Why - Until Now - Was The Hypothesis An Open Problem?

The discussion in this section is not intended to be comprehensive. However, it is intended to elucidate the behavior of the Riemann zeta function on the complex plane and to mention a few of the formulae that have been used over the years, albeit unsuccessfully, to resolve the hypothesis. In addition, a new approach to the hypothesis is introduced with some visual aids to help understand why the new approach is successful.

It's useful first to examine the zeta function and its regions of convergence on the complex plane with classical methods.

Note that since $n^{-s} = n^{-(\sigma+i\cdot t)} = n^{-\sigma} \cdot e^{-i\cdot t\cdot log(n)}$, and since $|\, n^{-\sigma}| = n^{-\sigma}$ and $|e^{-i\cdot\theta}| = 1$ for all real values of θ, it follows that $|e^{-i\cdot t\cdot log(n)}| = 1$, and therefore,

$$|\, n^{-s}| = \left| n^{-(\sigma+i\cdot t)} \right| = |\, n^{-\sigma}| \cdot \left|e^{-i\cdot t\cdot log(n)}\right| = n^{-\sigma}$$

Also note that, for decreasing real functions f,

$$f(n+1) \;\le\; \int_n^{n+1} f(x)\, dx \;\le\; f(n)$$

Consider the first inequality in this relationship and sum both sides over $n = 1, 2, 3, \dots, N-1$ to get

$$\sum_{n=1}^{N-1} f\{n+1) \;\le\; \sum_{n=1}^{N-1} \int_n^{n+1} f(x)\, dx$$

Exchanging the order of summation and integration gives

$$\sum_{n=1}^{N} f(n) \;\le\; f(1) \;+\; \int_1^N f(x)\, dx$$

Similarly, consider the second inequality in the relationship and sum both sides over $n = 1, 2, 3, \dots, N$ to get

$$\sum_{n=1}^{N} \int_n^{n+1} f(x)\, dx \;\le\; \sum_{n=1}^{N} f(n)$$

Exchanging the order of summation and integration gives

$$\int_1^{N+1} f(x)\, dx \;\le\; \sum_{n=1}^{N} f(n)$$

Combining these two results gives

$$\int_1^{N+1} f(x)\, dx \;\le\; \sum_{n=1}^{N} f(n) \;\le\; f(1) \;+\; \int_1^N f(x)\, dx$$

Now let's use these results to examine the complex plane for convergence or divergence of the Riemann zeta function.

Case 1: $s = 1$

Let $f(n) = 1/n$. The function $1/n$ is a decreasing function of n, and by the preceding result,

$$\int_1^{N+1} \frac{dx}{x} \le \sum_{n=1}^{N} \frac{1}{n} \le 1 + \int_1^{N} \frac{dx}{x}$$

or

$$Log(N+1) \le \sum_{n=1}^{N} \frac{1}{n} \le 1 + Log(N)$$

so that

$$\sum_{n=1}^{N} \frac{1}{n} \ge Log(N+1)$$

As $N \to \infty$, $Log(N+1)$ diverges, and therefore,

$$\zeta(1) \equiv \lim_{N\to\infty} \left\{ \sum_{n=1}^{N} \frac{1}{n} \right\}$$

diverges.

Case 2: $s = \sigma > 1$

Let $f(n) = 1/n^{\sigma}$. The function $1/n^{\sigma}$ is a decreasing function of n, and since

$$\int_1^{N+1} \frac{dx}{x^{\sigma}}\, dx \le \sum_{n=1}^{N} \frac{1}{n^{\sigma}} \le 1 + \int_1^{N} \frac{dx}{x^{\sigma}}$$

or

$$\left(\frac{1}{1-\sigma}\right) \cdot [(N+1)^{1-\sigma} - 1] \le \sum_{n=1}^{N} \frac{1}{n^{\sigma}} \le 1 + \left(\frac{1}{1-\sigma}\right) \cdot (N^{1-\sigma} - 1)$$

and

$$\sum_{n=1}^{N} \frac{1}{n^{\sigma}} \le 1 + \left(\frac{1}{1-\sigma}\right) \cdot (N^{1-\sigma} - 1) \le 1 + \left(\frac{1}{\sigma - 1}\right)$$

it follows that

$$\sum_{n=1}^{N} \frac{1}{n^{\sigma}} \le 1 + \left(\frac{1}{\sigma - 1}\right)$$

The sequence of partial sums

$$\sum_{n=1}^{N} \frac{1}{n^{\sigma}}$$

with $\sigma > 1$ is bounded by $1 + \left(\frac{1}{\sigma-1}\right)$ as N increases. Therefore

$$\zeta(\sigma) \equiv \lim_{N\to\infty}\left\{\sum_{n=1}^{N}\frac{1}{n^{\sigma}}\right\} \qquad \sigma > 1$$

converges.

