Borel Integral Summation Method

There are several variations of what is generally referred to as Emil Borel's integral summation method.

Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \cdot (-s)_{(2n-1)}$$

where B_n are the Bernoulli numbers:

$$B_0 = 1,\ B_1 = -1/2,\ B_2 = 1/6,\ B_3 = 0,\ B_4 = -1/30,\ B_5 = 0,\ B_6 = 1/42, \ldots$$

$$B_{2n} = \frac{(-1)^{n+1} \cdot 2 \cdot (2n)!}{(2\pi)^{2n}} \cdot \zeta(2n) \qquad n \geq 0$$

$$B_{2n+1} = 0 \qquad n \geq 1$$

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \cdot \frac{x^n}{n!}$$

and where $\zeta(n)$ is the zeta function for $n = 0, 1, 2, \ldots$, and the polynomials $(s)_{(n)}$ are the binomial, or falling, polynomials:

$$s_{(n)} = s \cdot (s-1) \cdot (s-2) \cdots (s-n+1) \qquad s_{(0)} \equiv 1 \qquad s_{(1)} = s$$

This series diverges everywhere in the critical strip. However, the Borel integral summation method can be used to obtain a closed-form expression representing the series. Note that

$$(-s)_{(2n-1)} = -s^{(2n-1)}$$

where $s^{(n)}$ are the Pochhammer, or rising, polynomials

$$s^{(n)} = s \cdot (s+1) \cdot (s+2) \cdots (s+n-1) \qquad s^{(0)} \equiv 1 \qquad s^{(1)} = s$$

Also note that

$$s^{(n)} = \frac{\Gamma(s+n)}{\Gamma(s)}$$

where

$$\Gamma(s) = \int_0^{\infty} x^{s-1} \cdot e^{-x} dx \qquad \mathrm{Re}(s) > 0$$

is the gamma function. Therefore, $s^{(2n-1)}$ can be represented as the ratio of the Mellin transform of $x^{2n-1} \cdot e^{-x}$ and the gamma function, or

$$s^{(2n-1)} = -\frac{1}{\Gamma(s)} \cdot \int_0^\infty x^{s+2n-2} \cdot e^{-x} dx \qquad \mathrm{Re}(s+2n) > 1$$

Substituting these formulae in the series above gives

$$\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \cdot (-s)_{(2n-1)} = -\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \cdot \frac{1}{\Gamma(s)} \cdot \int_0^\infty x^{s+2n-2} \cdot e^{-x} dx \qquad \mathrm{Re}(s+2n) > 1$$

Reversing the order of summation and integration and gives

$$\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \cdot (-s)_{(2n-1)} = -\frac{1}{\Gamma(s)} \cdot \int_0^\infty x^{s-2} \cdot e^{-x} \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} x^{2n}\, dx$$

Note that

$$\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} x^{2n} = \frac{x}{e^x - 1} + \frac{x}{2} - 1$$

$$\frac{\Gamma(s-1)}{\Gamma(s)} = \frac{1}{\mathrm{s} - 1}$$

and

$$\frac{1}{e^x \cdot (e^x - 1)} = \frac{1}{e^x - 1} - e^{-x}$$

Further note the Mellin transform representation of the Riemann zeta function

$$\zeta(s) = \frac{1}{\Gamma(s)} \cdot \int_0^\infty \frac{x^{s-1}}{e^x - 1} \cdot dx \qquad \mathrm{Re}(s) > 1 \quad \textit{a}\text{nd the roots of } \zeta(s) \text{ in the critical strip}$$

Substituting and simplifying gives

$$\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \cdot (-s)_{(2n-1)} = -\zeta(s) + \frac{1}{\mathrm{s} - 1} + \frac{1}{2} \qquad (\textit{Borel integral summation})$$

In summary, the Borel integral summation method was used to represent the divergent infinite series

$$\sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!} \cdot (-s)_{(2n-1)}$$

in the critical strip with the closed-form measure

$$-\zeta(s) + \frac{1}{\mathrm{s} - 1} + \frac{1}{2}$$

Euler-Maclaurin Summation Formula

The formula known as the Euler-Maclaurin summation formula was derived by Leonhard Euler and Colin Maclaurin in about 1735. The general formula is

$$\sum_{n=1}^{N} f(n) \;=\; \int_{1}^{N} f(x)dx \;+\; \frac{1}{2}\cdot f(1) \;+\; \frac{1}{2}\cdot f(N) \;+\; \sum_{k=1}^{\left\lceil\frac{(q+1)}{2}\right\rceil} \frac{B_{2k}}{(2k)!}\cdot\left[\,f^{(2k-1)}(N) \;-\; f^{(2k-1)}(1)\,\right] \;+\; R_q$$

$$R_q \;=\; \frac{(-1)^q}{(q+1)!}\cdot\int_{1}^{N} f^{(q+1)}(x)\cdot B_{q+1}(x)\,dx \qquad q \geq 0$$

