Divergent Series and Divergent Integrals

In this work, asymptotic relationships between the real and imaginary components of partial sums of the Riemann zeta function and the zeta function itself were derived independently using (1.) the Borel integral summation method and the Euler-Maclaurin summation formula on the real line, and (2.) the Cauchy residue theorem on the complex plane. The relationships are:

$$Re\left\{\sum_{n=1}^{N} n^{-s}\right\} \sim Re\left\{\frac{N^{1-s}}{1-s}\right\} + \left(\frac{1}{2}\right)\cdot Re\{N^{-s}\} + Re\{\zeta(s)\}$$

and

$$Im\left\{\sum_{n=1}^{N} n^{-s}\right\} \sim Im\left\{\frac{N^{1-s}}{1-s}\right\} + \left(\frac{1}{2}\right)\cdot Im\{N^{-s}\} + Im\{\zeta(s)\}$$

Roots of the Riemann zeta function in the critical strip occur when

$$Re\left\{\sum_{n=1}^{N} n^{-s}\right\} \quad \text{and} \quad Re\left\{\frac{N^{1-s}}{1-s}\right\} + \left(\frac{1}{2}\right)\cdot Re\{N^{-s}\}$$

and

$$Im\left\{\sum_{n=1}^{N} n^{-s}\right\} \quad \text{and} \quad Im\left\{\frac{N^{1-s}}{1-s}\right\} + \left(\frac{1}{2}\right)\cdot Im\{N^{-s}\}$$

diverge asymptotically and simultaneously, for arbitrarily large, finite values of integer N.

The formulae on the left and right-hand sides of the relationships above were calculated numerically and are illustrated in the graphs on the following pages.

The partial sum

$$Re\left\{\sum_{n=1}^{N} n^{-s}\right\}$$

on the left-hand side of the first relationship above is shown with “red points” in the graphs below.

The formula

$$Re\left\{\frac{N^{1-s}}{1-s}\right\} + \left(\frac{1}{2}\right)\cdot Re\{N^{-s}\}$$

on the right-hand side of the first relationship above is shown with “red curves” in the graphs.

Similarly, the partial sum

$$Im\left\{\sum_{n=1}^{N} n^{-s}\right\}$$

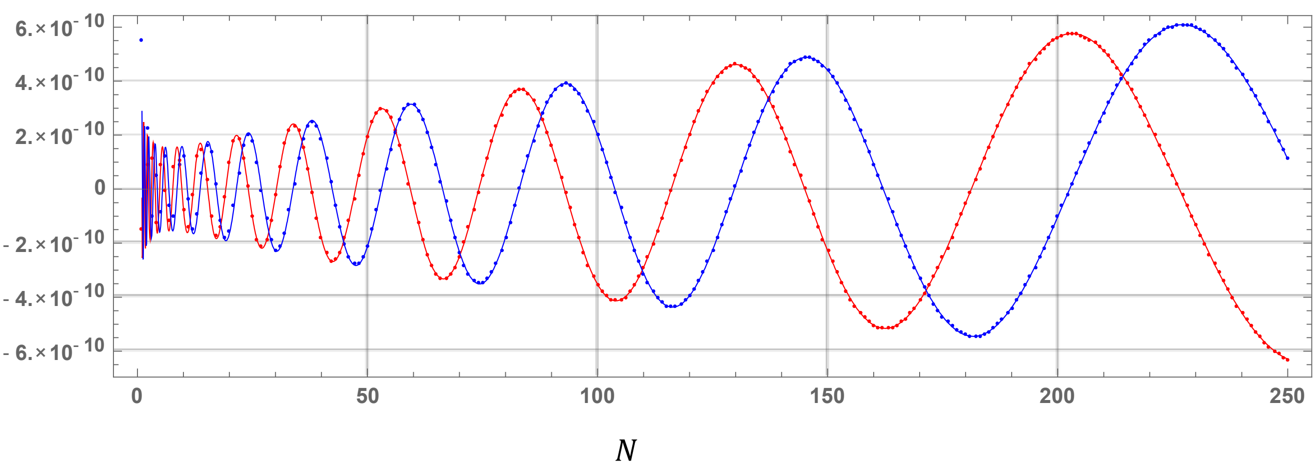
on the left-hand side of the second relationship above is shown with "blue points" in the graphs.

