Example 28. *A rod AB of weight W is movable about a point A and to B is attached a string whose other end is tied to a ring. The ring slides along a smooth horizontal wire passing through A. Prove that the horizontal force necessary to keep the ring at rest is* $\frac{W \cos\alpha \cos\beta}{2\sin(\alpha+\beta)}$, *where* α *and* β *are the inclination of the rod and the string to the horizontal.*

Sol. The rod AB is movable about the hinge at A and B is attached with the ring at C. Let P be the force required to keep the ring at rest. Let $\angle BAC = \alpha$ and $\angle ACB = \beta$. Consider the small vertical displacement such that lengths AB and BC are unchanged and angles α and β change.

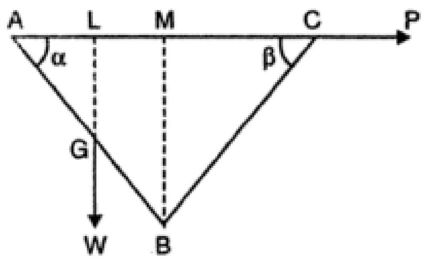


Fig. 30

The equation of virtual work is

$$W d(GL) + P d(AC) = 0 \quad ...(1)$$

Now, $GL = AG \sin\alpha = a \sin\alpha$, $AB = 2a$, $BC = l$

$$AC = AM + MC = 2a\cos\alpha + l\cos\beta$$

Putting GL and AC in (1), we get

$$W d(a \sin\alpha) + P d(2a\cos\alpha + l\cos\beta) = 0$$

$$Wa\cos\alpha\, d\alpha - P(2a\sin\alpha\, d\alpha + l\sin\beta\, d\beta) = 0$$

$$a(W\cos\alpha - 2P\sin\alpha)d\alpha = Pl\sin\beta\, d\beta \quad ...(2)$$

But $\quad BM = 2a\sin\alpha = l\sin\beta$

$$2a\cos\alpha\, d\alpha = l\cos\beta\, d\beta \quad ...(3)$$

Dividing (2) by (3), we get

$$\frac{W\cos\alpha - 2P\sin\alpha}{2\cos\alpha} = \frac{P\sin\beta}{\cos\beta}$$

or $$W\cos\alpha\cos\beta - 2P\sin\alpha\cos\beta = 2P\sin\beta\cos\alpha$$

or $$2P(\sin\alpha\cos\beta + \cos\alpha\sin\beta) = W\cos\alpha\cos\beta$$

or $$P = \frac{W}{2}\cdot\frac{\cos\alpha\cos\beta}{\sin(\alpha+\beta)}.$$

Example 36. *A heavy elcstic string of natural length 2a is placed round a smooth cone whose axis is vertical and whose semi vertical angle is α. If W be the weight and λ be the modulus of elasticity, prove that it will be in equilibrium when in the form of a circle its radius is* $a\left[1+\frac{W\cot\alpha}{2\pi\lambda}\right]$.

Sol. Refer figure 36.

Proceeding exactly as in Ex. 34, we have

$$T = W\cot\alpha/2\pi$$

But by Hooke's Law

$$T = \lambda\,\frac{(l-a)}{a} = \lambda\left(\frac{2\pi r - 2\pi a}{2\pi a}\right)$$

where natural length is $2\pi a$ and extended length is $2\pi r$.

Comparing (1) and (2)

$$\frac{W\cot\alpha}{2\pi} = \lambda\,\frac{(r-a)}{a}$$

$$r - a = \frac{aW\cot\alpha}{2\pi\lambda}$$

or $$r = a\left[1+\frac{W\cot\alpha}{2\pi\lambda}\right].$$

Example 37. *One end of a uniform rod AB, of length 2a and weight W, is attached by a frictionless joint to a smooth vertical wall and the other end B is smoothly joined to an equal rod BC. The middle points of the rods are joined by an elastic string of natural length a and modulus of elasticity 4W. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A and the angle between the rods is* $2 \sin^{-1}(3/4)$.

Sol. Let T be the tension in the string DE, joining middle points of rods AB and BC. 2W is the weight of the rods acting at G.

Let $\angle ABL = \theta$

$\because$ A is fixed, therefore the distances are measured from A.

Let the system be given a small virtual displacement such that θ changes to $\theta + \delta\theta$.

$\therefore$ Equation of virtual work is

$$2W d(AL) - T d(DE) = 0 \qquad ...(1)$$

$$AL = 2a \sin\theta \qquad \because AB = 2a$$

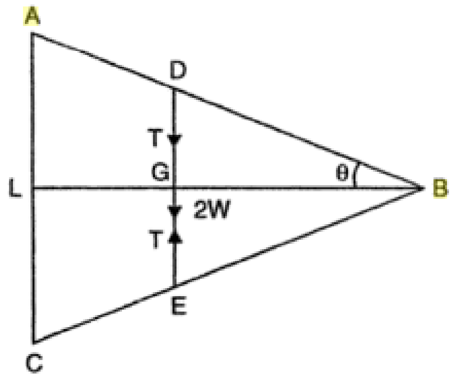


Fig. 38

$$DE = \frac{1}{2} AC = AL = 2a \sin\theta \qquad ...(2)$$

Putting in (1), we get

$\therefore \quad 2W d(2a \sin\theta) - T d(2a \sin\theta) = 0$

$\therefore \quad T = 2W \qquad ...(3)$

But by Hooke's Law

$$T = \lambda \frac{(DE - a)}{a}$$

$$T = 4W \frac{(2a \sin\theta - a)}{a} \qquad ...(4)$$

Comparing (3) and (4), we get

$$2W = 4W(2\sin\theta - 1)$$

$$6W = 8W \sin\theta \quad \text{or} \quad \sin\theta = \frac{3}{4}$$

$$\theta = \sin^{-1}\left(\frac{3}{4}\right).$$

Example 3. *The extremities of a heavy uniform string of length 2l and weight 2lω are attached to two small rings which can slide on a fixed horizontal wire. Each ring is acted on by a horizontal force lω. Show that the distance between the rings is 2l log $(1+\sqrt{2})$.*

Sol. Let weight/length = ω

A and B are the rings where the string is attached.

$$AC = l.$$

Considering the whole system and resolving vertically

$$R + R = 2l\omega \;\Rightarrow\; R = l\omega.$$

∴ Tension at A is the resultant of R and $l\omega$. If ψ is the angle which the tangent makes with X-axis then

$$T\cos\psi = l\omega$$

$$T\sin\psi = R$$

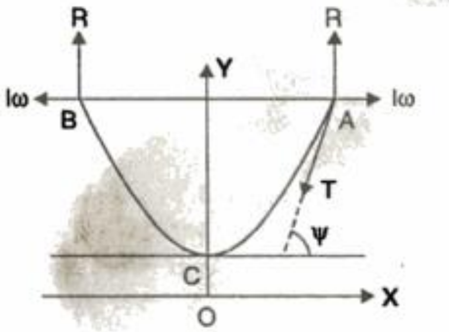


Fig. 6

$$\Rightarrow \quad \tan\psi = \frac{R}{l\omega} = \frac{l\omega}{l\omega} = 1$$

$$\therefore \quad \psi = \frac{\pi}{4}$$

Also, $\quad s = c\tan\psi$

At A, $\quad s = l \quad \therefore \quad l = c\tan\frac{\pi}{4} \;\Rightarrow\; c = l$

∴ Distance between the rings AB = 2x

$$2x = 2c\log(\tan\psi + \sec\psi) = 2l\log\left(\tan\frac{\pi}{4} + \sec\frac{\psi}{4}\right) = 2l\log(1+\sqrt{2}).$$

Example 4. *If tension at point A of a catenary is n times at the vertex then the span of the catenary ACB is* $\frac{2l}{\sqrt{n^2-1}}$ *log* $(n + \sqrt{n^2-1})$*. Where 2l is the length of the catenary.* (K.U. 2004)

Sol. Let length of the catenary ACB = 2l.

