 remains unchanged.

Once y_4 and f_4 are obtained to desired degree of accuracy, $y_5 = y(x_0 + 5h)$ is found from the *predictor* as

$$y_5^{(p)} = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4)$$

and $f_5 = f(x_0 + 5h, y_5)$ is calculated. Then a better approximation to the value of y_5 is obtained from the *corrector* as

$$y_5^{(c)} = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5).$$

We repeat this step till y_5 becomes stationary and we, then proceed to calculate y_6 as before.

This is *Milne's predictor-corrector method.* To ensure greater accuracy, we must first improve the accuracy of the starting values and then sub-divide the intervals.

Example 32.17. *Apply Milne's method, to find a solution of the differential equation $y' = x - y^2$ in the range $0 \le x \le 1$ for the boundary conditions $y = 0$ at $x = 0$.* (V.T.U., 2009, Anna, 2005, Rohtak, 2005)

Solution. Using Picard's method, we have

$$y = y(0) + \int_0^x f(x, y)\ dx, \text{ where } f(x, y) = x - y^2.$$

To get the first approximation, we put $y = 0$ in $f(x, y)$,

giving $$y_1 = 0 + \int_0^x x\, dx = \frac{x^2}{2}$$

To find the second approximation, we put $y = x^2/2$ in $f(x, y)$,

giving $$y_2 = \int_0^x \left(x - \frac{x^4}{4}\right) dx = \frac{x^2}{2} - \frac{x^5}{20}$$

Similarly, the third approximation is

$$y_3 = \int_0^x \left[x - \left(\frac{x^2}{2} - \frac{x^5}{20}\right)^2\right] dx = \frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400} \qquad ...(i)$$

Now let us determine the starting values of the Milne's method from (*i*), by choosing $h = 0.2$.

$\therefore$ $x_0 = 0.0,\quad y_0 = 0.0000,\quad f_0 = 0.0000$

$x_1 = 0.2,\quad y_1 = 0.020,\quad f_1 = 0.1996$

$x_2 = 0.4,\quad y_2 = 0.0795,\quad f_2 = 0.3937$

$x_3 = 0.6,\quad y_3 = 0.1762,\quad f_3 = 0.5689$

Using the *predictor*, $$y_4^{(p)} = y_0 \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$x = 0.8,\quad y_4^{(p)} = 0.3049,\quad f_4 = 0.7070$

and the *corrector*, $$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4), \text{ yields}$$

$$y_4^{(c)} = 0.3046,\quad f_4 = 0.7072 \qquad ...(ii)$$

Again using the *corrector*, $y_4^{(c)} = 0.3046$, which is same as in (*ii*)

Now using the *predictor*, $$y_5^{(p)} = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4),$$

$x = 1.0,\quad y_5^{(p)} = 0.4554,\quad f_5 = 0.7926$

and the *corrector*, $$y_5^{(c)} = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5), \text{ gives}$$

$$y_5^{(c)} = 0.4555,\quad f_5 = 0.7925$$

Again using the *corrector*,

$y_5^{(c)} = 0.4555$, a value which is the same as before.

Hence, $y(1) = 0.4555$.

Example 32.18. *Given $y' = x(x^2 + y^2)\, e^{-x}$, $y(0) = 1$, find y at $x = 0.1, 0.2$ and 0.3 by Taylor's series method and compute $y(0.4)$ by Milne's method.* (Anna, 2007)

Solution. Given $y(0) = 1$ and $h = 0.1$

We have $y'(x) = x(x^2 + y^2)e^{-x}$; $\qquad y'(0) = 0$

$$y''(x) = [(x^3 + xy^2)(-e^{-x}) + 3x^2 + y^2 + x(2y)y']e^{-x}$$

$$= e^{-x}[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy']; \qquad y''(0) = 1$$

$$y'''(x) = -e^{-x}[-x^3 - xy^2 + 3x^2 + y^2 + 2xyy' + 3x^2 + y^2 + 2xyy' - 6x - 2yy' - 2xy'^2 - 2xyy'] \qquad y'''(0) = -2$$

Substitute these values in the Taylor's series,

$$y(x) = y(0) + \frac{x}{1!}y'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \ldots$$

$$y(0.1) = 1 + (0.1)(0) + \frac{1}{2}(0.1)^2(1) + \frac{1}{6}(0.1)^3(-2) + \ldots$$

$$= 1 + 0.005 - 0.0003 = 1.0047 \quad i.e., \quad 1.005$$

Now taking $x = 0.1, y(0.1) = 1.005, h = 0.1$

$$y'(0.1) = 0.092, y''(0.1) = 0.849, y'''(0.1) = -1.247$$

Substituting these values in the Taylor's series about $x = 0.1$,

$$y(0.2) = y(0.1) + \frac{0.1}{1!}y'(0.1) + \frac{(0.1)^2}{2!}y''(0.1) + \frac{(0.1)^3}{3!}y'''(0.1) + \ldots$$

$$= 1.005 + (0.1)(0.092) + \frac{(0.1)^2}{2}(0.849) + \frac{(0.1)^3}{3}(-1.247) + \ldots$$

$$= 1.018$$

Now taking $x = 0.2, y(0.2) = 1.018, h = 0.1$

$$y'(0.2) = 0.176, y''(0.2) = 0.77, y'''(0.2) = 0.819$$

Substituting these values in the Taylor's series

$$y(0.3) = y(0.2) + \frac{0.1}{1!}y''(0.2) + \frac{(0.1)^2}{2!}y''(0.2) + \frac{(0.1)^3}{3!}y'''(0.2) + \ldots$$

$$= 1.018 + 0.0176 + 0.0039 + 0.0001 = 1.04$$

Thus the starting values of the Milne's method with $h = 0.1$ are

$x_0 = 0.0$	$y_0 = 1$	$f_0 = y'_0 = 0$
$x_1 = 0.1$	$y_1 = 1.005$	$f_1 = 0.092$
$x_2 = 0.2$	$y_2 = 1.018$	$f_2 = 0.176$
$x_3 = 0.3$	$y_3 = 1.04$	$f_3 = 0.26$

Using the predictor, $y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$

$$= 1 + \frac{4(0.1)}{3}[2(0.092) - (0.176) + 2(0.26)] = 1.09$$

$\therefore \quad x = 0.4 \qquad y_4^{(p)} = 1.09 \qquad f_4 = y'(0.4) = 0.362$

Using the corrector, $y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$

$$\therefore \quad y_4^{(c)} = 0.018 + \frac{0.1}{3}(0.176 + 4(0.26) + 0.362) = 1.071$$

Hence $y(0.4) = 1.071.$

Example 32.19. *Using Runge-Kutta method of order 4, find y for x = 0.1, 0.2, 0.3 given that $dy/dx = xy + y^2$, $y(0) = 1$. Continue the solution at x = 0.4 using Milne's method.*

(V.T.U., 2008 ; S.V.T.U., 2007 ; Madras, 2006)

Solution. We have $f(x, y) = xy + y^2$.