Case 3: $s = \sigma + i \cdot t \quad \sigma > 1$

Note from the previous analysis that $|n^{-s}| = n^{-\sigma}$, so that

$$\sum_{n=1}^{\infty} |n^{-s}| = \sum_{n=1}^{\infty} n^{-\sigma}$$

In Case 2 above, it was shown that the series on the right-hand side converges for all $\sigma > 1$. Therefore, the Riemann zeta function converges absolutely for $\sigma > 1$ everywhere in the half-plane. Since the Riemann zeta function converges absolutely for all $\sigma > 1$, it also converges on a non-absolute basis everywhere in the half-plane where $\sigma > 1$.

Case 4: The Critical Strip, $s = \sigma + \mathrm{i} \cdot t \quad 0 < \sigma < 1$

Similarly, let $f(n) = 1/n^{\sigma}$ where the function $f(n)$ is a decreasing function of n, with $0 < \sigma < 1$. Then

$$\int_{1}^{N+1} \frac{dx}{x^{\sigma}}\, dx \le \sum_{n=1}^{N}\frac{1}{n^{\sigma}} \le 1 + \int_{1}^{N}\frac{dx}{x^{\sigma}}$$

or

$$\left(\frac{1}{1-\sigma}\right)\cdot\left[(N+1)^{1-\sigma} - 1\right] \le \sum_{n=1}^{N}\frac{1}{n^{\sigma}} \le 1 + \left(\frac{1}{1-\sigma}\right)\cdot\left(N^{1-\sigma} - 1\right)$$

In the critical strip, $\frac{1}{1-\sigma} > 0$, and the quantities $[(N+1)^{1-\sigma} - 1]$ and $N^{1-\sigma} - 1$ are positive and diverge as $N \to \infty$. Therefore, the Riemann zeta function diverges absolutely everywhere in the critical strip. Although the function diverges absolutely, the analysis tells us nothing as to whether the function itself converges or diverges. In summary, the Riemann zeta function may either converge or diverge in the critical strip. The analysis using the classical methods is indeterminate.

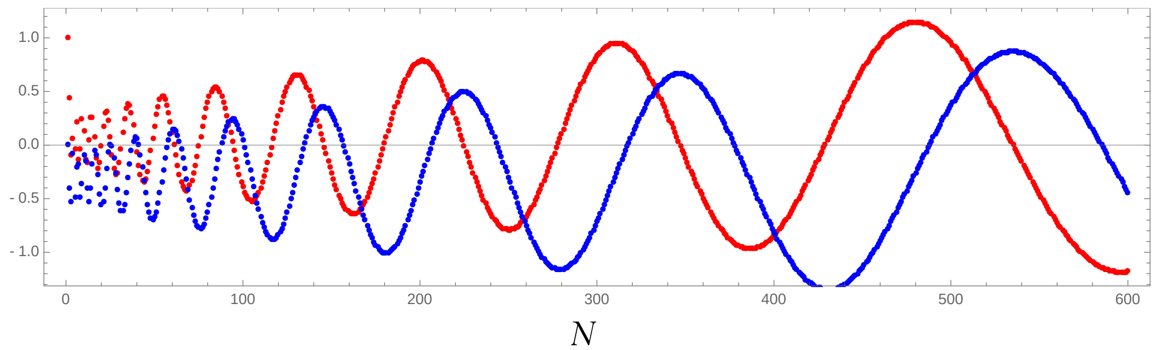
Furthermore, it is easy to understand from simple graphical methods why one could mistakenly conclude that the infinite series representation of the Riemann zeta function:

$$\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s} = Re\left\{\sum_{n=1}^{\infty} n^{-s}\right\} + i \cdot Im\left\{\sum_{n=1}^{\infty} n^{-s}\right\}$$

diverges everywhere in the critical strip. The following graph shows the real and imaginary parts of the partial sums of the function:

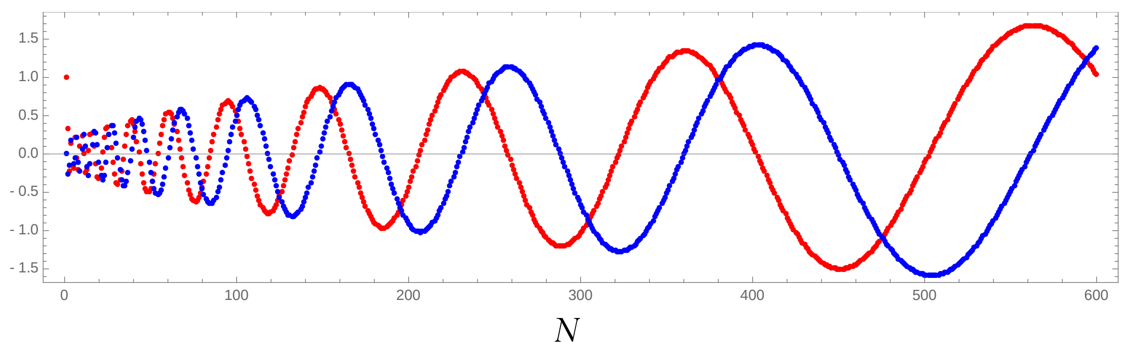
$$Re\left\{\sum_{n=1}^{N} n^{-s}\right\} \quad and \quad Im\left\{\sum_{n=1}^{N} n^{-s}\right\}$$

at the point in the critical strip $s = 0.55 + 14.50 \cdot i$, which is a point known not to be a root of the Riemann zeta function, for $N = 1$ to $N = 600$:



The red points are the real parts of the partial sums and the blue points are the imaginary parts of the partial sums. Both sums diverge as $N \to \infty$, and therefore, the zeta function would also appear to diverge.

The following graph shows the real and imaginary parts of the partial sums at a known root of the zeta function on the line $\sigma = 1/2$ ($s = \frac{1}{2} + 14.134725 \ldots \cdot i$):



As before, it appears that both sums diverge as $N \to \infty$, so evidently (but mistakenly), it would appear that the zeta function also diverges.

In these examples, the real and imaginary parts of the partial sums of the Riemann zeta function diverge as $N \to \infty$. In fact, the real and imaginary parts of the partial sums diverge as $N \to \infty$ everywhere in the critical strip. By this observation, it would appear that the series representation of the Riemann zeta function diverges everywhere in the critical strip and therefore is inapplicable for a potential resolution of the hypothesis. That is wrong, but that is where this aspect of the problem has remained for 165 years.

Riemann and virtually everyone else who has studied the hypothesis has assumed that the series representation of the zeta function diverges everywhere in the critical strip, so naturally, they have looked to other representations of the function to locate its roots.

Riemann derived an analytic continuation of the zeta function valid everywhere in the critical strip:

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \cdot \oint_C \frac{(-z)^{s-1}}{e^z - 1}\, dz$$

where the branch line lies along the positive real z axis, and contour C comes from $+\infty$ just below the branch line, encircles the branch point at $z = 0$ in clock-wise fashion and returns to $+\infty$ just above the branch line. The poles of the gamma function at $s = 2, 3, 4, \ldots$ are canceled by the zeros of the contour integral so that the formula is well-defined for all values of s, except for the singularity at $s = 1$.

Riemann also derived the functional equation of the zeta function in the complex plane:

$$\zeta(s) = 2^s \cdot \pi^{s-1} \cdot \sin\left(\frac{\pi s}{2}\right) \cdot \Gamma(1-s) \cdot \zeta(1-s)$$

which is, in effect, an extension by analytic continuation of the function via translation of the argument to the critical strip. The equation is valid everywhere in the complex plane except for $s = 1$.

Another analytic continuation of the zeta function, derived by summation by parts and valid in the half-plane $\sigma > 0$, including the critical strip except for $s = 1$, is:

$$\zeta(s) = \frac{s}{s-1} - s \cdot \int_1^\infty (x - \lfloor x \rfloor) \cdot x^{-s-1}\, dx$$

where $\lfloor x \rfloor$ is the integral floor of quantity x.