Finally, the function

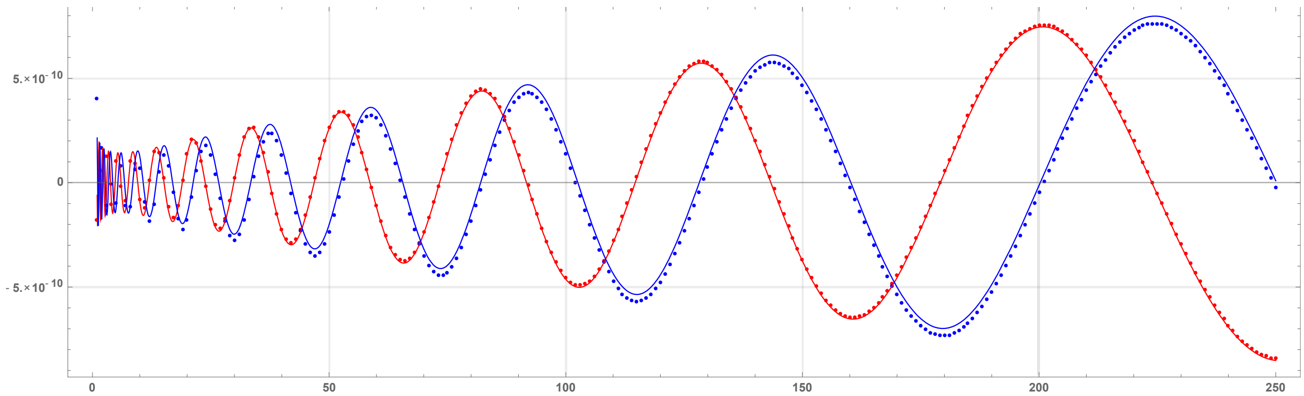
$$Im\left\{\frac{N^{1-s}}{1-s}\right\} + \left(\frac{1}{2}\right) \cdot Im\{N^{-s}\}$$

on the right-hand side of the second relationship above is shown with "blue curves" in the graphs.

$\sigma = 1/2,\ t = 14.134725\ldots$ A Root of the Riemann Zeta Function in the Critical Strip

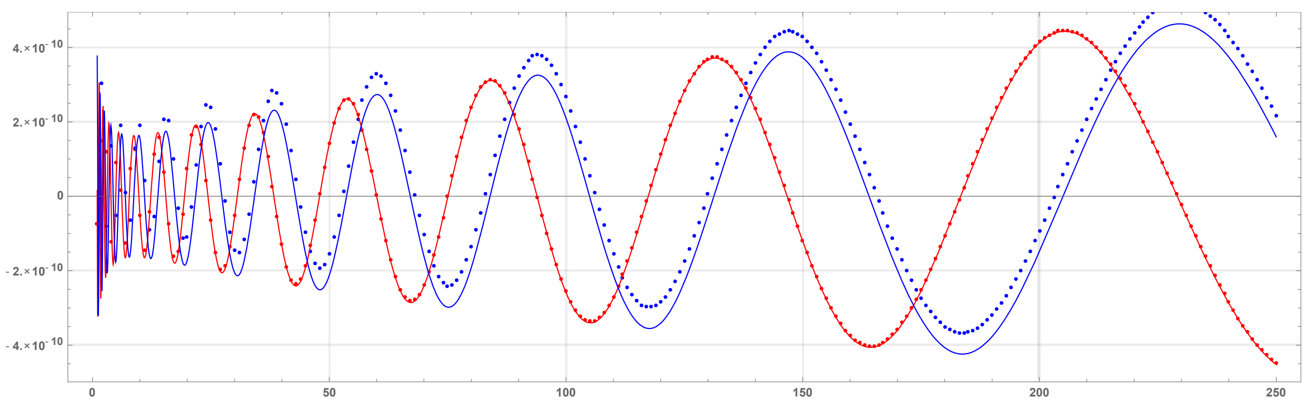


$\sigma = 2/5,\ t = 14.134725...$ Not a Root of the Riemann Zeta Function in the Critical Strip



