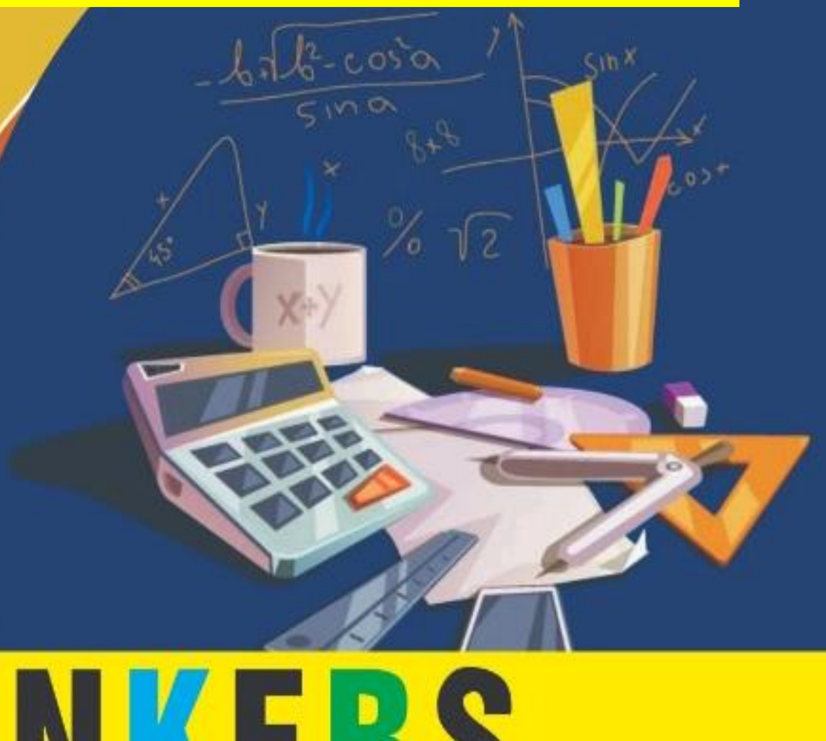
Rahul's
Science
Academy

# MATHEMATICS

For JEE / IIT ADVANCED / CET

Rankers

# Binomial Theorem



# RANKERS

# Binomial Theorem by Rahul sir

$(a+b)^0 = 1$

$(a+b)^1 = (a+b)$

$(a+b)^2 = (a^2+2ab+b^2)$

$(a+b)^3 = (a^3+3a^2b+3ab^2+b^3)$

$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$

$(a+b)^5 =$

$(a+b)^{17} =$

$$\underline{(a+b)^n} = \underbrace{{}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2}_{\text{Binomial Theorem}} + {}^nC_3 a^{n-3} b^3 + \dots\dots + \underline{{}^nC_n a^{n-n} b^n}$$

a, b – positive integer.
n – integer

${}^nC_0, {}^nC_1, {}^nC_2 \dots {}^nC_n$ – Binomial coefficient.

What is ${}^nC_r$?

$${}^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r} \approx \frac{n!}{r!\,(n-r)!}$$

$0 \leq r \leq n$

What is factorial?

$5! = 5\times4\times3\times2\times1$.

$9! = 9\times8\times7\times6\times5\times4\times3\times2\times1$

$9! = 9\times8!$

$= 9\times8\times7!$

$= 9\times8\times7\times6!$

$\hookrightarrow n! = n(n-1)(n-2)\dots\dots 3\times2\times1$.

$\boxed{\text{How to get } {}^nC_r}$ ??

Ex :- ${}^5C_2 = \frac{\lfloor 5}{\lfloor 2 \lfloor 5-2}$

$= \frac{\lfloor 5}{\lfloor 2 \lfloor 3} = \frac{5\times\cancel{4}^{2}\times\cancel{3}\times\cancel{2}\times1}{\cancel{2}\times1 \quad \cancel{3}\times\cancel{2}\times1} = 10$

${}^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r}$

${}^8C_3 = \frac{\lfloor 8}{\lfloor 3 \lfloor 5} = 56$,

${}^{11}C_9 = \frac{\lfloor 11}{\lfloor 9 \lfloor 2} = \frac{11\times10\times\cancel{\lfloor 9}}{\cancel{\lfloor 9}\times\lfloor 2}$

$= \frac{11\times\cancel{10}^{5}}{\cancel{2}\times1} = \underline{55}$.

✱ ${}^nC_1 = \frac{\lfloor n}{\lfloor 1 \lfloor n-1} = \frac{n(n-1)(n-2)\dots\dots 1}{\lfloor n-1}$

$= \frac{n\cancel{\lfloor(n-1)}}{1\ \cancel{\lfloor n-1}} = n$.

$\boxed{\text{Rem :- } \lfloor 0 = 1}$

$\rightarrow {}^{11}C_9 = 55$

$\rightarrow {}^{11}C_2 = \frac{\lfloor 11}{\lfloor 2 \lfloor 9} = \frac{11\times10\times\cancel{\lfloor 9}}{2\times1\times\cancel{\lfloor 9}}$

$= 55$

✱ ${}^nC_0 = \frac{\cancel{\lfloor n}}{\lfloor 0 \cancel{\lfloor n}} = \frac{\lfloor n}{1.\lfloor n} = 1$

✱ ${}^nC_n = \frac{\lfloor n}{\lfloor n \lfloor n-n} = \frac{\lfloor n}{\lfloor n \lfloor 0} = 1$

✱ $\boxed{{}^nC_r = {}^nC_{n-r}}$ ✱

${}^{11}C_9 = {}^{11}C_{11-9} = {}^{11}C_2 = 55$

✱ $\boxed{{}^nC_r = {}^nC_{n-r}}$ ✱✱ V.V.I

$$(a+b)^n = {}^nc_0 a^n b^0 + {}^nc_1 a^{n-1} b^1 + \dots \dots + {}^nc_n b^n. \leftarrow \text{Binomial Expansion}$$

$$(a+b)^3 = {}^3c_0\, a^3 . b^0 + {}^3c_1\, a^{3-1} . b^1 + {}^3c_2\, a^{3-2}\, b^2 + {}^3c_3\, a^{3-3}\, b^3 .$$

$$= 1 . a^3 . 1 + 3 . a^2 . b + 3 . a^1 . b^2 + 1 . 1\, b^3$$

$$(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$$ — proved from Binomial expansion

$${}^3c_1 = \frac{3!}{1!\,2!} = 3.$$

$${}^3c_2 = \frac{3!}{2!\,1!} = 3.$$

$${}^nc_n = \frac{n!}{0!\,n!} = 1.$$

**Sample**

* $\boxed{\text{Expand.} \sim (x^2+2y)^5.}$

$\rightarrow (a+b)^n = {}^nc_0 a^n b^0 + {}^nc_1 a^{n-1} b^1 + \dots \dots {}^nc_n a^0 b^n.$

Formula

$$= {}^5c_0 (x^2)^5 (2y)^0 + {}^5c_1 (x^2)^{5-1} (2y)^1 + {}^5c_2 (x^2)^{5-2} . (2y)^2 + {}^5c_3 (x^2)^{5-3} . (2y)^3 + {}^5c_4 . (x^2) . (2y)^4 + {}^5c_5 (x^2)^0 (2y)^5$$

$$= 1 . x^{10} . 1 + 5 . x^8 . 2y . + 10 . x^6\, 2^2 . y^2 + 10 . x^4 . 2^3 y^3 + 5 * x^2 . 2^4 . y^4 + 1 * (1) * 2^5 y^5 .$$

$$= x^{10} + 10 x^8 y + 40 x^6 y^2 + 80 x^4 y^3 + 80 x^2 y^4 + 32\, y^5 \quad \text{Ans}$$

**Few more formulas**

$${}^nc_r = \frac{n!}{r!\,(n-r)!}$$

$\hookrightarrow {}^nc_r = {}^nc_{n-r}$

$\rightarrow {}^nc_r + {}^nc_{r-1} = {}^{n+1}c_r$

$\rightarrow \boxed{{}^nc_x = {}^nc_y}$ ⊛

$\hookrightarrow \boxed{x=y}$ either

or $\boxed{x+y=n}$ — Example

${}^{11}c_9 = {}^{11}c_2 = 55$

${}^nc_x = {}^nc_y$

$\boxed{9+2=11}$

verify.

$$\boxed{{}^5c_3 + {}^5c_2 = {}^6c_3}$$

$$= \frac{5!}{3!\,2!} + \frac{5!}{3!\,2!} = \frac{6!}{3!\,3!}$$

$= 10 + 10$

$= 20$

$\boxed{20}$

${}^5c_3 = 10$

${}^5c_2 = 10$

${}^5c_3 = {}^5c_{5-3}$

${}^nc_r = {}^nc_{n-r}$

$$\rightarrow \frac{{}^nc_r}{{}^nc_{r-1}} = \frac{\dfrac{n!}{r!\,(n-r)!}}{\dfrac{n!}{(r-1)!\,(n-r+1)!}}$$

$$= \frac{\cancel{n!}}{r!\,(n-r)!} * \frac{(r-1)!\,(n-r+1)!}{\cancel{n!}}$$

$$= \frac{(r-1)!\,(n-r+1)!}{r!\,(n-r)!} = \frac{\cancel{(r-1)!}\,(n-r+1)\,\cancel{(n-r)!}}{r\,\cancel{(r-1)!} * \cancel{(n-r)!}} = \frac{n-r+1}{r}$$

$$\frac{{}^nc_r}{{}^nc_{r-1}} = \frac{n-r+1}{r}$$

Shortcut

$$\frac{{}^{10}c_5}{{}^{10}c_4} = \frac{10-5+1}{5} = \frac{6}{5}.$$

* $(x+y)^n = {}^nC_0\, x^n \cdot y^0 + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \cdots + {}^nC_n\, x^0 y^n$

* Let $x = y = 1$

* $\rightarrow\ 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots {}^nC_n$ ⊛ ← Formula

* $(1+x)^n = {}^nC_0\, 1^n \cdot x^0 + {}^nC_1\, 1^{n-1} \cdot x^1 + {}^nC_2\, 1^{n-2} \cdot x^2 + \cdots - + {}^nC_n\, 1^0 \cdot x^n$

$= 1 + nx + \dfrac{\lfloor n}{\lfloor 2 \lfloor n-2} \cdot x^2 + \cdots + x^n$

$= 1 + nx + \dfrac{n \cdot (n-1) \lfloor n-2}{\lfloor 2 \cdot \lfloor n-2} x^2 + \cdots + x^n$

* $\boxed{(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!} x^2 + \cdots + x^n}$ [Formula] *

* $\boxed{(1-x)^n = 1 - nx + \dfrac{n(n-1)}{2!} x^2 \cdots + (-x)^n}$ $\boxed{\text{Formula}}$ *

* $(x+a)^n + (x-a)^n = \left[{}^nC_0 x^n . a^0 + {}^nC_1 \cancel{x^{n-1}} . a^1 + {}^nC_2 x^{n-2} a^2 + \cdots {}^nC_n x^0 a^n\right]$
$+ \left[{}^nC_0 x^n (-a)^0 + {}^nC_1 \cancel{x^{n-1} (-a)^1} + \cdots + \cdots + {}^nC_n (x^0)(-a)^n\right]$

* $(x+a)^n + (x-a)^n = 2\left[{}^nC_0 x^n + {}^nC_2 x^{n-2} a^2 + {}^nC_4 x^{n-4} a^4 + \cdots\right]$
sum of "odd places".

** $(x+a)^n - (x-a)^n = 2\left[{}^nC_1 x^{n-1} . a^1 + {}^nC_3 x^{n-3} a^3 + \cdots\right]$
"sum of even place"

## Find

Q. $\boxed{(\sqrt{2}+1)^4 + (\sqrt{2}-1)^4 = ?}$

$= \left({}^4C_0 \sqrt{2}^4 . 1^0 + {}^4C_1 \cancel{\sqrt{2}^3 . 1^1} + {}^4C_2 \sqrt{2}^2 1^2 + {}^4C_3 \cancel{\sqrt{2}^1\, 1^3} + {}^4C_4 \sqrt{2}^0 1^4\right) + \left(\underline{{}^4C_0 (\sqrt{2})^4 . (-1)^0} + {}^4C_1 \cancel{(\sqrt{2})^3 (-1)^1}\right.$
$+ {}^4C_2 (\sqrt{2})^2 (-1)^2 + {}^4C_3 \cancel{(\sqrt{2})^1 (-1)^3}$
$\left. + {}^4C_4 (\sqrt{2})^0 (-1)^4\right)$

$= 2\left[{}^4C_0 (\sqrt{2})^4 . 1^0 + {}^4C_2 (\sqrt{2})^2 . 1^2 + {}^4C_4 (\sqrt{2})^0 1^4\right]$

Sum of odd terms

You can use formula directly

$= 2\left[1 . 4 + 6 . 2 + 1 * 1 * 1\right]$

$= 2\left[4 + 12 + 1\right] = 2 * 17 = \underline{34}$ Ans

$$(x+y)^n = {}^nC_0\, x^n y^0 + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 \dots \quad + {}^nC_n\, x^0 y^n$$

