

Why - Until Now - Was The Hypothesis An Open Problem?

The discussion in this section is not intended to be comprehensive. It is, in fact, intended to elucidate the behavior of the Riemann zeta function on the complex plane and to outline some of the approaches that have been used unsuccessfully over the years to resolve the hypothesis. Also, a completely new characterization of roots of the zeta function in the critical strip and a new approach to the hypothesis are introduced with some visual aids to help understand why this new characterization and approach are successful in resolving the hypothesis.

First, it's helpful to explore the zeta function and its regions of convergence on the complex plane with classical methods.

Note that since $n^{-s} = n^{-(\sigma+it)} = n^{-\sigma} \cdot e^{-i \cdot t \cdot \log(n)}$, and since $|n^{-\sigma}| = n^{-\sigma}$ and $|e^{-i\theta}| = 1$ for all real values of θ , it follows that $|e^{-i \cdot t \cdot \log(n)}| = 1$, and therefore,

$$|n^{-s}| = |n^{-(\sigma+it)}| = |n^{-\sigma}| \cdot |e^{-i \cdot t \cdot \log(n)}| = n^{-\sigma}$$

Also note that, for decreasing real functions f ,

$$f(n+1) \leq \int_n^{n+1} f(x) dx \leq f(n)$$

Consider the first inequality in this relationship and sum both sides over $n = 1, 2, 3, \dots, N-1$ to get

$$\sum_{n=1}^{N-1} f(n+1) \leq \sum_{n=1}^{N-1} \int_n^{n+1} f(x) dx$$

Exchanging the order of summation and integration gives

$$\sum_{n=1}^N f(n) \leq f(1) + \int_1^N f(x) dx$$

Similarly, consider the second inequality in the relationship and sum both sides over $n = 1, 2, 3, \dots, N$ to get

$$\sum_{n=1}^N \int_n^{n+1} f(x) dx \leq \sum_{n=1}^N f(n)$$

Exchanging the order of summation and integration gives

$$\int_1^{N+1} f(x) dx \leq \sum_{n=1}^N f(n)$$

The combination of these two results gives an application of the Integral Test for convergence of series of decreasing functions $f(x)$:

$$\int_1^{N+1} f(x) dx \leq \sum_{n=1}^N f(n) \leq f(1) + \int_1^N f(x) dx$$

This result can be used to examine the behavior of the Riemann zeta function on the complex plane.

Case 1: $s = 1$ harmonic Series

Let $f(n) = 1/n$. The function $f(x) = 1/x$ is a decreasing function of x , and by the preceding result,

$$\int_1^{N+1} \frac{dx}{x} \leq \sum_{n=1}^N \frac{1}{n} \leq 1 + \int_1^N \frac{dx}{x}$$

or

$$\text{Log}(N + 1) \leq \sum_{n=1}^N \frac{1}{n} \leq 1 + \text{Log}(N)$$

so that

$$\sum_{n=1}^N \frac{1}{n} \geq \text{Log}(N + 1)$$

As $N \rightarrow \infty$, $\text{Log}(N + 1)$ diverges, and therefore,

$$\zeta(1) \equiv \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N \frac{1}{n} \right\} = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Case 2: $s = \sigma > 1$

Let $f(n) = 1/n^\sigma$ where σ is real and greater than 1. The function $f(x) = 1/x^\sigma$ is a decreasing function of x , and since

$$\int_1^{N+1} \frac{dx}{x^\sigma} \leq \sum_{n=1}^N \frac{1}{n^\sigma} \leq 1 + \int_1^N \frac{dx}{x^\sigma}$$

or

$$\left(\frac{1}{1-\sigma} \right) \cdot [(N+1)^{1-\sigma} - 1] \leq \sum_{n=1}^N \frac{1}{n^\sigma} \leq 1 + \left(\frac{1}{1-\sigma} \right) \cdot (N^{1-\sigma} - 1)$$

and

$$\sum_{n=1}^N \frac{1}{n^\sigma} \leq 1 + \left(\frac{1}{1-\sigma} \right) \cdot (N^{1-\sigma} - 1) \leq 1 + \left(\frac{1}{\sigma-1} \right)$$