Function $f(x)$ is summable, integrable and continuously differentiable. The quantity R_q is a remainder term, B_k are the Bernoulli numbers, $B_q(x)$ are the Bernoulli polynomials, and $\left\lceil\frac{(q+1)}{2}\right\rceil$ is the ceiling of $\frac{(q+1)}{2}$, or the least integer greater than $\frac{(q+1)}{2}$.

The Euler-Maclaurin summation formula relating partial sums of the Riemann zeta function to integrals of the summand of the zeta function is:

$$\sum_{n=1}^{N} n^{-s} \;=\; \int_{1}^{N} x^{-s}dx \;+\; \frac{1}{2} \;+\; \frac{N^{-s}}{2} \;+\; \sum_{k=1}^{\infty}\frac{B_{2k}}{(2k)!}\cdot\left\{\left[\frac{d^{(2k-1)}}{dx^{(2k-1)}}x^{-s}\right]\Bigg|_{x=N} \;-\; \left[\frac{d^{(2k-1)}}{dx^{(2k-1)}}x^{-s}\right]\Bigg|_{x=1}\right\}$$

where the remainder term has been incorporated in the sum on the right-hand side of the formula.

The derivatives on the right-hand side are

$$\frac{d^{(2k-1)}}{dx^{(2k-1)}}x^{-s} \;=\; (-s)_{(2k-1)}\cdot x^{-s-2k+1}$$

where, as before, $s_{(k)}$ are the binomial polynomials

$$s_{(k)} \;=\; s\cdot(s-1)\cdot(s-2)\cdots(s-k+1) \qquad s_{(0)} \equiv 1, \qquad s_{(1)} = s$$

It follows that

$$\left[\frac{d^{(2k-1)}}{dx^{(2k-1)}}x^{-s}\right]\Bigg|_{x=N} \;=\; (-s)_{(2k-1)}\cdot N^{-s-2k+1}$$

and

$$\left[\frac{d^{(2k-1)}}{dx^{(2k-1)}}x^{-s}\right]\Bigg|_{x=1} \;=\; (-s)_{(2k-1)}$$

Therefore, the sum

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \cdot \left\{ \left[\frac{d^{(2k-1)}}{dx^{(2k-1)}} x^{-s} \right] \Bigg|_{x=N} \right\}$$

is equal to

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \cdot (-s)_{(2k-1)} \cdot N^{-s-2k+1} \;=\; N^{1-s} \cdot \left\{ \frac{B_2}{2!} \cdot (-s) \cdot N^{-2} + \frac{B_4}{4!} \cdot (-s)_{(3)} \cdot N^{-4} + \frac{B_6}{6!} \cdot (-s)_{(5)} \cdot N^{-6} + \cdots \right\}$$

For arbitrarily large values of integer N,

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \cdot (-s)_{(2k-1)} \cdot N^{-s-2k+1} \;\rightarrow\; 0$$

Note that although the series

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \cdot \left\{ \left[\frac{d^{(2k-1)}}{dx^{(2k-1)}} x^{-s} \right] \Bigg|_{x=1} \right\} \;=\; \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \cdot (-s)_{(2k-1)}$$

diverges, the Borel integral summation method gives:

$$\sum_{n=1}^{\infty} \frac{B_{2n}}{(2k)!} \cdot (-s)_{(2n-1)} \;=\; -\,\zeta(s) \;+\; \frac{1}{s-1} \;+\; \frac{1}{2} \qquad (\textit{Borel integral summation})$$

Substituting this sum and

$$\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \cdot (-s)_{(2k-1)} \cdot N^{-s-2k+1} \;\rightarrow\; 0$$

and

$$\int_{1}^{N} x^{-s} dx \;=\; \frac{N^{1-s}}{1-s} \;-\; \frac{1}{1-s}$$

in the Euler-Maclaurin summation formula and simplifying gives the asymptotic relationship relating the Riemann zeta function to its partial sums.

$$\sum_{n=1}^{N} n^{-s} \;\sim\; \frac{N^{1-s}}{1-s} \;+\; \frac{N^{-s}}{2} \;+\; \zeta(s)$$

Cauchy Residue Theorem

Augustin-Louis Cauchy's residue theorem (1831) states that the integral of an analytic function along a closed path in the complex plane is entirely dependent on the behavior of the function at its singularities in the interior of the path.

