Summary

The infinite series representation of the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \;=\; 1 + 2^{-s} + 3^{-s} + \cdots$$

diverges everywhere in the critical strip, except at its roots, where it converges to zero.

Borel integral summation, the Euler-Maclaurin summation formula, and the Cauchy residue theorem were used to show that the roots of the Riemann zeta function in the critical strip are solutions of the asymptotic relationship

$$\sum_{n=1}^{N} n^{-s} \;\sim\; \int_{0}^{N} x^{-s}\, dx$$

where

$$\sum_{n=1}^{N} (\, n^{-s} \,+\, n^{-\bar{s}}) \;\sim\; \int_{0}^{N} (\, x^{-s} \,+\, x^{-\bar{s}})\, dx$$

and

$$\sum_{n=1}^{N} (\, n^{-s} -\, n^{-\bar{s}}) \;\sim\; \int_{0}^{N} (\, x^{-s} -\, x^{-\bar{s}})\, dx$$

for arbitrarily large values of integer N.

These sums and integrals all individually diverge as $N \to \infty$ everywhere in the critical strip, including at the roots of the Riemann zeta function. However, the roots of the zeta function in the critical strip occur when the respective sums and integrals are asymptotic.

Substituting $N = \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor$ and $N = \left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}} \right\rfloor$ into the relationships above gave

$$Re\left\{ \sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor} n^{-s} \right\} \;\sim\; (-1)^{m+1} \cdot \left[\frac{1-\sigma}{(1-\sigma)^2 + t^2}\right] \cdot \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}} \right\rfloor^{1-\sigma}$$

and

$$Im\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} n^{-s}\right\} \sim -(-1)^{m+1}\cdot\left[\frac{1-\sigma}{(1-\sigma)^2+t^2}\right]\cdot\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor^{1-\sigma}$$

Division and simplification gave

$$Re\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} n^{-s}\right\} \sim -e^{-\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} n^{-s}\right\}$$

Bi-lateral integral transforms were used to represent the partials sums as:

$$Re\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} n^{-s}\right\} = \int_{-\infty}^{\infty} Re\left\{\frac{e^{-s\cdot x}}{\Gamma(s)}\right\}\cdot\left(\frac{1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}}{e^{e^{-x}}-1}\right)dx$$

and

$$Im\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} n^{-s}\right\} = \int_{-\infty}^{\infty} Im\left\{\frac{e^{-s\cdot x}}{\Gamma(s)}\right\}\cdot\left(\frac{1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}}{e^{e^{-x}}-1}\right)dx$$

Substitution gave

$$\int_{-\infty}^{\infty} Re\left\{\frac{e^{-s\cdot x}}{\Gamma(s)}\right\}\cdot\left(\frac{1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}}{e^{e^{-x}}-1}\right)dx \sim -e^{-\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot\int_{-\infty}^{\infty} Im\left\{\frac{e^{-s\cdot x}}{\Gamma(s)}\right\}\cdot\left(\frac{1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}}{e^{e^{-x}}-1}\right)dx$$

and therefore,

$$\int_{-\infty}^{\infty} sin(t\cdot x)\cdot \left\{\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Re[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}\right)+e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}\right)\right]\right\}dx$$

$$\sim$$

$$-\int_{-\infty}^{\infty} cos(t\cdot x)\cdot \left\{\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Im[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}\right)-e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Re[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}\right)\right]\right\}dx$$

It was proved that the sine and cosine integral transforms above, or

$$\int_{-\infty}^{\infty} sin(t \cdot x) \cdot \left\{\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Re[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}\right)+e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}\right)\right]\right\} dx$$

and

$$\int_{-\infty}^{\infty} cos(t \cdot x) \cdot \left\{\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Im[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}\right)-e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Re[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}\right)\right]\right\} dx$$

vanish asymptotically and simultaneously if and only if the functions in the integrands, or

$$\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Re[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}\right)+e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}\right)\right]$$

and

$$\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Im[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}\right)-e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Re[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}\right)\right]$$

are both even functions of the variable of integration x, where x is translated as $x + (m-1)\cdot\pi/t$ and as $x + (2\cdot m - 1)\cdot\pi/(2\cdot t)$, in the respective functions. The closest approximations of the translated functions to the even parts of the functions occur when $\sigma = ½$, for all values of the imaginary part of the argument, t.

It is important to note that the values of $N = \left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor$ and $N = \left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor$ were substituted in the respective partial sums

$$Re\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor} n^{-s}\right\} \sim (-1)^{m+1}\cdot\left[\frac{1-\sigma}{(1-\sigma)^2+t^2}\right]\cdot\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor^{1-\sigma}$$

and

$$Im\left\{\sum_{n=1}^{\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor} n^{-s}\right\} \sim -(-1)^{m+1}\cdot\left[\frac{1-\sigma}{(1-\sigma)^2+t^2}\right]\cdot\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor^{1-\sigma}$$

for a reason.