Tension at A = T

and Tension at C = T_0

$$T = nT_0$$

$$T = nT\cos\psi$$

$$\therefore \quad \cos\psi = \frac{1}{n}$$

$$\Rightarrow \quad \tan\psi = \frac{\sqrt{n^2-1}}{1}$$

$$\sec\psi = \frac{n}{1}$$

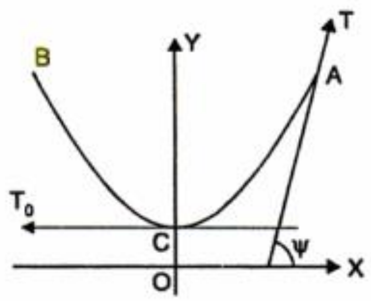


Fig. 7(*i*)

Also at A, $\quad s = c\tan\psi$

$$l = c\sqrt{n^2-1}$$

$$\Rightarrow \quad c = \frac{l}{\sqrt{n^2-1}}$$

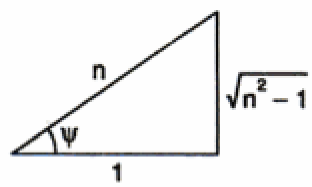


Fig. 7(*ii*)

Sag $\quad y - c = c \sec \psi - c = \dfrac{l(n-1)}{\sqrt{n^2-1}}$

$\therefore$ Span $\quad 2x = 2c \log(\tan\psi + \sec\psi) = \dfrac{2l}{\sqrt{n^2-1}} \log\left(\sqrt{n^2-1}+n\right)$.

Example 18. *The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is* $\mu \log\left[\dfrac{1+\sqrt{1+\mu^2}}{\mu}\right]$, *where* μ *is the coefficient of friction.*

Sol. Let the chain ACB is of length $2l$ hanging in the form of a catenery with C as the vertex. The end links of the chain slide on the rod AB.

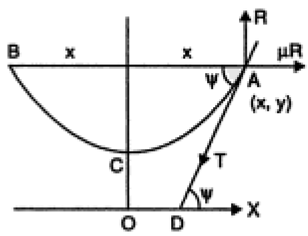


Fig. 20

Forces acting at A are :

(*i*) Tension T acting along the tangent AD.

(*ii*) Reaction R acting perpendicular to AB.

(*iii*) Friction force μR acting along BA.

Resolving the forces horizontally and vertically

$$T\cos\psi = \mu R$$
$$T\sin\psi = R$$

Eliminating T, we get

$$\tan\psi = \frac{1}{\mu} \qquad \therefore \quad \sec\psi = \sqrt{1+\tan^2\psi} = \sqrt{1+\frac{1}{\mu^2}} = \frac{\sqrt{1+\mu^2}}{\mu}$$

Now, $\quad s = c\tan\psi$

$$s = \overset{\frown}{AC} = l$$

$\therefore \quad l = c \cdot \dfrac{1}{\mu} \quad \Rightarrow \quad c = l\mu$

Now span $\quad AB = 2x = 2c\log(\sec\psi + \tan\psi)$

$$= 2l\mu \log\left(\frac{\sqrt{1+\mu^2}}{\mu} + \frac{1}{\mu}\right) = 2\mu l \log\left[\frac{1+\sqrt{1+\mu^2}}{\mu}\right]$$

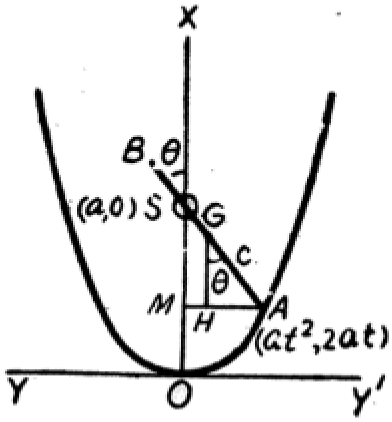
$$\therefore \quad \frac{\text{Maximum span}}{\text{Length of chain}} = \frac{AB}{2l} = \frac{2\mu l \log\left[\dfrac{1+\sqrt{1+\mu^2}}{\mu}\right]}{2l} = \mu\log\left[\frac{1+\sqrt{1+\mu^2}}{\mu}\right].$$

Ex. 24. *A uniform smooth rod passes through a ring at the focus of a fixed parabola whose axis is vertical and vertex below the focus, and rests with one end on the parabola. Prove that the rod will be in equilibrium if it makes with the vertical an angle θ given by the equation*

$$\cos^4 \tfrac{1}{2}\theta = a/2c$$

where 4a is the latus rectum and 2c the length of the rod. Investigate also the stability of equilibrium in this position. **[Lucknow 81]**

Sol. Let the equation of the parabola be $y^2 = 4ax$.



Let AB be the rod of length $2c$ with its end A on the parabola and passing through a ring at the focus S. Let the coordinates of A be $(at^2, 2at)$; the co-ordinates of the focus S are $(a, 0)$. If the rod AB makes an angle θ with the vertical OX, then

$\tan\theta$ = the gradient of the line AB

$$= \frac{2at - 0}{at^2 - a} = \frac{2t}{t^2 - 1} = \frac{-2t}{1 - t^2}.$$

$$\therefore \quad \frac{2\tan\frac{1}{2}\theta}{1 - \tan^2\frac{1}{2}\theta} = \frac{2(-t)}{1 - (-t)^2}, \quad \text{or} \quad \tan\tfrac{1}{2}\theta = -t.$$

Let z be the height of the centre of gravity G of the rod AB above the fixed horizontal line YOY'. Then

$$z = OM + HG = OM + AG\cos\theta$$
$$= at^2 + c\cos\theta$$

[∵ OM = x-coordinate of A and $AG = \frac{1}{2}AB$]

$$= a\tan^2\tfrac{1}{2}\theta + c\cos\theta.$$

$$\therefore \quad dz/d\theta = 2a\left(\tan\tfrac{1}{2}\theta \sec^2\tfrac{1}{2}\theta\right).\tfrac{1}{2} - c\sin\theta$$

$$= a \tan \tfrac{1}{2}\theta \sec^2 \tfrac{1}{2}\theta - c.2 \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta$$
$$= \sin \tfrac{1}{2}\theta \, [a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta].$$

For the equilibrium of the rod, we must have $dz/d\theta = 0$

i.e., $\sin \tfrac{1}{2}\theta \, (a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta) = 0.$

$\therefore$ either $\sin \tfrac{1}{2}\theta = 0$ *i.e.*, $\theta = 0$,

which gives the vertical position of equilibrium,

or $a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta = 0$ *i.a.*, $a \sec^3 \tfrac{1}{2}\theta = 2c \cos \tfrac{1}{2}\theta$

i.e., $\cos^4 \tfrac{1}{2}\theta = a/2c$, which gives the inclined position of rest of the rod.

Now

$$\frac{d^2 z}{d\theta^2} = \tfrac{1}{2} \cos \tfrac{1}{2}\theta \, [a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta]$$
$$+ \sin \tfrac{1}{2}\theta \left[\frac{3a}{2} \sec^3 \tfrac{1}{2}\theta \tan \tfrac{1}{2}\theta + c \sin \tfrac{1}{2}\theta\right]$$
$$= \tfrac{1}{2} \cos \tfrac{1}{2}\theta \, [a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta] + \sin^2 \tfrac{1}{2}\theta \, [\tfrac{3}{2} a \sec^4 \tfrac{1}{2}\theta + c],$$

which is >0 when $\cos^4 \tfrac{1}{2}\theta = a/2c$

i.e., when $a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta = 0.$

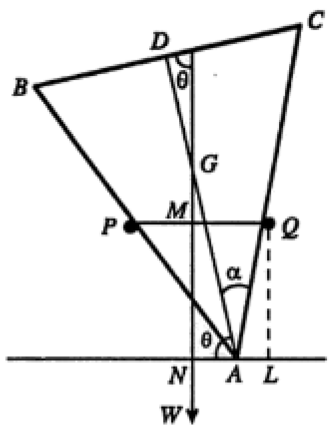
Ex. 2. *An isosceles triangular lamina, with its plane vertical rests with its vertex downwards, between two smooth pegs in the same horizontal line. Show that there will be equilibrium if the base makes an angle* $\sin^{-1}(\cos^2 \alpha)$ *with the vertical,* 2α *being the vertical angle of the lamina and the length of the base being three times the distance between the pegs.* **(Meerut 1994)**

Sol. ABC is an isosceles triangular lamina in which $AB = AC$. The sides AB and AC rest on two smooth pegs P and Q which are in the same horizontal line.