it follows that

$$\sum_{n=1}^N \frac{1}{n^\sigma} \leq 1 + \left(\frac{1}{\sigma-1} \right)$$

The sequence of partial sums

$$\sum_{n=1}^N \frac{1}{n^\sigma}$$

with $\sigma > 1$ is bounded by $1 + \left(\frac{1}{\sigma-1}\right)$ as N increases. Therefore

$$\zeta(\sigma) \equiv \lim_{N \rightarrow \infty} \left\{ \sum_{n=1}^N \frac{1}{n^\sigma} \right\} = \sum_{n=1}^{\infty} \frac{1}{n^\sigma}$$

converges for real $\sigma > 1$.

Case 3: $s = \sigma + i \cdot t$ $\sigma > 1$ t real

Let $f(n) = 1/n^{\sigma+i \cdot t}$ where σ is real and greater than 1. From the previous analysis, $|n^{-s}| = n^{-\sigma}$, and

$$\sum_{n=1}^{\infty} |n^{-s}| = \sum_{n=1}^{\infty} n^{-\sigma}$$

In Case 2 above, it was shown that this series converges for all real $\sigma > 1$. Therefore, the Riemann zeta function converges absolutely for $\sigma > 1$ everywhere in the complex half-plane. Since the Riemann zeta function converges absolutely for all $\sigma > 1$, it also converges on a non-absolute basis everywhere in the complex half-plane where $\sigma > 1$.

Case 4: the critical strip $s = \sigma + i \cdot t$ $0 < \sigma < 1$

Let $f(n) = 1/n^\sigma$ where $0 < \sigma < 1$. The function $f(x) = 1/x^\sigma$ is a decreasing function of x . Then

$$\int_1^{N+1} \frac{dx}{x^\sigma} \leq \sum_{n=1}^N \frac{1}{n^\sigma} \leq 1 + \int_1^N \frac{dx}{x^\sigma}$$

and

$$\left(\frac{1}{1-\sigma}\right) \cdot [(N+1)^{1-\sigma} - 1] \leq \sum_{n=1}^N \frac{1}{n^\sigma} \leq 1 + \left(\frac{1}{1-\sigma}\right) \cdot (N^{1-\sigma} - 1)$$

In the critical strip, $\frac{1}{1-\sigma} > 0$, and the quantities $[(N+1)^{1-\sigma} - 1]$ and $N^{1-\sigma} - 1$ are positive and diverge as $N \rightarrow \infty$. Therefore, the Riemann zeta function diverges absolutely everywhere in the critical strip. Although the function diverges absolutely, the analysis tells us nothing as to whether the function itself converges or diverges on a non-absolute basis. In summary, the Riemann zeta function may either converge or diverge in the critical strip. The analysis is indeterminate as to convergence or divergence.

Therefore, the most important question of the hypothesis has always been and remains: Exactly how does the series representation of the Riemann zeta function converge to zero at its roots in the critical strip?

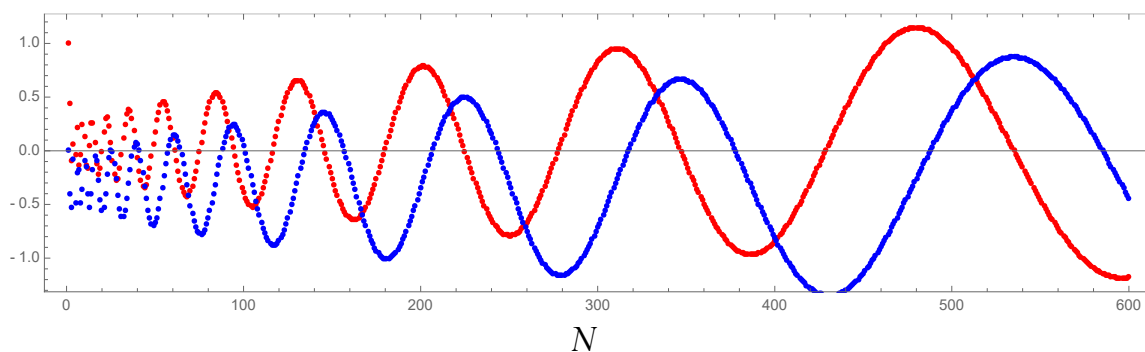
It is perhaps easy to understand why one could mistakenly conclude that the infinite series representation of the Riemann zeta function, or