Real analytic and complex analytic, or holomorphic, functions can be represented locally by convergent power series and are therefore infinitely differentiable. A complex function is holomorphic on region R if and only if it is complex differentiable at every point in region R.

Suppose that complex function $f(z)$ is holomorphic and that γ is a simple, closed curve in the complex plane where $f(z)$ exists. Further suppose that a finite number of points of singularity, or poles, of function $f(z)$ are located inside curve γ.

Poles are classified according to order. A pole of order 1 at point p is called simple if the function $f(z)$ can be written in the form

$$f(z) = \frac{g(z)}{(z-p)^n}$$

and if the smallest value of integer n satisfying this relationship is 1. It is also assumed that complex function $g(z)$ is holomorphic. Note that all the poles of the holomorphic function considered in this work are simple.

The residues of complex function $f(z)$ at its simple poles are defined by

$$Res(f, p_j) = \lim_{z \to p_j} \{(z - p_j) \cdot f(z)\}$$

Cauchy's residue theorem states that the contour integral of a real analytic function or a holomorphic function along a closed path in the complex plane is equal to the constant $2\pi i$ multiplied by the sum of the residues of the function at its singularities inside the path, or

$$\oint_\gamma f(z)\, dz = 2\pi i \cdot \sum_j Res(f, p_j)$$

Consider partial sums of the infinite series representation of the Riemann zeta function. Let $s = \sigma + i \cdot t$ and $z = x + i \cdot y$ where σ, t, x and y are real. The residues of the holomorphic function

$$\frac{\pi \cdot \cot(\pi \cdot z)}{z^s}$$

at its simple poles, p_j, are given by

$$\lim_{z\to p_j}\left\{(z-p_j)\cdot\left[\frac{\pi\cdot\cot(\pi\cdot z)}{z^s}\right]\right\} \; = \; p_j^{\,-s}$$

where $p_j = 1,2,3,\dots,N$.

Let γ be the counterclockwise, nearly semi-circular, closed path in the complex plane connecting points $a+i\cdot(N+½)$ and $a-i\cdot(N+½)$ with a line, and returning to point $a+i\cdot(N+½)$ along the circular arc K_p defined by

$$|z| \; = \; N \; + \; ½$$

where $0<a<1$ and $N=1,2,3,\dots$. All poles, p_j, at $z=1,2,3,\dots,N$ are located inside closed path γ.

Cauchy's residue theorem gives

$$\frac{1}{2i}\cdot\oint_{\gamma}\frac{\cot(\pi\cdot z)}{z^s}dz \; = \; 1+2^{-s}+3^{-s}+\cdots+N^{-s}$$

and therefore, partial sums of the infinite series representation of the Riemann zeta function are

$$\sum_{n=1}^{N}n^{-s} \; = \; \frac{1}{2i}\cdot\oint_{\gamma}\frac{\cot(\pi\cdot z)}{z^s}dz \; = \; \frac{1}{2i}\cdot\int_{a+i\cdot(N+½)}^{a-i\cdot(N+½)}\frac{\cot(\pi\cdot z)}{z^s}dz \; + \; \frac{1}{2i}\cdot\int_{K_p}\frac{\cot(\pi\cdot z)}{z^s}dz$$

$$(main\ formula)$$

The first integral in the main formula above can be decomposed into two integrals as

$$\frac{1}{2i}\cdot\int_{a+i\cdot(N+½)}^{a-i\cdot(N+½)}\frac{\cot(\pi\cdot z)}{z^s}dz \; = \; -\frac{1}{2i}\cdot\int_{a}^{a+i\cdot(N+½)}\frac{\cot(\pi\cdot z)}{z^s}dz \; + \; \frac{1}{2i}\cdot\int_{a}^{a-i\cdot(N+½)}\frac{\cot(\pi\cdot z)}{z^s}dz$$

Note the identities:

$$\frac{\cot(\pi\cdot z)}{2i} \; = \; -\left(\frac{1}{2}+\frac{1}{e^{-2\pi iz}-1}\right)$$

and

$$\frac{\cot(\pi\cdot z)}{2i} \; = \; \left(\frac{1}{2}+\frac{1}{e^{2\pi iz}-1}\right)$$

