Translation of the Variables of Integration

It was shown in this work that the real and imaginary components of the partial sums of the Riemann zeta function can be represented by the bi-lateral integral transforms, or

$$Re\left\{\sum_{n=1}^{N} n^{-s}\right\} = \int_{-\infty}^{\infty} Re\left\{\frac{e^{-s\cdot x}}{\Gamma(s)}\right\}\cdot\left(\frac{1-e^{-N\cdot e^{-x}}}{e^{e^{-x}}-1}\right)dx$$

and

$$Im\left\{\sum_{n=1}^{N} n^{-s}\right\} = \int_{-\infty}^{\infty} Im\left\{\frac{e^{-s\cdot x}}{\Gamma(s)}\right\}\cdot\left(\frac{1-e^{-N\cdot e^{-x}}}{e^{e^{-x}}-1}\right)dx$$

Substituting $\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor$ and $\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor$ and

$$Re\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} n^{-s}\right\} = \int_{-\infty}^{\infty} Re\left\{\frac{e^{-s\cdot x}}{\Gamma(s)}\right\}\cdot\left(\frac{1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}}{e^{e^{-x}}-1}\right)dx$$

and

$$Im\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} n^{-s}\right\} = \int_{-\infty}^{\infty} Im\left\{\frac{e^{-s\cdot x}}{\Gamma(s)}\right\}\cdot\left(\frac{1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}}{e^{e^{-x}}-1}\right)dx$$

into

$$Re\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} n^{-s}\right\} \sim -e^{-\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} n^{-s}\right\}$$

gives

$$\int_{-\infty}^{\infty} Re\left\{\frac{e^{-s\cdot x}}{\Gamma(s)}\right\}\cdot\left(\frac{1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}}{e^{e^{-x}}-1}\right)dx \sim -e^{-\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot\int_{-\infty}^{\infty} Im\left\{\frac{e^{-s\cdot x}}{\Gamma(s)}\right\}\cdot\left(\frac{1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}}{e^{e^{-x}}-1}\right)dx$$

Note that

$$Re\left[\frac{e^{-s\cdot x}}{\Gamma(s)}\right] = \left\{\frac{1}{\mathrm{Re}[\Gamma(s)]^2+Im[\Gamma(s)]^2}\right\}\cdot\{Re[\Gamma(s)]\cdot cos(t\cdot x)\cdot e^{-\sigma\cdot x} - Im[\Gamma(s)]\cdot sin(t\cdot x)\cdot e^{-\sigma\cdot x}\}$$

and

$$Im\left[\frac{e^{-s\cdot x}}{\Gamma(s)}\right] = \left\{\frac{1}{\mathrm{Re}[\Gamma(s)]^2 + Im[\Gamma(s)]^2}\right\} \cdot \{-Im[\Gamma(s)]\cdot cos(t\cdot x)\cdot e^{-\sigma\cdot x} - Re[\Gamma(s)]\cdot sin(t\cdot x)\cdot e^{-\sigma\cdot x}\}$$

Therefore,

$$\int_{-\infty}^{\infty} Re\left\{\frac{e^{-s\cdot x}}{\Gamma(s)}\right\}\cdot\left(\frac{1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}}{e^{e^{-x}}-1}\right)dx$$

$$= \left\{\frac{1}{\mathrm{Re}[\Gamma(s)]^2 + Im[\Gamma(s)]^2}\right\} \cdot\int_{-\infty}^{\infty}\{Re[\Gamma(s)]\cdot cos(t\cdot x)\cdot e^{-\sigma\cdot x} - Im[\Gamma(s)]\cdot sin(t\cdot x)\cdot e^{-\sigma\cdot x}\} \cdot\left(\frac{1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}}{e^{e^{-x}}-1}\right)dx$$

and

$$\int_{-\infty}^{\infty} Im\left\{\frac{e^{-s\cdot x}}{\Gamma(s)}\right\}\cdot\left(\frac{1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}}{e^{e^{-x}}-1}\right)dx$$

$$= \left\{\frac{1}{\mathrm{Re}[\Gamma(s)]^2 + Im[\Gamma(s)]^2}\right\} \cdot\int_{-\infty}^{\infty}\{-Im[\Gamma(s)]\cdot cos(t\cdot x)\cdot e^{-\sigma\cdot x} - Re[\Gamma(s)]\cdot sin(t\cdot x)\cdot e^{-\sigma\cdot x}\} \cdot\left(\frac{1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot x}}{e^{x}-1}\right)dx$$