Other analytic continuations of the zeta function to the critical strip include:

$$\zeta(s) = (1 - 2^{1-s})^{-1} \cdot \eta(s)$$

where

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot n^{-s} = 1 - 2^{-s} + 3^{-s} - 4^{-s} + \cdots$$

is the so-called eta function, and the combinatoric formulae derived by Knopp and Haase:

$$\zeta(s) = \left(\frac{1}{1 - 2^{1-s}}\right) \cdot \sum_{n=0}^{\infty} \frac{1}{2^{n-1}} \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{(-1)^k}{(k+1)^s}$$

and by Haase:

$$\zeta(s) = \left(\frac{1}{s-1}\right) \cdot \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} \cdot \frac{(-1)^k}{(k+1)^{s-1}}$$

These formulae are valid everywhere in the complex plane except for $s = 1$.

Other analytic continuations of the Riemann zeta function are discussed in the literature.

Although there are many analytic continuations of the zeta function that are valid in the critical strip, none have proved useful in locating all of the roots of the Riemann zeta function and thereby resolving the Riemann hypothesis.

In this work, we take another look at the series representation of the zeta function:

$$\zeta(s) \equiv \sum_{n=1}^{\infty} n^{-s} = Re\left\{\sum_{n=1}^{\infty} n^{-s}\right\} + i \cdot Im\left\{\sum_{n=1}^{\infty} n^{-s}\right\}$$

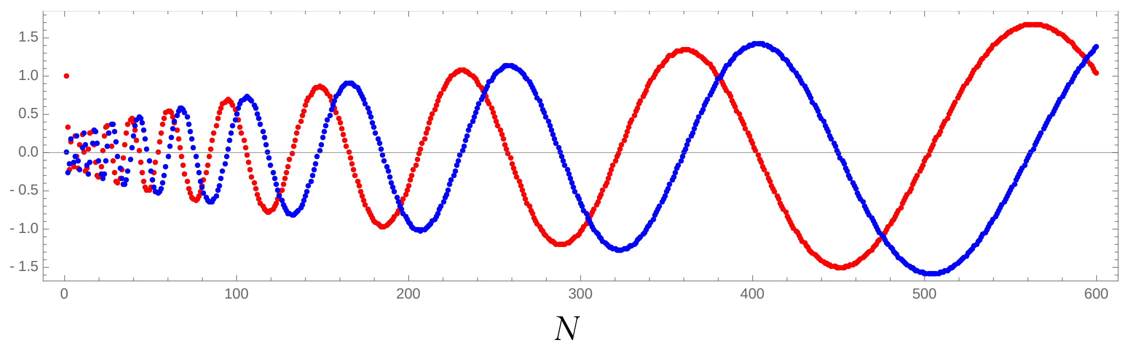
and the real and imaginary parts of its partial sums:

$$Re\left\{\sum_{n=1}^{N} n^{-s}\right\} \quad and \quad Im\left\{\sum_{n=1}^{N} n^{-s}\right\}$$

to locate the roots of the function in the critical strip.

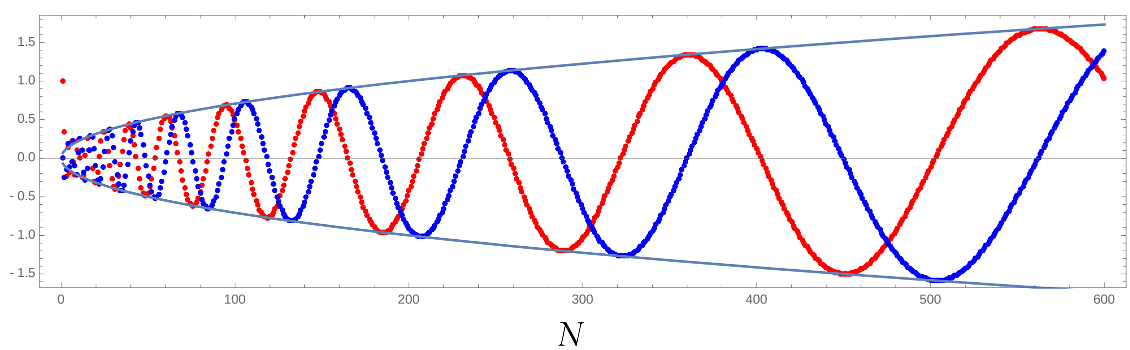
Question 1

What if the apparently divergent partial sums shown in the graph below for a known root of the Riemann zeta function (and shown above) diverge to zero in a "summable" sense?