Q. $\boxed{(\sqrt{2}+1)^6 - (\sqrt{2}-1)^6}$ ← Solve This

$$= (x+a)^n - (x-a)^n = 2\left[{}^nC_1\, x^{n-1} a^1 + {}^nC_3\, x^{n-3} a^3 + \dots\right]$$

$$= 2\left[{}^6C_1 (\sqrt{2})^{6-1} \cdot (1)^1 + {}^6C_3 (\sqrt{2})^3 (1)^3 + {}^6C_5 (\sqrt{2})^{6-5} \cdot (1)^5\right]$$

$$= 2\left[6 \cdot (\sqrt{2})^5 + 20(\sqrt{2})^3 + 6(\sqrt{2})^1\right]$$

$$= 2\sqrt{2}\left[6(\sqrt{2})^4 + 20(\sqrt{2})^2 + 6\right]$$

$$= 2\sqrt{2}\left[6 \times 4 + 20 \times 2 + 6\right] = \underline{140\sqrt{2}} \text{ Ans}$$

## General term of Binomial Expansion

$$(a+b)^n = {}^nC_0\, a^n + \underbrace{{}^nC_1\, a^{n-1}}\, b^1 + {}^nC_2\, a^{n-2} b^2 + \dots \quad + {}^nC_n\, a^0 b^n$$

First term $= {}^nC_0\, a^n$

2nd term $= {}^nC_1\, a^{n-1} b^1$

3rd term $= {}^nC_2\, a^{n-2} b^2$

4th term $= {}^nC_3\, a^{n-3} b^3$

(r+1) th term $= {}^nC\ a^{n-r} b^r$

$$\boxed{T_{r+1} = {}^nC_r\, a^{n-r} b^r}$$

Q. $\left(\frac{x}{3} - 3y\right)^7 \approx \left(\frac{x}{3} + (-3y)\right)^7$

Find 5th term of the expansion.

$$T_5 = T_{4+1} = {}^7C_4 \left(\frac{x}{3}\right)^{7-4} (-3y)^4 \quad \text{(General term)}$$

$$= \frac{\lfloor 7}{\lfloor 4 \lfloor 3} \cdot \left(\frac{x^3}{3^3}\right) (-3)^4 \cdot (y)^4$$

$$= \frac{7 \times 6 \times 5\, \cancel{\lfloor 4}}{\cancel{3 \times 2 \times 1} \times \cancel{\lfloor 4}} \cdot \frac{x^3}{3^3} \cdot 3^4 \cdot y^4$$

$$= 35 \cdot x^3 \cdot y^4 \times 3 = \underline{105\, x^3 y^4}$$

Coefficient of 5th $\underline{105}$ Ans

# How to find Middle term?

$(a+b)^2 = a^2 + 2ab + b^2$

Power 2 $\longrightarrow$ terms 3. $\longrightarrow$ Power even 2nd. $\left(\frac{n}{2}+1\right)$

Power 3 $\longrightarrow$ terms 4

For even Number

$(a+b)^6 = \times \quad \times \quad \times \quad \times \quad \times \quad \times \quad \times$ $\quad \frac{6}{2}+1$ $\quad \left(\frac{n}{2}+1\right)$ Middle Term.

4th

$12 \longrightarrow \frac{12}{2}+1 = 7th$

$(a+b)^7 = \underset{1}{\times} \quad \underset{2}{\times} \quad \underset{3}{\times} \quad \underset{4}{\times} \quad \underset{5}{\times} \quad \underset{6}{\times} \quad \underset{7}{\times} \quad \underset{8}{\times}$

Two middle

for odd case (odd power)

$\frac{7+1}{2}, \quad \frac{7+1}{2}+1,$

$\underbrace{\frac{n+1}{2}, \quad \frac{n+1}{2}+1}$ two Middle term

Middle term,

$(x+y)^{15} \longrightarrow \boxed{8,9}$ correct

$(x+y)^{20} \longrightarrow$ Middle term $\approx \frac{20}{2}+1 \approx 11$th term

Q. $\boxed{\text{Find Middle term} \sim (2x+3y)^9.}$

Soln Power ~ 9 (odd)

Middle term $= \frac{9+1}{2}, \frac{9+1}{2}+1$

$= 5, 6.$

$T_5 = T_{4+1} = {}^nC_r\, a^{n-r}\, b^r$

$= {}^9C_4\, (2x)^{9-4} \cdot (3y)^4$

$= 126 \quad 2^5\, x^5\, 3^4 \cdot y^4$

$= 326592\, x^5 y^4$. Ans

${}^9C_4 = \frac{9\times8\times7\times6}{4\times3\times2\times1}$

${}^{12}C_3 = \frac{12\times11\times10}{3\times2\times1}$

$T_6 = {}^9C_5\, (2x)^4\, (3y)^5 = 489888\, x^4 y^5$

Q. $\boxed{\text{Find Middle term} \sim \left(3 - \frac{x^3}{6}\right)^6.}$

Soln Middle term $T_4$

$\frac{6}{2}+1 \qquad \frac{n}{2}+1$

$3+1$

$T_4 = T_{3+1} = {}^nC_r\, a^{n-r}\, b^r$

$= {}^6C_3\, (3)^{6-3} \cdot \left(-\frac{x^3}{6}\right)^3$

$= \frac{6\times5\times4}{3\times2\times1}\, (3)^3 \cdot (-1)^3 \cdot \frac{x^9}{6^3} = -\frac{5}{2} \cdot x^9$

Q. $\boxed{\text{find The coefficient of } x^{32} \text{ in The expansion of } \left(x^4 - \frac{1}{x^3}\right)^{15}}$

lets find general term

$$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$$
$$= {}^{15}C_r \left(x^4\right)^{15-r} \cdot \left(-\frac{1}{x^3}\right)^r$$
$$= {}^{15}C_r\, x^{60-4r} \cdot (-1)^r \cdot \frac{1}{x^{3r}}$$
$$= {}^{15}C_r\, x^{60-4r-3r}\, (-1)^r = (-1)^r \cdot {}^{15}C_r \cdot x^{\boxed{60-7r}}$$

32

$60 - 7r = 32$

$60 - 32 = 7r$

$r = 4$

$T_{4+1} = T_5$ (5Th term will → us $x^{32}$)

Coefficient $= (-1)^r\; {}^{15}C_r$

$= (-1)^4\; {}^{15}C_4$

$= {}^{15}C_4 \approx 1365$ Ans

$\boxed{\begin{array}{l}\text{* Type ~2 ( Find term independent of variable)} \\ \text{Q. Find The term independent of } x \sim \left(2x - \frac{1}{x}\right)^{10}\end{array}}$

$$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$$
$$= {}^{10}C_r\, (2x)^{10-r} \cdot \left(-\frac{1}{x}\right)^r$$
$$= {}^{10}C_r\, 2^{10-r}\, x^{10-r}\, (-1)^r \cdot \frac{1}{x^r}$$
$$= {}^{10}C_r\, 2^{10-r} \cdot x^{10-r-r}\, (-1)^r = {}^{10}C_r\, 2^{10-r}\, x^{\boxed{10-2r}}\, (-1)^r$$

lets assume $T_{r+1}$ is independent of $x$

$\boxed{10 - 2r = 0}$

$10 = 2r$ $\quad \boxed{r = 5}$

$\boxed{T_6 \text{ th term is independent of } x}$

$$\hookrightarrow T_6 = T_{5+1} = {}^{10}C_5\, 2^{10-5} \cdot x^{10-2*5}\, (-1)^5$$
$$= {}^{10}C_5\, 2^5 \cdot x^0\, (-1)^5$$
$$= -8064 \text{ Ans}$$

RAHUL'S SCIENCE ACADEMY *PUNE* R

Show that the middle term in the expansion

$(1+x)^{2n}$ is $\frac{1 \cdot 3 \cdot 5 \cdot \ldots (2n-1)}{n!} 2^n \cdot x^n$. [n - integer +ve]

$(1+x)^{2n} \xrightarrow[\text{term}]{\text{Middle}} \left[\frac{\cancel{2}n}{\cancel{2}}+1\right] = (n+1)\text{th term}$

$$T_{n+1} = {}^nC_r\, a^{n-r} b^r$$

$$= {}^{2n}C_n (1)^{2n-n} \cdot (x)^n$$

$$= \boxed{\frac{\lfloor 2n}{\lfloor n \lfloor n}} \quad x^n$$

$$= \frac{[(2n)(2n-1).(2n-2)(2n-3) \cdots\cdots 4.3.2.1]}{\lfloor n \lfloor n.} x^n.$$

$$= \frac{[1.2.3.4.5 \cdots\cdots (2n-3).(2n-2).(2n-1).2n]}{(\lfloor n)^2} x^n$$

$$= \frac{[1.3.5 \quad (2n-1)].[2.4.6.\ldots(2n-2)2n]}{(\lfloor n)^2} x^n$$

$$= \frac{[1.3.5 \cdots (2n-1)] \quad [2\times1 * (2*2) * (2*3) \cdots (2\times n)]}{(\lfloor n)^2} x^n$$

$$= \frac{[1.3.5 \cdots (2n-1)\; 2^n\; [1.2.3 \cdots n]}{(\lfloor n)^2} . x^n$$

$$= \frac{(1.3.5 \cdots 2n-1)\; 2^n . [\lfloor n]}{(\lfloor n)^2} * x^n$$

$$= \left\{\frac{(1.3.5 \cdots 2n-1)\; 2^n}{\lfloor n}\right\} x^n$$

proved

Middle term

Q. Find the constant term in the expansion

$$\left(x-\frac{1}{x}\right)^{10}$$

Sol^n Const = independent of $x$

$$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$$

$$= {}^{10}C_r\,(x)^{10-r}\left(-\frac{1}{x}\right)^r$$

$$= {}^{10}C_r\, x^{10-r}(-1)^r\cdot\frac{1}{x^r} = {}^{10}C_r\, x^{10-2r}(-1)^r$$

$10-2r=0$ $\boxed{r=5}$

$T_6 = T_{5+1}$ ~ is independent of $x$

$$T_6 = T_{5+1} = {}^{10}C_5\, x^{10-2\times5}(-1)^5$$

$$= -{}^{10}C_5 = -252 \quad \text{Ans}$$

Q. Find a positive value of m for which the coefficient of $x^2$ in the expansion $(1+x)^m$ is 6

Sol^n :~

$$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$$

$$= {}^mC_r\, a^{m-r}\cdot(x)^r$$

$\boxed{r=2}$

$$T_3 = T_{2+1} = {}^mC_2\,(1)^{m-2}\,x^2$$

coefficient ~ ${}^mC_2 = 6$ (given)

$$\frac{m\times(m-1)}{2\times1} = 6$$

$$m(m-1) = 12$$

$$m(m-1) = 4\times3$$

$\boxed{m=4}$ Sol^n

RAHUL'S SCIENCE ACADEMY *PUNE*

1) The sum of the coefficient of all odd degree terms in the expansion of $\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5 (x>1)$ is **[2018]**

(a) 1 (b) 2

(c) –1 (d) 0

$$\cdot\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$$

$$=(x+a)^5+(x-a)^5$$

$$=2\left[{}^5C_0(x)^5+{}^5C_2\,x^{5-2}\left(\sqrt{x^3-1}\right)^2+{}^5C_4\,x^{5-4}\cdot\left(\sqrt{x^3-1}\right)^4\right]$$

$$=2\left[1.x^5+10\,x^3(x^3-1)+5x(x^3-1)^2\right]$$

$$=2\left[x^5+10(x^6-x^3)+5x\left[x^6+1-2.x^3\right]\right]$$

$$=2\left[x^5+10x^6-10x^3+5x^7+5x-10x^4\right]$$

$$=2x^5+20x^6-20x^3+10x^7+10x^1-20x^4$$

Sum of coeff of odd degree terms.

$$=2-20+10+10=2 \text{ Ans}$$

2) In the binomial expansion of $(a-b)^n, n\geq 5$, the sum of the $5^{th}$ and $6^{th}$ terms is zero. Then $\frac{a}{b}$ is equal to **[2007]**

(a) $\frac{1}{6}(n-5)$ (b) $\frac{1}{5}(n-4)$

(c) $\frac{5}{(n-4)}$ (d) $\frac{6}{(n-5)}$

$n\geq 5$

$$(a-b)^n={}^nC_0a^n+{}^nC_1a^{n-1}(-b)^1+{}^nC_2a^{n-2}(-b)^2+{}^nC_3a^{n-3}(-b)^3+\ldots+{}^nC_na^{n-n}b^n$$

$$T_5=T_{4+1}={}^nC_r a^{n-r}b^r={}^nC_4\,a^{n-4}(-b)^4$$

$$T_6=T_{5+1}={}^nC_5a^{n-5}\cdot(-b)^5$$

or, $T_5+T_6=0$

or, ${}^nC_4\,a^{n-4}\,b^4+{}^nC_5\,a^{n-5}(-b)^5=0$

or, ${}^nC_4\,a^{n-4}\,b^4={}^nC_5\,a^{n-5}\,b^5$

or, $\dfrac{a^{n-4}\cdot b^4}{a^{n-5}\,b^5}=\dfrac{{}^nC_5}{{}^nC_4}$

$$\left[\frac{{}^nC_r}{{}^nC_{r-1}}=\frac{n-r+1}{r}\right]$$

or, $\dfrac{a}{b}=\dfrac{{}^nC_5}{{}^nC_4}=\dfrac{n-5+1}{5}=\dfrac{n-4}{5}$ Ans

3) The coefficient of $x^7$ in the expansion of $(1-x-x^2+x^3)^6$ is **[2011]**

(a) 144 (b) – 132

(c) – 144 (d) 132

$$= \{(1-x) - x^2(1-x)\}^6$$

$$= (1-x)^6 (1-x^2)^6$$

$$= ({}^6C_0 - {}^6C_1 x^1 + {}^6C_2 x^2 - {}^6C_3 x^3 + {}^6C_4 x^4 - {}^6C_5 x^5 + {}^6C_6 x^6) *$$

$$({}^6C_0 - {}^6C_1 x^2 + {}^6C_2 x^4 - {}^6C_3 x^6 + {}^6C_4 x^8 - {}^6C_5 x^{10} + {}^6C_6 x^{12}]$$

