The Zeta Function and Integral Transforms

The theory of integral transform representations of the Riemann zeta function is well established. What follows is a brief outline of the application of Mellin, Laplace, and Fourier sine and cosine transforms to the the Riemann zeta function.

Let $f_M(x)$ be a real-valued function defined on the positive axis $0 < x < \infty$. The Mellin transform of $f_M(x)$ is the mapping of function $f_M(x)$ into the complex-valued function $F_M(s)$ by

$$F_M(s) = \int_0^\infty x^{s-1} \cdot f_M(x)\, dx$$

The function $F_M(s)$ is the Mellin transform of function $f_M(x)$. For most applications of the transform, but not including the Riemann zeta function, the Mellin transform $F_M(s)$ converges if the function $f_M(x)$ is continuous or piecewise continuous, and only in what is known as the fundamental strip, $\alpha < Re(s) < \beta$, where α and β are constants.

Mellin transforms exhibit the property of linearity

$$\theta \cdot F_{M1}(s) + \varphi \cdot F_{M2}(s) = \int_0^\infty x^{s-1} \cdot [\theta \cdot f_{M1}(x) + \varphi \cdot f_{M2}(x)]dx$$

where θ and φ are constants.

An example of a Mellin transform is the gamma function, where the complex-valued gamma function is the Mellin transform of e^{-x}:

$$\Gamma(s) = \int_0^\infty x^{s-1} \cdot e^{-x}\, dx$$

where $s = \sigma + i \cdot t$, and σ and t are real. The gamma function has no zeros in the complex plane and has simple poles at $s = -1, s = -3, s = -5, \ldots$

The Riemann zeta function is another example of a Mellin transform. The infinite series representation of the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

It will be shown in this work that, with the exceptions of its roots, the series representation of the Riemann zeta function diverges everywhere in the critical strip.

Changing the variable of integration from x to $(n \cdot x)$ in the Mellin transform for $\Gamma(s)$ above gives

$$\Gamma(s) \ = \int_0^{\infty} (n \cdot x)^{s-1} \cdot e^{-n \cdot x} \ \cdot n \, dx = \ n^s \cdot \int_0^{\infty} x^{s-1} \cdot (e^{-x})^n \, dx$$

Rearranging gives

$$n^{-s} \ = \ \frac{1}{\Gamma(s)} \cdot \int_0^{\infty} x^{s-1} \cdot (e^{-x})^n \, dx$$

Summing both sides over all positive integers, exchanging the order of summation and integration, and substituting the geometric series

$$\sum_{n=1}^{\infty} (e^{-x})^n \ = \ \frac{1}{e^x - 1} \qquad x > 0$$

gives the Mellin transform representation of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \ = \ \frac{1}{\Gamma(s)} \cdot \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx$$

which diverges everywhere in the critical strip, except at the roots of the function.

The conventional approach to the analysis of the singularity and the roots of the Riemann zeta function is as follows. Since the only source of singularity in the integral is at $x = 0$, the integral can be decomposed as

$$\zeta(s) \ = \ \frac{1}{\Gamma(s)} \cdot \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \ = \left[\frac{1}{\Gamma(s)} \cdot \int_0^{1} \frac{x^{s-1}}{e^x - 1} dx \ + \ \int_1^{\infty} \frac{x^{s-1}}{e^x - 1} dx \right]$$

Since $\Gamma(s)$ has no zeros in the complex plane, $1/\Gamma(s)$ has no singularities in the complex plane and is an entire function. The integral

$$\int_1^{\infty} \frac{x^{s-1}}{e^x - 1} dx$$

has no singularities in the complex plane is also entire. Therefore, any singularities in the Riemann zeta function must be contained in the first integral,

$$\int_0^{1} \frac{x^{s-1}}{e^x - 1} dx$$

The Laurent expansion of the integrand is

$$\frac{1}{e^x - 1} \ = \ \frac{1}{x} \ - \ \frac{1}{2} \ + \ \frac{x}{12} \ - \ \frac{x^3}{720} \ + \ \frac{x^5}{30240} \ - \ O(x^7)$$

Since

$$\int_0^1 x^{s-2}\,dx = \frac{1}{s-1} \qquad Re(s) > 1$$

$$\int_0^1 x^{s-1}\,dx = \frac{1}{s} \qquad Re(s) > 0$$

and

$$\int_0^1 x^{s-1+m}\,dx = \frac{1}{s+m} \qquad Re(s+m) > 0 \qquad m = 1, 2, 3, \ldots$$

it follows that

$$\int_0^1 \frac{x^{s-1}}{e^x - 1}dx = \frac{1}{s-1} - \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{12}\cdot\left(\frac{1}{s+1}\right) - \frac{1}{720}\cdot\left(\frac{1}{s+3}\right) + \frac{1}{30240}\cdot\left(\frac{1}{s+3}\right) - \cdots$$