Translating the variable of integration for each integral transform in the asymptotic relationship

$$\int_{-\infty}^{\infty} sin(t\cdot x)\cdot \left\{\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Re[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}\right)+e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}\right)\right]\right\}dx$$

$$\sim$$

$$-\int_{-\infty}^{\infty} cos(t\cdot x)\cdot \left\{\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Im[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-x}}\right)-e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Re[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-x}}\right)\right]\right\}dx$$

and substituting the identity

$$cos\left[t\cdot x+\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}\right] = -\,sin\left[t\cdot x+\frac{(m-1)\cdot\pi}{t}\right]$$

for cos $(t\cdot x)$ on the right-hand side gives

$$\int_{-\infty}^{\infty} sin\left[t\cdot x+\frac{(m-1)\cdot\pi}{t}\right]\cdot\left(\frac{e^{-\sigma\cdot\left[x+\frac{(m-1)\cdot\pi}{t}\right]}}{e^{e^{-\left[x+\frac{(m-1)\cdot\pi}{t}\right]}}-1}\right)\cdot \left[Re[\Gamma(s)]\cdot\left(1-e^{-\left\lfloor e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right\rfloor\cdot e^{-\left[x+\frac{(m-1)\cdot\pi}{t}\right]}}\right)+e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im[\Gamma(s)] \cdot\left(1-e^{-\left\lfloor e^{\frac{(m-1)\cdot\pi}{t}}\right\rfloor\cdot e^{-\left[x+\frac{(m-1)\cdot\pi}{t}\right]}}\right)\right]dx$$

$$\sim$$

$$\int_{-\infty}^{\infty} sin\left[t\cdot x+\frac{(m-1)\cdot\pi}{t}\right]\cdot\left(\frac{e^{-\sigma\cdot\left[x+\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}\right]}}{e^{e^{-\left[x+\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}\right]}}-1}\right)\cdot$$

$$\left[Im[\Gamma(s)]\cdot\left(1-e^{-\left[e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right]\cdot e^{-\left[x+\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}\right]}}\right)-e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Re[\Gamma(s)]\cdot\left(1-e^{-\left[e^{\frac{(m-1)\cdot\pi}{t}}\right]\cdot e^{-\left[x+\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}\right]}}\right)\right]dx$$

Note that sine functions in the integrands of both integral transforms are odd functions of the variable x. Furthermore, was shown that the translated functions

$$\left(\frac{e^{-\sigma\cdot\left[x+\frac{(m-1)\cdot\pi}{t}\right]}}{e^{e^{-\left[x+\frac{(m-1)\cdot\pi}{t}\right]}}-1}\right)\cdot$$

$$\left[Re[\Gamma(s)]\cdot\left(1-e^{-\left[e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right]\cdot e^{-\left[x+\frac{(m-1)\cdot\pi}{t}\right]}}\right)+e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im[\Gamma(s)]\cdot\left(1-e^{-\left[e^{\frac{(m-1)\cdot\pi}{t}}\right]\cdot e^{-\left[x+\frac{(m-1)\cdot\pi}{t}\right]}}\right)\right]$$

and

$$\left(\frac{e^{-\sigma\cdot\left[x+\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}\right]}}{e^{e^{-\left[x+\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}\right]}}-1}\right)\cdot$$

$$\left[Im[\Gamma(s)]\cdot\left(1-e^{-\left[e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right]\cdot e^{-\left[x+\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}\right]}}\right)-e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Re[\Gamma(s)]\cdot\left(1-e^{-\left[e^{\frac{(m-1)\cdot\pi}{t}}\right]\cdot e^{-\left[x+\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}\right]}}\right)\right]$$

are the best approximations of even functions of x, if and only if $\sigma = 1/2$, for all values of t. Since the products of odd and even functions are odd functions, and Fourier integrals of odd functions are identically zero, the asymptotic relationships above can be satisfied at the roots of the Riemann zeta function - if and only if $\sigma = 1/2$.

In conclusion, at the roots of the Riemann zeta function in the critical strip, (1.) the asymptotic relationship:

$$\int_{-\infty}^{\infty} sin(t\cdot x)\cdot \left\{\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Re[\Gamma(s)]\cdot\left(1-e^{-\left[e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right]\cdot e^{-x}}\right)+e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im[\Gamma(s)]\cdot\left(1-e^{-\left[e^{\frac{(m-1)\cdot\pi}{t}}\right]\cdot e^{-x}}\right)\right]\right\}dx$$

$$\sim$$

$$-\int_{-\infty}^{\infty} cos(t\cdot x)\cdot \left\{\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Im[\Gamma(s)]\cdot\left(1-e^{-\left[e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right]\cdot e^{-x}}\right)-e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Re[\Gamma(s)]\cdot\left(1-e^{-\left[e^{\frac{(m-1)\cdot\pi}{t}}\right]\cdot e^{-x}}\right)\right]\right\}dx$$

must be satisfied for arbitrarily large, finite values of integer m, and (2.) this relationship is satisfied only when the translated functions in the integrands of the integral transforms:

$$\left(\frac{e^{-\sigma\cdot\left[x+\frac{(m-1)\cdot\pi}{t}\right]}}{e^{e^{-\left[x+\frac{(m-1)\cdot\pi}{t}\right]}}-1}\right)\cdot$$

$$\left[Re[\Gamma(s)]\cdot\left(1-e^{-\left[e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right]\cdot e^{-\left[x+\frac{(m-1)\cdot\pi}{t}\right]}}\right)+\ e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im[\Gamma(s)]\cdot\left(1-e^{-\left[e^{\frac{(m-1)\cdot\pi}{t}}\right]\cdot e^{-\left[x+\frac{(m-1)\cdot\pi}{t}\right]}}\right)\right]$$

and

$$\left(\frac{e^{-\sigma\cdot\left[x+\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}\right]}}{e^{e^{-\left[x+\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}\right]}}-1}\right)\cdot$$

$$\left[Im[\Gamma(s)]\cdot\left(1-e^{-\left[e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right]\cdot e^{-\left[x+\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}\right]}}\right)-\ e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Re[\Gamma(s)]\cdot\left(1-e^{-\left[e^{\frac{(m-1)\cdot\pi}{t}}\right]\cdot e^{-\left[x+\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}\right]}}\right)\right]$$

most closely approximate even functions of the variable of integration, (3.) the translated functions most closely approximate even functions of the variable of integration - if and only if σ = ½ - for all values of t, and (4.) when σ = ½ and $x = (m-1)\cdot\pi/t$ and $x = (2\cdot m-1)\cdot\pi/(2\cdot t)$, or linear combinations of these respective general terms, the integral transforms vanish simultaneously, and the roots of the Riemann zeta function occur in the critical strip.

Therefore, all roots of the Riemann zeta function in the critical strip have real part equal to ½ and the Riemann hypothesis is correct.

It was mentioned in the Preface that the mathematics of the Riemann hypothesis bridges the discrete and the continuous. The series representation of the Riemann zeta function is an infinite sum of integers, each raised to a common, complex power, and it diverges everywhere in the critical strip, except at its roots. The zeta function can also be represented as Mellin and bi-lateral integral transforms. Furthermore, the partial sums of the zeta function and their bi-lateral integral transform representations incorporate an additional integer, further complicating the bridging of the discrete and continuous representations of the function. It was shown that when σ = ½, the functions:

$$\left\{\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Re[\Gamma(s)]\cdot\left(1-e^{-\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|\cdot e^{-x}}\right)+\ e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im[\Gamma(s)]\cdot\left(1-e^{-\left|e^{\frac{(m-1)\cdot\pi}{t}}\right|\cdot e^{-x}}\right)\right]\right\}$$

and

$$\left\{\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Im[\Gamma(s)]\cdot\left(1-e^{-\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|\cdot e^{-x}}\right)-\ e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Re[\Gamma(s)]\cdot\left(1-e^{-\left|e^{\frac{(m-1)\cdot\pi}{t}}\right|\cdot e^{-x}}\right)\right]\right\}$$

in the integrands of the respective bi-lateral sine and cosine transforms:

$$\int_{-\infty}^{\infty} sin(t\cdot x)\cdot$$
$$\left\{\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Re[\Gamma(s)]\cdot\left(1-e^{-\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|\cdot e^{-x}}\right)+\ e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Im[\Gamma(s)]\cdot\left(1-e^{-\left|e^{\frac{(m-1)\cdot\pi}{t}}\right|\cdot e^{-x}}\right)\right]\right\}dx$$

and

$$\int_{-\infty}^{\infty} cos(t\cdot x)\cdot$$
$$\left\{\left(\frac{e^{-\sigma\cdot x}}{e^{e^{-x}}-1}\right)\cdot\left[Im[\Gamma(s)]\cdot\left(1-e^{-\left|e^{\frac{(2\cdot m-1)\cdot\pi}{2\cdot t}}\right|\cdot e^{-x}}\right)-\ e^{\frac{\pi\cdot(1-\sigma)}{2\cdot t}}\cdot Re[\Gamma(s)]\cdot\left(1-e^{-\left|e^{\frac{(m-1)\cdot\pi}{t}}\right|\cdot e^{-x}}\right)\right]\right\}dx$$

are not precisely even functions of the translated variable of integration. As a result, the roots of the sine and cosine functions, *i.e.* when $x=(m-1)\cdot\pi/t$ and hence $sin(t\cdot x)=0$, and when $x=(2\cdot m-1)\cdot\pi/(2\cdot t)$ and hence $cos(t\cdot x)=0$, do not coincide precisely with the respective maxima or minima of the functions in the integrands of the transforms. However, when $\sigma=$ ½, and when $x=(m-1)\cdot\pi/t$ and $x=(2\cdot m-1)\cdot\pi/(2\cdot t)$, or linear combinations of these general terms, the integral transforms vanish simultaneously, and the roots of the Riemann zeta function occur in the critical strip.