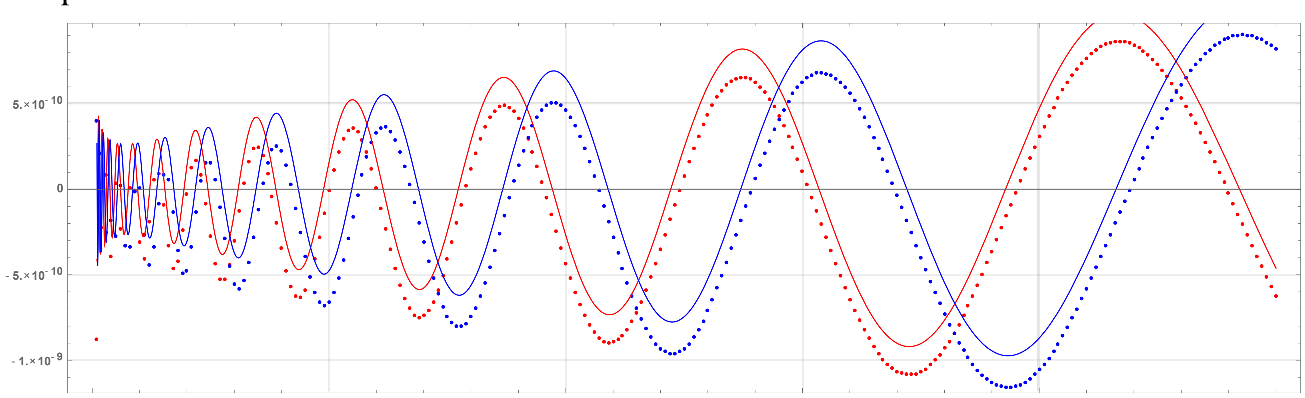
N

$\sigma = 3/5,\ t = 14.134725...$ Not a Root of the Riemann Zeta Function in the Critical Strip