To find y(0.1) :

Here $x_0 = 0, y_0 = 1, h = 0.1.$

$\therefore \quad k_1 = h\,f(x_0, y_0) = (0.1)\,f(0.1) \qquad = 0.1000$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1)\,f(0.05, 1.05) \qquad = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1)\,f(0.05, 1.0577) \qquad = 0.1172$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)\,f(0.1, 1.1172) \qquad = 0.13598$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.231 + 0.2348 + 0.13598) \qquad = 0.11687$$

Thus $y(0.1) = y_1 = y_0 + k = 1.1169$.

To find y(0.2) :

Here $x_1 = 0.1, y_1 = 1.1169, h = 0.1$.

$$k_1 = h\,f(x_1, y_1) = (0.1)\,f(0.1, 1.1169) \qquad = 0.1359$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1)\,f(0.15, 1.1848) \qquad = 0.1581$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1)\,f(0.15, 1.1959) \qquad = 0.1609$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)\,f(0.2, 1.2778) \qquad = 0.1888$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = 0.1605$$

Thus $y(0.2) = y_2 = y_1 + k = 1.2773$.

To find y(0.3) :

Here $x_2 = 0.2, y_2 = 1.2773, h = 0.1$.

$$k_1 = hf(x_2, y_2) = (0.1)\,f(0.2, 1.2773) \qquad = 0.1887$$

$$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1)\,f(0.25, 1.3716) \qquad = 0.2224$$

$$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1)\,f(0.25, 1.3885) \qquad = 0.2275$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1)\,f(0.3, 1.5048) \qquad = 0.2716$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = 0.2267$$

Thus $y(0.3) = y_3 = y_2 + k = 1.504$.

Now the starting values of the Milne's method are :

$x_0 = 0.0$	$y_0 = 1.0000$	$f_0 = 1.0000$
$x_1 = 0.1$	$y_1 = 1.1169$	$f_1 = 1.3591$
$x_2 = 0.2$	$y_2 = 1.2773$	$f_2 = 1.8869$
$x_3 = 0.3$	$y_3 = 1.5049$	$f_3 = 2.7132$

Using the *predictor*,

$$y_4^{(p)} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

$x_4 = 0.4 \qquad y_4^{(p)} = 1.8344 \qquad f_4 = 4.0988$

and the *corrector*,

$$y_4^{(c)} = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4) \text{ yields}$$

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4\,(2.7132) + 4.098]$$

$$= 1.8386 \qquad f_4 = 4.1159$$

Again using the *corrector*,

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1159]$$

$$= 1.8391 \qquad\qquad f_4 = 4.1182 \qquad ...(i)$$

Again using the *corrector*

$$y_4^{(c)} = 1.2773 + \frac{0.1}{3}[1.8869 + 4(2.7132) + 4.1182]$$

$= 1.8392$ which is same as (i).

Hence $y(0.4) = 1.8392$.

PROBLEMS 32.4

1. Given $\frac{dy}{dx} = x^3 + y, y(0) = 2$. The value of $y(0.2) = 2.073$, $y(0.4) = 2.452$, and $y(0.6) = 3.023$ are got by R.K. Method of 4th order. Find $y(0.8)$ by Milne's predictor-corrector method taking $h = 0.2$. *(Anna, 2004)*
2. Given $2\, dy/dx = (1 + x^2)y^2$ and $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$, evaluate $y(0.4)$ by Milne's predictor-corrector method. *(V.T.U., 2011 S ; Madras, 2003)*
3. From the data given below, find y at $x = 1.4$, using Milne's predictor-corrector formula :

$$\frac{dy}{dx} = x^2 + \frac{y}{2}$$

x :	1	1.1	1.2	1.3
y :	2	2.2156	2.4549	2.7514

(V.T.U., 2007)

4. Using Milne's method, find $y(4.5)$ given $5xy' + y^2 - 2 = 0$ given $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$, $y(4.3) = 1.0143$, $y(4.4) = 1.0187$. *(Anna, 2007)*
5. If $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.04$ and $y(0.3) = 2.09$; find $y(0.4)$ using Milne's predictor-corrector method. *(V.T.U., 2010)*
6. Using Runge-Kutta method, calculate $y(0.1)$, $y(0.2)$, and $y(0.3)$ given that $\frac{dy}{dx} - \frac{2xy}{1+x^2} = 1$, $y(0) = 0$. Taking these values as starting values, find $y(0.4)$ by Milne's method.

32.10 ADAMS-BASHFORTH METHOD

Given $\frac{dy}{dx} = f(x, y)$ and $y_0 = y(x_0)$, we compute

$$y_{-1} = y(x_0 - h), y_{-2} = y(x_0 - 2h), y_{-3} = y(x_0 - 3h)$$

by Taylor's series of **Euler's** method or Runge-Kutta method.

Next we calculate $f_{-1} = f(x_0 - h, y_{-1}), f_{-2} = f(x_0 - 2h, y_{-2}), f_{-3} = f(x_0 - 3h, y_{-3})$.

Then to find y_1, we substitute Newton's backward interpolation formula

$$f(x, y) = f_0 + n\,\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \frac{n(n+1)(n+2)}{6}\nabla^3 f_0 + \cdots$$

in
$$y_1 = y_0 + \int_{x_0}^{x_0 + h} f(x, y)\, dx \qquad ...(1)$$

$$\therefore \quad y_1 = y_0 + \int_{x_0}^{x_1}\left(f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \cdots\right) dx \qquad [\text{Put } x = x_0 + nh, dx = hdn]$$

$$= y_0 + h\int_0^1\left(f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \cdots\right) dn$$

$$= y_0 + h\left(f_0 + \frac{1}{2}\nabla f_0 + \frac{5}{12}\nabla^2 f_0 + \frac{3}{8}\nabla^3 f_0 + \cdots\right)$$

Neglecting fourth and higher order differences and expressing $\nabla f_0, \nabla^2 f_0$ and $\nabla^3 f_0$ in terms of function values, we get

$$y_1^{(p)} = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}) \qquad ...(2)$$

This is called *Adams-Bashforth predictor formula.*

Having found y_1, we find $f_1 = f(x_0 + h_1, y_1)$.