Now coeff of $x^7$

$$= +{}^6C_1 * {}^6C_3 - {}^6C_3 * {}^6C_2 + {}^6C_5 * {}^6C_1$$

$$= 6 * 20 - 20*15 + 6*6 = 156 - 300 = -144 \text{ Ans}$$

4) The coefficient of $t^{24}$ in the expansion of $(1+t^2)^{12}(1+t^{12})(1+t^{24})$ is **[2003]**

✓(a) ${}^{12}C_6 + 2$ (b) ${}^{12}C_5$

(c) ${}^{12}C_6$ (d) ${}^{12}C_7$



$$(1+t^2)^{12} . (1+t^{12}) . (1+t^{24})$$

$$= (1+t^2)^{12} . (1+t^{24}+t^{12}+t^{36})$$

$$= ({}^{12}C_0 + {}^{12}C_1 t^2 + {}^{12}C_2 t^4 + \dots \; {}^{12}C_4 t^8 + {}^{12}C_6 t^{12} + \dots$$

$$+ {}^{12}C_{10} t^{20} + {}^{12}C_{11} t^{22} + {}^{12}C_{12} t^{24})$$

$$(1+t^{24}+t^{12}+t^{36})$$

Coeff.

$$= [{}^{12}C_{12} + {}^{12}C_0 + {}^{12}C_6] \, t^{24}$$

Coeff of $t^{24}$ are

$= 1 + 1 + {}^{12}C_6 = {}^{12}C_6 + 2$ (a)

5) If the coefficients of $p^{th}$, $(p+1)^{th}$ and $(p+2)^{th}$ terms in the expansion of $(1+x)^n$ are in A.P., then [2005]

(a) $n^2 - 2np + 4p^2 = 0$

(b) $n^2 - n(4p+1) + 4p^2 - 2 = 0$

(c) $n^2 - n(4p+1) + 4p^2 = 0$

(d) None of these

Sol$^n$ $(1+x)^n$

$T_{r+1} = {}^nC_r (1)^{n-r} (x^r)$

General Term

$T_p = T_{(p-1)+1} = {}^nC_{p-1} \; x^{p-1}$

$T_{p+1} = {}^nC_p \; x^p$.

$T_{p+2} = T_{(p+1)+1} = {}^nC_{p+1} \; x^{p+1}$

Given,

${}^nC_{p-1}, {}^nC_p, {}^nC_{p+1}$ are in AP.

$\Rightarrow 2\,{}^nC_p = {}^nC_{p-1} + {}^nC_{p+1}$

$a, b, c$
$b - a = c - b$
$2b = a + c$

AP ka property

$\boxed{\text{let } p = 1}$

$\Rightarrow 2\,{}^nC_1 = {}^nC_0 + {}^nC_2$

$\Rightarrow 2 \cdot n = 1 + \frac{n(n-1)}{2}$.

$\Rightarrow 4n = 2 + n^2 - n$

$\Rightarrow n^2 - 5n + 2 = 0$

Check options
put $p = 1$

(b) $n^2 - n(5) + 2 = 0$

6) If the expansion in powers of $x$ of the function $\frac{1}{(1-ax)(1-bx)}$ is $a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$, then $a_n$ is [2006]

(a) $\frac{b^n - a^n}{b-a}$ (b) $\frac{a^n - b^n}{b-a}$

(c) $\frac{a^{n+1} - b^{n+1}}{b-a}$ (d) $\frac{b^{n+1} - a^{n+1}}{b-a}$

*

$$\underbrace{\frac{1}{(1-ax)(1-bx)}}_{\text{Find coeff of } x^n} = a_0 + a_1x + \cdots + a_nx^n$$

$\boxed{a_n = ?}$

b-

$= (1-ax)^{-1}(1-bx)^{-1}$

$\rightarrow (1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots$

put $n = -1$

$= (1 + ax + a^2x^2 + a^3x^3 + \ldots + a^nx^n)(1 + bx + b^2x^2 + \ldots + b^{n-1}x^{n-1} + b^nx^n)$

coeff of $x^n$

$= [b^n + a.b^{n-1} + a^2b^{n-2} + \ldots + a^n]$

check $n=2$

$= b^2 + a.b + a^2.b^0$

$= b^2 + ab + a^2$

check option

(d) $= \frac{b^{n+1} - a^{n+1}}{b-a}$

$= \frac{b^3 - a^3}{b-a}$

$= \frac{\cancel{(b-a)}(b^2+ab+b^2)}{\cancel{b-a}}$

$= a^2 + ab + b^2$

option (d)

7) The coefficient of $x^n$ in expansion of $(1+x)(1-x)^n$ is **[2004]**

(a) $(-1)^{n-1}n$ (b) $(-1)^n(1-n)$ ✓

(c) $(-1)^{n-1}(n-1)^2$ (d) $(n-1)$

$= (1+x)(1-x)^n$

$= 1(1-x)^n + x(1-x)^n$

$x^n$ coefficient

$x^{n-1}$ coefficient

${}^nC_n(1)^{n-n}\cdot(-x)^n$

${}^nC_{n-1}(1)^{n-n+1}(-x)^{n-1}$

${}^nC_r = {}^nC_{n-r}$

${}^nC_{n-1} = {}^nC_{n-n+1}$

$\Rightarrow {}^nC_1 = n$

Resultant coff ~. ${}^nC_n(-1)^n + {}^nC_{n-1}(-1)^{n-1}$

Need to add both

$= {}^nC_n(-1)^n + n.(-1)^n.\underline{-1}$

$= (-1)^n[1-n]$ Ans

8) If for positive integers $r > 1, n > 2$ the coefficient of the $(3r)^{th}$ and $(r+2)^{th}$ powers of $x$ in the expansion of $(1+x)^{2n}$ are equal, then **[2002]**

(a) $n = 2r$ (b) $n = 3r$

(c) $n = 2r+1$ (d) None of these

$r > 1$
$n > 2$

$(1+x)^{2n}$

Genral $\{ {}^nC_r a^{n-r} x^r \}$

$x^{3r}$ ~ coefficient- ${}^{2n}C_{3r}$

$x^{r+2}$ coefficient $= {}^{2n}C_{r+2}$

Formula
${}^nC_x = {}^nC_y$
$x = y$
$\boxed{x+y=n}$

Given,
$$\boxed{{}^{2n}C_{3r} = {}^{2n}C_{r+2}}$$

or $3r + r + 2 = 2n$

or $4r + 2 = 2n$

or $\boxed{2r+1 = n}$ (c)

9) The coefficient of the term independent of x in the expansion of $\left[\frac{(x+1)}{x^{2/3} - x^{1/3} + 1} - \frac{(x-1)}{x - x^{1/2}}\right]^{10}$ is **[2013]**

$x^0$

(a) 210 (b) 105

(c) 70 (d) 112

$\{ a^3 + b^3 = (a+b)(a^2 - ab + b^2) \}$

$$= \left[\frac{x+1}{(x^{2/3} - x^{1/3} + 1)} - \frac{(x-1)}{x - x^{1/2}}\right]^{10}$$

$$= \left[\frac{(x^{1/3})^3 + 1}{(x^{2/3} - x^{1/3} + 1)} - \frac{(x^{1/2})^2 - 1}{x - x^{1/2}}\right]^{10}$$

$$= \left[\frac{(x^{1/3}+1)\cancel{(x^{2/3} - x^{1/3} + 1)}}{\cancel{(x^{2/3} - x^{1/3} + 1)}} - \frac{(x^{1/2}+1)\cancel{(x^{1/2}-1)}}{x^{1/2}\cancel{(x^{1/2}-1)}}\right]^{10}$$

$$= \left[ (x^{1/3}+1) - \frac{(x^{1/2}+1)}{x^{1/2}} \right]^{10}$$

$$= \left[ x^{1/3} \cancel{+1} - \cancel{1} - x^{-1/2} \right]^{10} = \left[ x^{1/3} - x^{-1/2} \right]^{10}$$

Now we have to find the term independent of x.

$$T_{r+1} = \text{General term} = {}^nC_r\, a^{n-r}\, x^r$$

$$= {}^{10}C_r\, (x^{1/3})^{10-r}\, (-x^{-1/2})^r$$

$$= (-1)^r\, {}^{10}C_r\, x^{\frac{10-r}{3}} \cdot x^{-\frac{r}{2}}.$$

$$= (-1)^r\, {}^{10}C_r \cdot x^{\left(\frac{10-r}{3} - \frac{r}{2}\right)}$$

$$\frac{10-r}{3} - \frac{r}{2} = 0$$

$$\frac{20-2r-3r}{6} = 0$$

$$\boxed{r=4}.$$

$$\therefore \text{Coeff} = (-1)^4 \cdot {}^{10}C_4$$

$$= {}^{10}C_4 = 210 \quad \text{Ans}$$

10) 6. The term independent of $x$ in the binomial expansion of $\left(1-\frac{1}{x}+3x^5\right)\left(2x^2-\frac{1}{x}\right)^8$ is [2015]

(a) 400 (b) 496

(c) $-400$ (d) $-496$



Soln : $\left(1-\frac{1}{x}+3x^5\right)\left(2x^2-\frac{1}{x}\right)^8$ — independent of x.

General term ~ $T_{r+1} = {}^8C_r\,(2x^2)^{8-r} \cdot \left(-\frac{1}{x}\right)^r$

$$= {}^8C_r\, 2^{8-r} \cdot x^{16-2r} (-1)^r \cdot x^{-r}$$

$$\boxed{T_{r+1} = {}^8C_r\, 2^{8-r}\, x^{16-3r}\, (-1)^r}$$

$$\left(1-\frac{1}{x}+3x^5\right)\left(2x^2-\frac{1}{x}\right)^8$$

$$= 1 * \underbrace{\left(2x^2-\frac{1}{x}\right)^8}_{\text{General term}} - \frac{1}{x}\underbrace{\left(2x^2-\frac{1}{x}\right)^8}_{\text{Need to find term with } x^1} + 3x^5\underbrace{\left(2x^2-\frac{1}{x}\right)^8}_{\text{Need to find term of } x^{-5}}$$

General term

$^8C_r\, 2^{8-r}\, x^{16-3r}\, (-1)^r$

$16-3r=0$

$\boxed{r=16/3}$

N.P

Need to find term with $x^1$

$16-3r=1$

$16-1=3r$

$\boxed{r=5}$

Need to find term of $x^{-5}$

$16-3r=-5$

$16+5=3r$

$\boxed{r=7}$

The term independ of x is

$$=\left[-1\ {}^8C_5\, 2^{8-5}(-1)^5 + 3 * {}^8C_7\, 2^1 . (-1)^7\right]$$

$$= +56*8 + 3*8*2(-1)$$

$$= 56\times8 - 3*8*2$$

$$= 400 \text{ Ans}$$

11) The coefficient of the middle term in the binomial expansion in powers of $x$ of $(1+\alpha x)^4$ and of $(1-\alpha x)^6$ is the same if $\alpha$ equals [2004]

(a) $\frac{3}{5}$ (b) $\frac{10}{3}$

(c) $\frac{-3}{10}$ (d) $\frac{+3}{10}$

$(1+\alpha x)^4$ $(1-\alpha x)^6$

Middle term

even number

$\left(\frac{4}{2}+1\right)$

3rd

4th term

$T_3 = T_{2+1} = {}^4C_2 (1)^{4-2} (\alpha x)^2$

$= T_{r+1}$

$T_{3+1} = {}^6C_3 (1)^{6-3} (-\alpha x)^3$

Given coefficients are same.

or, $^4C_2\,\alpha^2 = {^6C_3}(-\alpha)^3$.

or, $\dfrac{^4C_2}{^6C_3} = -\alpha$ $\qquad \alpha = -3/10$ Ans

(a) $^nC_x = {^nC_y} \Rightarrow x = y$ or $x + y = n$

(b) $^nC_{r-1} + {^nC_r} = {^{n+1}C_r}$ $\qquad \boxed{^nC_r = \frac{n}{r}\,{^{n-1}C_{r-1}}}$

(c) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

(d) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} \ldots + \frac{(-1)^n C_n}{n+1} = \frac{1}{n+1}$

(e) $C_0 + C_1 + C_2 + \ldots = C_n = 2^n$

(f) $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$

(g) $C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = {^{2n}C_n} = \frac{(2n)!}{n!n!}$

(h) $C_0.C_r + C_1.C_{r+1} + C_2.C_{r+2} + \ldots + C_{n-r}C_n = \frac{(2n)!}{(n+r)!(n-r)!}$

**Remember :** $\boxed{(2n)! = 2^n.\, n!\,[1.3.5 \ldots (2n-1)]}$

12) The sum of the series

$^{20}C_0 - {^{20}C_1} + {^{20}C_2} - {^{20}C_3} + \ldots - \ldots + {^{20}C_{10}}$ is [2007]