For $y>0$: $z=i\cdot y$, $dz=i\cdot dy$, $z^{-s}=e^{-i\pi s/2}\cdot y^{-s}$, and

$$\frac{\cot(\pi\cdot z)}{2i} \; = \; -\left(\frac{1}{2}+\frac{1}{e^{2\pi y}-1}\right)$$

Similarly, for $y<0$: $z=-i\cdot y$, $dz=-i\cdot dy$, $z^{-s}=e^{i\pi s/2}\cdot y^{-s}$, and

$$\frac{\cot(\pi \cdot z)}{2i} = \left(\frac{1}{2} + \frac{1}{e^{2\pi y} - 1}\right)$$

Substituting these formulae into the right-hand side of the equation above, and applying the limit $a \to 0$ gives

$$\int_0^{(N+½)} \left(\frac{1}{2} + \frac{1}{e^{2\pi y} - 1}\right) \cdot i \cdot e^{-\frac{i\pi s}{2}} \cdot y^{-s}\, dy \; - \; \int_0^{(N+½)} \left(\frac{1}{2} + \frac{1}{e^{2\pi y} - 1}\right) \cdot i \cdot e^{\frac{i\pi s}{2}} \cdot y^{-s}\, dy$$

or

$$\left(\frac{i}{2}\right) \cdot \left(e^{-\frac{i\pi s}{2}} - e^{\frac{i\pi s}{2}}\right) \cdot \int_0^{(N+½)} y^{-s}\, dy \; + \; i \cdot \left(e^{-\frac{i\pi s}{2}} - e^{\frac{i\pi s}{2}}\right) \cdot \int_0^{(N+½)} \frac{y^{-s}}{e^{2\pi y} - 1} dy$$

Noting that

$$\sin(\theta) = \left(\frac{i}{2}\right) \cdot \left(e^{-i \cdot \theta} - e^{i \cdot \theta}\right)$$

and

$$\int_0^{(N+½)} y^{-s}\, dy = \frac{(N + ½)^{1-s}}{1 - s}$$

and changing the variable of integration in the second integral above from y to x, where $x = 2\pi y$, and $y = x/2\pi$, $dy = dx/2\pi$, $y^{-s} = (2\pi)^s \cdot x^{-s}$, it follows that

$$\lim_{a \to 0} \left\{\frac{1}{2i} \cdot \int_{a+i \cdot (N+½)}^{a-i \cdot (N+½)} \frac{\cot(\pi \cdot z)}{z^s} dz\right\} = \sin\left(\frac{\pi s}{2}\right) \cdot \frac{(N + ½)^{1-s}}{1 - s} \; + \; 2^s \cdot \pi^{s-1} \cdot \sin\left(\frac{\pi s}{2}\right) \cdot \int_0^{(N+½)} \frac{x^{-s}}{e^x - 1} dx$$

Also note that the Mellin transform representation of the Riemann zeta function is

$$\zeta(s) = \frac{1}{\Gamma(s)} \cdot \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \qquad Re(s) > 1 \;\; \textit{and at the roots of } \zeta(s) \textit{ in the critical strip}$$

and therefore,

$$\zeta(1 - s) = \frac{1}{\Gamma(1 - s)} \cdot \int_0^{\infty} \frac{x^{-s}}{e^x - 1} dx \qquad Re(s) > 1 \;\; \textit{and at the roots of } \zeta(s) \textit{ in the critical strip}$$

The functional equation of the Riemann zeta function is

$$2^s \cdot \pi^{s-1} \cdot \sin\left(\frac{\pi s}{2}\right) \cdot \Gamma(1 - s) \cdot \zeta(1 - s) = \zeta(s) \qquad s \neq 1$$

and hence, it follows that

$$\lim_{N \to \infty} \left\{\int_0^{(N+½)} \frac{x^{-s}}{e^x - 1} dx\right\} \sim \Gamma(1 - s) \cdot \zeta(1 - s)$$

$Re(s) > 1$ *and at the roots of* $\zeta(s)$ *in the critical strip*

Substituting these results into the first integral of the main formula above gives

$$\frac{1}{2i}\cdot\int_{a+i\cdot(N+½)}^{a-i\cdot(N+½)}\frac{\cot(\pi\cdot z)}{z^s}dz \;\sim\; \sin\left(\frac{\pi s}{2}\right)\cdot\frac{(N+½)^{1-s}}{1-s} + \zeta(s)$$

This relationship is valid everywhere in the critical strip for arbitrarily large values of integer N.