Then to find a better value of y_1, we derive a *corrector formula* by substituting Newton's backward formula at f_1 *i.e.*,

$$f(x, y) = f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \frac{n(n+1)(n+2)}{6}\nabla^3 f_1 + \cdots \text{ in (1).}$$

$$\therefore \quad y_1 = y_0 + \int_{x_0}^{x_1}\left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \cdots\right)dx \quad [\text{Put } x = x_1 + nh, dx = hdn]$$

$$= y_0 + \int_{-1}^{0}\left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2}\nabla^2 f_1 + \cdots\right)dn$$

$$= y_0 + h\left(f_1 - \frac{1}{2}\nabla f_1 - \frac{1}{12}\nabla^2 f_1 - \frac{1}{24}\nabla^3 f_1 - \cdots\right)$$

Neglecting fourth and higher order differences and expressing $\nabla f_1, \nabla^2 f_1$ and $\nabla^3 f_1$ in terms of function values, we obtain

$$y_1^{(c)} = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2}) \qquad ...(3)$$

which is called a *Adams-Moulton corrector formula.*

Then an improved value of f_1 is calculated and again the corrector (3) is applied to find a still better value of y_1. This step is repeated till y_1 remains unchanged and then proceed to calculate y_2 as above.

Obs. To apply both Milne and Adams-Bashforth methods, we require four starting values of y which are calculated by means of Picard's method or Taylor's series method or Euler's method or Runge-Kutta method. In practice, the Adams formulae (2) and (3) above together with fourth order Runge-Kutta formulae have been found to be most useful.

Example 32.20. *Given* $\frac{dy}{dx} = x^2(1 + y)$ *and* $y(1) = 1, y(1.1) = 1.233, y(1.2) = 1.548, y(1.3) = 1.979$, *evaluate* $y(1.4)$ *by Adams-Bashforth method.* (V.T.U., 2010 ; J.N.T.U., 2009 ; Anna, 2004)

Solution. Here $f(x, y) = x^2(1 + y)$.

Starting values of the Adams-Bashforth method with $h = 0.1$, are

$$x = 1.0, y_{-3} = 1.000, f_{-3} = (1.0)^2(1 + 1.000) = 2.000$$
$$= 1.1, y_{-2} = 1.233, f_{-2} = 2.702$$
$$= 1.2, y_{-1} = 1.548, f_{-1} = 3.669$$
$$= 1.3, y_0 = 1.979, f_0 = 5.035$$

Using th

$$\frac{h}{} (55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$$73, f_1 = 7.004$$

U

$$5f_{-1} + f_{-2})$$

$$19 \times 5.035 - 5 \times 3.669 + 2.702) = 2.575$$

Hence 5.

Example 32.21. *If $\frac{dy}{dx} = 2e^x y$, y(0) = 0, find y(4) using Adams predictor-corrector formula by calculating y(1), y(2) and y(3) using Euler's modified formula.* (J.N.T.U., 2006)

Solution. We have $f(x, y) = 2e^x y$.

To find 0.1 :

x	$2e^x\, y = y'$	*Mean slope*	*Old y + h (Mean slope) = new y*
0.0	4	—	2 + 0.1 (4) = 2.4
0.1	$2e^{0.1}$ (2.4) = 5.305	$\frac{1}{2}$(4 + 5.305) = 4.6524	2 + 0.1 (4.6524) = 2.465
0.1	$2e^{0.1}$ (2.465) = 5.449	$\frac{1}{2}$(4 + 5.449) = 4.7244	2 + 0.1 (4.7244) = 2.472
0.1	$2e^{0.1}$ (2.4724) = 5.465	$\frac{1}{2}$(4 + 5.465) = 4.7324	2 + 0.1 (4.7324) = 2.473
0.1	$2e^{0.1}$ (2.473) = 5.467	$\frac{1}{2}$(4 + 5.467) = 4.7333	2 + 0.1 (4.7333) = 2.473
0.1	5.467	—	2 + 0.1 (5.467) = 3.0199
0.2	$2e^{0.2}$ (3.0199) = 7.377	$\frac{1}{2}$(5.467 + 7.377) = 6.422	2.473 + 0.1 (6.422) = 3.1155
0.2	7.611	$\frac{1}{2}$(5.467 + 7.611) = 6.539	2.473 + 0.1 (6.539) = 3.127
0.2	7.639	$\frac{1}{2}$(5.467 + 7.639) = 6.553	2.473 + 0.1 (6.553) = 3.129
0.2	7.643	$\frac{1}{2}$(5.467 + 7.643) = 6.555	2.473 + 0.1 (6.455) = 3.129
0.2	7.643	—	3.129 + 0.1 (7.643) = 3.893
0.3	$2e^{0.3}$ (3.893) = 10.51	$\frac{1}{2}$(7.643 + 10.51) = 9.076	3.129 + 0.1 (9.076) = 4.036
0.3	10.897	$\frac{1}{2}$(7.643 + 10.897) = 9.266	3.129 + 0.1 (9.2696) = 4.056
0.3	10.949	$\frac{1}{2}$(7.643 + 10.949) = 9.296	3.129 + 0.1 (9.296) = 4.058
0.3	10.956	$\frac{1}{2}$(7.643 + 10.956) = 9.299	3.129 + 0.1 (9.299) = 4.0586

To find y(0.4) by Adam's method, the starting values with $h = 0.1$ are

$x = 0.0$ $\quad y_{-3} = 2.4$ $\quad f_{-3} = 4$

$x = 0.1$ $\quad y_{-2} = 2.473$ $\quad f_{-2} = 5.467$

$x = 0.2$ $\quad y_{-1} = 3.129$ $\quad f_{-1} = 7.643$

$x = 0.3$ $\quad y_0 = 4.059$ $\quad f_{-0} = 10.956$

Using the predictor formula

$$y_1^{(p)} = y_0 + \frac{h}{24}\,(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$$= 4.059 + \frac{0.1}{24}\,(55 \times 10.957 - 59 \times 7.643 + 37 \times 5.467 - 9 \times 4)$$

$$= 5.383$$

Now $x = 0.4$ $\quad y_1 = 5.383$ $\quad f_1 = 2e^{0.4}\,(5.383) = 16.061$

Using the corrector formula,

$$y_1^{(c)} = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$$= 4.0586 + \frac{0.1}{24}(9 \times 6.061 + 19 \times 10.956 - 5 \times 7.643 + 5.467) = 5.392$$

Hence $y(0.4) = 5.392$.

Example 32.22. *Solve the initial value problem $dy/dx = x - y^2$, $y(0) = 1$ to find $y(0.4)$ by Adam's method. Starting solutions required are to be obtained using Runge-Kutta method of order 4 using step value $h = 0.1$.* (P.T.U., 2003)

Solution. We have $f(x, y) = x - y^2$.