Question 2

What if an envelope can be superimposed on the graph above so that:



where the real and imaginary parts of the function are "summable" and equal to zero:

$$Re\{\zeta(s)\} = \lim_{N\to\infty}\left\{Re\left[\sum_{n=1}^{N} n^{-s}\right]\right\} \rightarrow 0 \quad summable$$

and

$$Im\{\zeta(s)\} = \lim_{N\to\infty}\left\{Im\left[\sum_{n=1}^{N} n^{-s}\right]\right\} \rightarrow 0 \quad summable$$

and where the local minima and maxima of the partial sums of the function are coincident with the envelope defined by

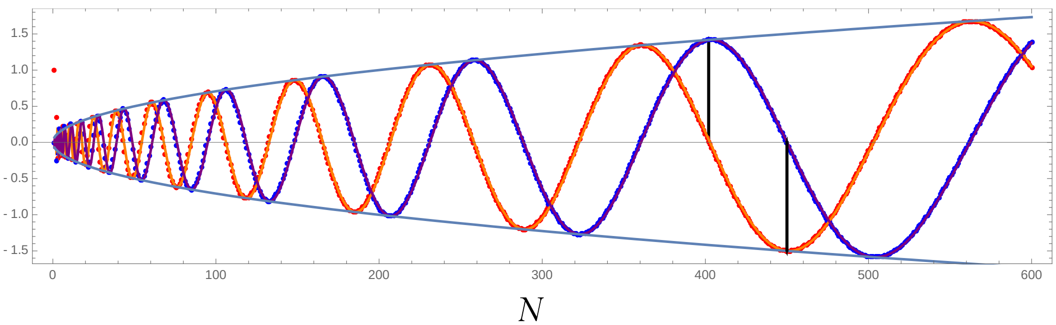
$$\pm\frac{N^{\sigma}}{t} \quad where \quad \sigma = 1/2$$

for large values of integer N (generally greater than 10^2 and dependent on the value of t at the roots)?

And furthermore, what if all roots of the Riemann zeta function in the critical strip can be represented similarly where $\sigma = 1/2$?

<u>Question 3</u>

What if the same root of the zeta function in the critical strip can also be represented by:



where $s = \frac{1}{2} + 14.134725\ldots \cdot i$ and the two vertical black lines relate the local minima and maxima of the partial sums of the zeta function to the envelope so that (in this example)

$$Re\left[\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} n^{-s}\right] \sim 0 \quad and \quad Im\left[\sum_{n=1}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} n^{-s}\right] \sim +\frac{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor^{\sigma}}{t}$$

and

$$Im\left[\sum_{n=1}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} n^{-s}\right] \sim 0 \quad and \quad Re\left[\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} n^{-s}\right] \sim -\frac{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor^{\sigma}}{t}$$

with $\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor = 403$ and $\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor = 451$, and the sufficiently large but otherwise arbitrary value of $m = 25$.

<u>Important Point</u>

It is not possible by any known method to sum the divergent real and imaginary parts of the Riemann zeta function, so that:

$$Re\left[\sum_{n=1}^{\infty} n^{-s}\right] \sim 0 \quad and \quad Im\left[\sum_{n=1}^{\infty} n^{-s}\right] \sim 0$$

and thereby resolve the Riemann hypothesis.

Therefore, in this work, the real and imaginary parts of the partial sums of the function are related to integrals where:

$$Re\left[\sum_{n=1}^{N} n^{-s}\right] \sim Re\left[\int_{0}^{N} x^{-s}\,dx\right]$$

and

$$Im\left[\sum_{n=1}^{N} n^{-s}\right] \sim Im\left[\int_{0}^{N} x^{-s}\,dx\right]$$

for arbitrarily large values of $N = 1, 2, 3, \dots$, or equivalently,

$$Re\left[\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} n^{-s}\right] \sim Re\left[\int_{0}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} x^{-s}\,dx\right]$$

and

$$Im\left[\sum_{n=1}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} n^{-s}\right] \sim Im\left[\int_{0}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} x^{-s}\,dx\right]$$

for arbitrarily large values of $m = 1, 2, 3, \dots$

Note that the second pair of asymptotic relationships above converge much faster than the first pair and are therefore much easier to work with computationally and graphically.