The second integral in the main formula above, or

$$\frac{1}{2i}\cdot\int_{K_p}\frac{\cot(\pi\cdot z)}{z^s}dz$$

can be decomposed into two integrals as

$$\lim_{a\to 0}\left\{\frac{1}{2i}\cdot\int_{K_p}\frac{\cot(\pi\cdot z)}{z^s}dz\right\} \;=\; -\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}+\frac{1}{e^{-2\pi i\cdot z}-1}\right)\cdot z^{-s}\,dz \;+\; \int_{-\frac{\pi}{2}}^{0}\left(\frac{1}{2}+\frac{1}{e^{2\pi i\cdot z}-1}\right)\cdot z^{-s}\,dz$$

Again note the identities:

$$\frac{\cot(\pi\cdot z)}{2i} \;=\; -\left(\frac{1}{2}+\frac{1}{e^{-2\pi i\cdot z}-1}\right)$$

and

$$\frac{\cot(\pi\cdot z)}{2i} \;=\; \left(\frac{1}{2}+\frac{1}{e^{2\pi i\cdot z}-1}\right)$$

so that for $\theta > 0$,

$$\frac{\cot(\pi\cdot z)}{2i} \;=\; -\left(\frac{1}{2}+\frac{1}{e^{-2\pi i\cdot r\cdot e^{i\theta}}-1}\right)$$

and for $\theta < 0$,

$$\frac{\cot(\pi\cdot z)}{2i} \;=\; \left(\frac{1}{2}+\frac{1}{e^{2\pi i\cdot r\cdot e^{i\theta}}-1}\right)$$

where $z = r\cdot e^{i\theta}$, $dz = i\cdot r\cdot e^{i\theta}d\theta$, and $z^{-s} = r^{-s}e^{-is\theta}$.

Substituting in the right-hand side of the formula above,

$$\int_{-\frac{\pi}{2}}^{0}\left(\frac{1}{2}+\frac{1}{e^{2\pi i\cdot r\cdot e^{i\theta}}-1}\right)\cdot r^{-s}\cdot e^{-is\theta}\cdot i\cdot r\cdot e^{i\theta}\,d\theta \;-\; \int_0^{\frac{\pi}{2}}\left(\frac{1}{2}+\frac{1}{e^{-2\pi i\cdot r\cdot e^{i\theta}}-1}\right)\cdot r^{-s}\cdot e^{-is\theta}\cdot i\cdot r\cdot e^{i\theta}\,d\theta$$

and rearranging gives

$$\left(\frac{i}{2}\right)\cdot r^{1-s}\cdot\left\{\int_{-\frac{\pi}{2}}^{0}e^{i(1-s)\theta}\,d\theta - \int_0^{\frac{\pi}{2}}e^{i(1-s)\theta}\,d\theta\right\} + i\cdot r^{1-s}$$
$$\cdot\left\{\int_{-\frac{\pi}{2}}^{0}\left(\frac{e^{i(1-s)\theta}}{e^{2\pi i\cdot r\cdot e^{i\theta}}-1}\right)d\theta - \int_0^{\frac{\pi}{2}}\left(\frac{e^{i(1-s)\theta}}{e^{-2\pi i\cdot r\cdot e^{i\theta}}-1}\right)d\theta\right\}$$

Substituting $(N+½)$ for r and noting that

$$\int_{-\frac{\pi}{2}}^{0} e^{i(1-s)\theta}\, d\theta \;=\; -\,i\cdot\left(\frac{1}{1-\mathrm{s}}\right)\cdot\left[1 - e^{-\frac{i\pi(1-s)}{2}}\right]$$

and

$$\int_{0}^{\frac{\pi}{2}} e^{i(1-s)\theta}\, d\theta \;=\; -\,i\cdot\left(\frac{1}{1-\mathrm{s}}\right)\cdot\left[e^{\frac{i\pi(1-s)}{2}} - 1\right]$$

gives

$$\left(\frac{1}{2}\right)\cdot\frac{(N+½\,)^{1-s}}{1-s}\cdot\left\{2-\left[e^{-\frac{i\pi(1-s)}{2}}+e^{-\frac{i\pi(1-s)}{2}}\right]\right\}$$
$$-\,i\cdot(N+½\,)^{1-s}\cdot\left\{\int_{-\frac{\pi}{2}}^{0}\left(\frac{e^{i(1-s)\theta}}{e^{2\pi i\cdot(N+½\,)\cdot e^{i\theta}}-1}\right)d\theta\;-\int_{0}^{\frac{\pi}{2}}\left(\frac{e^{i(1-s)\theta}}{e^{-2\pi i\cdot(N+½\,)\cdot e^{i\theta}}-1}\right)d\theta\right\}$$