(a) $-{^{20}C_{10}}$ (b) $\frac{1}{2}\,{^{20}C_{10}}$ ✓

(c) 0 (d) $^{20}C_{10}$

$^{20}C_0 - {^{20}C_1} + {^{20}C_2} - {^{20}C_3} - \ldots\ {^{20}C_{10}} = ?$

$(1+x)^{20} = {^{20}C_0} + {^{20}C_1}x + {^{20}C_2}x^2 + \ldots\ {^{20}C_{20}}x^{20}$

Put $\longrightarrow$ $\boxed{x = -1}$

$0 = {^{20}C_0} - {^{20}C_1} + {^{20}C_2} - {^{20}C_3} + \ldots + {^{20}C_{20}}$

$0 = {^{20}C_0} - {^{20}C_1} + {^{20}C_2} - \ldots - {^{20}C_9} + {^{20}C_{10}} - {^{20}C_{11}} + \ldots + {^{20}C_{20}}$

$$\left[\begin{aligned} {^{20}C_{11}} &= {^{20}C_9} \\ {^{20}C_{12}} &= {^{20}C_8} \\ {^{20}C_{13}} &= {^{20}C_7} \\ {^{20}C_{14}} &= {^{20}C_6} \\ {^{20}C_{20}} &= {^{20}C_0} \end{aligned}\right]$$

$0 = 2({^{20}C_0} - {^{20}C_1} + {^{20}C_2} - \ldots - {^{20}C_9}) + {^{20}C_{10}}$

$-{^{20}C_{10}} = 2({^{20}C_0} - {^{20}C_1} + \ldots - {^{20}C_9})$

$\frac{1}{2}(-{^{20}C_{10}}) = {^{20}C_0} - {^{20}C_1} - \ldots - {^{20}C_9}.$

RAHUL'S SCIENCE ACADEMY • PUNE • R

$^{20}C_{10} - \frac{1}{2}\,^{20}C_{10} = \,^{20}C_0 - \,^{20}C_1 \; - - - \; - \,^{20}C_9 + \,^{20}C_{10}$

$\boxed{\frac{1}{2}\,^{20}C_{10}} = \,^{20}C_0 - \,^{20}C_1 \; - - - \; + \,^{20}C_{10}$

Ans

13) If $x$ is positive, the first negative term in the expansion of $(1+x)^{27/5}$ is [2003]

(a) $7^{th}$ term
(b) $5^{th}$ term
(c) $8^{th}$ term ✓
(d) $6^{th}$ term

$n = \frac{27}{5}$
$= 5.4$

General term, $T_{r+1} = \,^nC_r \, a^{n-r} x^r$

$\frac{\lfloor n}{\lfloor r \lfloor n-r} < 0$

or $\frac{n(n-1)(n-2)\cdots(n-r+1)\cancel{\lfloor n-r}}{\lfloor r \; \cancel{\lfloor n-r}} < 0$

or $\frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r} < 0$

or $n - r + 1 < 0$

or $\frac{27}{5} + 1 < r$

or $\frac{27+5}{5} < r$

or $\frac{32}{5} < r \quad 6.4 < r \quad \boxed{r = 7}$

$T_{r+1}$ th term $= T_8$

8th term

14) The coefficient of $t^4$ in the expansion of $\left(\frac{1-t^6}{1-t}\right)^3$ is [2019]

(a) 12  (b) 14

(c) 10  (d) 15

$$= (1-t^6)^3 (1-t)^{-3}$$

$$= (1 - 3.t^6 + 3t^{12} - t^{18}) \boxed{(1-t)^{-3}}$$

$t^4$

$t^{-2}$, $t^{-8}$, $t^{-14}$ — NP

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots \; x^3 + \quad x^4$$

$(1-t)^{-3}$ = we will not get $t^{-2}$, $t^{-8}$, $t^{-14}$ like term.

$$(1-t)^{-3} = 1 + 3t + \frac{-3(-3-1)}{2!}(-t)^2 + \frac{(-3)(-3-1)(-3-2)}{3!}(-t)^3 + \underbrace{\frac{(-3)(-3-1)(-3-2)(-3-3)}{4!}}(-t)^4 \cdots$$

$$\text{coff} = \frac{(-3)(-4)(-5)(-6)}{4!} = \frac{3 \times 4 \times 5 \times 6}{4!} = 15 \text{ An}$$

15) 14. The number of terms in the expansion of $(a+b+c)^n$, where $n \in N$ is

✓(a) $\frac{(n+1)(n+2)}{2}$ (b) $n+1$

(c) $n+2$ (d) $(n+1)n/2$

DPP

Soln

$= (a+b+c)^n$

$= [a + (b+c)]^n$

$= \underbrace{a^n}_{1\text{ term}} + \underbrace{{}^nC_1\, a^{n-1}(b+c)}_{2\text{ term}} + \underbrace{{}^nC_2\,(a)^{n-2}(b+c)^2}_{3\text{ term}} + \underbrace{\cdots}_{4\text{ term}} + \underbrace{{}^nC_n\, a^0 (b+c)^n}_{(n+1)\text{ term}}$

sum of total term $= 1 + 2 + 3 + 4 + \cdots (n+1)$

$= \frac{(n+1)(n+1+1)}{2} = \frac{(n+1)(n+2)}{2}$ Ans

16) 15. If $(1-x+x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}$, then $a_0 + a_2 + a_4 + \ldots + a_{2n}$ equals

~~(a)~~ $\frac{3^n+1}{2}$ (b) $\frac{3^n-1}{2}$ (c) $\frac{1-3^n}{2}$ (d) $3^n + \frac{1}{2}$

0, 2, 4
1 3 5

$(1-x+x^2)^n = a_0 + a_1x + a_2x^2 + \cdots a_{2n}x^{2n}$.

$x = 0, 1, -1, \ldots$

$\boxed{x=1}$ put in eqn.

$(1-1+1)^n = a_0 + a_1 + a_2 \cdots + a_{2n}$

$1 = a_0 + a_1 + a_2 \cdots + a_{2n}$ — (1)

$\boxed{x=-1}$

$(1+1+1)^n = a_0 - a_1 + a_2 - a_3 \cdots + a_{2n}$

$3^n = a_0 - a_1 + a_2 - a_3 \cdots \cdots a_{2n}$ — (2)

$\boxed{\text{Add (1)+(2)}}$

$1 + 3^n = 2a_0 + 2a_2 + 2a_4 \cdots + 2a_{2n}$

$1 + 3^n = 2(a_0 + a_2 + a_4 \cdots a_{2n})$

$a_0 + a_2 + a_4 \cdots a_{2n} = \frac{3^n+1}{2}$ Ans ✓

17) 16. If $(2x^2-x-1)^5 = a_0 + a_1x + a_2x^2 + \dots + a_{10}x^{10}$, then, $a_2 + a_4 + a_6 + a_8 + a_{10} =$

(a) 15 (b) 30 (c) 16 (d) 17

DPP

$$(2x^2-x-1)^5 = a_0 + a_1x + a_2x^2 + \dots a_{10}x^{10}$$

$x=1$

$$0 = a_0 + a_1 + a_2 - - - + a_{10} \quad —(1)$$

$\boxed{x=-1}$

$$(2*1+\cancel{1}-\cancel{1})^5 = a_0 - a_1 + a_2 - a_3 - - - + a_{10}$$

$$32 = a_0 - a_1 + a_2 \dots + a_{10} \quad —(2)$$

(*) $\boxed{\text{Put } x=0}$ $-1 = a_0$

(1) + (2)

$$32 = 2(a_0) + 2[a_2 + a_4 + a_6 \dots a_{10}]$$

$$32 = \underbrace{2*-1} + 2[a_2 + a_4 + \dots a_{10}]$$

$$34 = +2[a_2 + a_4 \dots a_{10}]$$

$$a_2 + a_4 + \dots a_{10} = 17 \text{ Ans}$$

18) 19. The sum of the series

$$1 + \frac{1}{3^2} + \frac{1\cdot4}{1\cdot2}\cdot\frac{1}{3^4} + \frac{1\cdot4\cdot7}{1\cdot2\cdot3} * \frac{1}{3^6} + \dots \text{ is}$$

(a) $\sqrt{\frac{3}{2}}$ ~~(b)~~ $\left(\frac{3}{2}\right)^{1/3}$ (c) $\sqrt{\frac{1}{3}}$ (d) $\sqrt[3]{\frac{2}{3}}$

$$= 1 + \frac{1}{3^2} + \frac{1\cdot4}{1\cdot2}\cdot\frac{1}{3^4} + \frac{1\cdot4\cdot7}{1\cdot2\cdot3}\,\frac{1}{3^6} \quad —(1)$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots \quad \text{—— (2)}$$

Compare (1) & (2)

$$nx = \frac{1}{9}; \quad \frac{n(n-1)}{2} \cdot x^2 = \frac{1 \cdot 4}{1 \cdot 2} * \frac{1}{3^4}.$$

or, $$\frac{(nx)(nx - x)}{2} = \frac{4}{2} \frac{1}{3^4}.$$

or, $$\frac{\frac{1}{9} \cdot \left(\frac{1}{9} - x\right)}{2} = \frac{4}{2} \cdot \frac{1}{3^4} \Rightarrow \boxed{x = -1/3} \quad \boxed{n = -\frac{1}{3}}$$

or, $$(1+x)^n = \left(1 - \frac{1}{3}\right)^{-1/3} = \left(\frac{2}{3}\right)^{-1/3} = \left(\frac{3}{2}\right)^{1/3} \sim \text{(b)}$$

19) 20. The last four digits of the natural number $3^{100}$ are
(a) 7231 (b) 1231 (c) 3451 (d) 2001

$$\Rightarrow 3^{100}$$

$$\Rightarrow 9^{50}$$

$$= (1-10)^{50}$$

$$= 1 - 50 \times 10 + \frac{50 \cdot (49)}{2} \cdot 10^2 - \frac{50 \cdot 49 \cdot 48}{6} \cdot 10^3 + (\;\;) 10^4 + \cdots$$

$$= \underbrace{1 - 500 + 122500}_{} + (\;\;) \times 1000 + (\;\;) 10000 + \cdots$$

$$= 122001 + \text{Multiple of } 10^4$$

Last four digit $\sim$ $\boxed{2001}$

20) **22.** The digit at the units place in the number $19^{2005} + 11^{2005} - 9^{2005}$ is
(a) 2 (~~b~~) 1 (c) 0 (d) 8

$9 \times 9 \times 9$
$81 \times 9 = 729$

$= 19^{2005}$

unit place
$19^1 = 9$
$19^2 \approx 1$
$19^3 \sim 9$

$\underline{19^{2005} \sim 9}$

$11^{2005}$

unit place
$11^1 \longrightarrow 1$
$11^2 = 1$
$11^3 \to 1$

$11^{2005} \longrightarrow 1$

$9^{2005}$

unit place
$9^1 \longrightarrow 9$
$9^2 \longrightarrow 1$
$9^3 \longrightarrow 9$
$9^4 \longrightarrow 1$

$9^{2005} \longrightarrow 9$

So digits at unit place :~

$9 + 1 - 9 \approx 1$

21) **24.** The value of $^{14}C_1 + {}^{14}C_3 + {}^{14}C_5 + \ldots + {}^{14}C_{11}$ is
(a) $2^{14} - 1$ (b) $2^{14} - 14$ (c) $2^{12}$ (~~d~~) $2^{13} - 14$

$^{14}C_1 + {}^{14}C_3 + {}^{14}C_5 \; - - \; {}^{14}C_{11}$

$(x+y)^n + (x-y)^n = 2\left[{}^nC_0\, x^n + {}^nC_2\, x^{n-2} y^2 + \ldots\right]$

$(x+y)^n - (x-y)^n = 2\left[{}^nC_1\, x^n + {}^nC_3\, x^{n-3} y^3 + \ldots\right] - 2$

$x = 1, y = 1$

$2^n - 0 = 2\left[{}^nC_1 + {}^nC_3 + \ldots\right]$

$2^{14} = 2\left[{}^{14}C_1 + {}^{14}C_3 + {}^{14}C_5 + {}^{14}C_7 + {}^{14}C_9 + {}^{14}C_{11} + {}^{14}C_{13}\right]$

$2^{13} = \left[{}^{14}C_1 + {}^{14}C_3 + \ldots {}^{14}C_{11}\right] + {}^{14}C_{13}$

$2^{13} - {}^{14}C_{13} = \underbrace{{}^{14}C_1 + {}^{14}C_3 + \cdots \quad {}^{14}C_{11}}$

$\underbrace{2^{13} - 14} = {}^{14}C_1 + \cdots \quad + {}^{14}C_{11}$

22) **26.** If $n$ is a positive integer, then $\underline{5^{2n+2} - 24n - 25}$ is divisible by
(a) 574 (b) 575 (c) 675 (d) 576

shortcut
$n = 1$
$576$ (✓)

$= 5^{2n+2} - 24n - 25$ remove

$= (5^2)^{n+1} - 24n - 25$

$= 25 \cdot 25^n - 24n - 25$

$= 25 \cdot (1+24)^n - 24n - 25$

$= 25\left[1^n + {}^nC_1 24 + {}^nC_2 24^2 + \cdots \quad {}^nC_n 24^n\right] - 24n - 25$

$= 25\left[1 + n \cdot 24 + {}^nC_2 24^2 + \cdots \quad {}^nC_n 24^n\right] - 24n - 25$

$= \cancel{25} + \underline{n \cdot 25 \cdot 24} + 25\left[{}^nC_2 24^2 + \cdots \quad {}^nC_n 24^n\right] - 24n - \cancel{25}$

$= \quad n \cdot \underline{24 \cdot 24} + 25\left[{}^nC_2\, 24^2 + \cdots \quad 24^n\right]$

$= \underbrace{24 \times 24}\left[n + 25\left[{}^nC_2 + {}^nC_3 \cdot 24 \cdots \quad 24^{n-2}\right]\right]$

$= \underbrace{576}\left[n + 25\left( \cdots \quad \right)\right]$

eqn can be divided by $\underline{576}$.