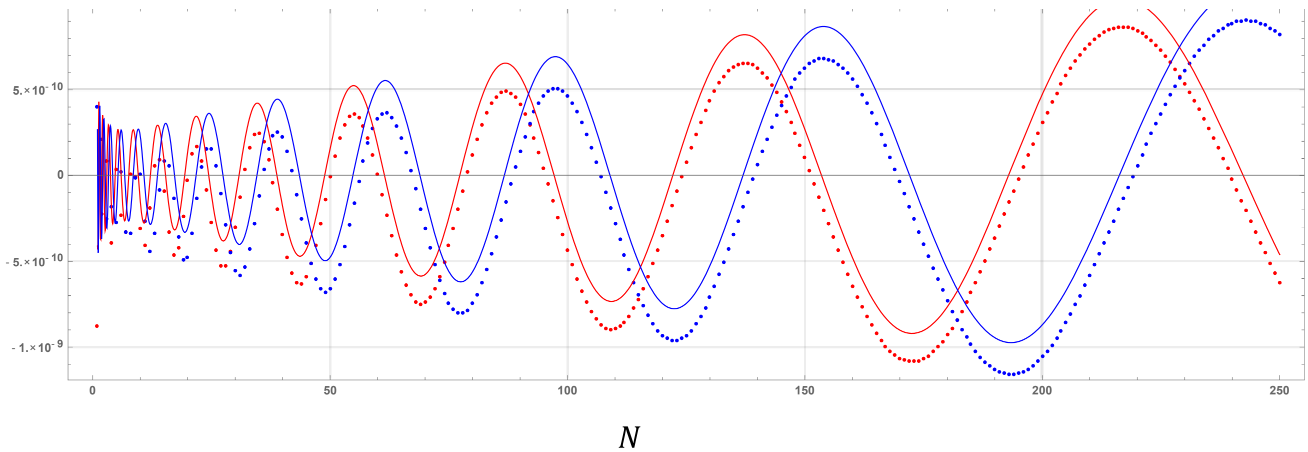
N

$\sigma = 1/2,\ t = 13.800000$ Not a Root of the Riemann Zeta Function in the Critical Strip

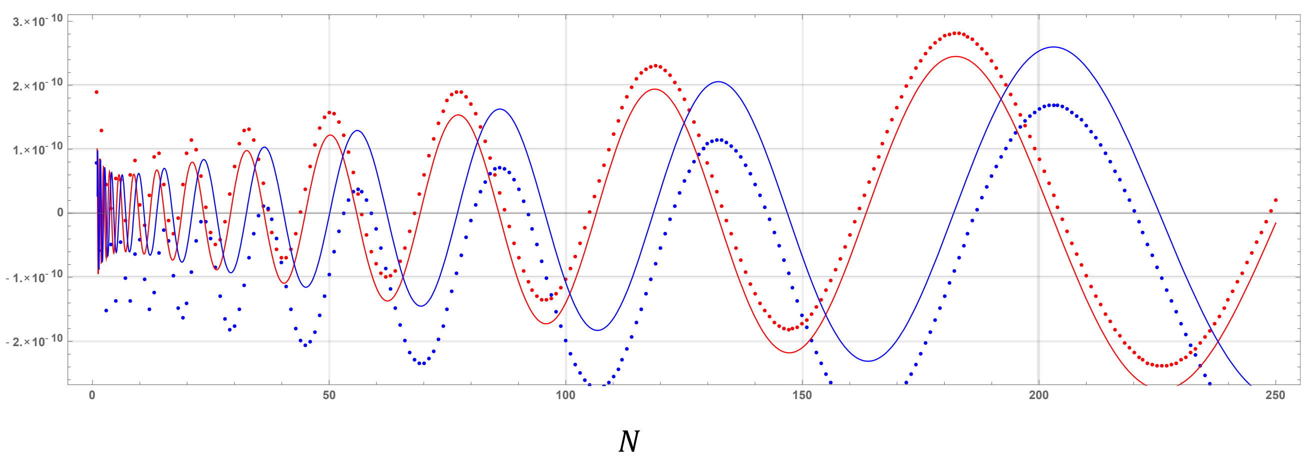


N

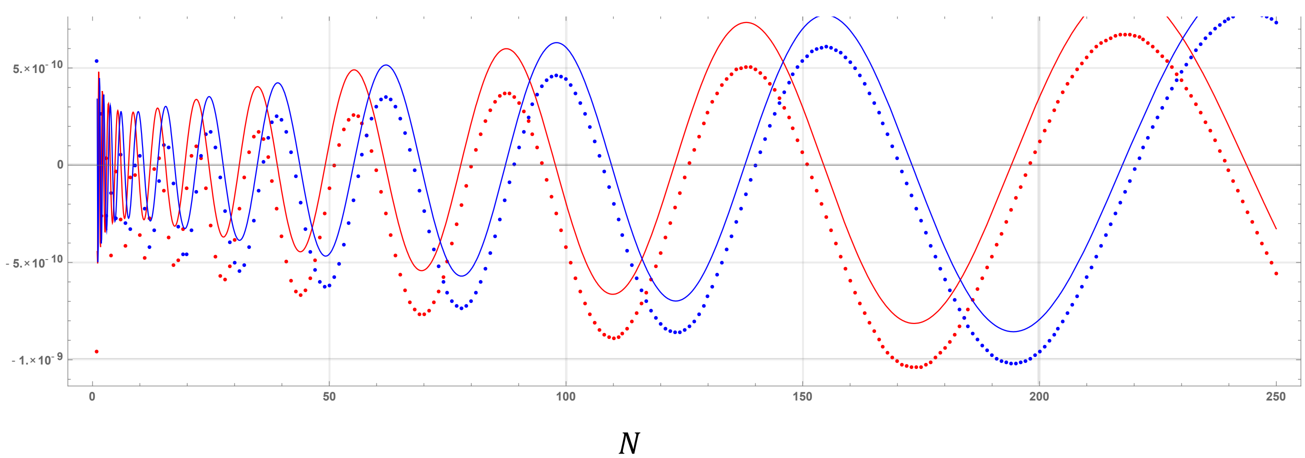
$\sigma = 1/2, \; t = 14.400000$ Not a Root of the Riemann Zeta Function in the Critical Strip