$$\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s} = \operatorname{Re} \left\{ \sum_{n=1}^{\infty} n^{-s} \right\} + i \cdot \operatorname{Im} \left\{ \sum_{n=1}^{\infty} n^{-s} \right\}$$

diverges everywhere in the critical strip. The following graph shows the real and imaginary parts of the partial sums of the function:

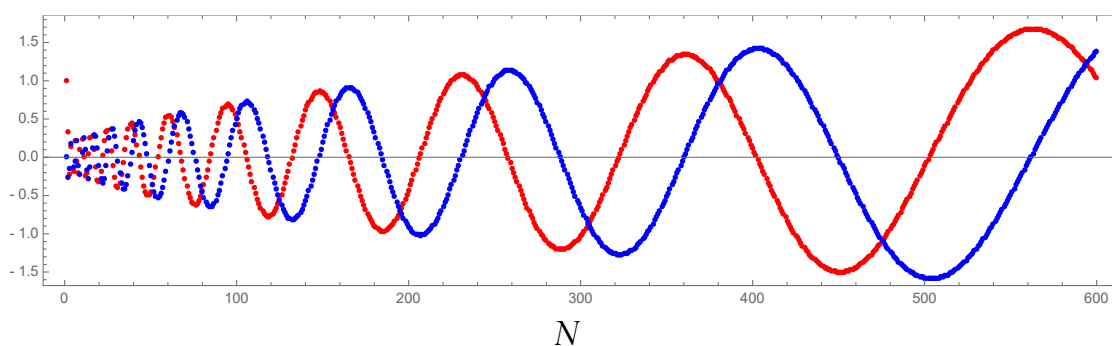
$$\operatorname{Re} \left\{ \sum_{n=1}^N n^{-s} \right\} \quad \text{and} \quad \operatorname{Im} \left\{ \sum_{n=1}^N n^{-s} \right\}$$

at an arbitrary point in the critical strip, $s = 0.55 + 14.50 \cdot i$, - a point known to not be a root of the Riemann zeta function, for $N = 1$ to $N = 600$:



The red points in the graph above are the real parts of the partial sums of the zeta function and the blue points are the imaginary parts of the partial sums of the zeta function. Both sums diverge as $N \rightarrow \infty$. The zeta function appears to diverge and, in fact, that is correct.

The following graph shows the real and imaginary parts of the partial sums of the zeta function at a known root in the critical strip: $s = \frac{1}{2} + 14.134725 \dots \cdot i$



As before, both sums appear to diverge as $N \rightarrow \infty$, so that it could be assumed that the zeta function would also diverge. That assumption is incorrect. In fact, the zeta function converges to zero at the point.

In these examples, the real and imaginary parts of the partial sums of the Riemann zeta function diverge as $N \rightarrow \infty$. In fact, all real and imaginary parts of the zeta function diverge

as $N \rightarrow \infty$ everywhere in the critical strip. Based on this observation, it may appear that the series representation of the Riemann zeta function also diverges everywhere in the critical strip and therefore the series representation is inapplicable for a resolution of the hypothesis. That assumption is wrong, and that is where characterizations of the infinite series representation of the zeta function in the critical strip and the Riemann hypothesis have remained for 165 years – and with essentially no further progress.

Riemann and virtually everyone else who has studied the hypothesis have assumed that the infinite series representation of the zeta function diverges everywhere in the critical strip, so logically, they looked to other representations of the zeta function to locate its roots.