Further noting that

$$\cos(\theta) \;=\; \left(\frac{1}{2}\right)\cdot\left(e^{i\cdot\theta}\;+\;e^{-i\cdot\theta}\right)$$

it follows that

$$\left\{1-\cos\left[\frac{\pi(1-s)}{2}\right]\right\}\cdot\frac{(N+½\,)^{1-s}}{1-s}$$
$$-\,i\cdot(N+½\,)^{1-s}\cdot\left\{\int_{-\frac{\pi}{2}}^{0}\left(\frac{e^{i(1-s)\theta}}{e^{2\pi i\cdot(N+½\,)\cdot e^{i\theta}}-1}\right)d\theta\;-\int_{0}^{\frac{\pi}{2}}\left(\frac{e^{i(1-s)\theta}}{e^{-2\pi i\cdot(N+½\,)\cdot e^{i\theta}}-1}\right)d\theta\right\}$$

and finally,

$$\frac{1}{2i}\cdot\int_{K_p}\frac{\cot(\pi\cdot z)}{z^s}dz \;=\; \left\{1\;-\;\cos\left[\frac{\pi(1-s)}{2}\right]\right\}\cdot\frac{(N+½\,)^{1-s}}{1-s}$$
$$-\;\;i\cdot(N+½\,)^{1-s}$$
$$\cdot\left\{\int_{0}^{\frac{\pi}{2}}\left(\frac{e^{i(1-s)\theta}}{e^{-2\pi i\cdot(N+½)\cdot e^{i\theta}}-1}\right)d\theta\;+\;\int_{-\frac{\pi}{2}}^{0}\left(\frac{e^{i(1-s)\theta}}{e^{2\pi i\cdot(N+½)\cdot e^{i\theta}}-1}\right)d\theta\right\}$$

The first integral,

$$\int_{0}^{\frac{\pi}{2}}\left(\frac{e^{i(1-s)\theta}}{e^{-2\pi i\cdot(N+½)\cdot e^{i\theta}}-1}\right)d\theta$$

on the right-hand side of the equation above can be simplified as follows. Changing the variable of integration from θ to x, where $\theta = -\,i\cdot Log\,[i\cdot x/2\pi\cdot(N+½\,)]$ and $d\theta = -\,i\cdot(dx/x)$, gives

$$\int_{0}^{\frac{\pi}{2}}\left(\frac{e^{i(1-s)\theta}}{e^{-2\pi i\cdot(N+½)\cdot e^{i\theta}}-1}\right)d\theta \;=\; (2\pi)^{s-1}\cdot e^{-i\pi s/2}\cdot r^{s-1}\cdot\int_{0}^{\frac{\pi}{2}}\frac{x^{-s}}{e^{x}-1}dx$$

The second integral above, or

$$\int_{-\frac{\pi}{2}}^{0}\left(\frac{e^{i(1-s)\theta}}{e^{2\pi i\cdot(N+½)\cdot e^{i\theta}}-1}\right)d\theta$$

can also be simplified. Changing the variable of integration from θ to x, where $\theta = -i \cdot Log\,[-i \cdot x/2\pi \cdot (N + ½\,)]$ and $d\theta = -i \cdot (dx/x)$ as before, gives

$$\int_{-\frac{\pi}{2}}^{0} \left(\frac{e^{i(1-s)\theta}}{e^{-2\pi i \cdot (N+½) \cdot e^{i\theta}} - 1} \right) d\theta \;=\; -(2\pi)^{s-1} \cdot e^{i\pi s/2} \cdot (N + ½\,)^{s-1} \cdot \int_{-\frac{\pi}{2}}^{0} \frac{x^{-s}}{e^{x} - 1} dx$$