Substituting these formulae into

$$\int_{-\infty}^{\infty} Re\left\{\frac{e^{-s\cdot x}}{\Gamma(s)}\right\}\cdot\left(\frac{1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}}{e^{e^{-x}}-1}\right)dx \sim -\, e^{-\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot\int_{-\infty}^{\infty} Im\left\{\frac{e^{-s\cdot x}}{\Gamma(s)}\right\}\cdot\left(\frac{1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}}{e^{e^{-x}}-1}\right)dx$$

and canceling the term

$$\frac{1}{\mathrm{Re}[\Gamma(s)]^2 + Im[\Gamma(s)]^2}$$

on each side gives

$$\int_{-\infty}^{\infty} \{Re[\Gamma(s)] \cdot cos(t \cdot x) \cdot e^{-\sigma \cdot x} - Im[\Gamma(s)] \cdot sin(t \cdot x) \cdot e^{-\sigma \cdot x}\} \cdot \left(\frac{1 - e^{-\left| e^{\frac{(m-1)\cdot\pi}{t}} \right| \cdot e^{-x}}}{e^{e^{-x}} - 1} \right) dx$$

$$\sim - e^{-\frac{\pi\cdot(1-\sigma)}{2\cdot t}} \cdot \int_{-\infty}^{\infty} \{- Im[\Gamma(s)] \cdot cos(t \cdot x) \cdot e^{-\sigma \cdot x} - Re[\Gamma(s)] \cdot sin(t \cdot x) \cdot e^{-\sigma \cdot x}\} \cdot \left(\frac{1 - e^{-\left| e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}} \right| \cdot x}}{e^{x} - 1} \right) dx$$

or

$$\left\{ Re[\Gamma(s)] \cdot \int_{-\infty}^{\infty} cos(t \cdot x) \cdot \left(\frac{e^{-\sigma \cdot x}}{e^{e^{-x}} - 1} \right) \cdot \left(1 - e^{-\left| e^{\frac{(m-1)\cdot\pi}{t}} \right| \cdot e^{-x}} \right) dx - Im[\Gamma(s)] \cdot \int_{-\infty}^{\infty} sin(t \cdot x) \cdot \left(\frac{e^{-\sigma \cdot x}}{e^{e^{-x}} - 1} \right) \cdot \left(1 - e^{-\left| e^{\frac{(m-1)\cdot\pi}{t}} \right| \cdot e^{-x}} \right) dx \right\}$$

$$\sim$$

$$- e^{-\frac{\pi\cdot(1-\sigma)}{2\cdot t}} \cdot \left\{ - Im[\Gamma(s)] \cdot \int_{-\infty}^{\infty} cos(t \cdot x) \cdot \left(\frac{e^{-\sigma \cdot x}}{e^{e^{-x}} - 1} \right) \cdot \left(1 - e^{-\left| e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}} \right| \cdot e^{-x}} \right) dx - Re[\Gamma(s)] \cdot \int_{-\infty}^{\infty} sin(t \cdot x) \cdot \left(\frac{e^{-\sigma \cdot x}}{e^{e^{-x}} - 1} \right) \cdot \left(1 - e^{-\left| e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}} \right| \cdot e^{-x}} \right) dx \right\}$$

Multiplying both sides by $e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}$ and rearranging gives

$$\int_{-\infty}^{\infty} sin(t \cdot x) \cdot \left\{ \left(\frac{e^{-\sigma \cdot x}}{e^{e^{-x}} - 1} \right) \cdot \left[Re[\Gamma(s)] \cdot \left(1 - e^{-\left| e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}} \right| \cdot e^{-x}} \right) + e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}} \cdot Im[\Gamma(s)] \cdot \left(1 - e^{-\left| e^{\frac{(m-1)\cdot\pi}{t}} \right| \cdot e^{-x}} \right) \right] \right\} dx$$

$$\sim$$

$$- \int_{-\infty}^{\infty} cos(t \cdot x) \cdot \left\{ \left(\frac{e^{-\sigma \cdot x}}{e^{e^{-x}} - 1} \right) \cdot \left[Im[\Gamma(s)] \cdot \left(1 - e^{-\left| e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}} \right| \cdot e^{-x}} \right) - e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}} \cdot Re[\Gamma(s)] \cdot \left(1 - e^{-\left| e^{\frac{(m-1)\cdot\pi}{t}} \right| \cdot e^{-x}} \right) \right] \right\} dx$$

In summary, at the roots of the Riemann zeta function in the critical strip, partial sums of the series converge proportionally and asymptotically when

$$\int_{-\infty}^{\infty} sin(t\cdot x)\cdot\left\{\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Re[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}\right)+e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}\right)\right]\right\}dx$$

$$\sim$$

$$-\int_{-\infty}^{\infty} cos(t\cdot x)\cdot\left\{\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Im[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}\right)-e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Re[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}\right)\right]\right\}dx$$

for arbitrarily large, finite values of integer m.