Riemann derived an analytic continuation of the zeta function valid everywhere in the critical strip:

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \cdot \oint_C \frac{(-z)^{s-1}}{e^z - 1} dz$$

where the branch line lies along the positive real z axis, and contour C comes from $+\infty$ just below the branch line, encircles the branch point at $z = 0$ in clockwise fashion and returns to $+\infty$ just above the branch line. The poles of the gamma function at $s = 2, 3, 4, \dots$ are canceled by the zeros of the contour integral so that the formula is well-defined for all values of s , except for the singularity at $s = 1$.

Riemann also derived the functional equation of the zeta function in the complex plane:

$$\zeta(s) = 2^s \cdot \pi^{s-1} \cdot \sin\left(\frac{\pi s}{2}\right) \cdot \Gamma(1-s) \cdot \zeta(1-s)$$

which is, in effect, an extension by analytic continuation of the function via translation of the argument to the critical strip. The equation is valid everywhere in the complex plane except for $s = 1$.

Another analytic continuation of the zeta function, derived by summation by parts and valid in the half-plane $\sigma > 0$, including the critical strip - except for $s = 1$, is:

$$\zeta(s) = \frac{s}{s-1} - s \cdot \int_1^{\infty} (x - [x]) \cdot x^{-s-1} dx$$

where $[x]$ is the integral floor of quantity x .

Other analytic continuations of the zeta function to the critical strip include:

$$\zeta(s) = (1 - 2^{1-s})^{-1} \cdot \eta(s)$$

where

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot n^{-s} = 1 - 2^{-s} + 3^{-s} - 4^{-s} + \dots$$

is the so-called eta function, or alternating Riemann zeta function, and the combinatoric formulae derived by Knopp and Haase:

$$\zeta(s) = \left(\frac{1}{1 - 2^{1-s}} \right) \cdot \sum_{n=0}^{\infty} \frac{1}{2^{n-1}} \sum_{k=0}^n \binom{n}{k} \cdot \frac{(-1)^k}{(k+1)^s}$$

and Haase:

$$\zeta(s) = \left(\frac{1}{s-1} \right) \cdot \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n \binom{n}{k} \cdot \frac{(-1)^k}{(k+1)^{s-1}}$$

All these formulae are valid everywhere in the complex plane except for $s = 1$.

Other analytic continuations of the Riemann zeta function are discussed in the literature.

Although there exist many analytic continuations of the zeta function that are valid everywhere in the critical strip, except for $s = 1$, none have proved useful in locating all the roots of the Riemann zeta function and thereby resolving the Riemann hypothesis.

In this work, we take another look at the infinite series representation of the zeta function:

$$\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s} = \operatorname{Re} \left\{ \sum_{n=1}^{\infty} n^{-s} \right\} + i \cdot \operatorname{Im} \left\{ \sum_{n=1}^{\infty} n^{-s} \right\}$$

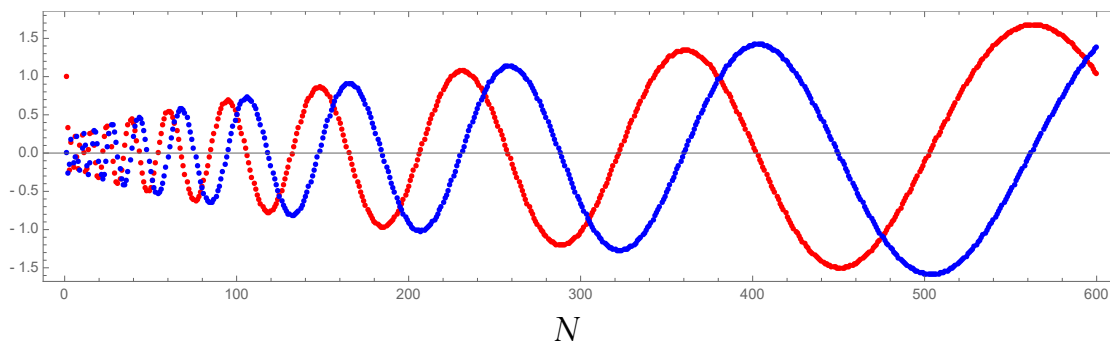
and the real and imaginary parts of its partial sums:

$$\operatorname{Re} \left\{ \sum_{n=1}^N n^{-s} \right\} \quad \text{and} \quad \operatorname{Im} \left\{ \sum_{n=1}^N n^{-s} \right\}$$

to locate the roots of the function in the critical strip.