To find y (0.1) :

Here $x_0 = 0, y_0 = 1, h = 0.1$

$$\therefore \quad k_1 = hf(x_0, y_0) = (0.1) f(0, 1) \qquad = -0.1000$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right) = (0.1) f(0.05, 0.95) \qquad = -0.08525$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right) = (0.1) f(0.05, 0.9574) \qquad = -0.0867$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1) f(0.1, 0.9137) \qquad = -0.07341$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = -0.0883$$

Thus $\quad y(0.1) = y_1 = y_0 + k = 1 - 0.0883 \qquad = 0.9117$

To find y(0.2) :

Here $x_1 = 0.1, y_1 = 0.9117, h = 0.1$.

$$\therefore \quad k_1 = hf(x_1, y_1) = (0.1) f(0.1, 0.9117) \qquad = -0.0731$$

$$k_2 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1\right) = (0.1) f(0.15, 0.8751) \qquad = -0.0616$$

$$k_3 = hf\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2\right) = (0.1) f(0.15, 0.8809) \qquad = -0.0626$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1) f(0.2, 0.8491) \qquad = -0.0521$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad = -0.0623$$

Thus $\quad y(0.2) = y_2 = y_1 + k = 0.8494$.

To find y (0.3) :

Here $x_2 = 0.2, y_2 = 0.8494, y = 0.1$

$$k_1 = hf(x_2, y_2) = (0.1) f(0.2, 0.8494) \qquad = -0.0521$$

$$k_2 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_1\right) = (0.1) f(0.25, 0.8233) \qquad = -0.0428$$

$$k_3 = hf\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}k_2\right) = (0.1) f(0.25, 0.828) \qquad = -0.0436$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1) f(0.3, 0.8058) \qquad = -0.0349$$

$$k = \frac{1}{6}(k_1 + 2y_2 + 2k_3 + k_4) \qquad = -0.0438$$

Thus $\quad y(0.3) = y_3 = y_2 + k = 0.8061$

Now the starting values of Adam's method with $h = 0.1$ are :

$x = 0.0$	$y_{-3} = 1.0000$	$f_{-3} = 0.0 - (1.0)^2$	$= -1.0000$
$x = 0.1$	$y_{-2} = 0.9117$	$f_{-2} = 0.1 - (0.9117)^2$	$= -1.7312$
$x = 0.2$	$y_{-1} = 0.8494$	$f_{-1} = 0.2 - (0.8494)^2$	$= -0.5215$
$x = 0.3$	$y_0 = 0.8061$	$f_0 = 0.3 - (0.8061)^2$	$= -0.3498$

Using the *predictor*,

$$y_1^{(p)} = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

$x = 0.4$ $$y_1^{(p)} = 0.8061 + \frac{0.1}{24}[55(-0.3498) - 59(-0.5215) + 37(-0.7312) - 9(-1)]$$

$$= 0.7789 \qquad f_1 = -0.2067$$

Using the *corrector*,

$$y_1^{(c)} = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

$$y_1^{(c)} = 0.8061 + \frac{0.1}{24}[9(-0.2067) + 19(-0.3498) - 5(-0.5215) - 0.7312] = 0.7785$$

Hence $y(0.4) = 0.7785$.

PROBLEMS 32.5

1. Using Adams-Bashforth method, obtain the solution of $dy/dx = x - y^2$ at $x = 0.8$, given the values

x :	0	0.2	0.4	0.6
y :	0	0.0200	0.0795	0.1762

(*Bhopal, 2002*)

2. Using Adams-Bashforth formulae, determine $y(0, 4)$ given the differential equation $dy/dx = \frac{1}{2}xy$ and the data

x :	0	0.1	0.2	0.3
y :	1	1.0025	1.0101	1.0228

3. Given $y' = x^2 - y$, $y(0) = 1$ and the starting values $y(0.1) = 0.90516$, $y(0.2) = 0.82127$, $y(0.3) = 0.74918$, evaluate $y(0, 4)$ using Adams-Bashforth method. (*S.V.T.U., 2007*)
4. Using Adams-Bashforth method, find $y(4.4)$ given $5xy' + y^2 = 2$, $y(4) = 1$, $y(4, 1) = 1.0049$, $y(4, 2) = 1.0097$ and $y(4.3) = 1.0143$.
5. Given the differential equation $dy/dx = x^2y + x^2$ and the data :

x :	1	1.1	1.2	1.3
y :	1	1.233	1.548488	1.978921

(*Indore, 2003 S*)

6. Using Adams-Bashforth method, evaluate $y(1.4)$, if y satisfies $dy/dx + y/x = 1/x^2$ and $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$. (*Madras, 2003*)

32.11 SIMULTANEOUS FIRST ORDER DIFFERENTIAL EQUATIONS

The simultaneous differential equations of the type

$$\frac{dy}{dx} = f(x, y, z) \quad \text{...(1)}$$

and $$\frac{dz}{dx} = \phi(x, y, z) \quad \text{...(2)}$$

with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ can be solved by the methods discussed in the preceding sections, especially by Picard's or Runge-Kutta methods.

(*i*) *Picard's method gives*

$$y_1 = y_0 + \int f(x, y_0, z_0)\,dx, \; z_1 = z_0 + \int \phi(x, y_0, z_0)\,dx$$

$$y_2 = y_0 + \int f(x, y_1, z_1)\,dx, \; z_2 = z_0 + \int \phi(x, y_1, z_1)\,dx$$

$$y_3 = y_0 + \int f(x, y_2, z_2)\,dx, \; z_3 = z_0 + \int \phi(x, y_2, z_2)\,dx$$

and so on.

(*ii*) *Taylor's series* method is used as follows :

If h be the step-size, $y_1 = y(x_0 + h)$ and $z_1 = z(x_0 + h)$. Then Taylor's algorithm for (1) and (2) gives

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \quad \ldots(3)$$

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} + z_0''' + \ldots \quad \ldots(4)$$

Differentiating (1) and (2) successively, we get y'', z'', etc. So the values $y_0', y_0'', y_0''' \ldots$ and $z_0', z_0'', z_0''' \ldots$ are known. Substituting these in (3) and (4), we obtain y_1, z_1 for the next step.

Similarly, we have the algorithms

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots \quad \ldots(5)$$

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!} z_1'' + \frac{h^3}{3!} z_1''' + \ldots \quad \ldots(6)$$

Since y_1 and z_1 are known, we can calculate $y_1', y_1'', \ldots$ and $z_1', z_1'' \ldots$. Substituting these in (5) and (6), we get y_2 and z_2.

Proceeding further, we can calculate the other values of y and z step by step.

(iii) Runge-Kutta method is applied as follows :

Starting at (x_0, y_0, z_0) and taking the step-sizes for x, y, z to be h, k, l respectively, the Runge-Kutta method gives,

$$k_1 = hf(x_0, y_0, z_0) \qquad l_1 = h\phi(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right) \qquad l_2 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right) \qquad l_3 = h\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2, z_0 + \frac{1}{2}l_2\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) \qquad l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

Hence $\quad y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad$ and $\quad z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

To compute y_2 and z_2, we simply replace x_0, y_0, z_0 by x_1, y_1, z_1 in the above formulae.