Let $PQ = a$ so that $BC = 3a$.

If D is the middle point of BC, then the centre of gravity G of the lamina lies on the median AD and is such that

$$AG = \tfrac{2}{3} AD.$$



The weight W of the lamina acts vertically downwards at G. We have

$$\angle BAD = \angle CAD = \alpha.$$

Suppose in equilibrium the base BC of the lamina makes an angle θ with the vertical. Since the angle between two lines is equal to the angle between their perpendicular lines, therefore $\angle DAN = \theta$. [Note that DA is perpendicular to BC and AN is perpendicular to the vertical line NMG].

Now $\angle QPA = \angle PAN = \theta - \alpha$, and $\angle QAL = \pi - (\theta + \alpha)$.

Give the lamina a small displacement in which θ changes to $\theta + \delta\theta$. The line PQ joining the pegs remains fixed and the distances will be measured from this line. The angle α remains fixed. The only force contributing to the sum of virtual works is the weight W of the lamina acting at G. We have, the height of G above the fixed line PQ

$$= MG = NG - NM = NG - LQ$$
$$= AG \sin\theta - AQ \sin\{\pi - (\theta + \alpha)\}$$
$$= \frac{2}{3} AD \sin\theta - AQ \sin(\theta + \alpha).$$

Now $AD = CD \cot\alpha = \frac{3}{2} a \cot\alpha$. Also from the ΔAQP, by the sine theorem of trigonometry, we have

$$\frac{AQ}{\sin APQ} = \frac{PQ}{\sin PAQ} \quad i.e., \quad \frac{AQ}{\sin(\theta - \alpha)} = \frac{a}{\sin 2\alpha}.$$

$\therefore$ $$AQ = \frac{a}{\sin 2\alpha} \sin(\theta - \alpha).$$

$\therefore$ $$MG = \frac{2}{3} \cdot \frac{3}{2} a \cot\alpha \sin\theta - \frac{a}{\sin 2\alpha} \sin(\theta - \alpha) \sin(\theta + \alpha)$$
$$= a \cot\alpha \sin\theta - \frac{a}{2 \sin 2\alpha} 2 \sin(\theta - \alpha) \sin(\theta + \alpha)$$
$$= a \cot\alpha \sin\theta - \frac{a}{4 \sin\alpha \cos\alpha} (\cos 2\alpha - \cos 2\theta)$$
$$= a \cot\alpha \sin\theta - \frac{a \cos 2\alpha}{4 \sin\alpha \cos\alpha} + \frac{a \cos 2\theta}{4 \sin\alpha \cos\alpha}.$$

The equation of virtual work is

$$-W\delta(MG) = 0, \quad \text{or} \quad \delta(MG) = 0$$

or $$\delta\left[a \cot\alpha \sin\theta - \frac{a \cos 2\alpha}{4 \sin\alpha \cos\alpha} + \frac{a \cos 2\theta}{4 \sin\alpha \cos\alpha}\right] = 0$$

or $$\left[a \cot\alpha \cos\theta - \frac{2a \sin 2\theta}{4 \sin\alpha \cos\alpha}\right]\delta\theta = 0$$

or $$a \cot\alpha \cos\theta - \frac{4a \sin\theta \cos\theta}{4 \sin\alpha \cos\alpha} = 0 \qquad [\because \delta\theta \neq 0]$$

or $$a \cos\theta\left(\cot\alpha - \frac{\sin\theta}{\sin\alpha \cos\alpha}\right) = 0.$$

$\therefore$ either $\cos\theta = 0$ *i.e.*, $\theta = \frac{\pi}{2}$, giving one position of equilibrium in which the lamina rests symmetrically on the pegs

or $$\cot\alpha - \frac{\sin\theta}{\sin\alpha \cos\alpha} = 0 \quad i.e., \quad \sin\theta = \cos^2\alpha$$

i.e., $\theta = \sin^{-1}(\cos^2\alpha)$, giving the other position of equilibrium.

Ex. 3. *A square of side $2a$ is placed with its plane vertical between two smooth pegs which are in the same horizontal line at a distance c apart ; show that it will be in equilibrium when the inclination of one of its edges to the horizon is either*

$$\frac{\pi}{4} \quad or \quad \frac{1}{2} \sin^{-1}\left(\frac{a^2 - c^2}{c^2}\right).$$

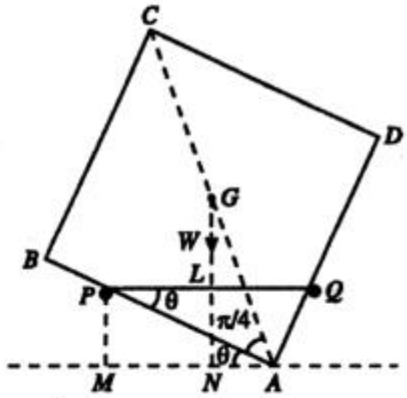
(Meerut 1988; Gorakhpur 86; Jiwaji 92; Garhwal 93)

Sol. The sides AB and AD of the square lamina $ABCD$ rest on two smooth pegs P and Q which are in the same horizontal line. It is given that $PQ = c$ and $AB = 2a$.

The weight W of the lamina acts at G, the middle point of the diagonal AC. Suppose in the position of equilibrium the side AB of the lamina makes an angle θ with the horizontal so that

$$\angle PAM = \theta = \angle QPA.$$

We have $\angle BAC = \frac{1}{4}\pi = \text{constant}$.



Give the lamina a small displacement in which θ changes to $\theta + \delta\theta$. The line PQ joining the pegs remains fixed. The only force contributing to the sum of virtual works is the weight W of the lamina acting at G. We have, the height of G above the fixed line PQ

$$= LG = NG - NL = NG - MP$$

$$= AG \sin\left(\frac{1}{4}\pi + \theta\right) - AP \sin\theta$$

$$= a\sqrt{2} \sin\left(\frac{1}{4}\pi + \theta\right) - PQ \cos\theta \sin\theta$$

$$[\because \ AG = \tfrac{1}{2}AC = \tfrac{1}{2} 2a\sqrt{2} = a\sqrt{2}, \quad \text{and} \quad AP = PQ\cos\theta]$$

$$= a\sqrt{2}\left(\sin\tfrac{1}{4}\pi\cos\theta + \cos\tfrac{1}{4}\pi\sin\theta\right) - c\cos\theta\sin\theta$$

$$= a(\cos\theta + \sin\theta) - c\cos\theta\sin\theta.$$

The equation of virtual work is

$$-W\delta(LG) = 0, \quad \text{or} \quad \delta(LG) = 0$$

or $\quad \delta\,[a(\cos\theta + \sin\theta) - c\cos\theta\sin\theta] = 0$

or $\quad [a(-\sin\theta + \cos\theta) - c(\cos^2\theta - \sin^2\theta)]\,\delta\theta = 0$

or $\quad a(\cos\theta - \sin\theta) - c(\cos^2\theta - \sin^2\theta) = 0 \qquad [\because \ \delta\theta \neq 0]$

or $\quad (\cos\theta - \sin\theta)\,[a - c(\cos\theta + \sin\theta)] = 0.$

$\therefore \quad$ either $\quad \cos\theta - \sin\theta = 0$

i.e., $\quad \sin\theta = \cos\theta \quad$ *i.e.,* $\quad \tan\theta = 1 \quad$ *i.e.,* $\quad \theta = \frac{1}{4}\pi,$

giving one position of equilibrium in which the lamina rests symmetrically on the pegs

or $\quad a - c(\cos\theta + \sin\theta) = 0$

i.e., $\quad c^2(\cos\theta + \sin\theta)^2 = a^2 \quad$ *i.e.,* $\quad c^2(1 + \sin 2\theta) = a^2$

i.e., $\quad \sin 2\theta = \dfrac{a^2}{c^2} - 1 = \dfrac{a^2 - c^2}{c^2} \quad$ *i.e.,* $\quad \theta = \dfrac{1}{2}\sin^{-1}\left(\dfrac{a^2 - c^2}{c^2}\right),$

giving the other position of equilibrium.