Question #1

What if the apparently divergent partial sums shown in the graph below for a known root of the Riemann zeta function (and previously discussed) diverge to zero in a “summable” sense?

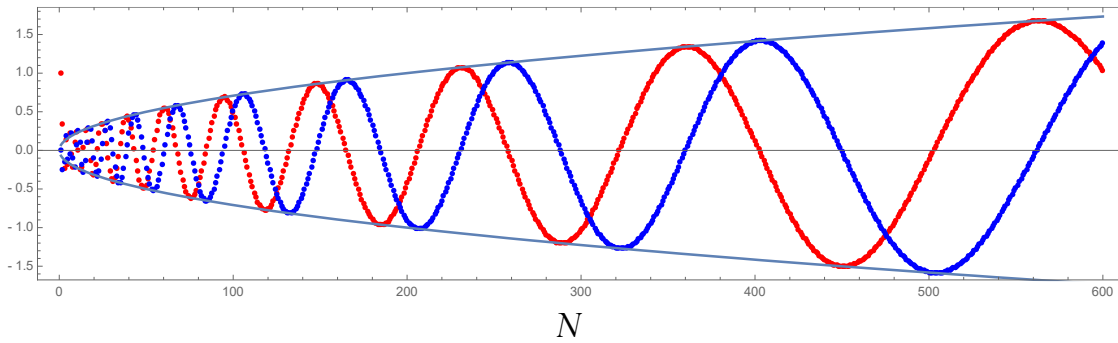


Question #2

What if the envelope defined by

$$\pm \frac{N^\sigma}{t} \quad \text{where} \quad \sigma = 1/2$$

can be superimposed on the graph above so that:



where the real and imaginary parts of the function are “summable” and equal to zero as $N \rightarrow \infty$, where

$$\lim_{N \rightarrow \infty} \left\{ \operatorname{Re} \left[\sum_{n=1}^N n^{-s} \right] \right\} = \operatorname{Re} \left[\sum_{n=1}^{\infty} n^{-s} \right] = \operatorname{Re}\{\zeta(s)\} = 0 \quad \text{summable}$$

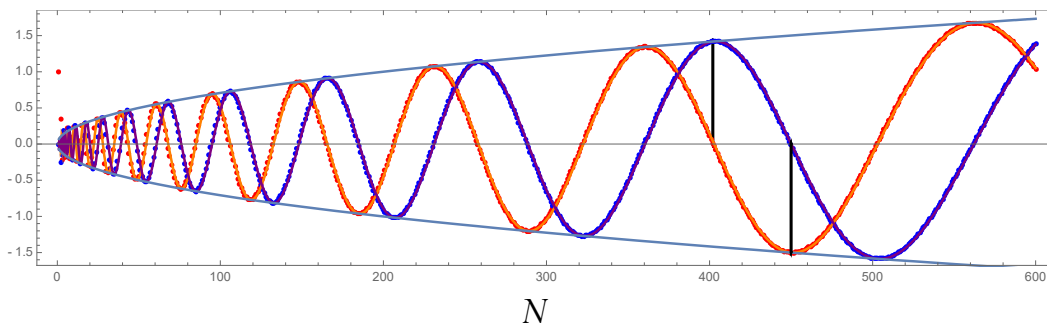
and

$$\lim_{N \rightarrow \infty} \left\{ \operatorname{Im} \left[\sum_{n=1}^N n^{-s} \right] \right\} = \operatorname{Im} \left[\sum_{n=1}^{\infty} n^{-s} \right] = \operatorname{Im}\{\zeta(s)\} = 0 \quad \text{summable}$$

and where the local minima and maxima of the partial sums of the function are coincident with the envelope for defined values of integer N generally greater than an order of magnitude 10^2 , depending on the value of the imaginary part of the argument t at the roots?