Example 32.23. *Using Picard's method find approximate values of y and z corresponding to $x = 0.1$, given that $y(0) = 2$, $z(0) = 1$ and $dy/dx = x + z$, $dz/dx = x - y^2$.*

Solution. Here $x_0 = 0, y_0 = 2, z_0 = 1$,

$$\frac{dy}{dx} = f(x, y, z) = x + z; \qquad \text{and} \qquad \frac{dz}{dx} = \phi(x, y, z) = x - y^2$$

$$\therefore \qquad y = y_0 + \int_{x_0}^{x} f(x, y\, z)\, dx \qquad \text{and} \qquad z = z_0 + \int_{x_0}^{x} \phi(x, y\, z)\, dx.$$

First approximations $\quad y_1 = y_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)\, dx = 2 + \int_0^x (x+1)\, dx = 2 + x + \frac{1}{2}x^2$

$$z_1 = z_0 + \int_{x_0}^{x} \phi(x, y_0, z_0)\, dx = 1 + \int_0^x (x-4)\, dx = 1 - 4x + \frac{1}{2}x^2$$

Second approximations $\quad y_2 = y_0 + \int_{x_0}^{x} f(x, y_1, z_1)\, dx = 2 + \int_0^x \left(x + 1 - 4x + \frac{1}{2}x^2\right) dx$

$$= 2 + x - \frac{3}{2}x^2 + \frac{x^3}{6}$$

$$z_2 = z_0 + \int_{x_0}^{x} \phi(x, y_1, z_1)\, dx$$

$$= 1 + \int_{x_0}^{x} \left[x - \left(2 + x + \frac{1}{2}x^2\right)^2\right] dx = 1 - 4x + \frac{3}{2}x^2 - x^3 - \frac{x^4}{4} - \frac{x^5}{20}.$$

Third approximations $y_3 = y_0 + \int_{x_0}^{x} f(x, y_2, z_2)\,dx$

$$= 2 + x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{4}x^4 - \frac{1}{20}x^5 - \frac{1}{120}x^6$$

$$z_3 = z_0 + \int_{x_0}^{x} \phi(x, y_2, z_2)\,dx$$

$$= 1 - 4x - \frac{3}{2}x^2 + \frac{5}{3}x^3 + \frac{7}{12}x^4 - \frac{31}{60}x^5 + \frac{1}{12}x^6 - \frac{1}{252}x^7$$

and so on.

When $x = 0.1$,

$$y_1 = 2.105, y_2 = 2.08517, y_3 = 2.08447$$

$$z_1 = 0.605, z_2 = 0.58397, z_3 = 0.58672.$$

Hence $y(0.1) = 2.0845, z(0.1) = 0.5867$

correct to four decimal places.

Example 32.24. *Solve the differential equations*

$$\frac{dy}{dx} = 1 + xz, \quad \frac{dz}{dx} = -xy \text{ for } x = 0.3,$$

using fourth order Runge-Kutta method. Initial values are $x = 0, y = 0, z = 1$.

Solution. Here $f(x, y, z) = 1 + xz$, $\phi(x, y, z) = -xy$

$x_0 = 0, y_0 = 0, z_0 = 1$. Let us take $h = 0.3$.

$\therefore$ $k_1 = h\,f(x_0, y_0, z_0) = 0.3\,f(0, 0, 1) = 0.3\,(1 + 0) = 0.3$

$$l_1 = h\,\phi(x_0, y_0, z_0) = 0.3\,(-0 \times 0) = 0$$

$$k_2 = h\,f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$= (0.3)\,f(0.15, 0.15, 1) = 0.3\,(1 + 0.15) = 0.345$$

$$l_2 = h\,\phi\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1, z_0 + \frac{1}{2}l_1\right)$$

$$= 0.3\,[-(0.15)\,(0.15)] = -0.00675.$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= (0.3)\,f(0.15, 0.1725, 0.996625)$$

$$= 0.3\,[1 + 0.996625 \times 0.15] = 0.34485$$

$$l_3 = h\,\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= 0.3\,[-(0.15)\,(0.1725)] = -0.007762$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= (0.3)\,f(0.3, 0.34485, 0.99224) = 0.3893$$

$$l_4 = h\,\phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= 0.3\,[-(0.3)\,(0.34485)] = -0.03104$$

Hence $y(x_0 + h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

i.e., $y(0.3) = 0 + \frac{1}{6}[0.3 + 2\,(0.345) + 2\,(0.34485) + 0.3893] = 0.34483$

and $z(x_0 + h) = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

i.e., $z(0.3) = 1 + \frac{1}{6}[0 + 2 + (-0.00675) + 2\,(-0.0077625) + (-0.03104)] = 0.98999$

32.12 SECOND ORDER DIFFERENTIAL EQUATIONS

Consider the second order differential equation $\dfrac{d^2y}{dx^2} = f\left(x, y, \dfrac{dy}{dx}\right)$

By writing $dy/dx = z$, it can be reduced to two first order simultaneous differential equations

$$\frac{dy}{dx} = z, \qquad \frac{dz}{dx} = f(x, y, z)$$

These equations can be solved as explained above.

Example 32.25. *Using Runge-Kutta method, solve $y'' = xy'^2 - y^2$ for $x = 0.2$ correct to 4 decimal places. Initial conditions are $x = 0, y = 1, y' = 0$.* *(Delhi, 2002)*

Solution. Let $dy/dx = z = f(x, y, z)$. Then $dz/dx = xz^2 - y^2 = \phi(x, y, z)$

We have $x_0 = 0, y_0 = 1, z_0 = 0, h = 0.2$.

Using k_1, k_2 ... for $f(x, y, z)$ and l_1, l_2, ... for $\phi(x, y, z)$, Runge-Kutta formulae become

$$k_1 = hf(x_0, y_0, z_0) = 0.2\,(0) = 0$$
$$l_1 = h\phi(x_0, y_0, z_0) = 0.2(-1) = -0.2$$
$$k_2 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1, z_0 + \tfrac{1}{2}l_1) = 0.2\,(-0.1) = -0.02$$
$$l_2 = h\phi(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1, z_0 + \tfrac{1}{2}l_1) = 0.2(-0.999) = -0.1998$$
$$k_3 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2, z_0 + \tfrac{1}{2}l_2) = 0.2\,(-0.0999) = -0.02$$
$$l_3 = h\phi(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2, z_0 + \tfrac{1}{2}l_2) = 0.2(-0.9791) = -0.1958$$
$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2\,(-0.1958) = -0.0392$$
$$l_4 = h\phi(x_0 + h, y_0 + k_3, z_0 + l_3) = 0.2(0.9527) = -0.1905$$
$$\therefore \quad k = \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.0199$$
$$l = \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + 2l_4) = -0.1970$$

Hence at $x = 0.2$,

$$y = y_0 + k = 1 - 0.0199 = 0.9801$$

and $$y' = z = z_0 + l = 0 - 0.1970 = -0.1970.$$

Example 32.26. *Given $y'' + xy' + y = 0$, $y(0) = 1$, $y'(0) = 0$, obtain y for $x = 0(0.1)\,0.3$ by any method. Further, continue the solution by Milne's method to calculate $y(0.4)$.* *(Anna, 2004; Madras, 2003 S)*

Solution. Putting $y' = z$, the given equation reduces to the simultaneous equations

$$z' + xz + y = 0, \; y' = z \qquad \text{...}(i)$$

We employ Taylor's series method to find y.