23)

**28.** For all $n \in N, 2^{3n+3} - 7n - 8$ is divisible by
(a) 49 (b) 17 (c) 343 (d) 81

Shortcut → $2^{3n+3} - 7n - 8 \longrightarrow n = 1 \longrightarrow \boxed{49}$

Sol$^n$

$$= 2^{3n+3} - 7n - 8.$$

$$= (2^3)^{n+1} - 7n - 8$$

$$= 8^n \cdot 8 - 7n - 8$$

$$= 8(1+7)^n - 7n - 8$$

$$= 8\left[1 + 7n + \frac{n(n-1)}{2} \cdot 7^2 + \cdots\right] - 7n - 8$$

$$= \left[\cancel{8} + 8 \cdot 7 \cdot n + 8 \cdot \frac{n \cdot (n-1)}{2} \cdot 7^2 + \cdots\cdots\right] - 7n - \cancel{8}$$

$$= 7 \cdot 7 \cdot n + 8 \cdot \frac{n(n-1)}{2} \cdot 7^2 + 8 \cdot \frac{n(n-1)(n-2)}{3!} 7^3 + \cdots$$

$$= 49\left[n + \frac{8(n)(n-1)}{2} + 8 \cdot \frac{(n)(n-1)(n-2)}{3!} \cdot 7 + \cdots\right]$$

Divisible by 49 Ans

24)

**29.** The remainder when $23^{23}$ is divided by 53, is
(a) 17 (b) 21 (c) 30 (d) 43

$$\frac{23^{23}}{53} = \frac{(23)^{22+1}}{53} = \frac{23\ (23)^{22}}{53} = \frac{23\ (23)^{2 \times 11}}{53}$$

$$= \frac{23\ (529)^{11}}{53}.$$

$$\frac{23\cdot(530-1)^{11}}{53}$$

$$\frac{23\left[{}^{11}C_0\, 530^{11} + {}^{11}C_1 (530)^{10}(-1) + {}^{11}C_2 (530)^{9}(-1)^{2} + \dots (-1)^{11}\right]}{53}$$

$$= \frac{23\cdot\left[530^{11} - {}^{11}C_1 (530)^{10} + \dots + {}^{11}C_{10}(530)^{1}(-1)^{10} + (-1)^{11}\right]}{53}$$

$$= \frac{\left[23\cdot(530)^{11} \dots + 23\cdot {}^{11}C_{10}\, 530\right]}{53} \quad -23$$

$-23+53 \sim 30$ emindee

Example $\left[\frac{14}{3}\right] \rightarrow \frac{3\times 4+2}{3} \rightarrow 2$

$\rightarrow \frac{3\times 5-1}{3} \rightarrow 2 \rightarrow (-1+3) = 2$

25)

**37.** Find the term independent of $x$ in the expansion of $\left(\sqrt[3]{x}+\frac{1}{2\sqrt[3]{x}}\right)^{18}, x>0$.

✓(a) ${}^{18}C_9 \frac{1}{2^9}$ (b) ${}^{18}C_8 \frac{1}{2^8}$

(c) ${}^{18}C_7 \frac{1}{2^7}$ (d) None of these

$$T_{r+1} = {}^nC_r\, a^{n-r} x^r$$

$$= {}^{18}C_r \cdot \left(\sqrt[3]{x}\right)^{18-r} \cdot \left(\frac{1}{2\sqrt[3]{x}}\right)^{r}$$

$$= {}^{18}C_r\, x^{\frac{18-r}{3}} \cdot \frac{1}{2^r \cdot x^{r/3}}$$

$$\boxed{\frac{1}{x} = x^{-1}}$$

$$= {}^{18}C_r\, x^{\frac{18-r}{3}} \cdot x^{-\frac{r}{3}} \cdot \frac{1}{2^r}$$

$$= {}^{18}C_r\, x^{\frac{18-2r}{3}} \cdot \frac{1}{2^r}$$

$T_{9+1}$

Terms are independent of $x$ $\quad \frac{18-2r}{3} = 0 \quad \boxed{r=9} \quad T_{10}$

$$T_{10} = T_{9+1} = {}^{18}C_9 \cdot \frac{1}{2^9} \quad \text{Ans}$$

independent of $x$

26)

**46.** Find numerically the greatest term in the expansion of $(2+3x)^9$, where $x = \frac{3}{2}$.

(a) $\frac{(7\times 13^3)}{2}$ (b) $\frac{(7\times 3^2)}{2}$

~~(c)~~ $\frac{(7\times 3^{13})}{2}$ (d) None of these

Soln

$$(2+3x)^9 = 2^9\Big(\underbrace{1+\frac{3}{2}x}\Big)^9$$

$$\rightarrow \boxed{\frac{T_{r+1}}{T_r} \geq 1}$$

$$\frac{T_{r+1}}{T_r} = \frac{\cancel{2^9}\left[{}^9C_r \cdot (1)^{\cancel{9-r}} \cdot \left(\frac{3x}{2}\right)^r\right]}{\cancel{2^9}\left[{}^9C_{r-1}\,(1)^{\cancel{9-r+1}}\left(\frac{3}{2}x\right)^{r-1}\right]}$$

$T_{r-1+1}$

$$= \frac{{}^9C_r}{{}^9C_{r-1}} * \frac{\frac{\cancel{3^r}}{\cancel{2^r}}\cdot \cancel{x^r}}{\frac{\cancel{3^r}}{\cancel{2^r}} \cdot \frac{3^{-1}}{2^{-1}} \cdot \cancel{x} \cdot x^{-1}}$$

$$\frac{{}^9C_r}{{}^9C_{r-1}} = \frac{n-r+1}{r}.$$

$$= \frac{9-r+1}{r} \cdot \frac{3x}{2}$$

$$= \frac{10-r}{r} \cdot \frac{9}{4}.$$

$$\frac{T_{r+1}}{T_r} \geq 1$$

$$\frac{10-r}{r} * \frac{9}{4} \geq 1$$

$$90 - 9r \geq 4r$$

$$r \leq \frac{90}{13}. \quad r \leq 6.9$$

$\boxed{r \approx 6}$ $\boxed{T_7}$ Maximum value

$$T_7 = T_{6+1} = 2^9 \cdot {}^9C_6 \left(\frac{9}{4}\right)^6 \quad [x = 3/2] \qquad 2^9\left(1+\frac{3}{2}x\right)^9$$

$$= \frac{7 \times 3^{13}}{2} \text{ Ans}$$

27)

49. The coefficient of $x^p$ and $x^q$ ($p$ and $q$ are positive integers) in the expansion of $(1+x)^{p+q}$ are
(a) equal
(b) equal with opposite signs
(c) reciprocal of each other
(d) none of these

Soln

$$= (1+x)^{p+q}$$

$$T_{p+1} = {}^{p+q}C_p \cdot (1)^{p+q-p} \cdot (x)^p \quad \left[\text{coeff} = {}^{p+q}C_p\right]$$

$$T_{q+1} = {}^{p+q}C_q (1)^{p+q-q} (x)^q \quad \left[\text{coeff} = {}^{p+q}C_q\right]$$

$$\boxed{{}^{p+q}C_p = {}^{p+q}C_q} \quad \text{(a)}$$

$$^nC_r = {}^nC_{n-r}$$

28)

51. The sixth term in the expansion of

$$\left[2^{\log_2\sqrt{9^{x-1}+7}}+\frac{1}{2^{\frac{1}{5}\log_2(3^{x-1}+1)}}\right]^7$$ is 84.

Then the number of values of $x$ is

(a) 0 (b) 1 (c) 2 (d) 3

Property.

$$\boxed{2^{\log_2 x} = x}$$

$$\underbrace{\log x^n = n\log x}$$

$$2^{\log_2\sqrt{9^{x-1}+7}} = \sqrt{9^{x-1}+7}.$$

$$2^{\frac{1}{5}\log_2(3^{x-1}+1)} = 2^{\log_2(3^{x-1}+1)^{1/5}} = (3^{x-1}+1)^{1/5}.$$

$$T_6 = T_{5+1} = \left[\sqrt{9^{x-1}+7}+\frac{1}{(3^{x-1}+1)^{1/5}}\right]^7 = \underline{\underline{84}}$$

$$^7C_5\left(\sqrt{(9^{x-1}+7)}\right)^{7-5}\cdot\left[\frac{1}{(3^{x-1}+1)^{1/5}}\right]^5 = 84.$$

or $$\frac{9^{x-1}+7}{3^{x-1}+1} = \frac{84}{21}$$

or $$\frac{3^{2x-2}+7}{3^{x-1}+1} = \frac{84}{21}$$ | $3^{x-1} = t$

or $$\frac{t^2+7}{t+1} = \frac{84}{21}$$ $\underset{\text{Solve}}{\equiv}$ $[t = 3 \text{ and } 1]$

$3^{x-1} = 3 \rightarrow \boxed{x=2}$

$$\left\{ 3^{x-1} = 1,\quad x-1=0,\quad x=1 \right\}$$

$x=1$ and $x=2$

(c)

29)

58. The greatest value of the term independent of $x$ in the expansion of $\left(x\sin\alpha+\dfrac{\cos\alpha}{x}\right)^{10}$ is

(a) 25 (b) $\begin{pmatrix}10\\5\end{pmatrix}$ (c) $\begin{pmatrix}10\\5\end{pmatrix}\dfrac{1}{2^5}$ (d) $2^{10}$

Soln

$$\left(x\sin\alpha+\frac{\cos\alpha}{x}\right)^{10}.$$

$$T_{r+1} = {}^nC_r\, a^{n-r}\, x^r.$$

$$= {}^{10}C_r . (x\sin\alpha)^{10-r} . \left(\frac{\cos\alpha}{x}\right)^r .$$

$$= {}^{10}C_r . x^{10-r} . \frac{\sin^{10-r}\alpha}{x^r} \cos^r\alpha$$

$$= {}^{10}C_r\, x^{10-2r} . \sin^{10-r}\alpha\, \cos^r.\alpha$$

Independent of $x$ $\quad 10-2r=0 \quad \boxed{r=5}$

$$T_{5+1} = T_6 = {}^{10}C_5\, x^0 . \sin^5\alpha\cos^5\alpha$$

$$= {}^{10}C_5\, \frac{1}{2^5}\, .2^5 \sin^5\alpha\cos^5\alpha \text{ — Trigo}$$

$$= {}^{10}C_5 \frac{1}{2^5}\left[\frac{2\sin\alpha\cos\alpha}{}\right]^5$$

$$= {}^{10}C_5 \frac{1}{2^5} \cdot [\sin 2\alpha]^5$$

Max ①

$T_{5+1} \sim$ Max value

$${}^{10}C_5 \frac{1}{2^5} \cdot \text{ Ans}$$

$$\binom{10}{5}\frac{1}{2^5}\text{ A}$$

30)

**64.** If the sum of the coefficients in the expansion of $(p + q)^n$ is 1024, then the greatest coefficient in the expansion is,

~~(a)~~ ${}^{10}C_5$ (b) ${}^{10}C_4$ (c) ${}^{10}C_2$ (d) ${}^{10}C_9$

$$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + \dots \quad {}^nC_n x^0 y^n$$

put.