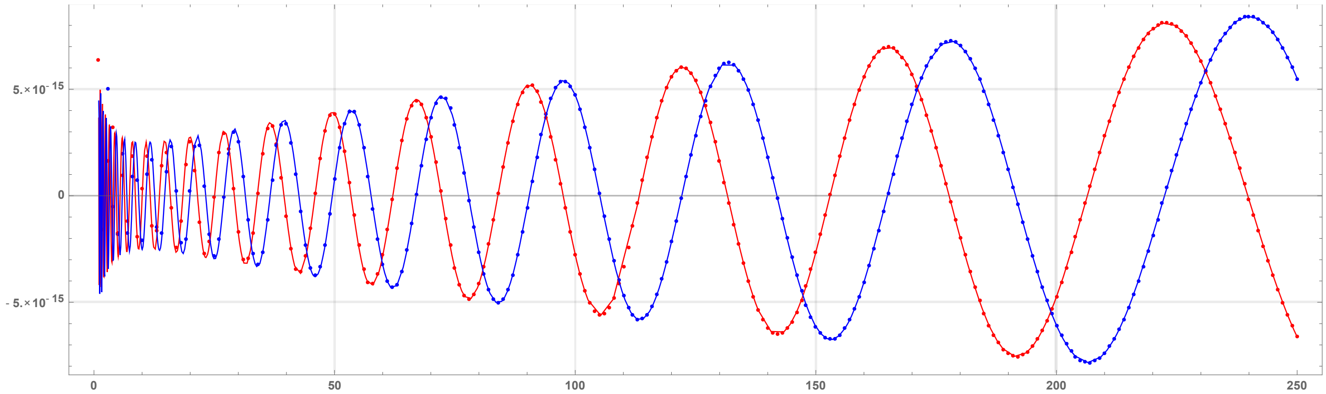
$\sigma = 9/20, \; t = 14.700000$ Not a Root of the Riemann Zeta Function in the Critical Strip



$\sigma = 11/20, \; t = 13.800000$ Not a Root of the Riemann Zeta Function in the Critical Strip



$\sigma = 1/2,\ t = 21.022040\ldots$ A Root of the Riemann Zeta Function in the Critical Strip



N

The graphs illustrate that the roots of the Riemann zeta function in the critical strip occur when

$$Re\left\{\sum_{n=1}^{N} n^{-s}\right\} \quad \text{and} \quad Re\left\{\frac{N^{1-s}}{1-s}\right\} + \left(\frac{1}{2}\right)\cdot Re\{N^{-s}\}$$

and

$$Im\left\{\sum_{n=1}^{N} n^{-s}\right\} \quad \text{and} \quad Im\left\{\frac{N^{1-s}}{1-s}\right\} + \left(\frac{1}{2}\right)\cdot Im\{N^{-s}\}$$

diverge asymptotically and simultaneously for large, finite values of integer N.