This series representation of the integral has simple poles at $s = 1$, $s = 0$, and $s = -1$, $s = -3$, $s = -5$, ... Furthermore, since the gamma function has simple poles at all negative integers, or $s = -1$, $s = -2$, $s = -3$, ..., and the integral is multiplied by $1/\Gamma(s)$, it follows that the Riemann zeta function has a simple pole at $s = 1$ with residue $\Gamma(1) = 1$, and is holomorphic everywhere else in the complex plane. Also, since the roots of $1/\Gamma(s)$ are not cancelled by the poles of the integral of the Laurent expansion, the Riemann zeta function has roots at all negative even integers.

Similarly, let $f_L(x)$ be a real-valued function defined on the positive axis $-\infty < x < \infty$. The bi-lateral Laplace transform of $f_L(x)$ is defined by the mapping of function $f_L(x)$ to the complex-valued function $F_L(s)$ by

$$F_L(s) = \int_{-\infty}^{\infty} e^{-s\cdot x}\cdot f_L(x)\,dx$$

The function $F_L(s)$ is the bi-lateral Laplace transform of function $f_L(x)$. Generally, the bi-lateral Laplace transform $F_L(s)$ converges absolutely if $f_L(x)$ is continuous or piecewise continuous in the strip $a < Re(s) < b$, and possibly including the lines $Re(s) = a$ and $Re(s) = b$, where a and b are constants. Bi-lateral Laplace transforms exhibit the property of linearity

$$\theta\cdot F_{L1}(s) + \varphi\cdot F_{L2}(s) = \int_0^{\infty} x^{s-1}\cdot[\theta\cdot f_{L1}(x) + \varphi\cdot f_{L2}(x)]dx$$

where θ and φ are constants.

Let $f_{FS}(x)$ be a real-valued function defined on the positive axis $-\infty < x < \infty$. The Fourier sine transform of $f_{FS}(x)$ is defined by the mapping of function $f_{FS}(x)$ to the real-valued function $F_{FS}(\sigma, t)$ by

$$F_{FS}(\sigma, t) = \int_{-\infty}^{\infty} sin(t\cdot x)\cdot e^{-\sigma\cdot x}\cdot f_{FS}(x)\,dx$$

The function $F_{FS}(\sigma, t)$ is the Fourier sine transform of function $f_{FS}(x)$.

Similarly, let $f_{FC}(x)$ be a real-valued function defined on the positive axis -∞ < x < ∞. The Fourier cosine transform of $f_{FC}(x)$ is defined by the mapping of function $f_{FC}(x)$ into the real-valued function $F_{FC}(\sigma, t)$ by

$$F_{FC}(\sigma, t) = \int_{-\infty}^{\infty} cos(t \cdot x) \cdot e^{-\sigma \cdot x} \cdot f_{FC}(x)\, dx$$

Likewise, the function $F_{FC}(\sigma, t)$ is the Fourier cosine transform of function $f_{FC}(x)$.

The Fourier sine and cosine transforms in this work are derived directly from bi-lateral Laplace transforms, and therefore converge under the same conditions as the bi-lateral Laplace transforms.

Fourier sine and cosine transforms also exhibit the property of linearity:

$$\theta \cdot F_{FS1}(s) + \varphi \cdot F_{FS2}(s) = \int_{-\infty}^{\infty} x^{s-1} \cdot [\theta \cdot f_{FS1}(x) + \varphi \cdot f_{FS2}(x)]dx$$

and

$$\theta \cdot F_{FC1}(s) + \varphi \cdot F_{FC2}(s) = \int_{-\infty}^{\infty} x^{s-1} \cdot [\theta \cdot f_{FC1}(x) + \varphi \cdot f_{FC2}(x)]dx$$

where θ and φ are constants.

Every real-valued function $f(x)$ can be represented as the sum of an odd function and an even function:

$$f(x) = f_{odd}(x) + f_{even}(x)$$

where

$$f_{odd}(x) = \left(\frac{1}{2}\right) \cdot [f(x) - f(-x)]$$

$$f_{even}(x) = \left(\frac{1}{2}\right) \cdot [f(x) + f(-x)]$$

and

$$f_{odd}(-x) = -f_{odd}(x)$$

$$f_{even}(-x) = f_{even}(x)$$

The sum of two odd functions is an odd function and the sum of two even functions is an even function.

The product of an odd function and an even function is an odd function. Also, the product of two odd functions is an even function and the product of two even functions is an even function.