$x = 1 \quad y = 1$

$$2^n = \underbrace{{}^nC_0 + {}^nC_1 + {}^nC_2 - \dots \quad {}^nC_n}$$

sum of coefficients

$n = 10$

$\therefore \; 2^n = 1024$

$\boxed{n = 10}$

$n = 10$ even.

middle term

${}^{10}C_5$ Ans

$${}^nC_{n/2} \propto {}^{10}C_5$$

31)

**84.** In the expansion of $(1+x)^n$,

$\frac{C_1}{C_0}+2\frac{C_2}{C_1}+3\frac{C_3}{C_2}+\ldots+n\frac{C_n}{C_{n-1}}$ is equal to

(a) $\frac{(n+1)}{2}$ (b) $\frac{n}{2}$

(c) $\frac{n(n+1)}{2}$ (d) $3n(n+1)$

(a) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(b) ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

$\boxed{{}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}}$

(c) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

(d) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} \ldots\ldots + \frac{(-1)^n C_n}{n+1} = \frac{1}{n+1}$

(e) $C_0 + C_1 + C_2 + \ldots\ldots = C_n = 2^n$

(f) $C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots = 2^{n-1}$

(g) $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{n!n!}$

(h) $C_0.C_r + C_1.C_{r+1} + C_2.C_{r+2} + \ldots\ldots + C_{n-r}C_n = \frac{(2n)!}{(n+r)!(n-r)!}$

**Remember :** $\boxed{(2n)! = 2^n.\, n!\,[1.3.5 \ldots\ldots (2n-1)]}$

$$\approx \frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \ldots\; n\frac{C_n}{C_{n-1}}$$

$$\boxed{\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}}$$

$$\frac{{}^nC_1}{{}^nC_0} = \frac{n-\cancel{1}+\cancel{1}}{1} = n.$$

or

$$\frac{C_2}{C_1} = \frac{n-2+1}{2} = \frac{n-1}{2}.$$

or

$$\frac{C_3}{C_2} = \frac{n-3+1}{3} = \frac{n-2}{3}.$$

$$= \frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \ldots\; n\frac{C_n}{C_{n-1}}$$

$$= n + \cancel{2}.\frac{n-1}{\cancel{2}} + \cancel{3}.\frac{(n-2)}{\cancel{3}} + \ldots\; \cancel{n}.\frac{n-\overline{(n-1)}}{\cancel{n}}.$$

$$= n + (n-1) + (n-2) + \ldots 3 + 2 + 1.$$

Reverse

$$= 1 + 2 + 3 + \ldots\ldots\ldots (n)$$

$$\underline{n(n+1)}$$

32) Ans

85. $(C_0 + C_1)(C_1 + C_2) \ldots (C_{n-1} + C_n)$ is equal to

(a) $(C_0C_1C_2 \ldots C_{n-1})(n+1)$ (b) $(C_0C_1C_2 \ldots C_{n-1})(n+1)^n$

~~(c)~~ $\dfrac{(C_0C_1C_2 \ldots C_{n-1})(n+1)^n}{n!}$

(d) None of these

$$= (C_0 + C_1)(C_1 + C_2) \ldots (C_{n-1} + C_n)$$

$$= C_0\left(1 + \frac{C_1}{C_0}\right)C_1\left(1 + \frac{C_2}{C_1}\right) \ldots C_{n-1}\left(1 + \frac{C_n}{C_{n-1}}\right)$$

$$= \underbrace{(C_0 C_1 C_2 \ldots C_{n-1})}\left[\left(1 + \frac{C_1}{C_0}\right)\left(1 + \frac{C_2}{C_1}\right) \ldots \left(1 + \frac{C_n}{C_{n-1}}\right)\right]$$

$$= (C_0 C_1 C_2 \ldots C_{n-1})(1+n)\left(1 + \frac{n-1}{2}\right) \ldots \left(1 + \frac{1}{n}\right)$$

$$= (C_0 C_1 \ldots C_{n-1})\,\frac{(1+n)}{1}\cdot\frac{(n+1)}{2} \ldots \frac{(n+1)}{n}$$

$$= \frac{(C_0 C_1 \ldots C_{n-1})(1+n)^n}{n!}$$ Ans

(a) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(b) ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$

$\boxed{{}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}}$

(c) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

(d) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} \ldots + \frac{(-1)^n C_n}{n+1} = \frac{1}{n+1}$

(e) $C_0 + C_1 + C_2 + \ldots = C_n = 2^n$

(f) $C_0 + C_2 + C_4 + \ldots = C_1 + C_3 + C_5 + \ldots = 2^{n-1}$

(g) $C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{n!n!}$

(h) $C_0.C_r + C_1.C_{r+1} + C_2.C_{r+2} + \ldots + C_{n-r}C_n = \frac{(2n)!}{(n+r)!(n-r)!}$

**Remember** : $\boxed{(2n)! = 2^n.\,n!\,[1.3.5 \ldots (2n-1)]}$

33

The sum of the coefficients of the first 50 terms in the binomial expansion of $(1-x)^{100}$, is equal to

(a) ${}^{101}C_{50}$ (b) $-{}^{99}C_{49}$

(c) $-{}^{101}C_{50}$ (d) ${}^{99}C_{49}$

*(12th April 1st Shift 2023)*

$$(1-x)^{100} = {}^{100}C_0 - {}^{100}C_1 x + {}^{100}C_2 x^2 - {}^{100}C_3 x^3 \dots \dots + {}^{100}C_{100} x^n$$

$$(1-x)^{100} = C_0 - C_1 x + C_2 x^2 - C_3 x^3 \dots + C_{100} x^n$$

$\boxed{x=1}$

$$0^{100} = C_0 - C_1 + C_2 - C_3 \dots + C_{100}$$

$$0 = C_0 - C_1 + C_2 \dots + C_{50} - C_{51} + \dots + C_{100}$$

$$= C_0 - C_1 + C_2 \dots + C_{50} - C_{49} + C_{48} \dots + C_0$$

$$= 2[C_0 - C_1 + C_2 - C_{49}] + [C_0 - C_1 \dots + C_{49}] + C_{50}$$

$$C_0 - C_1 + C_2 \dots - C_{49} = -\frac{C_{50}}{2}$$

$$= -\frac{1}{2}\frac{100!}{50!\,50!} = -\frac{99!}{50!\,49!}$$

(b)

34)

Fractional part of the number $\frac{4^{2022}}{15}$ is equal to

~~(a)~~ $\frac{1}{15}$ (b) $\frac{4}{15}$ (c) $\frac{14}{15}$ (d) $\frac{8}{15}$

*(13th April 1st Shift 2023)*

$$\frac{4^{2022}}{15} = \frac{(4^2)^{1011}}{15} = \frac{(16)^{1011}}{15} = \frac{(1+15)^{1011}}{15}$$

$$= \frac{1 + {}^{1011}C_1 15 + {}^{1011}C_2 15^2 + \dots}{15}$$

$\boxed{1/15}$ fraction $= \underbrace{\frac{1}{15}} + [\underbrace{{}^{1011}C_1 + {}^{1011}C_2 15}_{\text{integer}} + \cdots]$

35)

The value of

$$\frac{1}{1!\,50!}+\frac{1}{3!\,48!}+\frac{1}{5!\,46!}+\ldots+\frac{1}{49!\,2!}+\frac{1}{51!\,1!}$$ is

(a) $\frac{2^{51}}{50!}$ (b) $\frac{2^{50}}{50!}$ (c) $\frac{2^{51}}{51!}$ (d) $\frac{2^{50}}{51!}$

*(1st Feb 1st Shift 2023)*

$$= \frac{1}{1!\,50!} + \frac{1}{3!\,48!} + \frac{1}{5!\,46!} + \cdots + \frac{1}{51!\,1!}$$

$$= \frac{1}{51!}\left[\frac{51!}{1!\,50!} + \frac{51!}{3!\,48!} + \cdots + \frac{51!}{51!\,1!}\right]$$

$$= \frac{1}{51!}\left[{}^{51}C_1 + {}^{51}C_3 + \cdots {}^{51}C_{51}\right]$$

$$= \frac{1}{\lfloor 51}\left[2^{51-1}\right] = 2^{50}/\lfloor 51$$

36)

The remainder when $\underbrace{(11)^{1011} + (1011)^{11}}$ is divided by 9 is

(a) 1 (b) 4 (c) 6 (d) 8

*(25th July 2nd Shift 2022)*

$$= (11)^{1011} + (1011)^{11}$$

$$= (9+2)^{1011} + (1008+3)^{11}$$

$$= \underbrace{(9^{1011} + {}^{n}C_1 9^{1010}\cdot 2 + \cdots + 2^{1011})}_{\text{if we devide by 9 remainder } 2^{1011}} + \underbrace{(1008^{11} + {}^{n}C_1 \cdot 1008^{10}\cdot 3 + \cdots + 3^{11})}_{\text{Remainder } 3^{11}}$$

$2^{1011} + 3^{11}$

$=$

$= (8)^{337} + 3^{11}$

$= (9-1)^{337} + (3^9 \cdot 3^2)$

$= (9-1)^{337} + \{9\}\{9^3\}$ No remainder

remainder $= -1$.

actual remainder $-1+9 = 8$

37

The remainder when $(2021)^{2022} + (2022)^{2021}$ is divided by 7 is

(a) 0 (b) 1 (c) 2 (d) 6

*($27^{th}$ July $1^{st}$ Shift 2022)*

$$7\overline{)2023}\ 289$$

$(2021)^{2022} + (2022)^{2021}$

$= (2023-2)^{2022} + (2023-1)^{2021}$

$= \{(2023)^{2022} + {}^{2022}C_1 (2023)^1 (-2) + \dots + (-2)^{2022}\}$

$+ \{(2023)^{2021} + {}^{2021}C_1 (2023)^{2020} \cdot (-1) + \dots \dots (-1)^{2021}\}$

$= 7\lambda + (2)^{2022} + 7\beta - 1$

$= 7\lambda + (8)^{674} + 7\beta - 1$

$= 7\lambda + (7+1)^{674} + 7\beta - 1$

$= 7\lambda + (\underbrace{7^{674} + {}^{674}C_1 7^{673} \cdot 1 + \dots} + 1^{674}) + 7\beta - 1$

$= 7\lambda + 7\gamma + \cancel{1} + 7\beta - \cancel{1}$

remainder zero

38

The remainder when $3^{2022}$ is divided by 5 is

(a) 1 (b) 2 (c) 3 (d) 4

*(24th June 1st Shift 2022)*

$$\frac{3^{2022}}{5} = \frac{(3^2)^{1011}}{5} = \frac{9^{1011}}{5} = \frac{(10-1)^{1011}}{5} = \frac{10 + {}^{1011}C_1\, 10^9 \cdot(-1) + \dots (-1)^{1011}}{5}$$

$= 10\beta - 1$

$= 5\cdot(2\beta) - 1$

$= 5(8) - 1$

$= 5(8') + 4$ — Remainder

it will be $10\lambda$ type, $105 - 1$, $\therefore$ 4 remainder

39

$\sum_{k=0}^{20} \left({}^{20}C_k\right)^2$ is equal to

(a) ${}^{40}C_{21}$ (b) ${}^{41}C_{20}$ (c) ${}^{40}C_{20}$ (d) ${}^{40}C_{19}$

*(27th Aug 1st Shift 2021)*

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots {}^nC_n x^n$

$(x+1)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} + \dots + {}^nC_n 1^n$

Multiply

$(1+x)^n(x+1)^n = ({}^nC_0 + {}^nC_1 x + \dots {}^nC_n x^n)({}^nC_0 x^n + \dots + {}^nC_n 1^n)$

comparing coefficients of $x^n$ to both sides.

$${}^{2n}C_n = {}^nC_0^2 + {}^nC_1^2 + \dots {}^nC_n^2$$

$$= \sum_{r=0}^{n} ({}^nC_r)^2 = {}^{2n}C_n$$

$$\sum_{r=0}^{20} ({}^{20}C_k)^2 = {}^{40}C_{20} \quad \text{Ans}$$

(40)

The value of $\sum_{r=0}^{6} ({}^6C_r \cdot {}^6C_{6-r})$ is equal to

(a) 1324 (b) 1124 (c) 1024 (d) 924

*(17th March 2nd Shift 2021)*

$$= {}^6C_0\,{}^6C_6 + {}^6C_1\,{}^6C_5 + {}^6C_2\,{}^6C_4 + {}^6C_3\,{}^6C_3 + {}^6C_4\,{}^6C_2 + {}^6C_5\,{}^6C_1 + {}^6C_6\,{}^6C_0$$

$$= 2({}^6C_0\,{}^6C_6 + {}^6C_1\,{}^6C_5 + {}^6C_2\,{}^6C_4) + [{}^6C_3]^2$$

$$= 2[({}^6C_0)^2 + ({}^6C_1)^2 + ({}^6C_2)^2] + [{}^6C_3]^2$$

$$= 2[1 + 6^2 + (15)^2] + [20]^2$$

$$= 2[1 + 36 + 225] + 400 = 924$$

41)

If $\{p\}$ denotes the fractional part of the number $p$, then $\left\{\frac{3^{200}}{8}\right\}$, is equal to

(a) $\frac{5}{8}$ (b) $\frac{7}{8}$ (c) $\frac{3}{8}$ ✓(d) $\frac{1}{8}$

*(6th Sept 1st Shift 2020)*

$$\frac{3^{200}}{8} = \frac{(3^2)^{100}}{8} = \frac{9^{100}}{8} = \frac{(1+8)^{100}}{8} = \frac{1 + {}^{100}C_1 8 + \cdots {}^{100}C_{100} 8^{100}}{8}$$

$$= \frac{1+8\lambda}{8} = 8\lambda' + 1/8$$

$\therefore$ Remainder $1/8$

42

If ${}^{20}C_1 + (2^2)\ {}^{20}C_2 + (3^2)\ {}^{20}C_3 + \ldots + (20^2)\ {}^{20}C_{20} = A(2^{\beta})$, then the ordered pair $(A, \beta)$ is equal to

(a) (420, 19) (b) (420, 18)

(c) (380, 19) (d) (380, 18)

*(12th April 2nd Shift 2019)*

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$

Differentiating both sides.

$n(1+x)^{n-1} = {}^nC_1 + {}^nC_2(2x) + {}^nC_3 \cdot 3x^2 \quad + {}^nC_n\, n x^{n-1}$

multiply both side by $x$,

$nx(1+x)^{n-1} = {}^nC_1 x + {}^nC_2\, 2x^2 + {}^nC_3\, 3x^3 + \ldots\ {}^nC_n n x^n$

Again Differentiating w.r.t to $x$.