Example 4. *A heavy hemispherical shell of radius r has a particle attached to a point on its rim and rests on a rough sphere of radius R at the highest point. If their curved surfaces are in contact then equilibrium is stable if* $\frac{R}{r} > \sqrt{5} - 1$.

Sol. Let the weight hemispherical shell be W acting at C.G. G′.

$\therefore$ $O'G' = \frac{r}{2}$ where O′ is the centre of the shell. Let ω be the weight of the particle attached at the rim. Let G be the C.G. of the hemisphere after attaching the weight at A.

The resultant of these two parallel forces lies on AG′ at G.

Let C the point of contact of hemisphere and sphere. Let $\angle O'AG' = \theta$

$$\tan\theta = \frac{O'G'}{O'A} = \frac{\frac{r}{2}}{r} = \frac{1}{2}, \sin\theta = \frac{1}{\sqrt{5}}$$

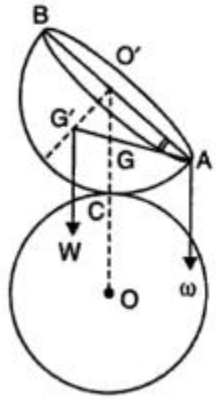


Fig. 12

Let height of G above C $= h$

$$h = O'C - O'G = r - r\sin\theta = r - r\cdot\frac{1}{\sqrt{5}}$$

The equilibrium is stable

if $$\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$$

if $$h < \frac{rR}{r+R}$$

or $$r - \frac{r}{\sqrt{5}} < \frac{rR}{r+R}$$

or if $$\frac{r}{\sqrt{5}} > r - \frac{rR}{r+R}$$

or if $$\frac{1}{\sqrt{5}} > 1 - \frac{R}{r+R}$$

or if $$\frac{r+R}{\sqrt{5}} > r + R - R$$

or if $$\frac{r+R}{r} > \sqrt{5} \Rightarrow \frac{R}{r} > \sqrt{5} - 1$$

$\because$ The result is independent of the weight attached at the rim.

Therefore, the equilibrium is stable for any weight attached.

Example 5. *A body consists of a cone and a hemisphere on the same base, rests on a rough horizontal table with curved surface of hemisphere in contact with the table. Show that greatest height of the cone so that the equilibrium may be stable is* $\sqrt{3}$ *times the radius of hemisphere.*

Sol. Let H and r be the height and radius of base of the cone respectively. If C.G. of hemisphere is at G_1 and C.G. of cone is at G_2 then

$$OG_1 = \frac{3r}{8}, OG_2 = \frac{H}{4}.$$

C.G. of the combined body is given by

$$\bar{x} = \frac{w_1 \bar{x}_1 + w_2 \bar{x}_2}{w_1 + w_2}$$

Height of C.G. from the fixed plane h

$$h = \frac{\left(\frac{2}{3}\pi r^3\right).\frac{5r}{8} + \left(\frac{1}{3}\pi r^2 H\right)\left(r + \frac{H}{4}\right)}{\frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 H}$$

$$= \frac{\frac{5r^2}{4} + Hr + \frac{H^2}{4}}{2r + H} = \frac{5r^2 + 4rH + H^2}{4(H + 2r)}$$

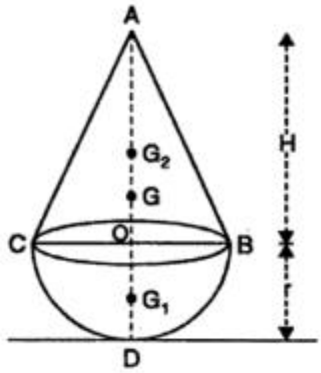


Fig. 13

The equilibrium is stable if

$$\frac{1}{h} > \frac{1}{r} + \frac{1}{R} \quad R = \infty, \quad r = r$$

if $$\frac{4(H + 2r)}{5r^2 + 4rH + H^2} > \frac{1}{r}$$

if $$4Hr + 8r^2 > 5r^2 + 4rH + H^2$$

if $$H^2 < 3r^2$$

$$H < r\sqrt{3}$$

Hence the greatest height of the cone consistent with the stable equilibrium is $\sqrt{3}$ times the radius of the hemisphere.

Example 7. *A solid hemisphere rests on a plane inclined to the horizon at an angle $sin^{-1}\left(\frac{3}{8}\right)$. The plane is rough enough to prevent sliding. Find the position of equilibrium and show that it is stable.*

Sol. Let O and G be the centre and C.G. of the hemisphere of radius r

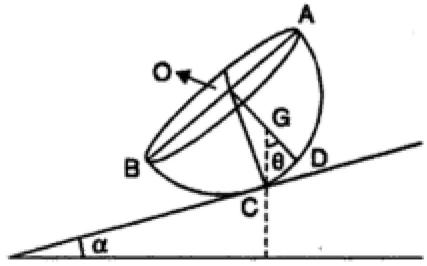


Fig. 15

$$OC = OD = r$$

$$OG = \frac{3r}{8}$$

Let α be the inclination of the plane with horizontal

$\therefore \quad \sin\alpha = \frac{3}{8}$

$\angle OCG = \alpha$, Let $\angle CGD = \theta$

From ΔOGC

$$\frac{OG}{\sin\alpha} = \frac{OG}{\sin(\pi-\theta)} \Rightarrow \frac{3r}{8\sin\alpha} = \frac{r}{\sin\theta}$$

$\therefore \quad \sin\theta = \frac{8}{3}\sin\alpha$ ...(1)

$\because \quad \sin\theta < 1 \Rightarrow$ Equilibrium is possible if

$$\frac{8}{3}\sin\alpha < 1$$

or $\quad \sin\alpha < \frac{3}{8}$ or $\alpha < \sin^{-1}\left(\frac{3}{8}\right)$ ...(2)

For stability of equilibrium we find height of C.G. above the point of contact.

Let $\quad CG = h$

From $\Delta OGC \quad \frac{OG}{\sin\alpha} = \frac{CG}{\sin(\theta-\alpha)}$

$$\frac{3r}{8\sin\alpha} = \frac{h}{\sin(\theta-\alpha)} \Rightarrow h = \frac{3r\sin(\theta-\alpha)}{8\sin\alpha} \quad ...(3)$$

For stability, $\quad \frac{1}{h} > \left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\frac{1}{\cos\alpha} \qquad \rho_2 = \infty$

$\Rightarrow \quad \frac{1}{h} > \frac{1}{r\cos\alpha} \qquad \rho_1 = r$

$\Rightarrow \qquad h < r\cos\alpha$

Putting the value of h from (3), we have

$\Rightarrow \qquad \dfrac{3r\sin(\theta-\alpha)}{8\sin\alpha} < r\cos\alpha$

$\Rightarrow \qquad 3\sin(\theta-\alpha) < 8\sin\alpha\cos\alpha$

$\Rightarrow \qquad 3[\sin\theta\cos\alpha - \cos\theta\sin\alpha] < 8\sin\alpha\cos\alpha$

$\Rightarrow \qquad 3\left[\dfrac{8}{3}\sin\alpha\cos\alpha - \sqrt{1-\dfrac{64}{9}\sin^2\alpha}\,\sin\alpha\right] < 8\sin\alpha\cos\alpha$ Using (1)

$\Rightarrow \qquad -3\,.\,\sqrt{9-64\sin^2\alpha}\,.\,\sin\alpha < 0 \qquad \Rightarrow \sin^2\alpha\,(9-64\sin^2\alpha) > 0$

$\Rightarrow \qquad 64\sin^2\alpha < 9 \qquad \Rightarrow \sin\alpha < \dfrac{3}{8}$

$\Rightarrow \qquad \alpha < \sin^{-1}\left(\dfrac{3}{8}\right)$

which is true for equilibrium from (2).