Differentiating the given equation n times, we get

$$y_{n+2} + xy_{n+1} + ny_n + y_n = 0$$

At $\quad x = 0, (y_{n+2})_0 = -(n+1)(y_n)_0$

$\therefore \quad y(0) = 1$, gives $y_2(0) = -1, y_4(0) = 32, y_6(0) = -5 \times 3, \ldots$

and $\quad y_1(0) = 0$ yields $y_3(0) = y_5(0) = \ldots = 0$.

Expanding $y(x)$ by Taylor's series, we have

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \ldots$$

$$\therefore \quad y(x) = 1 - \frac{x^2}{2!} + \frac{3}{4!}x^4 - \frac{5 \times 3}{6!}x^6 + \ldots \qquad \text{...}(ii)$$

and $$z(x) = y'(x) = -x + \frac{1}{2}x^3 - \frac{1}{8}x^5 = \ldots = -xy \qquad \text{...}(iii)$$

From (ii), we have

$$y(0.1) = 1 - \frac{(0.1)^2}{2} + \frac{1}{8}(0.1)^4 - \ldots = 0.995$$

$$y(0.2) = 1 - \frac{(0.2)^2}{2} + \frac{(0.2)^4}{8} - \ldots = 0.9802$$

$$y(0.3) = 1 - \frac{(0.3)^2}{2} + \frac{(0.3)^4}{8} - \frac{(0.3)^6}{48} + \ldots = 0.956$$

From (*iii*), we have

$$z(0.1) = -0.0995, z(0.2) = -0.196, z(0.3) = -0.2863.$$

Also from (*i*), $z'(x) = -(xz + y)$ $\therefore$ $z'(0.1) = 0.985, z'(.2) = -0.941, z'(0.3) = -0.87$.

Applying Milne's predictor formula, first to z and then to y, we obtain

$$z(0.4) = z(0) + \frac{4}{3}(0.1)\{2z'(0.1) - z'(0.2) + 2z'(0.3)\}$$

$$= 0 + \left(\frac{0.4}{3}\right)\{-1.79 + 0.941 - 1.74\} = -0.3692$$

and
$$y(0.4) = y(0) + \frac{4}{3}(0.1)\{2y'(0.1) - y'(0.2) + 2y'(0.3)\}$$

$$= 0 + \left(\frac{0.4}{3}\right)\{-0.199 + 0.196 - 0.5736\} = 0.9231 \qquad [\because \; y' = z]$$

Also $z'(0.4) = -\{x(0.4)z(0.4) + y(0.4)\} = \{0.4(-0.3692) + 0.9231\} = -0.7754$.

Now applying Milne's corrector formula, we get

$$z(0.4) = z(0.2) + \frac{h}{3}\{z'(0.2) + 4z'(0.3) + z'(0.4)\}$$

$$= -0.196 + \left(\frac{0.1}{3}\right)\{-0.941 - 3.48 - 0.7754\} = -0.3692$$

and
$$y(0.4) = y(0.2) + \frac{h}{3}\{y'(0.2) + 4y'(0.3) + y'(0.4)\}$$

$$= 0.9802 + \left(\frac{0.1}{3}\right)\{-0.196 - 1.1452 - 0.3692\} = 0.9232$$

Hence $\quad y(0.4) = 0.9232$ and $z(0.4) = -0.3692$.

PROBLEMS 32.6

1. Apply Picard's method to find the third approximation to the values of y and z, given that $dy/dx = z$, $dz/dx = x^3(y + z)$, given $y = 1$, $z = \frac{1}{2}$ when $x = 0$.
2. Solve the following differential equations using Taylor series method of the 4th order, for $x = 0.1$ and 0.2, $\frac{dy}{dx} = xz + 1, \frac{dz}{dy} = -xy$; $y(0) = 0$ and $z(0) = 1$.
3. Find $y(0.1)$, $z(0.1)$, $y(0.2)$ and $z(0.2)$ from the system of equations $y' = x + z$, $z' = x - y^2$ given $y(0) = 0$, $z(0) = 1$ using Runge-Kutta of 4th order. (*J.N.T.U., 2009*)
4. Using Picard's method, obtain the second approximation to the solution of
$$\frac{d^2y}{dx^2} = x^3\frac{dy}{dx} + x^3y \text{ so that } y(0) = 1, y'(0) = \frac{1}{2}.$$
5. Use Picard's method to approximate y when $x = 0.1$, given that $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + y = 0$ and $y = 0.5$, $\frac{dy}{dx} = 0.1$, when $x = 0$.
6. Using Runge-Kutta method of order four, solve $y'' = y + xy'$, $y(0) = 1$, $y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$.
7. Consider the second order value problem $y'' - 2y' + 2y = e^{2t}\sin t$ with $y(0) = -0.4$ and $y'(0) = -0.6$. Using the fourth order Runge-Kutta method, find $y(0.2)$. (*Anna, 2003*)

8. The angular displacement θ of a simple pendulum is given by the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l} \sin \theta = 0$$

where $l = 98$ cm and $g = 980$ cm/sec^2. If $\theta = 0$ and $d\theta/dt = 4.472$ at $t = 0$, use Runge-Kutta method to find θ and $d\theta/dt$ when $t = 0.2$ sec.

32.13 BOUNDARY VALUE PROBLEMS

Such a problem requires the solution of a differential equation in a region R subject to the various conditions on the boundary of R. Practical applications give rise to many such problems. We shall discuss two-point linear boundary value problems of the following types :

(*i*) $\dfrac{d^2y}{dx^2} + \lambda(x)\dfrac{dy}{dx} + \mu(x)y = \gamma(x)$ with the conditions $y(x_0) = a$, $y(x_n) = b$.

(*ii*) $\dfrac{d^4y}{dx^4} + \lambda(x)y = \mu(x)$ with the conditions $y(x_0) = y'(x_0) = a$ and $y(x_n) = y'(x_n) = b$.

While there exist many numerical methods for solving such boundary value problems, the method of finite-differences is most commonly used. We shall explain this method in the next section.

32.14 FINITE-DIFFERENCE METHOD

In this method, the derivatives appearing in the differential equation and the boundary conditions are replaced by their finite-difference approximations and the resulting linear system of equations are solved by any standard procedure. These roots are the values of the required solution at the pivotal points.