Substituting these formulae into

$$\left\{ 1 - \cos\left[\frac{\pi(1-s)}{2}\right] \right\} \cdot \frac{(N + ½\,)^{1-s}}{1-s}$$
$$- \; i \cdot (N + ½\,)^{1-s} \cdot \left\{ \int_{0}^{\frac{\pi}{2}} \left(\frac{e^{i(1-s)\theta}}{e^{-2\pi i \cdot (N+½) \cdot e^{i\theta}} - 1} \right) d\theta \;+\; \int_{-\frac{\pi}{2}}^{0} \left(\frac{e^{i(1-s)\theta}}{e^{2\pi i \cdot (N+½) \cdot e^{i\theta}} - 1} \right) d\theta \right\}$$

from above and cancelling the term $(N + ½\,)^{1-s}$ in the numerator and denominator gives

$$(2\pi)^{s-1} \left\{ e^{i\pi(1+s)/2} \cdot \int_{-\frac{\pi}{2}}^{0} \frac{x^{-s}}{e^{x} - 1} dx \;-\; e^{i\pi(1-s)/2} \cdot \int_{0}^{\frac{\pi}{2}} \frac{x^{-s}}{e^{x} - 1} dx \right\}$$

Finally, the second integral of the main formula is

$$\frac{1}{2i} \cdot \int_{K_p} \frac{\cot(\pi \cdot z)}{z^{s}} dz$$
$$= \left\{ 1 - \cos\left[\frac{\pi(1-s)}{2}\right] \right\} \cdot \frac{(N + ½\,)^{1-s}}{1-s}$$
$$+ \; (2\pi)^{s-1} \left\{ e^{i\pi(1+s)/2} \cdot \int_{-\frac{\pi}{2}}^{0} \frac{x^{-s}}{e^{x} - 1} dx \;-\; e^{i\pi(1-s)/2} \cdot \int_{0}^{\frac{\pi}{2}} \frac{x^{-s}}{e^{x} - 1} dx \right\}$$

and the main formula, or

$$\sum_{n=1}^{N} n^{-s} \;\sim\; \frac{1}{2i} \cdot \oint_{\gamma} \frac{\cot(\pi \cdot z)}{z^{s}} dz \;=\; \frac{1}{2i} \cdot \int_{a+i\cdot(N+½)}^{a-i\cdot(N+½)} \frac{\cot(\pi \cdot z)}{z^{s}} dz \;+\; \frac{1}{2i} \cdot \int_{K_p} \frac{\cot\,(\pi \cdot z)}{z^{s}} dz$$

is

$$\sum_{n=1}^{N} n^{-s} \;\sim\; \frac{1}{2i} \cdot \oint_{\gamma} \frac{\cot(\pi \cdot z)}{z^{s}} dz \;\sim\; \left\{ 1 + \sin\left(\frac{\pi s}{2}\right) - \cos\left[\frac{\pi(1-s)}{2}\right] \right\} \cdot \frac{(N + ½)^{1-s}}{1-s} \;+\; \zeta(s)$$
$$+ \; (2\pi)^{s-1} \left\{ e^{i\pi(1+s)/2} \cdot \int_{-\frac{\pi}{2}}^{0} \frac{x^{-s}}{e^{x} - 1} dx \;-\; e^{i\pi(1-s)/2} \cdot \int_{0}^{\frac{\pi}{2}} \frac{x^{-s}}{e^{x} - 1} dx \right\}$$

Note the identity

$$1 + \sin\left(\frac{\pi s}{2}\right) - \cos\left[\frac{\pi(1-s)}{2}\right] \;=\; 1$$

and therefore,

$$\sum_{n=1}^{N} n^{-s} \sim \frac{(N+½)^{1-s}}{1-s} + \zeta(s)$$
$$+ (2\pi)^{s-1}\left\{e^{i\pi(1+s)/2}\cdot\int_{-\frac{\pi}{2}}^{0}\frac{x^{-s}}{e^{x}-1}dx - e^{i\pi(1-s)/2}\cdot\int_{0}^{\frac{\pi}{2}}\frac{x^{-s}}{e^{x}-1}dx\right\}$$

For very large values of integer N, $N + ½ \approx N$, so that

$$\sum_{n=1}^{N} n^{-s} \sim \frac{N^{1-s}}{1-s} + \zeta(s) + (2\pi)^{s-1}\left\{e^{i\pi(1+s)/2}\cdot\int_{-\frac{\pi}{2}}^{0}\frac{x^{-s}}{e^{x}-1}dx - e^{i\pi(1-s)/2}\cdot\int_{0}^{\frac{\pi}{2}}\frac{x^{-s}}{e^{x}-1}dx\right\}$$