Hence the equilibrium is stable.

Example 25. *A heavy uniform rod is in equilibrium with one end resting against a smooth vertical wall and the other end against a smooth plane inclined to the wall at an angle θ. If α be the inclination of the rod to the horizontal then 2 tan α = tan θ.*

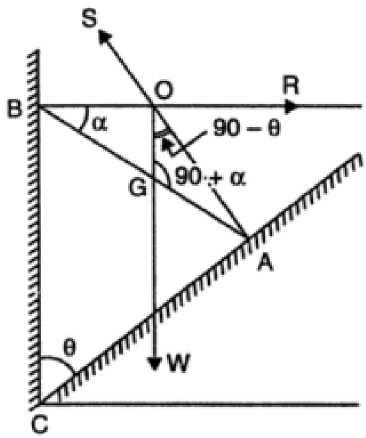


Fig. 26

Sol. Rod AB is in equilibrium under three concurrent forces.

(*i*) Reaction R at B ⊥ wall

(*ii*) Reaction S at A ⊥ inclined plane.

(*iii*) Weight W at G vertically downwards.

Forces are concurrent at O

$\angle ABO = \alpha,\ \angle ACB = \theta$

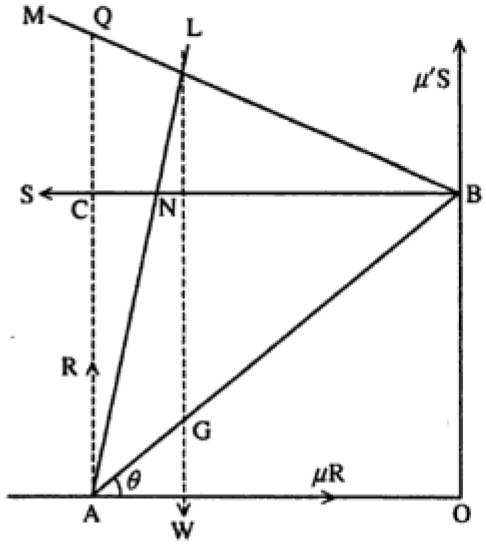
Applying $m : n$ theorem in ΔOAB

$(1+1)\cot(90+\alpha) = 1\cot 90 - 1\cot(90-\theta)$

$-2\tan\alpha = -\tan\theta$

or $\qquad 2\tan\alpha = \tan\theta.$

13. A ladder AB whose C.G. G divides it into two portion of length GA = a and GB = b, rests with end A on a rough horizontal floor and the other end B against a rough vertical wall. If the co-efficient of friction at the floor and the wall he μ and μ' respectively, show that the inclination of the ladder to the floor when it is in equilibrium, $\tan^{-1}\left\{\frac{(a-b\mu\mu')}{(a+b)\mu}\right\}$. **[C.H., 2004]**



Solution : The ladder AB is with its one end on the rough horizontal floor and other against a rough vertical plane. The reactions at A is R and μR and at B be S & μ'S as shown in the figure. and AG = a, GB = b.

If W be the weight of the ladder and θ be the inclination to the floor then we can write for equilibrium, $\mu'S + R = W$ and $\mu R = S$ ... (i)

Also taking moment about G, we get

$\mu'S\, b\cos\theta + S.b\sin\theta - R.a\cos\theta + \mu R.a\sin\theta = 0$

$\sin\theta\,(S.b + \mu R.a) = \cos\theta(Ra - \mu' Sb)$

Using (1),

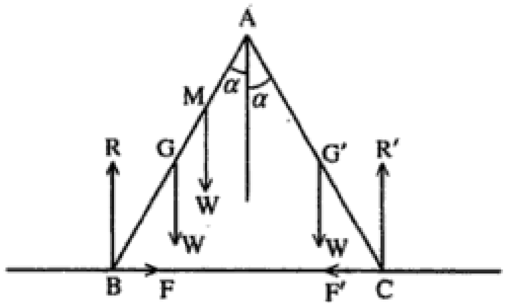
$$\therefore \tan\theta = \frac{Ra-\mu' Sb}{(a+b)\mu R} = \frac{Ra-\mu'\mu R.b}{(a+b)\mu.R} = \frac{a-b\mu\mu'}{(a+b)\mu}.$$

$$i.e.,\ \tan\theta = \frac{a-b\mu\mu'}{\mu(a+b)}.$$

$$\therefore\ \theta = \tan^{-1}\left\{\frac{(a-b\mu\mu')}{(a+b)\mu}\right\}.$$

14. Two equal uniform ladder are joined at one end and stand with other ends on a rough horizontal plane. A man whose weight is equal to that of a ladder ascends on one of them. Prove that the other ladder will slip first. It begins to slip when he has ascended a distance d, prove that the coefficient of friction is $\left(\frac{l+d}{2l+d}\right)\tan\alpha$, l being the length of each ladder and 'α' then angle each makes with vertical. **[C.H., 2001]**

Solution : Let M be the position of of the man on the ladder AB at any instant when BM = d, R, R′ the normal reactions of the ground, and F, F′ the magnitudes of frictions at the instant at B and C respectively. Let W be the weight of the ladder.



Now considering the two ladders AB and AC as forming one system, action and reaction at A on the two neutralise each other.

Hence resolving horizontally and vertically and taking moments about B, for the equilibrium of the combined system, we get

$$F = F' \quad \ldots (1)$$

$$R + R' = 3W \quad \ldots (2)$$

$$R'.2l\sin\alpha = W.\frac{l}{2}\sin\alpha + W.d\sin\alpha + W.\frac{3l}{2}\sin\alpha \quad \ldots (3)$$

Again considering the equilibrium of the ladder AC separately taking moments about A, we get

$$F'.l\cos\alpha = R'.l\sin\alpha - W.\frac{l}{2}\sin\alpha \quad \ldots (4)$$

From (3) $R' = W\,\frac{2l+d}{2l}$ and from (2) $R = W\,\frac{4l-d}{2l}$.

Hence $F = F' = \tan\alpha\left(W\frac{2l+d}{2l} - \frac{W}{2}\right) = W\tan\alpha\frac{l+d}{2l}$.

$\therefore\ \frac{F}{R} = \frac{d+l}{4l-d}\tan\alpha$ and $\frac{F'}{R'} = \frac{l+d}{2l+d}\tan\alpha$

Now $4l - d - (2l + d) = 2l - 2d > 0 \quad \because d < l$

$\therefore\ \frac{F'}{R'} > \frac{F}{R}$ for all values of d.

One of the ladders will slip when either $\frac{F}{R}$ or $\frac{F'}{R'}$ is equal to the co-efficient of friction μ and as $\frac{F'}{R'}$ is greater it will attain the value μ first and $\mu = \frac{F'}{R'} = \frac{l+d}{2l+d}\tan\alpha$.

18. A uniform ladder is in equilibrium with one end resting on the ground and the other against a vertical wall; if the ground and the wall be both rough, the coefficients of friction being μ and μ' respectively and if the ladder be at the point of slipping at both ends, show that the inclination θ of the ladder to the horizon is given by $\tan\theta = \frac{1-\mu\mu'}{2\mu}$.

Solution : Let AB be the ladder, of length $2l$ and weight W, whose one end A rests on the rough horizontal ground AC and the other end B rests against a vertical wall BC. Let AB makes an angle θ with the horizontal ground. Let the normal reactions at A and B be R and R′. Also weight of the ladder is acting vertically down wards through G which is middle point of the ladder AB. Frictional forces at A and B are shown in figure. Let S and S′ be the resultant of the normal reaction and friction at A and B respectively.