In this work, the partial sums in the second pair of relationships are represented by bi-lateral integral transforms. Furthermore, the integrals in the relationships are represented by functions that are proportional to $\lfloor e^{(m-1)\cdot\pi/t}\rfloor^{1-\sigma}$ and $\lfloor e^{(2\cdot m-1)\cdot\pi/(2\cdot t)}\rfloor^{1-\sigma}$, respectively. Since the partial sums and integrals are asymptotic at the roots of the Riemann zeta function, and the limiting ratio of the integrals is $e^{\pi\cdot(1-\sigma)/(2\cdot t)}$ for arbitrarily large values of integer m, it follows that the ratio of bi-lateral integral transforms is asymptotically proportional to $e^{\pi\cdot(1-\sigma)/(2\cdot t)}$. By separating the bi-lateral transforms into their real and imaginary components, it is shown that bi-lateral sine and cosine integral transforms vanish simultaneously at the roots of the Riemann zeta function in the critical strip. In fact, the integral transforms vanish if and only if the functions in the integrands of the two transforms most closely approximate even functions when translated with respect to the variables of integration. Also, it is shown that the two translated functions most closely approximate even functions if and only if σ = ½. The integral transforms vanish and the roots of Riemann zeta function occur when σ = ½ and when the transform kernels $sin(t\cdot x)$ and $cos(t\cdot x)$ exhibit roots at the maxima and/or minima of the even functions in the integrands of the transforms. Only when these conditions are simultaneously satisfied do the roots of the zeta function occur in the critical strip.

Therefore, roots of the Riemann zeta function in the critical strip all have real part equal to ½ and the Riemann hypothesis is correct.

For an answer to the question at the title of this section, "Why – Until Now – Was the Hypothesis An Open Problem", consider the following:

1. The classical analysis tells us nothing definitive as to where or even if the Riemann zeta function converges in the critical strip.

2. Simple graphical analyses of the real and imaginary parts of partial sums of the Riemann zeta function are misleading. Apparently, the sums diverge. However, the sums are "summable" in the sense of the asymptotic relationships:

$$Re\left[\sum_{n=1}^{N} n^{-s}\right] \sim Re\left[\int_{0}^{N} x^{-s}\,dx\right]$$

and

$$Im\left[\sum_{n=1}^{N} n^{-s}\right] \sim Im\left[\int_{0}^{N} x^{-s}\,dx\right]$$

for arbitrarily large values of $N = 1, 2, 3, \dots$, or

$$Re\left[\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} n^{-s}\right] \sim Re\left[\int_{0}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} x^{-s}\,dx\right]$$

and

$$Im\left[\sum_{n=1}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} n^{-s}\right] \sim Im\left[\int_{0}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} x^{-s}\,dx\right]$$

for arbitrarily large values of $m = 1, 2, 3, \dots$, at the roots of the Riemann zeta function in the critical strip.

3. Solutions of these asymptotic relationships are found using bi-lateral integral transforms where:

$$Re\left\{\sum_{n=1}^{N} n^{-s}\right\} = \int_{-\infty}^{\infty} Re\left\{\frac{e^{-s\cdot x}}{\Gamma(s)}\right\}\cdot\left(\frac{1-e^{e^{-N\cdot x}}}{e^{e^{-x}}-1}\right)dx$$

and

$$Im\left\{\sum_{n=1}^{N} n^{-s}\right\} = \int_{-\infty}^{\infty} Im\left\{\frac{e^{-s\cdot x}}{\Gamma(s)}\right\}\cdot\left(\frac{1-e^{e^{-N\cdot x}}}{e^{e^{-x}}-1}\right)dx$$