Note that for large values of integer N,

$$Re\left\{\frac{N^{1-s}}{1-s}\right\} >> \left(\frac{1}{2}\right)\cdot Re\{N^{-s}\}$$

and

$$Im\left\{\frac{N^{1-s}}{1-s}\right\} >> \left(\frac{1}{2}\right)\cdot Im\{N^{-s}\}$$

and at the roots of the Riemann zeta function, where $\zeta(s) = \zeta(\bar{s}) = 0$:

$$Re\left\{\sum_{n=1}^{N} n^{-s}\right\} \sim Re\left\{\frac{N^{1-s}}{1-s}\right\}$$

and

$$Im\left\{\sum_{n=1}^{N} n^{-s}\right\} \sim Im\left\{\frac{N^{1-s}}{1-s}\right\}$$

These relationships are valid everywhere in the critical strip for arbitrarily large, finite values of integer N.

Therefore, the roots of the Riemann zeta function in the critical strip occur when

$$Re\left\{\sum_{n=1}^{N} n^{-s}\right\} \quad \text{and} \quad Re\left\{\frac{N^{1-s}}{1-s}\right\}$$

and

$$Im\left\{\sum_{n=1}^{N} n^{-s}\right\} \quad \text{and} \quad Im\left\{\frac{N^{1-s}}{1-s}\right\}$$

diverge asymptotically and simultaneously.

It will be evident later in this work why the respective substitutions of $\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor$ and $\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor$ for N in the partial sums

$$Re\left\{\sum_{n=1}^{N} n^{-s}\right\} \quad \text{and} \quad Im\left\{\sum_{n=1}^{N} n^{-s}\right\}$$

are important.

Substituting $N = \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor$ in the first asymptotic relationship, or

$$Re\left\{\sum_{n=1}^{N} n^{-s}\right\} \quad \text{and} \quad Re\left\{\frac{N^{1-s}}{1-s}\right\} + \left(\frac{1}{2}\right)\cdot Re\{N^{-s}\}$$

gives

$$Re\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} n^{-s}\right\} \sim Re\left\{\frac{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor^{1-s}}{1-s}\right\} + \left(\frac{1}{2}\right)\cdot Re\left\{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor^{-s}\right\}$$

where

$$Re\left\{\frac{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor^{1-s}}{1-s}\right\}$$

$$= \left(\frac{1}{2}\right)\cdot\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor^{1-\sigma}\cdot\left[\frac{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor^{-i\cdot t}}{1-s} + \frac{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor^{i\cdot t}}{1-\bar{s}}\right]$$

$$= \left(\frac{1}{2}\right) \cdot \left[\frac{1}{(1-\sigma)^2 + t^2}\right] \cdot \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{1-\sigma}$$

$$\cdot \left\{ [(1-\sigma) + i \cdot t] \cdot \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{-i\cdot t} + [(1-\sigma) - i \cdot t] \cdot \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{i\cdot t} \right\}$$

$$\approx \left(\frac{1}{2}\right) \cdot \left[\frac{1}{(1-\sigma)^2 + t^2}\right] \cdot \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{1-\sigma}$$

$$\cdot \left\{ [(1-\sigma) + i \cdot t] \cdot e^{(m-1)\cdot\pi\cdot(-i)} + [(1-\sigma) - i \cdot t] \cdot e^{(m-1)\cdot\pi\cdot(i)} \right\}$$

$$= \left(\frac{1}{2}\right) \cdot \left[\frac{1}{(1-\sigma)^2 + t^2}\right] \cdot \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{1-\sigma}$$

$$\cdot \{ [(1-\sigma) + i \cdot t] \cdot (-1)^{m+1} + [(1-\sigma) - i \cdot t] \cdot (-1)^{m+1} \}$$

$$= (-1)^{m+1} \cdot \left[\frac{1-\sigma}{(1-\sigma)^2 + t^2}\right] \cdot \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{1-\sigma}$$

so that

$$Re\left\{ \frac{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{1-s}}{1-s} \right\} \approx (-1)^{m+1} \cdot \left[\frac{1-\sigma}{(1-\sigma)^2+t^2}\right] \cdot \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{1-\sigma}$$