Now taking moments about A, we get

$-W.l\cos\theta + R'.2l\sin\theta + \mu'R'.2l\cos\theta = 0$

or, $2R'.\tan\theta = W - 2\mu' R'$ ... (1)

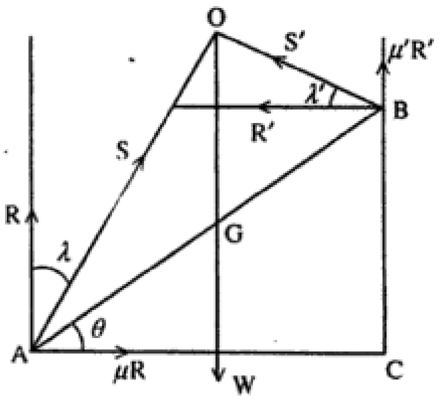
Also for equilibrium

$R' = \mu R$ and $R + \mu'R' = W$

Eliminating R, $R' + \mu\mu'R' = \mu.W$

or, $R'(1 + \mu\mu') = \mu W$

$R' = \frac{\mu}{1+\mu\mu'}.W$

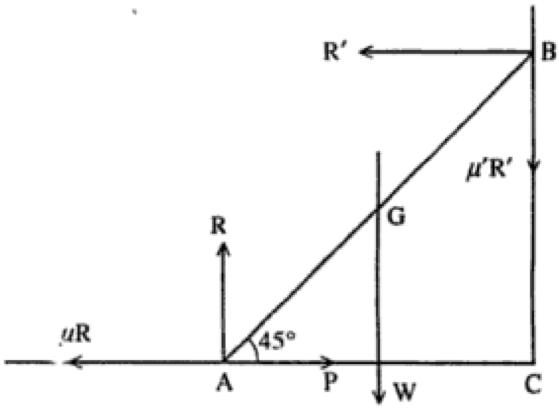


$\therefore$ From (1) $2\frac{\mu.W}{1+\mu\mu'}(\tan\theta + \mu') = W$

or, $\tan\theta + \mu' = \frac{1+\mu\mu'}{2\mu}$ or, $\tan\theta = \frac{1+\mu\mu'}{2\mu} - \mu' = \frac{1-\mu\mu'}{2\mu}$

19. A uniform ladder of weight W, inclined to the horizon at 45°, rests with its upper extremity against a rough vertical wall and its lower extremity on the rough horizontal ground. Prove that the least horizontal force which will move the lower end towards the wall is just greater than $\frac{W}{2}.\frac{1+2\mu-\mu\mu'}{1-\mu'}$

where μ and μ' are the coefficients of friction at the lower and upper end respectively.



Solution : Let AB be the ladder of length $2l$ which makes angle 45° with the horizon and P be the horizontal force in the position when A is just on the point of slipping towards C along AC. Let R and R′ be the normal reactions at the ends A and B. Since A is on the point of slipping towards C, μR acts in the direction of CA, and since B is on the point of slipping up, μ'R′ acts in the direction BC. Weight of the ladder is acting at G which is middle point of AB.

Now for equilibrium,

$P = R' + \mu R$ and $R = W + \mu' R'$ ... (i)

Taking moments about A, we get

$-W.l\cos 45° - \mu' R'.2l\cos 45° + R'.2l\sin 45° = 0$

or, $-W.\frac{1}{\sqrt{2}} - \mu' R'\, 2.\frac{1}{\sqrt{2}} + R'.2.\frac{1}{\sqrt{2}} = 0$ or, $2R'(1-\mu') = W$

$\therefore R' = \frac{W}{2(1-\mu')}$ ... (2)

Form (1) $P = R' + \mu R = R' + \mu W + \mu\mu' R'$

or, $P = R'(1+\mu\mu') + \mu W$

or, $P = \frac{W}{2}.\frac{(1+\mu\mu')}{1-\mu'} + \mu W = \frac{W}{2}.\frac{1+\mu\mu'+2\mu-2\mu\mu'}{1-\mu'}$.

$\therefore P = \frac{W}{2}.\frac{1+2\mu-\mu\mu'}{1-\mu'}$.

20. A solid homogeneous hemisphere rests on a rough horizontal plane and against a rough vertical wall, the coefficients of friction being μ and μ' respectively. Show that the least angle that the base of the hemisphere can make with the vertical is $\cos^{-1}\left(\frac{8\mu}{3}.\frac{1+\mu'}{1+\mu\mu'}\right)$.

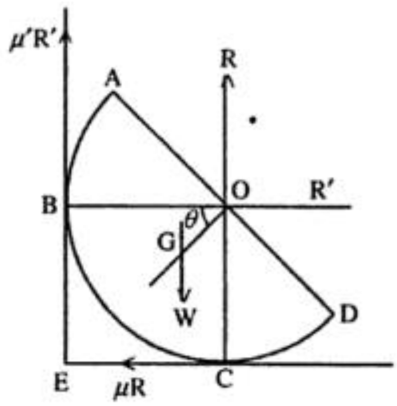
Solution : Let ABCD be the solid homogeneous hemisphere rests on a rough horizontal plane and against a rough vertical wall.

Let a be the radius of the hemisphere. Now weight of the hemisphere is acting at G, where OG = $\frac{3}{8}a$ and OG is always perpendicular to the diameter AD. Now reactions and frictions at B and C are as shown in figure.

Now for equilibrium, $R + \mu' R' = W$

and $R' = \mu R$ ... (1)

When hemisphere is in limiting equilibrium, let the line OG makes θ with the horizontal direction.



Now taking moments about 0, we get

$$W.\frac{3a}{8}.\cos\theta - \mu R.a - \mu' R' a = 0$$

or, $\frac{3}{8}.\cos\theta.W = \mu R + \mu' R'$ ... (2)

From (1), $R' + \mu\mu' R' = \mu W$

$$\therefore R' = \frac{\mu W}{1+\mu\mu'}$$

From (2), $\frac{3}{8}\cos\theta.W = R' + \mu' R' = (1+\mu')\frac{W.\mu}{1+\mu\mu'}$

$$\therefore \frac{3}{8}\cos\theta = \frac{\mu(1+\mu')}{1+\mu\mu'}.$$

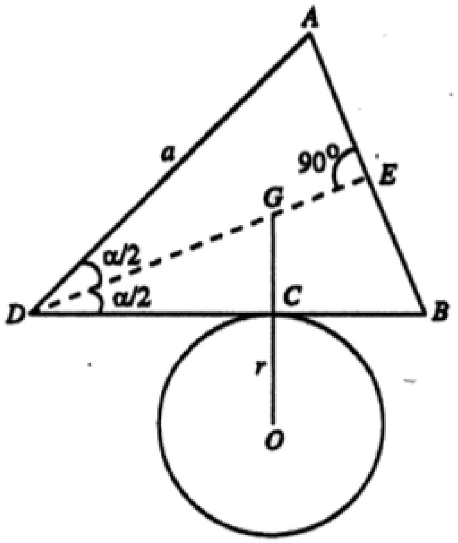
$\therefore$ Least angle the plane face can make with the vertical

$$= \theta = \cos^{-1}\left(\frac{8\mu}{3}.\frac{1+\mu'}{1+\mu\mu'}\right).$$

Ex. 3. *A lamina in the form of an isosceles triangle, whose vertical angle is α, is placed on a sphere, of radius r, so that its plane is vertical and one of its equal sides is in contact with the sphere; show that, if the triangle be slightly displaced in its own plane, the equilibrium is stable if $\sin\alpha < 3r/a$, where a is one of the equal sides of the triangle.*

Sol. DAB is an isosceles triangular lamina in which

$$DA = DB = a \text{ and } \angle ADB = \alpha.$$



The centre of gravity G of the lamina lies on its median DE which is perpendicular to AB and also bisects the angle ADB. We have

$$DG = \frac{2}{3} DE = \frac{2}{3} a \cos \frac{1}{2} \alpha.$$

The lamina rests on a fixed sphere whose centre is O and radius r. Their point of contact is C. For equilibrium the line OCG must be vertical.