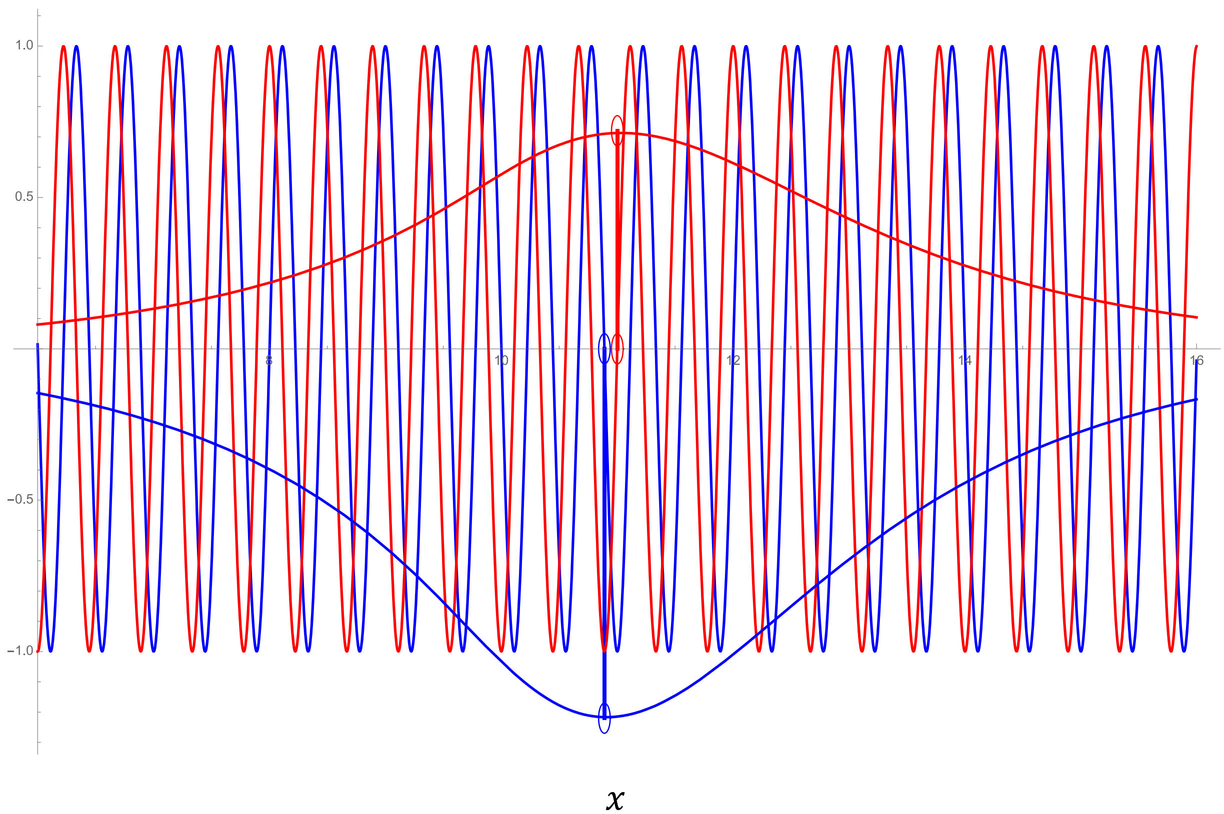
and the well-known property of bi-lateral sine integral transforms:

$$\int_{-\infty}^{\infty} sin(x) \cdot f_{even}(x)\, dx \; = \; 0$$

4. Integral transforms vanish and the roots of Riemann zeta function occur in the critical strip when σ = ½ and when the transform kernel $sin(t \cdot x)$ exhibits roots at the maxima and/or minima of the even functions in the integrands of the transforms:
5.

a root of the Riemann zeta function with $\sigma = 1/2,\ t = 14.1347$, and $m = 51$

(functions are scaled by $\pm 1\text{x}10^{7}$)



Notes:

Blue Vertical Lines

$$x \;\approx\; 10.8908 \;=\; \frac{(m-2)\cdot\pi}{t} \quad\Longrightarrow\quad t \;\approx\; \frac{(51-2)\cdot\pi}{10.8908} \;\approx\; 14.1347$$

Red Vertical Lines

$$x \;\approx\; 11.0019 \;=\; \frac{(2\cdot m-3)\cdot\pi}{2\cdot t} \quad\Longrightarrow\quad t \;\approx\; \frac{(2\cdot 51-3)\cdot\pi}{2\cdot 11.0019} \;\approx\; 14.1347$$

The value of *t* from the literature for the root of the Riemann zeta function is approximately 14.134725...

6. It is important to note that the functions in the graphs above with single maxima and minima are scaled by a factor of $\pm 1 \times 10^7$ whereas the sinusoidal functions in the graphs are unscaled. Without a computer mathematics program such as Mathematica™, it would be virtually impossible to compare these functions conceptually or graphically, and thereby locate roots of the Riemann zeta function in the critical strip.

There are several reasons why a resolution of the Riemann hypothesis has been so elusive. The approach taken in this work considers all the reasons together and provides a completely new methodology and a resolution of the problem.