Also note that,

$$Re\left\{ \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{-s} \right\}$$

$$= \left(\frac{1}{2}\right) \cdot \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{-\sigma} \cdot \left[\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{-i\cdot t} + \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{i\cdot t} \right]$$

$$\approx \left(\frac{1}{2}\right) \cdot \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{-\sigma} \cdot \left[e^{(m-1)\cdot\pi\cdot(-i)} + e^{(m-1)\cdot\pi\cdot(i)} \right]$$

$$= \left(\frac{1}{2}\right) \cdot \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{-\sigma} \cdot \{ (-1)^{m+1} + (-1)^{m+1} \}$$

$$= (-1)^{m+1} \cdot \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{-\sigma}$$

and therefore,

$$Re\left\{ \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{-s} \right\} \approx (-1)^{m+1} \cdot \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{-\sigma}$$

Substituting these formulae into the first relationship above, or

$$Re\left\{ \sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor} n^{-s} \right\} \sim Re\left\{ \frac{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{1-s}}{1-s} \right\} + \left(\frac{1}{2}\right) \cdot Re\left\{ \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{-s} \right\}$$

gives

$$Re\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} n^{-s}\right\} \sim (-1)^{m+1}\cdot\left[\frac{1-\sigma}{(1-\sigma)^2+t^2}\right]\cdot\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor^{1-\sigma} + (-1)^{m+1}\cdot\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor^{-\sigma}$$

or

$$Re\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} n^{-s}\right\} \sim (-1)^{m+1}\cdot\left\{\left[\frac{1-\sigma}{(1-\sigma)^2+t^2}\right]\cdot\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor^{1-\sigma} + \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor^{-\sigma}\right\}$$

Note that for large values of integer m,

$$\left[\frac{1-\sigma}{(1-\sigma)^2+t^2}\right]\cdot\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor^{1-\sigma} \gg \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor^{-\sigma}$$

and therefore,

$$Re\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} n^{-s}\right\} \sim (-1)^{m+1}\cdot\left[\frac{1-\sigma}{(1-\sigma)^2+t^2}\right]\cdot\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor^{1-\sigma}$$

Similarly, substituting $N = \left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor$ in the second relationship above gives

$$Im\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} n^{-s}\right\} \sim Im\left\{\frac{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor^{1-s}}{1-s}\right\} + \left(\frac{1}{2}\right)\cdot Im\left\{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor^{-s}\right\}$$

where

$$Im\left\{\frac{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor^{1-s}}{1-s}\right\}$$

$$= -\left(\frac{i}{2}\right)\cdot\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor^{1-\sigma}\cdot\left[\frac{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor^{-i\cdot t}}{1-s} - \frac{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor^{i\cdot t}}{1-\bar{s}}\right]$$

$$= -\left(\frac{i}{2}\right)\cdot\left[\frac{1}{(1-\sigma)^2+t^2}\right]\cdot\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor^{1-\sigma}$$

$$\cdot\left\{[(1-\sigma)+i\cdot t]\cdot\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor^{-i\cdot t} - [(1-\sigma)-i\cdot t]\cdot\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor^{i\cdot t}\right\}$$