Consider the functions in the integral transforms above:

$$\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Re[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}\right)+e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}\right)\right]$$

and

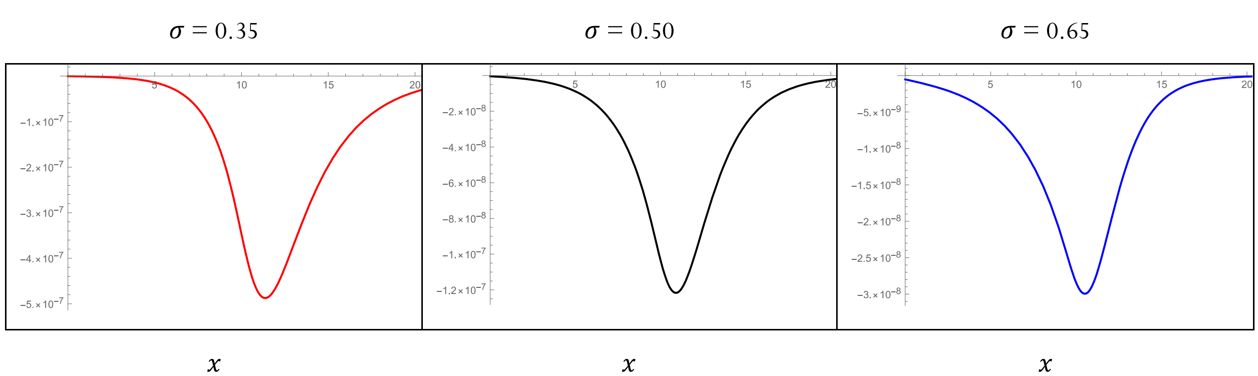
$$\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Im[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}\right)-e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Re[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}\right)\right]$$

The first function above, or

$$\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Re[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}\right)+e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}\right)\right]$$

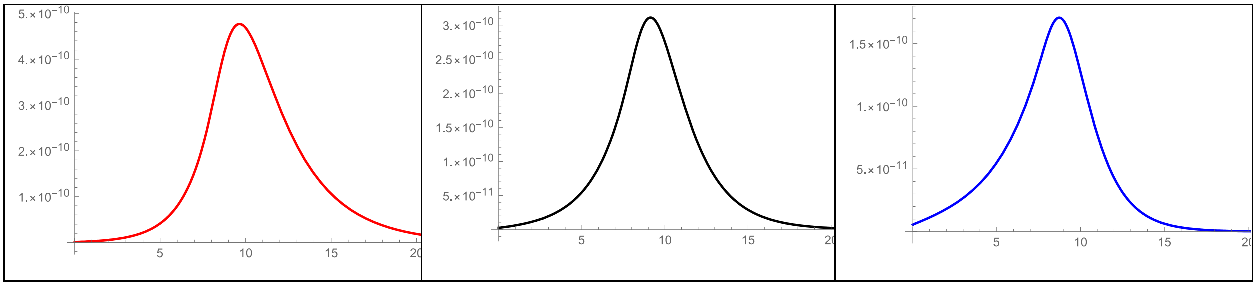
is shown in the graphs below for $t = 14.134725...$ and $t = 17$, with $m = 51$, for $\sigma = 0.35$, $\sigma = 0.50$, and $\sigma = 0.65$.