The finite-difference approximations to the various derivatives are derived as under :

If $y(x)$ and its derivatives are single-valued continuous functions of x then by Taylor's expansion, we have

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + \frac{h^2}{3!}y'''(x) + \ldots \qquad \ldots(1)$$

and
$$y(x-h) = y(x) - hy'(x) + \frac{h^2}{2!}y''(x) - \frac{h^3}{3!}y'''(x) + \ldots \qquad \ldots(2)$$

Equation (1) gives $y'(x) = \dfrac{1}{h}[y(x+h) - y(x)] - \dfrac{h}{2}y''(x) - \ldots$

i.e.,
$$y'(x) = \frac{1}{h}[y(x+h) - y(x)] + O(h)$$

which is the *forward difference approximation of* $y'(x)$ with an error of the order h.

Similarly (2) gives $y'(x) = \dfrac{1}{h}[y(x) - y(x-h)] + O(h)$

which is the *backward difference approximation of* $y'(x)$ with an error of the order h.

Subtracting (2) from (1), we obtain

$$y'(x) = \frac{1}{2h}[y(x+h) - y(x-h)] + O(h^2)$$

which is the *central-difference approximation of* $y'(x)$ with an error of the order h^2. Clearly this central difference approximation to $y'(x)$ is better than the forward or backward difference approximations and hence should be preferred.

Adding (1) and (2), we get

$$y''(x) = \frac{1}{h^2}[y(x+h) - 2y(x) + y(x-h)] + O(h^2)$$

which is the central difference approximation of $y''(x)$. Similarly we can derive central difference approximations to higher derivatives.

Hence *the working expressions for the central difference approximations to the first four derivatives of* y_i *are as under :*

$$y'_i = \frac{1}{2h}(y_{i+1} - y_{i-1}) \quad ...(3)$$

$$y''_i = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \quad ...(4)$$

$$y'''_i = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}) \quad ...(5)$$

$$y_i^{iv} = \frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i+1} + y_{i-2}) \quad ... (6)$$

Obs. The accuracy of this method depends on the size of the sub-interval h and also on the order of approximation. As we reduce h, the accuracy improves but the number of equations to be solved also increases.

Example 32. 27. *Solve the equation $y'' = x + y$ with the boundary conditions $y(0) = y(1) = 0$.* (*Calicut, 1999*)

Solution. We divide the interval (0, 1) into four sub-intervals so that $h = 1/4$ and the pivot points are $x_0 = 0, x_1 = 1/4, x_2 = 1/2, x_3 = 3/4$ and $x_4 = 1$.

The differential equation is approximated as

$$\frac{1}{h^2}[y_{i+1} - 2y_i + y_{i-1}] = x_i + y_i$$

or
$$16y_{i+1} - 33y_i + 16y_{i-1} = x_i,\ i = 1, 2, 3.$$

Using $y_0 = y_4 = 0$, we get the system of equations

$$16y_2 - 33y_1 = \frac{1}{4}$$

$$16y_3 - 33y_2 + 16y_1 = \frac{1}{2}$$

$$-33y_3 + 16y_2 = \frac{3}{4}$$

Their solution gives

$$y_1 = -0.03488,\ y_2 = -0.05632,\ y_3 = -0.05003.$$

Obs. The exact solution being $y(x) = \dfrac{\sinh x}{\sinh 1} - x$, the error at each nodal point is given in the table:

x	*Computed value y(x)*	*Exact value y(x)*	*Error*
0.25	− 0.03488	− 0.03505	0.00017
0.5	− 0.05632	− 0.05659	0.00027
0.75	− 0.05003	− 0.05028	0.00025

Example 32.28. *Determine values of y at the pivotal points of the interval (0, 1), if y satisfies the boundary value problem $y^{iv} + 81y = 81x^2$, $y(0) = y(1) = y''(0) = y''(1) = 0$. (Take $n = 3$).*

Solution. Here $h = 1/3$ and the pivotal points are $x_0 = 0, x_1 = 1/3, x_2 = 2/3, x_3 = 1$. The corresponding y-values are $y_0 (= 0), y_1, y_2, y_3 (= 0)$.

Replacing y^{iv} by its central difference approximation, the differential equation becomes

$$\frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) + 81y_i = 81x_i^2$$

or
$$y_{i+2} - 4y_{i+1} + 7y_i - 4y_{i-1} + y_{i-2} = x_i^2,\ i = 1, 2$$

At $i = 1$, $\quad y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = 1/9$

At $i = 2$, $\quad y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = 4/9$

Using $y_0 = y_3 = 0$, we get $\quad -4y_2 + 7y_2 + y_{-1} = 1/9 \quad ...(i)$

$$y_4 + 7y_1 - 4y_1 = 4/9 \quad ...(ii)$$

Regarding the conditions $y''_0 = y''_3 = 0$, we know that

$$x_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$

At $i = 0$, $\quad y_0'' = 9(y_1 - 2y_0 + y_{-1})$

or $\quad y_{-1} = -y_1 \qquad [\because \; y_0 = y_0'' = 0] \quad ...(iii)$

At $i = 3$, $\quad y_3'' = 9(y_4 - 2y_3 + y_2)$

$$y_4 = -y_2 \qquad [\because \; y_3 = y_3'' = 0] \quad ...(iv)$$

Using (*iii*), the equation (*i*) becomes

$$-4y_2 + 6y_1 = 1/9 \quad ...(v)$$

Using (*iv*), the equation (*ii*) reduces to

$$6y_2 - 4y_1 = 4/9 \quad ...(vi)$$

Solving (*v*) and (*vi*), we obtain

$$y_1 = 11/90 \text{ and } y_2 = 7/45.$$

Hence $y(1/3) = 0.1222$ and $y(2/3) = 0.1556$.

Example 32.29. *The deflection of a beam is governed by the equation*

$$\frac{d^4y}{dx^4} + 81y = \phi(x)$$

where $\phi(x)$ is given by the table

x :	1/3	2/3	1,
$\phi(x)$:	81	162	243,

and boundary condition $y(0) = y'(0) = y''(1) = y'''(1) = 0$. Evaluate the deflection at the pivotal points of the beam using three sub-intervals.

Solution. Here $h = 1/3$ and the pivotal points are $x_0 = 0, x_1 = 1/3, x_2 = 2/3, x_3 = 1$. The corresponding y-values are $y_0 (= 0), y_1, y_2, y_3$.