The integral of an odd function $f_{odd}(x)$ over the domain $-\infty < x < \infty$ is zero:

$$\int_{-\infty}^{\infty} f_{odd}(x)\,dx \;=\; 0$$

and the integral of an even function $f_{even}(x)$ over the domain $-\infty < x < \infty$ is:

$$\int_{-\infty}^{\infty} f_{even}(x)\,dx \;=\; 2 \cdot \int_{0}^{\infty} f_{even}(x)\,dx$$

Therefore, the Fourier sine integral of an even function $f_{even}(x)$ is zero:

$$\int_{-\infty}^{\infty} sin(x) \cdot f_{even}(x)\,dx \;=\; 0$$

and the Fourier cosine integral of an odd function $f_{odd}(x)$ is also zero:

$$\int_{-\infty}^{\infty} cos(x) \cdot f_{odd}(x)\,dx \;=\; 0$$

If the variable of integration of the Mellin transform representation of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \;=\; \frac{1}{\Gamma(s)} \cdot \int_{0}^{\infty} \frac{x^{s-1}}{e^{x} - 1}\,dx$$

is changed from x to $y = -\log(x)$, where $x = e^{-y}$, and $dx = -e^{-y}dy$, then the zeta function can be represented by the bi-lateral Laplace transform

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \;=\; \frac{1}{\Gamma(s)} \cdot \int_{-\infty}^{\infty} \frac{e^{-s \cdot x}}{e^{e^{-x}} - 1}\,dx$$

Furthermore, if the bilateral Laplace transform is separated into its real and complex components, then the real and complex components of the Riemann zeta function are given by:

$$Re\{\zeta(s)\} \;=\; Re\left\{\sum_{n=1}^{\infty} n^{-s}\right\} \;=\; \int_{-\infty}^{\infty} Re\left\{\frac{e^{-s \cdot x}}{\Gamma(s)}\right\} \cdot \left(\frac{1}{e^{e^{-x}} - 1}\right) dx$$

and

$$Im\{\zeta(s)\} \;=\; Im\left\{\sum_{n=1}^{\infty} n^{-s}\right\} \;=\; \int_{-\infty}^{\infty} Im\left\{\frac{e^{-s \cdot x}}{\Gamma(s)}\right\} \cdot \left(\frac{1}{e^{e^{-x}} - 1}\right) dx$$

Note that

$$Re\left\{\frac{e^{-s \cdot x}}{\Gamma(s)}\right\} = \left[\frac{1}{Re\Gamma(s)^2 + Im\Gamma(s)^2}\right] \cdot [Re\Gamma(s) \cdot cos(t \cdot x) \cdot e^{-\sigma \cdot x} \;-\; Im\Gamma(s) \cdot sin(t \cdot x) \cdot e^{-\sigma \cdot x}]$$

and

$$Im\left\{\frac{e^{-s \cdot x}}{\Gamma(s)}\right\} = \left[\frac{1}{Re\Gamma(s)^2 + Im\Gamma(s)^2}\right] \cdot [-Im\Gamma(s) \cdot cos(t \cdot x) \cdot e^{-\sigma \cdot x} \;-\; Re\Gamma(s) \cdot sin(t \cdot x) \cdot e^{-\sigma \cdot x}]$$

Therefore,

$$Re\{\zeta(s)\} = Re\left\{\sum_{n=1}^{\infty} n^{-s}\right\} = \left[\frac{1}{Re\Gamma(s)^2 + Im\Gamma(s)^2}\right]$$
$$\cdot\left\{Re\Gamma(s)\cdot\int_{-\infty}^{\infty} cos(t\cdot x)\cdot\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)dx - Im\Gamma(s)\right.$$
$$\left.\cdot\int_{-\infty}^{\infty} sin(t\cdot x)\cdot e^{-\sigma\cdot x}\cdot\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)dx\right\}$$

and

$$Im\{\zeta(s)\} = Re\left\{\sum_{n=1}^{\infty} n^{-s}\right\} = \left[\frac{1}{Re\Gamma(s)^2 + Im\Gamma(s)^2}\right]$$
$$\cdot\left\{- Im\Gamma(s)\cdot\int_{-\infty}^{\infty} cos(t\cdot x)\cdot\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)dx - Re\Gamma(s)\right.$$
$$\left.\cdot\int_{-\infty}^{\infty} sin(t\cdot x)\cdot e^{-\sigma\cdot x}\cdot\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)dx\right\}$$

Note again that changing the variable of integration from x to $(n\cdot x)$ in the Mellin transform for $\Gamma(s)$ above gives

$$\Gamma(s) = \int_0^{\infty}(n\cdot x)^{s-1}\cdot e^{-n\cdot x}\cdot n\,dx = n^s\cdot\int_0^{\infty} x^{s-1}\cdot(e^{-x})^n\,dx$$

and

$$n^{-s} = \frac{1}{\Gamma(s)}\cdot\int_0^{\infty} x^{s-1}\cdot(e^{-x})^n\,dx$$

Summing both sides over the positive integers from 1 to N, exchanging the order of summation and integration, and substituting

$$\sum_{n=1}^{N}(e^{-x})^n = \frac{1-e^{-N\cdot x}}{e^{x}-1} \qquad x > 0$$

gives the Mellin transform representation of the partial sums of the Riemann zeta function

$$\sum_{n=1}^{N} n^{-s} = \frac{1}{\Gamma(s)}\cdot\int_0^{\infty} x^{s-1}\cdot\left(\frac{1-e^{-N\cdot x}}{e^{x}-1}\right)dx$$

which converges everywhere in the critical strip, including at the roots of the zeta function.