If h be the height of the C.G. of the lamina above the point of contact C, then

$$h = GC = DG \sin \frac{1}{2} \alpha$$

$$= \frac{2}{3} a \cos \frac{1}{2} \alpha \sin \frac{1}{2} \alpha = \frac{1}{3} a \sin \alpha.$$

Here ρ_1 = the radius of curvature of the upper body at the point of contact C $= \infty$

and ρ_2 = the radius of curvature of the lower fixed body at the point $C = r$.

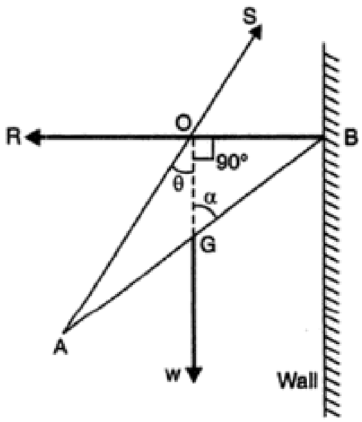
The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \quad i.e., \quad \frac{1}{h} > \frac{1}{\infty} + \frac{1}{r} \quad i.e., \quad \frac{1}{h} > \frac{1}{r}$$

i.e., $h < r$ *i.e.,* $\frac{1}{3} a \sin \alpha < r$

i.e., $\sin \alpha < 3r/a.$

Example 14. *A uniform rod AB movable about a hinge at A, rests with the other end against a smooth vertical wall. If α be the inclination of the rod to the vertical, prove that the magnitude of the reaction of the hinge is $\frac{1}{2} w \sqrt{4 + \tan^2 \alpha}$.*



Sol. Let w be the weight of the rod acting vertically downwards through G, the mid-pt. of AB.

Let R be the normal reaction of the wall.

Let the line of action of w and R meet in O. Then S, the reaction of the hinge at A must also pass through O and act along AO.

Let $\angle AOG = \theta$ and $\angle OGB = \alpha$

By "m–n theorem" in ΔAOB

$(a + a) \cot \alpha = a \cot \theta - a \cot 90° \Rightarrow 2 \cot \alpha = \cot \theta$

$\therefore \quad \tan \theta = \frac{1}{2} \tan \alpha \quad ...(1)$

By Lami's Theorem to forces at O, we have

$$\frac{S}{\sin 90°} = \frac{w}{\sin (90° + \theta)}$$

$$\therefore \quad S = \frac{w}{\cos \theta} = w \sec \theta = w\sqrt{1 + \tan^2 \theta} = w \sqrt{1 + \frac{1}{4} \tan^2 \alpha} \quad | \because \text{ of (1)}$$

$$= \frac{w}{2} \sqrt{4 + \tan^2 \alpha} \quad \left[\text{or } \frac{w}{2} \sqrt{3 + \sec^2 \alpha}\right]$$

Example 7. *Two weights P and Q are suspended from a fixed point O by strings OA, OB and are kept apart by a light rod AB. If the strings make angles α and β with the rod, show that the angle θ which the rod makes with the vertical is given by $\tan \theta = \frac{P + Q}{P \cot \alpha - Q \cot \beta}$.*

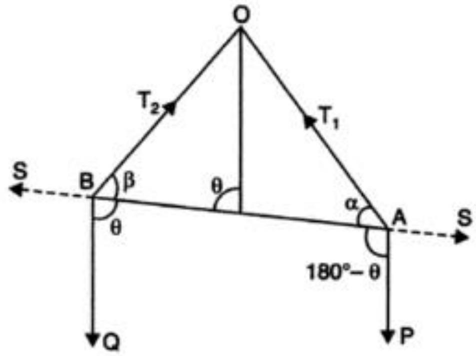
Sol. Let T_1, T_2 be the tensions in the strings AO and BO respectively. The thrust S of the rod at A will be equal and opposite to that at B in directions as shown in the figure.

Forces T_1, S, P acting at A are in equilibrium.

∴ By Lami's Theorem, we have

$$\frac{T_1}{\sin\theta} = \frac{S}{\sin(180° - \theta + \alpha)} = \frac{P}{\sin(180° - \alpha)}$$

or $$\frac{T_1}{\sin\theta} = \frac{S}{\sin(\theta - \alpha)} = \frac{P}{\sin\alpha} \quad ...(1)$$



Forces T_2, S, Q acting at B are in equilibrium.

∴ By Lami's Theorem, we have

$$\frac{T_2}{\sin(180° - \theta)} = \frac{S}{\sin(\theta + \beta)} = \frac{Q}{\sin(180° - \beta)}$$

or $$\frac{T_2}{\sin\theta} = \frac{S}{\sin(\theta + \beta)} = \frac{Q}{\sin\beta} \quad ...(2)$$

From (1), $$S = \frac{P\sin(\theta - \alpha)}{\sin\alpha}$$

From (2), $$S = \frac{Q\sin(\theta + \beta)}{\sin\beta}$$

∴ $$\frac{P\sin(\theta - \alpha)}{\sin\alpha} = \frac{Q\sin(\theta + \beta)}{\sin\beta}$$

or $$P \cdot \frac{\sin\theta\cos\alpha - \cos\theta\sin\alpha}{\sin\alpha} = Q \cdot \frac{\sin\theta\cos\beta + \cos\theta\sin\beta}{\sin\beta}$$

or $$P[\sin\theta\cot\alpha - \cos\theta] = Q[\sin\theta\cot\beta + \cos\theta]$$

Dividing both sides by cos θ,

$$P[\tan\theta\cot\alpha - 1] = Q[\tan\theta\cot\beta + 1]$$

or $$\tan\theta[P\cot\alpha - Q\cot\beta] = P + Q$$

∴ $$\tan\theta = \frac{P + Q}{P\cot\alpha - Q\cot\beta}.$$

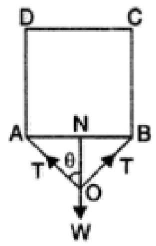
Example 8. (*i*) *A square board ABCD, the length of whose side is 'a' is fixed in a vertical plane with two of its sides horizontal. An endless string of length l [> 4a] passes over four pegs at the angles of the board and through a ring of weight W which is hanging vertically. Show that tension of the string is* $\dfrac{W(l-3a)}{2\sqrt{l^2-6la+8a^2}}$.

(*ii*) *A fine light string AOB of length l is passed at O through a small smooth ring of no appreciable weight and is attached at its extremities to two fixed points A and B at a distance c apart. A given force P is applied to the ring in a direction making an angle α with BA. Show that in the position of equilibrium, the two parts of the string are inclined to each other at an angle* $2\sin^{-1}\left[\dfrac{c\sin\alpha}{l}\right]$.

Show also that the tension of the string is $\dfrac{lP}{2\sqrt{l^2-c^2\sin^2\alpha}}$.

Sol. (*i*) Let OADCBO be the string of length l.

Draw ON ⊥ on AB, then OA = OB.



$$AN = NB = \frac{a}{2}$$

$$\therefore \quad l = OA + AD + DC + CB + BO$$
$$= OA + a + a + a + OA = 2OA + 3a$$

$$\therefore \quad OA = \frac{l-3a}{2}$$

$$ON = \sqrt{OA^2 - AN^2} = \sqrt{\frac{(l-3a)^2}{4} - \frac{a^2}{4}}$$
$$= \sqrt{\frac{l^2-6al+9a^2-a^2}{4}} = \frac{1}{2}\sqrt{l^2-6al+8a^2}$$

Also, ON bisects ∠AOB.

Let ∠AON = θ, then $\cos\theta = \dfrac{ON}{OA} = \dfrac{\frac{1}{2}\sqrt{l^2-6al+8a^2}}{\frac{1}{2}(l-3a)} = \dfrac{\sqrt{l^2-6al+8a^2}}{l-3a}$.

Let T be the tension in the parts OA and OB of the string. The tensions in the two parts are equal because the line of action of W bisects the ∠AOB.