$$\approx -\left(\frac{i}{2}\right)\cdot\left[\frac{1}{(1-\sigma)^2+t^2}\right]\cdot\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{1-\sigma}$$
$$\cdot\left\{[(1-\sigma)+i\cdot t]\cdot e^{\frac{(2\cdot m-1)\cdot\pi\cdot(-i)}{2}} - [(1-\sigma)-i\cdot t]\cdot e^{\frac{(2\cdot m-1)\cdot\pi\cdot(i)}{2}}\right\}$$
$$= -\left(\frac{i}{2}\right)\cdot\left[\frac{1}{(1-\sigma)^2+t^2}\right]\cdot\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{1-\sigma}$$
$$\cdot\{[(1-\sigma)+i\cdot t]\cdot(-1)^{m+1}\cdot(-i) - [(1-\sigma)-i\cdot t]\cdot(-1)^{m+1}\cdot(i)\}$$
$$= -\left(\frac{1}{2}\right)\cdot\left[\frac{1}{(1-\sigma)^2+t^2}\right]\cdot\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{1-\sigma}$$
$$\cdot\{[(1-\sigma)+i\cdot t]\cdot(-1)^{m+1} + [(1-\sigma)-i\cdot t]\cdot(-1)^{m+1}\}$$
$$= -(-1)^{m+1}\cdot\left(\frac{1}{2}\right)\cdot\left[\frac{1}{(1-\sigma)^2+t^2}\right]\cdot\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{1-\sigma}\cdot\{[(1-\sigma)+i\cdot t] + [(1-\sigma)-i\cdot t]\}$$
$$= -(-1)^{m+1}\cdot\left[\frac{1-\sigma}{(1-\sigma)^2+t^2}\right]\cdot\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{1-\sigma}$$

and therefore,

$$Im\left\{\frac{\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{1-s}}{1-s}\right\} \approx -(-1)^{m+1}\cdot\left[\frac{1-\sigma}{(1-\sigma)^2+t^2}\right]\cdot\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{1-\sigma}$$

Also note that

$$Im\left\{\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{-s}\right\}$$
$$= -\left(\frac{i}{2}\right)\cdot\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{-\sigma}\cdot\left[\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{-i\cdot t} - \left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{i\cdot t}\right]$$
$$\approx -\left(\frac{i}{2}\right)\cdot\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{-\sigma}\cdot\left[e^{\frac{(2\cdot m-1)\cdot\pi\cdot(-i)}{2}} - e^{\frac{(2\cdot m-1)\cdot\pi\cdot(i)}{2}}\right]$$
$$= -\left(\frac{i}{2}\right)\cdot\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{-\sigma}\cdot\{(-i)\cdot(-1)^{m+1} - (i)\cdot(-1)^{m+1}\}$$
$$= -(-1)^{m+1}\cdot\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{-\sigma}$$

so that

$$Im\left\{\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{-s}\right\} \approx -(-1)^{m+1}\cdot\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{-\sigma}$$

For large values of integer m,

$$\left[\frac{1-\sigma}{(1-\sigma)^2+t^2}\right]\cdot\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{1-\sigma} \gg \left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|^{-\sigma}$$

Substituting these formulae into the second relationship, or

$$Im\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} n^{-s}\right\} \sim Im\left\{\frac{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor^{1-s}}{1-s}\right\} + \left(\frac{1}{2}\right)\cdot Im\left\{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor^{-s}\right\}$$

gives

$$Im\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} n^{-s}\right\} \sim -(-1)^{m+1}\cdot\left[\frac{1-\sigma}{(1-\sigma)^2+t^2}\right]\cdot\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor^{1-\sigma}$$

Therefore, at the roots of the Riemann zeta function in the critical strip,

$$Re\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} n^{-s}\right\} \sim (-1)^{m+1}\cdot\left[\frac{1-\sigma}{(1-\sigma)^2+t^2}\right]\cdot\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor^{1-\sigma}$$

and

$$Im\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} n^{-s}\right\} \sim -(-1)^{m+1}\cdot\left[\frac{1-\sigma}{(1-\sigma)^2+t^2}\right]\cdot\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor^{1-\sigma}$$

for arbitrarily large, finite values of integer m.

Dividing the two asymptotic relationships and simplifying gives

$$Re\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} n^{-s}\right\} \sim -\left(\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\Big/\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\right)^{1-\sigma}\cdot Im\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} n^{-s}\right\}$$

Note that for large values of integer m,

$$\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\Big/\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor \approx e^{\frac{(m-1)\cdot\pi}{t}}/e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}} = e^{-\frac{\pi}{2\cdot t}}$$

and therefore,

$$Re\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} n^{-s}\right\} \sim -e^{-\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} n^{-s}\right\}$$