t = 14.134725



$t = 17.$

$\sigma = 0.35$ $\sigma = 0.50$ $\sigma = 0.65$



x x x

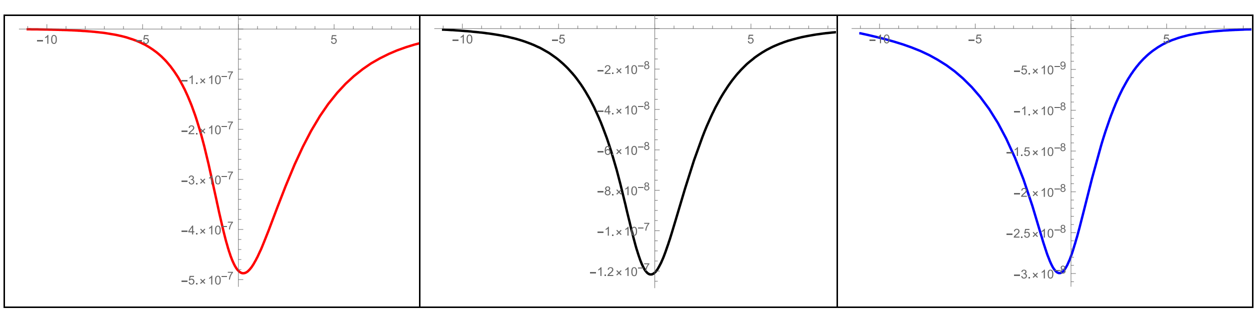
If the first function above is translated linearly with respect to the x axis, where the variable x is translated so that $x \rightarrow x + (m-1) \cdot \pi/t$, the closest approximation of the translated function to the even part of the function occurs when $\sigma = ½$, for all values of t. The translated first function, or

$$\left(\frac{e^{-\sigma\cdot\left[x+\frac{(m-1)\cdot\pi}{t}\right]}}{e^{e^{-\left[x+\frac{(m-1)\cdot\pi}{t}\right]}}-1}\right) \cdot\left[Re[\Gamma(s)]\cdot\left(1-e^{-\left[e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right]\cdot e^{-\left[x+\frac{(m-1)\cdot\pi}{t}\right]}}\right)+e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im[\Gamma(s)] \cdot\left(1-e^{-\left[e^{\frac{(m-1)\cdot\pi}{t}}\right]\cdot e^{-\left[x+\frac{(m-1)\cdot\pi}{t}\right]}}\right)\right]$$

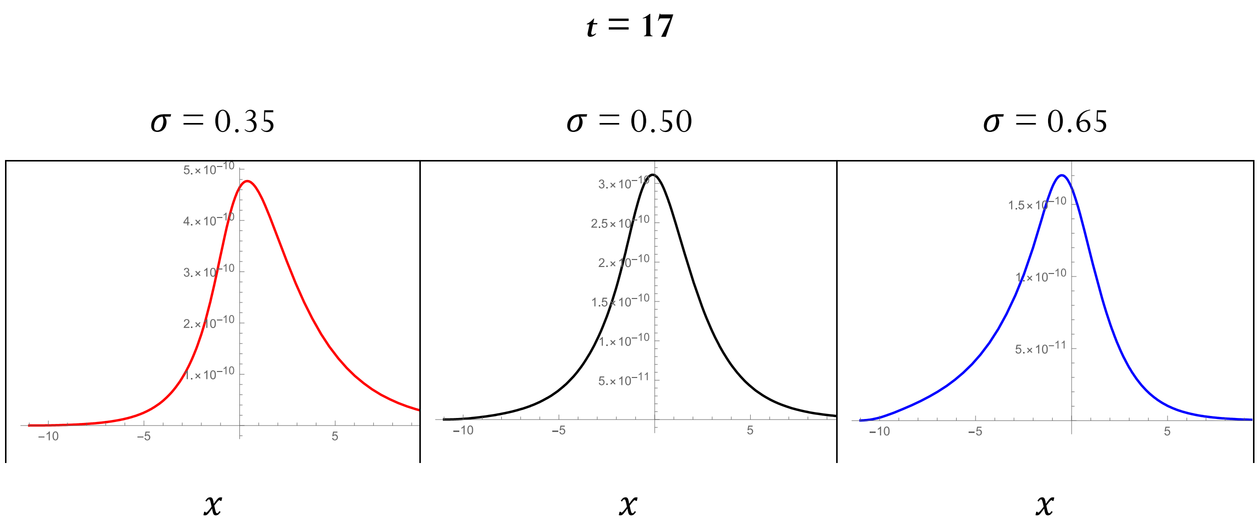
is shown in the graphs below for $t = 14.134725$ and $t = 17$, with $m = 51$, for $\sigma = 0.35$, $\sigma = 0.50$, and $\sigma = 0.65$.

$t = 14.134725$

$\sigma = 0.35$ $\sigma = 0.50$ $\sigma = 0.65$



x x x



t = 17
σ = 0.35
σ = 0.50
σ = 0.65
x
x
x