Also, for equilibrium at O, W must balance the resultant of equal forces T, T inclined at an angle AOB = 2θ.

$$\therefore \quad W = 2T\cos\theta$$

$$\therefore \quad T = \frac{W}{2}\cdot\frac{1}{\cos\theta} = \frac{W(l-3a)}{2\sqrt{l^2-6la+8a^2}}.$$

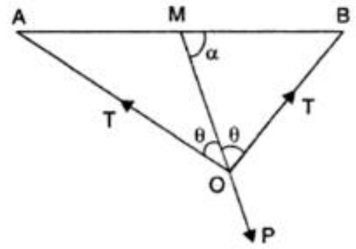
(*ii*) As the string passes through a small smooth ring at O, therefore, tension in both the parts is the same, say T.

Also, for equilibrium at O, the resultant of two equal forces T, T must be balanced by P.

⇒ Line of action P must bisect ∠AOB.

Let $\angle AOM = \angle MOB = \theta$

Then, $P = 2T\cos\theta$ ...(1)



Now in ΔOAM, $\dfrac{\sin A\hat{M}O}{OA} = \dfrac{\sin\theta}{AM}$

$$\Rightarrow \quad \frac{\sin(180^\circ-\alpha)}{OA} = \frac{\sin\theta}{AM}$$

$$\therefore \quad \frac{\sin\theta}{\sin\alpha} = \frac{AM}{OA} \quad ...(2)$$

In ΔOMB, $\dfrac{\sin\alpha}{OB} = \dfrac{\sin\theta}{MB}$

$$\therefore \quad \frac{\sin\theta}{\sin\alpha} = \frac{MB}{OB} \quad ...(3)$$

From (2) and (3), $\dfrac{\sin\theta}{\sin\alpha} = \dfrac{AM}{OA} = \dfrac{MB}{OB} = \dfrac{AM+MB}{OA+OB}$ | ∵ If $\dfrac{a}{b} = \dfrac{c}{d}$, then each $= \dfrac{a+c}{b+d}$

$$= \frac{AB}{l} = \frac{c}{l} \quad | \because AB = c$$

$$\therefore \quad \sin\theta = \frac{c\sin\alpha}{l}, \ \theta = \sin^{-1}\left[\frac{c\sin\alpha}{l}\right]$$

$$\therefore \quad \angle AOB = 2\theta = 2\sin^{-1}\left[\frac{c\sin\alpha}{l}\right]$$

From (1), $T = \dfrac{P}{2\cos\theta} = \dfrac{P}{2\sqrt{1-\sin^2\theta}}$

$$= \frac{P}{2\sqrt{1-\dfrac{c^2\sin^2\alpha}{l^2}}} = \frac{Pl}{2\sqrt{l^2-c^2\sin^2\alpha}}.$$

Ex. 6. *A uniform chain of length l hangs between two points A and B which are at a horizontal distance a from one another, with B at a vertical distance b above A. Prove that the parameter of the catenary is given by*

$$2c \sinh (a/2c) = \sqrt{(l^2 - b^2)}.$$

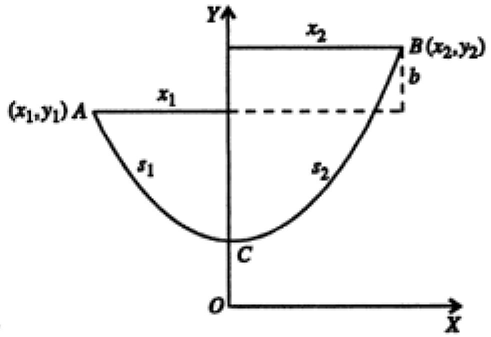
Prove also that, if the tensions at A and B are T_1 and T_2 respectively,

$$T_1 + T_2 = W\sqrt{\left(1 + \frac{4c^2}{l^2 - b^2}\right)}$$

and $$T_2 - T_1 = Wb/l,$$

where W is the weight of the chain. **(Rohilkhand 1982)**

Sol. A uniform chain of length l and weight W hangs between two points A and B. Let C be the vertex, OX the directrix, OY the axis and c the parameter of the catenary in which the chain hangs. Let (x_1, y_1) and (x_2, y_2) be the coordinates of the points A and B respectively and let arc $CA = s_1$ and arc $CB = s_2$.



We have $s_1 + s_2 = l.$

Since the horizontal distance between A and B is a, therefore

$$x_1 + x_2 = a.$$

Again since the vertical distance of B above A is b, therefore

$$y_2 - y_1 = b.$$

Let w be the weight per unit length of the chain. Then

$$W = lw, \quad \text{or} \quad w = W/l.$$

By the formula $s = c \sinh (x/c)$, we have

$$s_1 = c \sinh (x_1/c) \quad \text{and} \quad s_2 = c \sinh (x_2/c).$$

$$\therefore \quad l = s_1 + s_2 = c\,[\sinh (x_1/c) + \sinh (x_2/c)]. \qquad ...(1)$$

Again by the formula $y = c \cosh (x/c)$, we have

$$y_1 = c \cosh (x_1/c) \text{ and } y_2 = c \cosh (x_2/c).$$

$$\therefore \quad b = y_2 - y_1 = c\,[\cosh (x_2/c) - \cosh (x_1/c)]. \qquad ...(2)$$

Squaring and subtracting (1) and (2), we have

$$l^2 - b^2 = c^2\,[-\{\cosh^2 (x_1/c) - \sinh^2 (x_1/c)\} - \{\cosh^2 (x_2/c) - \sinh^2 (x_2/c)\}$$
$$+ 2\,\{\cosh (x_1/c) \cosh (x_2/c) + \sinh (x_1/c) \sinh (x_2/c)\}]$$
$$= c^2\,[-1-1 + 2 \cosh (x_1/c + x_2/c)]$$
$$= c^2\,[-2 + 2 \cosh \{(x_1 + x_2)/c\}]$$
$$= 2c^2 \left\{\cosh \frac{a}{c} - 1\right\} = 2c^2 \left\{1 + 2 \sinh^2 \frac{a}{2c} - 1\right\}$$
$$= 4c^2 \sinh^2 \frac{a}{2c}. \qquad ...(3)$$

$\therefore$ c is given by

$$2\,c \sinh (a/2c) = \sqrt{(l^2 - b^2)}.$$

[Remember that $\cosh (\alpha + \beta) = \cosh \alpha \cosh \beta + \sinh \alpha \sinh \beta$, and $\cosh 2\alpha = 1 + 2 \sinh^2 \alpha$]

Now let T_1 and T_2 be the tensions at the points A and B respectively. Then by the formula $T = wy$, we have

$$T_1 = wy_1,\ T_2 = wy_2.$$

$$\therefore \quad T_2 - T_1 = w(y_2 - y_1) = wb = (W/l)\,b = Wb/l.$$

Also
$$T_1 + T_2 = w(y_1 + y_2) = \frac{W}{l}(y_1 + y_2) = W\frac{y_1 + y_2}{s_1 + s_2}$$

$$= W\frac{c\cosh(x_1/c) + c\cosh(x_2/c)}{c\sinh(x_1/c) + c\sinh(x_2/c)}$$

$$= W\frac{\cosh(x_1/c) + \cosh(x_2/c)}{\sinh(x_1/c) + \sinh(x_2/c)}$$

$$= W\frac{2\cosh\frac{1}{2}(x_1/c + x_2/c)\cosh\frac{1}{2}(x_1/c - x_2/c)}{2\sinh\frac{1}{2}(x_1/c + x_2/c)\cosh\frac{1}{2}(x_1/c - x_2/c)}$$

$$= W\coth\left(\frac{x_1 + x_2}{2c}\right) = W\coth\frac{a}{2c}$$

$$= W\sqrt{\left[1 + \operatorname{cosech}^2\frac{a}{2c}\right]} \qquad [\because \coth^2\alpha = 1 + \operatorname{cosech}^2\alpha]$$

$$= W\sqrt{\left[1 + \frac{4c^2}{l^2 - b^2}\right]},$$

substituting for $\operatorname{cosech}^2(a/2c)$ from (3).