The given differential equation is approximated to

$$\frac{1}{h^4}(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}) + 81y_i = \phi(x_i)$$

At $i = 1$, $\quad y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = 1 \quad ...(i)$

At $i = 2$, $\quad y_4 - 4y_3 + 7y_2 - 4y_1 + y_0 = 2 \quad ...(ii)$

At $i = 3$, $\quad y_5 - 4y_4 + 7y_3 - 4y_2 + y_1 = 3 \quad ...(iii)$

We have $\quad y_0 = 0 \quad ...(iv)$

Since $\quad y_i' = \frac{1}{2h}(y_{i+1} - y_{i-1})$

$\therefore$ for $i = 0$, $\quad 0 = y_0' = \frac{1}{2h}(y_1 - y_{-1})$ *i.e.* $y_{-1} = y_1 \quad ...(v)$

Since $\quad y_i'' = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$

$\therefore$ for $i = 3$, $\quad 0 = y_3'' = \frac{1}{h^2}(y_4 - 2y_3 + y_2)$, *i.e.* $y_4 = 2y_3 - y_2 \quad ...(vi)$

Also $\quad y_i''' = \frac{1}{2h^3}(y_{i+2} - 2y_{i+1}\, 2y_{i-1} - y_{i-2})$

$\therefore$ for $i = 3$, $\quad 0 = y_3''' = \frac{1}{2h^3}(y_5 - 2y_4 + 2y_2 - y_1)$

$$y_5 = 2y_4 - 2y + y_1 \quad ...(vii)$$

Using (*iv*) and (*v*), the equation (*i*) reduces to

$$y_3 - 4y_2 + 8y_1 = 1 \quad ...(viii)$$

Using (*iv*) and (*vi*), the equation (*ii*) becomes

$$-y_3 + 3y_2 - 2y_1 = 1 \quad ...(ix)$$

Using (*vi*) and (*vii*), the equation (*iii*) reduces to

$$3y_3 - 4y_2 + 2y_1 = 3 \quad \text{...}(x)$$

Solving (*viii*), (*ix*) and (*x*), we get

$$y_1 = 8/13, y_2 = 22/13, y_3 = 37/13.$$

Hence $y(1/3) = 0.6154, y(2/3) = 1.6923, y(1) = 2.8462.$

PROBLEMS 32.7

1. Solve the boundary value problem for $x = 0.5$:
$$\frac{d^2y}{dx^2} + y + 1 = 0, y(0) = y(1) = 0. \qquad \text{(Take } n = 4)$$
2. Find an approximate solution of the boundary value problem :
$$y'' + 8(\sin^2 \pi y)y = 0, 0 \le x \le 1, y(0) = y(1) = 1. \qquad \text{(Take } n = 4)$$
3. Solve the boundary value problem
$$xy'' + y = 0, y(1) = 1, y(2) = 2. \qquad \text{(Take } n = 4)$$
4. Solve the equation
$$y'' - 4y' + 4y = e^{3x}, \text{ with the conditions } y(0) = 0, y(1) = -2, \text{ taking } n = 4.$$
5. Solve the boundary value problem $y'' - 64y + 10 = 0$ with $y(0) = y(1) = 0$ by the finite difference method. Compute the value of $y(0.5)$ and compare with the true value.
6. Solve the boundary value problem
$$y'' + xy' + y = 3x^2 + 2, y(0) = 0, y(1) = 1.$$
7. The boundary value problem governing the deflection of a beam of length 3 metres is given by
$$\frac{d^4y}{dx^4} + 2y = \frac{1}{9}x^2 + \frac{2}{3}x + 4, y(0) = y'(0) = y(3) = y'(3) = 0.$$
The beam is built-in at the left end ($x = 0$) and simply supported at the right end ($x = 3$). Determine y at the pivotal points $x = 1$ and $x = 2$.
8. Solve the boundary value problem,
$$\frac{d^4y}{dx^4} + 81y = 729x^2, y(0) = y'(0) = y''(1) = y'''(1) = 0. \qquad \text{(Use } n = 3)$$
9. Solve the equation $y^{iv} - y''' + y = x^2$ subject to the boundary conditions
$$y(0) = y'(0) = 0 \text{ and } y(1) = 2, y'(1) = 0. \qquad \text{(Take } n = 5)$$

32.15 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 32.8

Select the correct answer or fill up the blanks in the following questions :

1. Which of the following is a step by step method :
(*a*) Taylor' s (*b*) Adams-Bashforth (*c*) Picard's (*d*) None.
2. The finite difference scheme for the equation $2y'' + y = 5$ is ....
3. If $y'' = x + y, y(0) = 1$ and $y^{(1)}(x) = 1 + x + x^2/2$, then by Picard's method, the value of $y^{(2)}(x)$ is ....
4. The iterative formula of Euler's method for solving $y' = f(x, y)$ with $y(x_0) = y_0$, is ....
5. Taylor's series for solution of first order ordinary differential equations is ....
6. Using Runge-Kutta method of order four, the value of $y(0.1)$ for $y' = x - 2y, y(0) = 1$ taking $h = 0.1$ is
(*a*) 0.813 (*b*) 0.825 (*c*) 0.0825 (*d*) none.
7. Given y_0, y_1, y_2, y_3, Milne's corrector formula to find y_4 for $dy/dx = f(x, y)$, is ....
8. The second order Runge-Kutta formula is ....
9. Adams-Bashforth predictor formula to solve $y' = f(x, y)$ given $y_0 = y(x_0)$ is ....
10. The multi-step methods available for solving ordinary differential equations are ....
11. To predict Adam's method atleast ...... values of y, prior to the desired value, are required.
12. Taylor's series solution of $y' = -xy, y(0) = 1$ upto x^4 is ....

13. Using modified Euler's method, the value of $y(0.1)$ for $\frac{dy}{dx} = x - y$, $y(0) = 1$ is ....
(a) 0.809 (b) 0.909 (c) 0.0809 (d) none.
14. Milne's Predictor formula is ....
15. Adam's corrector formula is ....
16. Using Euler's method, $dy/dx = (y - 2x)/y$, $y(0) = 1$; gives $y(0.1) =$ ....
17. $\frac{d^2y}{dx^2} + y^2 \frac{dy}{dx} + y = 0$ is equivalent to a set of two first order differential equations ...... and ....
18. The formula for the 4th order Runge-Kutta method is ....
19. Taylor's series method will be useful to give some .... of Milne's method.
20. The name of two self-starting methods to solve $y' = f(x, y)$ given $y(x_0) = y_0$ are ......
21. In the derivation of fourth order Runge-Kutta formula, it is called fourth order because ......
22. If $y' = x$, $y(0) = 1$ then by Picard's method, the value of $y(1)$ is ......
(a) 0.915 (b) 0.905 (c) 0.981 (d) none.
23. The finite difference scheme of the differential equation $y'' + 2y = 0$ is ......
24. If $y' = -y$, $y(0) = 1$, the Euler's method, the value of $y(1)$ is
(a) 0.99 (b) 0.999 (c) 0.981 (d) none.
25. In Euler's method if h is small the method is too slow, if h is large, it gives inaccurate value. (True or False)
26. Runge-Kutta method is a self-starting method. (True or False)
27. Predictor-corrector methods are self-starting methods. (True or False)