$n\cdot(1+x)^{n-1} + nx\cdot(n-1)(1+x)^{n-2} = {}^nC_1 + {}^nC\ 2^2\cdot x + {}^nC_3 3^2 x^2$
$+ \ldots\ {}^nC_n n^2 x^{n-1}$

let $\boxed{x=1}$

$n(2)^{n-1} + n(n-1)2^{n-2} = {}^nC_1 + {}^nC_2 2^2 + {}^nC_3 3^2 + \ldots {}^nC_n n^2$

let put $n = 20$

${}^{20}C_1 + {}^{20}C_2\, 2^2 + \ldots {}^{20}C_{20}\, 20^2 = 20\cdot 2^{19} + 20\cdot 19\cdot 2^{18}$
$= 2^{18}\{40 + 20\cdot 19\}$
$= 2^{18}\{420\}$

$A = 420$
$B = 18.$

(b)

43

Let $(x + 10)^{50} + (x - 10)^{50} = a_0 + a_1x + a_2x^2 + \ldots + a_{50}x^{50}$, for all $x \in R$; then $\frac{a_2}{a_0}$ is equal to

(a) 12.00 (b) 12.75 (c) 12.25 (d) 12.50

*(11th Jan 2nd Shift 2019)*

$$(x+10)^{50} + (x-10)^{50} = a_0 + a_1 x + a_2 x^2 - \cdots + a_{50} x^{50}.$$

$$2\left[{}^{50}C_0 x^{50} + {}^{50}C_2 x^{48} \cdot 10^2 + \cdots\cdots + {}^{50}C_{50}\, 10^{50}\right]$$

$$= a_0 + a_1 x + \cdots + a_{50} x^{50}$$

$$a_0 = 2 \cdot 10^{50}$$

$$a_2 = 2 \cdot {}^{50}C_{48}\, 10^{48}.$$

$$\frac{a_2}{a_0} = \frac{{}^{50}C_{48}}{10^2} = 12.25$$

44)

The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \ldots\ldots\ldots + ({}^{21}C_{10} - {}^{10}C_{10})$ is

(a) $2^{21} - 2^{10}$ (b) $2^{20} - 2^9$

(c) $2^{20} - 2^{10}$ (d) $2^{21} - 2^{11}$ *(2017)*

$$= \left(\underbrace{{}^{21}C_1 + {}^{21}C_2 \cdots}\ + {}^{21}C_{10}\right) - \left(\underbrace{{}^{10}C_1 + {}^{10}C_2 \cdots + {}^{10}C_{10}}\right)$$

$$= \quad (1+x)^{21} = {}^{21}C_0 + {}^{21}C_1 x + {}^{21}C_2 x^2 \cdots \quad {}^{21}C_{21} x^{21}$$

$$2^{21} = 1 + [\underbrace{{}^{21}C_1 \cdots {}^{21}C_{10}} + \underbrace{{}^{21}C_{10} + {}^{21}C_9 \cdots {}^{21}C_{20}} + {}^{21}C_{21}]$$

$$2 + 2[{}^{21}C_1 + \cdots + {}^{21}C_{10}] = 2^{21} \qquad \therefore {}^{21}C_1 + \cdots \ {}^{21}C_{10} = \frac{2^{21}}{2} - 1$$

$$= \frac{2^{21}}{2} - 1 - [2^{10} - 1]$$

$$= \frac{2^{21}}{2} - 2^{10} = 2^{20} - 2^{10}$$

45

The remainder left out when $8^{2n} - (62)^{2n+1}$ is divided by 9 is

(a) 2 (b) 7 (c) 8 (d) 0 *(2009)*

$$(8^2)^n - 62^{2n+1} = 64^n - 62^{2n+1}$$

$$= (1+63)^n - (63-1)^{2n+1}.$$

$$= [1 + {}^nC_1\, 63 + {}^nC_2\, 63^2 \ldots 63^n] + [1-63]^{2n+1}$$

$$= [1 + {}^nC_1\, 63 \ldots + 63^n] + [1 - {}^{2n+1}C_1\, 63 + \cdots (-)^{2n}C_1 (63)]^{2n+1}$$

$$= 2 + 63K \qquad \therefore \text{ Remainder } \underline{\underline{2}}$$

46

Let the sum of the coefficients of the first three terms in the expansion of $\left(x - \frac{3}{x^2}\right)^n, x \neq 0, n \in N,$ be 376. Then the coefficient of $x^4$ is ________.

*($24^{th}$ Jan $2^{nd}$ Shift 2023)*

$${}^nC_0 - {}^nC_1\, 3 + {}^nC_2\, 3^2 = 376$$

$$1 - n3 + \frac{n(n-1)}{2} 9 = 376$$

$$3n^2 - 5n - 250 = 0$$

$$n = 10$$

$$T_{r+1} = {}^{10}C_r\, x^{10-r} \cdot \left(\frac{-3}{x^2}\right)^r$$

$$= {}^{10}C_r (-3)^r \cdot x^{10-3r}$$

$$= {}^{10}C_r (-3)^r x^{10-3r}$$

$10 - 3r = 4$

$r = 2$

$\therefore$ coeff of $x^4 = {}^{10}C_2(-3)^2$

$= 405$

47) The remainder when $(2023)^{2023}$ is divided by 35 is ______. (25th Jan 2nd Shift 2023)

$= \frac{(2023)^{2023}}{35}$

$2023^{2023} = (\underbrace{2030}_{\text{Divisible by 35}} - 7)^{2023}$

$= (35K - 7)^{2023}$

$= \left[(35k)^{2023} + {}^{2023}C_1 (35k)^{2022} \cdot (-7) + \dots + (-7)^{2023}\right]$

$= \underbrace{35\lambda}_{\text{Divisible by 35}} - \underbrace{7^{2023}}$

Remainder $- 7^{2023}$.

$= -7 \times 7^{2022}$

$= -7\,(49)^{1011}$

$= -7\,(50-1)^{1011}$

$= -7\left[{}^{1011}C_0\, 50^{1011} - {}^{11}C_1\, 50^{1010} \dots - {}^{1011}C_{1011}\right]$

$= -7\,[5\lambda - 1]$

$= -35\lambda + 7$

remainder

48

$$1+\left(2+\underbrace{{}^{49}C_1 \cdots +{}^{49}C_{49}}\right) * \left(\underbrace{{}^{50}C_2 + {}^{50}C_4 \cdots {}^{50}C_{50}}\right)$$

If $1+(2+{}^{49}C_1+{}^{49}C_2+\ldots+{}^{49}C_{49})({}^{50}C_2+{}^{50}C_4+\ldots+{}^{50}C_{50})$ is equal to $2^n \times m$, where $m$ is odd, then $n+m$ is equal to ______. *(28th July 2nd Shift 2022)*

$$= 1+\left(2+2^{49}-1\right) * \left[\frac{2^{50}}{2}-1\right]$$

$$= 1+(1+2^{49})\left(\frac{2^{50}}{2}-1\right) = 1+2^{98}-1 = \underline{2^{98}}$$

49

If $\sum_{k=1}^{10} k^2\left({}^{10}C_k\right)^2 = 22000L$, then $L$ is equal to ______. *(29th July 2nd Shift 2022)*

$$\sum_{k=1}^{10} k^2\left({}^{10}C_k\right)^2 = \underline{22000L}$$

$${}^nC_r = \frac{n}{r}\cdot {}^{n-1}C_{r-1}$$

$$\boxed{r\cdot {}^nC_r = n\,{}^{n-1}C_{r-1}}$$

$$\sum_{k=1}^{10} k^2\left({}^{10}C_k\right)^2 = \sum_{k=1}^{10}\left(\underbrace{k\,{}^{10}C_k}\right)^2$$

$$= \sum_{k=1}^{10}\left(10\,{}^{9}C_{k-1}\right)^2$$

$$= 100\sum_{k=1}^{10}\left({}^{9}C_{k-1}\right)^2 = 100\left[\underbrace{\left({}^9C_0\right)^2+\left({}^9C_1\right)^2+\cdots\left({}^9C_9\right)^2}\right]$$

$$= 100\cdot{}^{18}C_9 = 486200$$

$$\therefore \underline{L = 221}$$

(a) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(b) ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r$ $\quad \boxed{{}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}}$

(c) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \ldots\ldots \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

(d) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} \ldots\ldots + \frac{(-1)^n C_n}{n+1} = \frac{1}{n+1}$

(e) $C_0 + C_1 + C_2 + \ldots\ldots = C_n = 2^n$

(f) $C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots = 2^{n-1}$

(g) $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = {}^{2n}C_n = \frac{(2n)!}{n!n!}$

(h) $C_0.C_r + C_1.C_{r+1} + C_2.C_{r+2} + \ldots\ldots + C_{n-r}C_n = \frac{(2n)!}{(n+r)!(n-r)!}$

**Remember :** $\boxed{(2n)! = 2^n . n! [1.3.5 \ldots\ldots (2n-1)]}$

$$r\,{}^nC_r = n\,{}^{n-1}C_{r-1}$$

$$\sum r\,{}^nC_r = n \sum {}^{n-1}C_{r-1}$$

Now

$$\sum_{r=1}^{n} r\,{}^nC_r = n \sum_{r=1}^{n} {}^{n-1}C_{r-1}$$

$$= n\left[{}^{n-1}C_0 + {}^{n-1}C_1 + \ldots {}^{n-1}C_{n-1}\right]$$

$$\boxed{\sum_{r=0}^{n} r\,{}^nC_r = n \cdot 2^{n-1}}$$

Imp

50) The remainder on dividing $1 + 3 + 3^2 + 3^3 + \ldots. + 3^{2021}$ by 50 is ________. *(24th June 2nd Shift 2022)*

$$GP = 1 + 3 + 3^2 + 3^3 \ldots + 3^{2021}$$

$$S = 1 . \frac{3^{2022} - 1}{2} = \frac{(3^2)^{1011} - 1}{2} = \frac{(10-1)^{1011} - 1}{2}$$

$$= \frac{\left[10^{1011} + {}^{1011}C_1\, 10^{1010}(-1) + \ldots + (-1)^{1011}\right] - 1}{2}$$

$$= \frac{10^{1011} - {}^{1011}C_1\, 10^{1010} \ldots + {}^{1011}C_{1009}\, 10^2 (-1)^{1009} + {}^{1011}C_{1010} . 10 (-1)^{1010} + (-1)^{1011} - 1}{2}$$

$$= \frac{100(\lambda) + 1011 \times 10 - 2}{2}$$

$$= \frac{100\lambda + 10108}{2} = 50\lambda + 5054$$

$$= 50\lambda + 50\lambda' + 4$$

Remainder 4

51)

Let $n \in N$ and $[x]$ denote the greatest integer less than or equal to $x$. If the sum of $(n+1)$ terms ${}^nC_0, 3.{}^nC_1, 5.{}^nC_2, 7.{}^nC_3, \ldots$ is equal to $2^{100}.101$, then $2\left[\frac{n-1}{2}\right]$ is equal to ______. *(25th July 2nd Shift 2021)*

$${}^nC_0 + 3.{}^nC_1 + 5.{}^nC_2 \cdots + (2n+1)\,{}^nC_n = 2^{100}.101.$$

$$T_r = (2r+1)\,{}^nC_r$$

$$\sum_{r=0}^{n} T_r = \sum (2r+1)\,{}^nC_r = \sum_{r=0}^{n} 2r.{}^nC_r + \sum_{r=0}^{n} {}^nC_r$$

$$= 2(n\,2^{n-1}) + 2^n.$$

$$= 2^n(n+1)$$

$$2^n(n+1) = 2^{100}.101.$$

$$\boxed{n = 100}$$

$$\therefore\ 2\left[\frac{n-1}{2}\right] = 2\frac{99}{2} = 98$$

Imp.

$$\sum_{r=0}^{n} r.{}^nC_r = n\,2^{n-1}$$

52

If $(2021)^{3762}$ is divided by 17, then the remainder is ______. *(17th March 1st Shift 2021)*

$$\frac{(2021)^{3762}}{17} = (2023-2)^{3762}$$

$$= 17K + 2^{3762}$$

Remainder

$\frac{2^{3762}}{17}$ ← Find Remainder

$2^{3762} = 4 \cdot 16^{940}$

$= 4(17-1)^{940}$

$= 4\{17K + (1)^{940}\}$

$= 17 \cdot 4K + \boxed{4}$ — Remainder

53

If $C_r \equiv {}^{25}C_r$ and $C_0 + 5C_1 + 9C_2 + .... + (101) \cdot C_{25} = 2^{25} \cdot k$, then $k$ is equal to ______. (9th Jan 2nd Shift 2020)

$C_0 + 5C_1 + 9C_2 + \dots + 101\, C_{25}$

$= \sum_{r=0}^{25} (4r+1) C_r$

$= 4 \sum_{r=0}^{25} r\, {}^{25}C_r + \sum_{r=0}^{25} {}^{25}C_r$

$= 4\left[25 \cdot 2^{24}\right] + 2^{25}$

$= 2^{25} \cdot 51 = 2^{25} \cdot K$

$K = 51$

54

If the coefficients of $x^7$ in $\left(ax^2 + \frac{1}{2bx}\right)^{11}$ and $x^{-7}$ in $\left(ax - \frac{1}{3bx^2}\right)^{11}$ are equal, then

(a) $64ab = 243$
(b) $729ab = 32$
(c) $32ab = 729$
(d) $243ab = 64$

(6th April 2nd Shift 2023)

$T_{r+1} = {}^{11}C_r (ax^2)^{11-r} \cdot \left(\frac{1}{2bx}\right)^r$

$= {}^{11}C_r \frac{a^{11-r}}{(2b)^r} \cdot x^{22-3r}$

coeff of $x^7 \Rightarrow 22 - 3r = 7 \rightarrow \boxed{r = 5}$

$$T_6 = {}^{11}C_5 \frac{a^6}{2^5 b^5} x^7.$$

Similarly

$$T_{r+1} = {}^{11}C_r (ax)^{11-r} \left(\frac{-1}{3bx^2}\right)^r$$

$$= {}^{11}C_r \frac{a^{11-r}}{(-3b)^r} x^{11-3r}$$

coeff of $x^{-7}$ $\quad 11 - 3r = -7$

$$\boxed{r = 6}$$

$$T_7 = {}^{11}C_6 \frac{a^5}{3^6 b^6} x^{-7}$$

$${}^{11}C_5 \frac{a^6}{2^5 b^5} = {}^{11}C_6 \frac{a^5}{3^6 \cdot b^6} \Rightarrow \boxed{729ab = 32}$$