Question #3

What if the same root of the zeta function in the critical strip can only be represented by:



where $s = \frac{1}{2} + 14.134725 \dots \cdot i$ and the two vertical black lines relate the local minima and maxima of the partial sums of the zeta function to the envelope where (in this example)

$$\operatorname{Re} \left[\sum_{n=1}^{\left\lceil e^{\frac{(m-1)\pi}{t}} \right\rceil} n^{-s} \right] \sim 0 \quad \text{and} \quad \operatorname{Im} \left[\sum_{n=1}^{\left\lceil e^{\frac{(2m-1)\pi}{2t}} \right\rceil} n^{-s} \right] \sim + \frac{\left\lceil e^{\frac{(m-1)\pi}{t}} \right\rceil^{\sigma}}{t}$$

and

$$Im \left[\sum_{n=1}^{\left\lfloor e^{\frac{(2m-1)\pi}{2t}} \right\rfloor} n^{-s} \right] \sim 0 \quad \text{and} \quad Re \left[\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\pi}{t}} \right\rfloor} n^{-s} \right] \sim - \frac{\left\lfloor e^{\frac{(2m-1)\pi}{2t}} \right\rfloor^\sigma}{t}$$

with $\left\lfloor e^{\frac{(m-1)\pi}{t}} \right\rfloor = 403$ and $\left\lfloor e^{\frac{(2m-1)\pi}{2t}} \right\rfloor = 451$, and the sufficiently large but otherwise arbitrary value of $m = 25$.

Question #4

What if all roots of the Riemann zeta function in the critical strip can be similarly represented only when $\sigma = 1/2$?

Important Point

It is not possible using any known classical, or formal method to sum the divergent real and imaginary parts of the Riemann zeta function, so that:

$$Re \left[\sum_{n=1}^{\infty} n^{-s} \right] = 0 \quad \text{and} \quad Im \left[\sum_{n=1}^{\infty} n^{-s} \right] = 0$$

and thereby resolve the Riemann hypothesis.

Therefore, in this work, the real and imaginary parts of the partial sums of the zeta function are related to integrals such that:

$$Re \left[\sum_{n=1}^N n^{-s} \right] \sim Re \left[\int_0^N x^{-s} dx \right]$$

and

$$Im \left[\sum_{n=1}^N n^{-s} \right] \sim Im \left[\int_0^N x^{-s} dx \right]$$

for arbitrarily large values of $N = 1, 2, 3, \dots$, or equivalently,

$$Re \left[\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\pi}{t}} \right\rfloor} n^{-s} \right] \sim Re \left[\int_0^{\left\lfloor e^{\frac{(m-1)\pi}{t}} \right\rfloor} x^{-s} dx \right]$$

and

$$Im \left[\sum_{n=1}^{\left\lfloor e^{\frac{(2m-1)\pi}{2t}} \right\rfloor} n^{-s} \right] \sim Im \left[\int_0^{\left\lfloor e^{\frac{(2m-1)\pi}{2t}} \right\rfloor} x^{-s} dx \right]$$

for arbitrarily large values of $m = 1, 2, 3, \dots$

In this work, the partial sums in the second pair of relationships above are represented as bi-lateral integral transforms. The integrals in the relationships are represented by functions that are proportional to $[e^{(m-1)\pi/t}]^{1-\sigma}$ and $[e^{(2\cdot m-1)\pi/(2\cdot t)}]^{1-\sigma}$, respectively. Since the partial sums and integrals are asymptotic at the roots of the Riemann zeta function in the critical strip, and the limiting ratio of the integrals is $e^{\pi\cdot(1-\sigma)/(2\cdot t)}$ for arbitrarily large values of integer m , it follows that the ratio of bi-lateral integral transforms is asymptotically proportional to $e^{\pi\cdot(1-\sigma)/(2\cdot t)}$. By separating the bi-lateral transforms into their real and imaginary components, it is shown that bi-lateral sine and cosine integral transforms vanish simultaneously at the roots of the Riemann zeta function in the critical strip. And in fact, the integral transforms vanish if and only if the functions in the integrands most closely approximate even functions of the translated variables of integration. Furthermore, it is shown that the two translated functions most closely approximate even functions only when $\sigma = 1/2$. The integral transforms vanish and the roots of Riemann zeta function occur only when $\sigma = 1/2$ and the transform kernels $\sin(t \cdot x)$ and $\cos(t \cdot x)$ exhibit roots at the maxima or minima of the functions in the transforms. Only when these conditions are simultaneously satisfied do the roots of the zeta function occur in the critical strip. Thus, thus the roots of the Riemann zeta function in the critical strip all have real part equal to $1/2$ and the Riemann hypothesis is correct.