Furthermore, the last term on the right-hand side is independent of integer N and is relatively small in comparison with the terms

$$\sum_{n=1}^{N} n^{-s} \quad and \quad \frac{N^{1-s}}{1-s}$$

for all values of argument s everywhere in the critical strip, including at the roots of the Riemann zeta function. Therefore, for large values of integer N and everywhere in the critical strip:

$$\sum_{n=1}^{N} n^{-s} \sim \frac{N^{1-s}}{1-s} + \zeta(s)$$

Similarly, partial sums of the infinite series representation of the Riemann zeta function with complex conjugate argument, $\bar{s} = \sigma - i \cdot t$, are given by

$$\sum_{n=1}^{N} n^{-\bar{s}} = \frac{1}{2i}\cdot\oint_{\gamma}\frac{\cot(\pi\cdot z)}{z^{\bar{s}}}dz = \frac{1}{2i}\cdot\int_{a+i\cdot(N+½)}^{a-i\cdot(N+½)}\frac{\cot(\pi\cdot z)}{z^{\bar{s}}}dz + \frac{1}{2i}\cdot\int_{K_p}\frac{\cot(\pi\cdot z)}{z^{\bar{s}}}dz$$

where

$$\frac{1}{2i}\cdot\int_{a+i\cdot(N+½)}^{a-i\cdot(N+½)}\frac{\cot(\pi\cdot z)}{z^{\bar{s}}}dz \sim \sin\left(\frac{\pi\bar{s}}{2}\right)\cdot\frac{(N+½)^{1-\bar{s}}}{1-\bar{s}} + \zeta(\bar{s})$$

$$\frac{1}{2i}\cdot\int_{K_p}\frac{\cot(\pi\cdot z)}{z^{\bar{s}}}dz \sim \left\{1 - \cos\left[\frac{\pi(1-\bar{s})}{2}\right]\right\}\cdot\frac{(N+½)^{1-\bar{s}}}{1-\bar{s}}$$

and

$$\sum_{n=1}^{N} n^{-\bar{s}} \sim \frac{(N+½)^{1-\bar{s}}}{1-\bar{s}} + \zeta(\bar{s})$$

This final relationship is also valid for arbitrarily large, finite values of integer N, everywhere in the critical strip.

In summary, Cauchy's residue theorem was used to derive an asymptotic relationship between the partial sums of the Riemann zeta function and the zeta function for arbitrarily large, finite values of integer N, as well as an analogous relationship for the complex conjugate value of the argument:

$$\sum_{n=1}^{N} n^{-s} \sim \frac{N^{1-s}}{1-\mathrm{s}} + \zeta(s)$$

and

$$\sum_{n=1}^{N} n^{-\bar{s}} \sim \frac{N^{1-\bar{s}}}{1-\bar{s}} + \zeta(\bar{s})$$

Adding and subtracting the relationships above, and multiplying by $1/2$ and $i/2$, respectively, gives the identical results derived using Borel's integral summation method and the Euler-Maclaurin summation formula:

$$\left(\frac{1}{2}\right)\cdot\sum_{n=1}^{N}\left(n^{-s}+n^{-\bar{s}}\right) \sim \left(\frac{1}{2}\right)\cdot\left(\frac{N^{1-s}}{1-\mathrm{s}}+\frac{N^{1-\bar{s}}}{1-\bar{s}}\right) + \left(\frac{1}{2}\right)\cdot\left[\zeta(s) + \zeta(\bar{s})\right]$$

and

$$\left(\frac{i}{2}\right)\cdot\sum_{n=1}^{N}\left(n^{-s} - n^{-\bar{s}}\right) \sim \left(\frac{i}{2}\right)\cdot\left(\frac{N^{1-s}}{1-\mathrm{s}}-\frac{N^{1-\bar{s}}}{1-\bar{s}}\right) + \left(\frac{i}{2}\right)\cdot\left[\zeta(s) - \zeta(\bar{s})\right]$$

or equivalently,

$$Re\left\{\sum_{n=1}^{N} n^{-s}\right\} \sim Re\left\{\frac{N^{1-s}}{1-s}\right\} + Re\{\zeta(s)\}$$

and

$$Im\left\{\sum_{n=1}^{N} n^{-s}\right\} \sim Im\left\{\frac{N^{1-s}}{1-s}\right\} + Im\{\zeta(s)\}$$

These relationships are valid for arbitrarily large, finite values of integer N, everywhere in the critical strip.