5) If the coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio 1 : 5 : 20, then the coefficient of the fourth term is

(a) 1827 (b) 5481 (c) 2436 (d) 3654

*($8^{th}$ April $1^{st}$ Shift 2023)*

$${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 1 : 5 : 20$$

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{1}{5}$$

$$\frac{r}{n-r+1} = \frac{1}{5} \Rightarrow 5r = n-r+1$$

$$\boxed{n = 6r-1}$$

$$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{5}{20} = \frac{1}{4}.$$

$$n = 5r+4$$

Solve

$6r-1=5r+4$

$r=5 \quad \therefore n=29.$

coefficient of 4th term $= {}^{29}C_3 = 3654.$

56

If the coefficients of $x$ and $x^2$ in $(1+x)^p(1-x)^q$ are 4 and $-5$ respectively, then $2p+3q$ is equal to

(a) 66 (b) 60 (c) 63 (d) 69

*(10th April 2nd Shift 2023)*

$= (1+x)^p (1-x)^q$

$= \left[1+px+\frac{p(p-1)}{2}x^2+\ldots\right]\left[1-qx+\frac{q(q-1)}{2}x^2+\ldots\right]$

coefficient of $x = p-q = 4$

coeff of $x^2 = \frac{q(q-1)}{2} - pq + \frac{p(p-1)}{2} = -5$

$(p-q)^2 - (p+q) = -10$

$p-q = 26$

$\therefore p = 15 \quad q = 11$

$\therefore \boxed{2p+3q = 63}.$

57

The sum of the coefficients of three consecutive terms in the binomial expansion of $(1+x)^{n+2}$, which are in the ratio 1 : 3 : 5, is equal to

(a) 41 (b) 92 (c) 25 (d) 63

*(11th April 2nd Shift 2023)*

${}^{n+2}C_{r-1} : {}^{n+2}C_r : {}^{n+2}C_{r+1}$

$= 1 : 3 : 5$

$\frac{{}^{n+2}C_{r-1}}{{}^{n+2}C_r} = \frac{1}{3}$

Solve ~ $\boxed{n = 4r-3}$ — ①

$\frac{{}^{n+2}C_r}{{}^{n+2}C_{r+1}} = \frac{3}{5}$

$8r-1 = 3n$ — ②

$$4r-3 = \frac{8r-1}{3}$$

$$r = 2 \text{ and } n = 5$$

$$\text{Required sum} = {}^7C_1 + {}^7C_2 + {}^7C_3$$

$$= 63.$$

58

The coefficient of $x^5$ in the expansion of $\left(2x^3 - \frac{1}{3x^2}\right)^5$ is

(a) $\frac{26}{3}$ (b) 9 (c) 8 (d) $\frac{80}{9}$

*(13th April 2nd Shift 2023)*

$$T_{r+1} = {}^5C_r \,(2x^3)^{5-r} \cdot \left(-\frac{1}{3x^2}\right)^r$$

$$= {}^5C_r \; 2^{5-r} \left(-\frac{1}{3}\right)^r x^{15-3r-2r}$$

$$15 - 5r = 5 \quad \therefore \; r = 2$$

$$\text{coeff} \Rightarrow {}^5C_2 \; 2^3 \cdot (-1/3)^2 = 80/9$$

59

The coefficient of $x^{301}$ in $(1+x)^{500} + x(1+x)^{499} + x^2(1+x)^{498} + \ldots + x^{500}$ is

(a) ${}^{500}C_{301}$ (b) ${}^{501}C_{200}$ (c) ${}^{500}C_{300}$ (d) ${}^{501}C_{302}$

*(30th Jan 1st Shift 2023)*

$$\sim (1+x)^{500} + x(1+x)^{499} + x^2(1+x)^{498} + \ldots \quad x^{500}$$

coef of $x^{500}$

$$= {}^{500}C_{301} + {}^{499}C_{300} + {}^{498}C_{299} + \ldots \; {}^{199}C_0$$

$$\approx {}^{500}C_{199} + {}^{499}C_{199} + {}^{498}C_{199} + \ldots + {}^{199}C_{199}$$

Formula

$$\Rightarrow \; {}^{n}C_r + {}^{n-1}C_r + {}^{n-2}C_r \cdots + {}^{r}C_r = {}^{n+1}C_{r+1}$$

$${}^{r}C_r + {}^{r+1}C_r + \cdots\cdots {}^{n}C_r = {}^{n+1}C_{r+1}$$

$$= {}^{501}C_{200}$$

60

The term independent of $x$ in the expansion of

$(1 - x^2 + 3x^3)\left(\frac{5}{2}x^3 - \frac{1}{5x^2}\right)^{11}$, $x \neq 0$ is :

(a) $\frac{7}{40}$ (b) $\frac{33}{200}$ (c) $\frac{39}{200}$ (d) $\frac{11}{50}$

*(28th June 2nd Shift 2022)*

$$= (1 - x^2 + 3x^3)\left(\left(\frac{5}{2}x^3 - \frac{1}{5x^2}\right)^{11}\right.$$

$$= \underbrace{\left(\frac{5}{2}x^3 - \frac{1}{x^2}\right)^{11}} - x^2\left(\left(\frac{5}{2}x^3 - \frac{1}{5x^2}\right)^{11} + 3x^3\left(\frac{5}{2}x^3 - \frac{1}{5x^2}\right)^{11}\right.$$

$$T_{r+1} = {}^{11}C_r\left(\frac{5}{2}x^3\right)^r \cdot \left(-\frac{1}{5x^2}\right)^{11-r}$$

$${}^{11}C_r\left(\frac{5}{2}\right)^r\left(-\frac{1}{5}\right)^{11-r} x^{5r-22}$$

$5r - 22 = 0$

$r = 22/5$ Fraction

N.P

For independent term

$5r - 22 = -2$

$5r = 20$

$r = 20/5$

$= 4$

$5r - 22 = -3$

$5r = 22 - 3$

$5r = 18$

$r = 18/5$

$$T_{r+1} = -{}^{11}C_4\left(\frac{5}{2}\right)^4\left(-\frac{1}{5}\right)^{11-4} \cdot$$

$$(-)\,{}^{11}C_4\left(\frac{5}{2}\right)^4(-)^7 \cdot \left(\frac{1}{5}\right)^7 = 33/200$$

(61)

The sum of all those terms which are rational numbers in the expansion of $(2^{1/3} + 3^{1/4})^{12}$ is

(a) 27 (b) 89 (c) 35 (d) 43

*(25th July 2nd Shift 2021)*

$$T_{r+1} = {}^{12}C_r \left(2^{1/3}\right)^{12-r} \left(3^{1/4}\right)^{r}$$

$$= {}^{12}C_r \; 2^4 . \; 2^{-r/3} . \; 3^{r/4} .$$

$T_{r+1}$ will be rational if $r = 0, 12.$

$\therefore T_1 + T_{13} = 16 + 27 = \underline{43}$

(62)

If $n$ is the number of irrational terms in the expansion of $\left(3^{\frac{1}{4}} + 5^{\frac{1}{8}}\right)^{60}$, then $(n-1)$ is divisible by

(a) 26 (b) 7 (c) 8 (d) 30

*(16th March 1st Shift 2021)*

$$T_{r+1} = {}^{60}C_r \left(3^{1/4}\right)^{60-r} . \left(5^{1/8}\right)^{r}$$

$$= {}^{60}C_r \; 3^{\frac{60-r}{4}} . \; 5^{r/8}$$

For rational terms, $r$ should be a multiple of 8 and less than 60.

$r \sim 0, 8, 16, \ldots \; 56.$

Number of irrational terms $61 - 8 = \underline{53}$

$n = 53.$ $n - 1 = 52$ which is divisible by $\underline{26}$.

63

If the fourth term in the expansion of $(x + x^{\log_2 x})^7$ is 4480, then the value of $x$ where $x \in N$ is equal to

(a) 3 (b) 4 ~~(c)~~ 2 (d) 1

*(17th March 1st Shift 2021)*

$$T_4 = {}^7C_3 \, x^4 \cdot \left(x^{\log_2 x}\right)^3 = 4480$$

$$35 \, x^4 \, x^{3\log_2 x} = 4480$$

$$x^{4+3\log_2 x} = 2^7$$

$$x = 2.$$

64

The maximum value of the term independent of $t$ in the expansion of $\left(tx^{1/5} + \frac{(1-x)^{1/10}}{t}\right)^{10}$, where $x \in (0, 1)$ is

(a) $\frac{2 \cdot 10!}{3(5!)^2}$ (b) $\frac{10!}{\sqrt{3}(5!)^2}$

(c) $\frac{2 \cdot 10!}{3\sqrt{3}(5!)^2}$ (d) $\frac{10!}{3(5!)^2}$

*(26th Feb 1st Shift 2021)*

$$T_{r+1} = {}^{10}C_r \left(t x^{1/5}\right)^{10-r} \left(\frac{(1-x)^{1/10}}{t}\right)^r$$

independent of t.

$$10 - r - r = 0$$

$$r = 5$$

$$T_6 = {}^{10}C_5 \; x (1-x)^{1/2}$$

$$\frac{d}{dx}(T_6) = {}^{10}C_5 \left[(1-x)^{1/2} - \frac{x}{2\sqrt{1-x}}\right] = 0$$

$$2(1-x) - x = 0$$

$$2 - 3x = 0 \quad x = 2/3$$

$$T_6 = {}^{10}C_5 \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^{1/2} = \frac{2 \times 10!}{3\sqrt{3}\,(5!)^2}$$

65

If the term independent of $x$ in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ is $k$, then $18k$ is equal to

(a) 5 (b) 9 (c) 7 (d) 11

*(3rd Sept 2nd Shift 2020)*

$$T_{r+1} = {}^9C_r \left(\frac{3}{2}x^2\right)^{9-r} \cdot \left(-\frac{1}{3x}\right)^r$$

Independent of $x$.

$$(x^2)^{9-r} \; x^{-r} = x^0$$

$$r = 6$$

$$\therefore \; T_7 = {}^9C_6 \left(\frac{3}{2}\right)^3 \left(-\frac{1}{3}\right)^6$$

$$= 84 \cdot \frac{27}{8} \cdot \frac{1}{729} = \frac{7}{18}$$

$$k = \frac{7}{18}$$

$$18k = 7$$

66

The total number of irrational terms in the binomial expansion of $(7^{1/5} - 3^{1/10})^{60}$ is

(a) 55 (b) 54 (c) 48 (d) 49

*(12th Jan 2nd Shift 2019)*

$$T_{r+1} = (-1)^r \cdot {}^{60}C_r \; 7^{\frac{60-r}{5}} \cdot 3^{r/10}$$

For rational terms, $r = 0, 10, 20, 30, 40, 50, 60$

So, number of rational terms $= 7$

Number of irrational terms $= 61 - 7$

$= \underline{54}$

# Multinominal Theorem

## MULTINOMIAL THEOREM

Concepts And Questions Solving Techniques

JEE (MAINS & ADVANCED)

$$(x_1 + x_2 + x_3 + \ldots\ldots x_n)^n$$

$$= \sum \frac{n!}{(\alpha_1)!\,(\alpha_2)!\ldots(\alpha_n)!} \times x_1^{\alpha_1} x_2^{\alpha_2} \ldots\ldots x_n^{\alpha_n}$$

$$\alpha_1 + \alpha_2 + \alpha_3 - - - + \alpha_n = n$$

67

The constant term in the expansion of $\left(2x + \frac{1}{x^7} + 3x^2\right)^5$ is ______.

*(25th Jan 1st Shift 2023)*

General term $= \frac{5!}{r_1!\,r_2!\,r_3!} (2x)^{r_1} \left(\frac{1}{x^7}\right)^{r_2} (3x^2)^{r_3}$

$= \frac{5!}{r_1!\,r_2!\,r_3!} 2^{r_1} 3^{r_3} x^{r_1 - 7r_2 + 2r_3}$

for constant $r_1 - 7r_2 + 2r_3 = 0$

$r_1 + r_2 + r_3 = 5$

Hit & trial, $r_1 = 1 \quad r = 1 \quad r_3 = 3$

constant term

$$\frac{5!}{1!\,1!\,3!}\,2^1\,3^3 = 1080$$

The coefficient of $x^7$ in $(1 - x + 2x^3)^{10}$ is _____.

jee main 2023

$$\text{General term} = \frac{10!}{r_1!.r_2!.r_3!}(-1)^{r_2}.(2)^{r_3}\,x^{r_2+3r_3}$$

where $r_1 + r_2 + r_3 = 10$ and $r_2 + 3r_3 = 7$

| $r_1$ | $r_2$ | $r_3$ |
|---|---|---|
| 3 | 7 | 0 |
| 5 | 4 | 1 |
| 7 | 1 | 2 |

Required coefficient

$$= \frac{10!}{3!.7!}(-1)^7 + \frac{10!}{5!.4!}(-1)^4(2) + \frac{10!}{7!.2!}(-1)^1(2)^2$$

$$= -120 + 2520 - 1440 = 960$$