For an answer to the question at the title of this section, “Why – Until Now – Was the Hypothesis An Open Problem”, consider the following:

1. Classical analysis tells us nothing definitive as to whether the infinite series representation of the Riemann zeta function converges anywhere in the critical strip.
2. Simple graphical analyses of the real and imaginary parts of partial sums of the Riemann zeta function are potentially misleading. At first glance, the sums appear to diverge as $N \rightarrow \infty$ everywhere in the critical strip. However, the sums are “summable” at the roots of the function in the sense of the asymptotic relationships:

$$Re \left[\sum_{n=1}^N n^{-s} \right] \sim Re \left[\int_0^N x^{-s} dx \right]$$

and

$$Im \left[\sum_{n=1}^N n^{-s} \right] \sim Im \left[\int_0^N x^{-s} dx \right]$$

for arbitrarily large values of $N = 1, 2, 3, \dots$, or

$$Re \left[\sum_{n=1}^{\left\lceil e^{\frac{(m-1)\pi}{t}} \right\rceil} n^{-s} \right] \sim Re \left[\int_0^{\left\lceil e^{\frac{(m-1)\pi}{t}} \right\rceil} x^{-s} dx \right]$$

and

$$Im \left[\sum_{n=1}^{\left\lceil e^{\frac{(2m-1)\pi}{2t}} \right\rceil} n^{-s} \right] \sim Im \left[\int_0^{\left\lceil e^{\frac{(2m-1)\pi}{2t}} \right\rceil} x^{-s} dx \right]$$

for arbitrarily large values of $m = 1, 2, 3, \dots$, at the roots of the Riemann zeta function in the critical strip.

3. These asymptotic relationships are represented using bi-lateral integral transforms where:

$$Re \left\{ \sum_{n=1}^N n^{-s} \right\} = \int_{-\infty}^{\infty} Re \left\{ \frac{e^{-s \cdot x}}{\Gamma(s)} \right\} \cdot \left(\frac{1 - e^{e^{-N \cdot x}}}{e^{e^{-x}} - 1} \right) dx$$

and

$$Im \left\{ \sum_{n=1}^N n^{-s} \right\} = \int_{-\infty}^{\infty} Im \left\{ \frac{e^{-s \cdot x}}{\Gamma(s)} \right\} \cdot \left(\frac{1 - e^{e^{-N \cdot x}}}{e^{e^{-x}} - 1} \right) dx$$