If the variable of integration is changed from x to $y = -\,log\,(x)$, where $x = e^{-y}$, and $dx = -\,e^{-y}dy$, then partial sums of the zeta function are given by the bi-lateral Laplace transform as

$$\sum_{n=1}^{N} n^{-s} = \frac{1}{\Gamma(s)} \cdot \int_{-\infty}^{\infty} e^{-s\cdot x} \cdot \left(\frac{1 - e^{e^{-N\cdot x}}}{e^{e^{-x}} - 1} \right) dx$$

which converges everywhere in the critical strip, including at the roots of the zeta function.

Furthermore, if the bilateral Laplace transform is separated into its real and complex components, then the real and complex components of the partial sums of the Riemann zeta function are given by:

$$Re\left\{ \sum_{n=1}^{N} n^{-s} \right\} = \int_{-\infty}^{\infty} Re\left\{ \frac{e^{-s\cdot x}}{\Gamma(s)} \right\} \cdot \left(\frac{1 - e^{e^{-N\cdot x}}}{e^{e^{-x}} - 1} \right) dx$$

and

$$Im\left\{ \sum_{n=1}^{N} n^{-s} \right\} = \int_{-\infty}^{\infty} Im\left\{ \frac{e^{-s\cdot x}}{\Gamma(s)} \right\} \cdot \left(\frac{1 - e^{e^{-N\cdot x}}}{e^{e^{-x}} - 1} \right) dx$$

Note again that

$$Re\left\{ \frac{e^{-s\cdot x}}{\Gamma(s)} \right\} = \left[\frac{1}{Re\Gamma(s)^2 + Im\Gamma(s)^2} \right] \cdot \left[Re\Gamma(s) \cdot cos(t \cdot x) \cdot e^{-\sigma\cdot x} - Im\Gamma(s) \cdot sin(t \cdot x) \cdot e^{-\sigma\cdot x} \right]$$

and

$$Im\left\{ \frac{e^{-s\cdot x}}{\Gamma(s)} \right\} = \left[\frac{1}{Re\Gamma(s)^2 + Im\Gamma(s)^2} \right] \cdot \left[- Im\Gamma(s) \cdot cos(t \cdot x) \cdot e^{-\sigma\cdot x} - Re\Gamma(s) \cdot sin(t \cdot x) \cdot e^{-\sigma\cdot x} \right]$$

and therefore,

$$\begin{aligned} Re\left\{ \sum_{n=1}^{N} n^{-s} \right\} = & \left[\frac{1}{Re\Gamma(s)^2 + Im\Gamma(s)^2} \right] \\ & \cdot \left\{ Re\Gamma(s) \cdot \int_{-\infty}^{\infty} cos(t \cdot x) \cdot e^{-\sigma\cdot x} \cdot \left(\frac{1 - e^{e^{-N\cdot x}}}{e^{e^{-x}} - 1} \right) dx - Im\Gamma(s) \right. \\ & \left. \cdot \int_{-\infty}^{\infty} sin(t \cdot x) \cdot e^{-\sigma\cdot x} \cdot \left(\frac{1 - e^{e^{-N\cdot x}}}{e^{e^{-x}} - 1} \right) dx \right\} \end{aligned}$$

and

$$\begin{aligned} Im\left\{ \sum_{n=1}^{N} n^{-s} \right\} = & \left[\frac{1}{Re\Gamma(s)^2 + Im\Gamma(s)^2} \right] \\ & \cdot \left\{ - Im\Gamma(s) \cdot \int_{-\infty}^{\infty} cos(t \cdot x) \cdot e^{-\sigma\cdot x} \cdot \left(\frac{1 - e^{e^{-N\cdot x}}}{e^{e^{-x}} - 1} \right) dx - Re\Gamma(s) \right. \\ & \left. \cdot \int_{-\infty}^{\infty} sin(t \cdot x) \cdot e^{-\sigma\cdot x} \cdot \left(\frac{1 - e^{e^{-N\cdot x}}}{e^{e^{-x}} - 1} \right) dx \right\} \end{aligned}$$