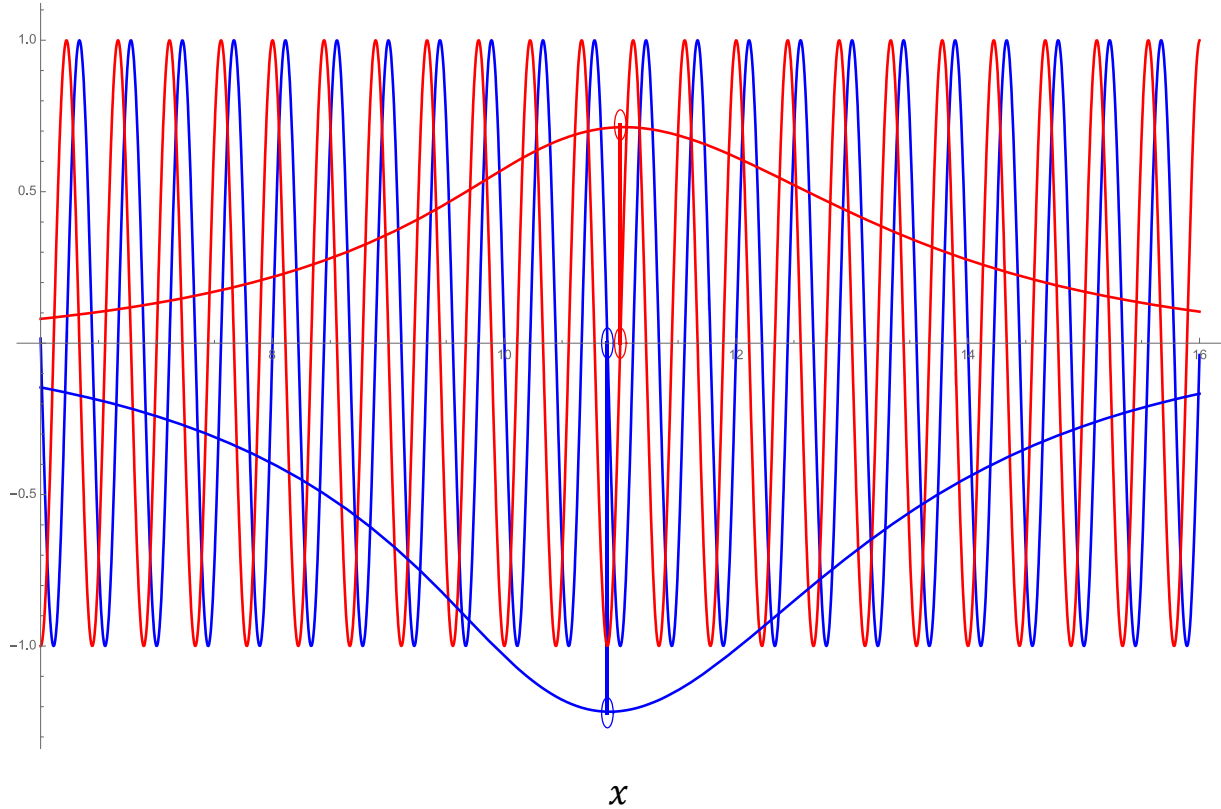
and the well-known property of bi-lateral sine integral transforms:

$$\int_{-\infty}^{\infty} \sin(x) \cdot f_{even}(x) dx = 0$$

4. The integral transforms vanish and the roots of Riemann zeta function occur in the critical strip when $\sigma = \frac{1}{2}$ and when the transform kernel $\sin(t \cdot x)$ exhibits roots at the maxima and/or minima of the functions in the integrands of the transforms:

5. Consider:

a root of the Riemann zeta function with $\sigma = 1/2$, $t = 14.1347$, and $m = 51$ (*arbitrary*)
 (non-sinusoidal functions are scaled by $\pm 1 \times 10^7$)



Notes:

Blue Vertical Line

$$x \approx 10.8908 = \frac{(m-2) \cdot \pi}{t} \Rightarrow t \approx \frac{(51-2) \cdot \pi}{10.8908} \approx 14.1347 \text{ (approximately)}$$

Red Vertical Line

$$x \approx 11.0019 = \frac{(2 \cdot m - 3) \cdot \pi}{2 \cdot t} \Rightarrow t \approx \frac{(2 \cdot 51 - 3) \cdot \pi}{2 \cdot 11.0019} \approx 14.1347 \text{ (approximately)}$$

The value of t from the literature for the root of the Riemann zeta function is approximately 14.134725...

It is important to note that the functions in the graphs above with single maxima and minima are scaled by a factor of $\pm 1 \times 10^7$, whereas the sinusoidal functions in the graphs are unscaled.

6.

There are several reasons why a resolution of the Riemann hypothesis has been so elusive. The approach taken in this work provides a completely new methodology and a resolution of the problem.