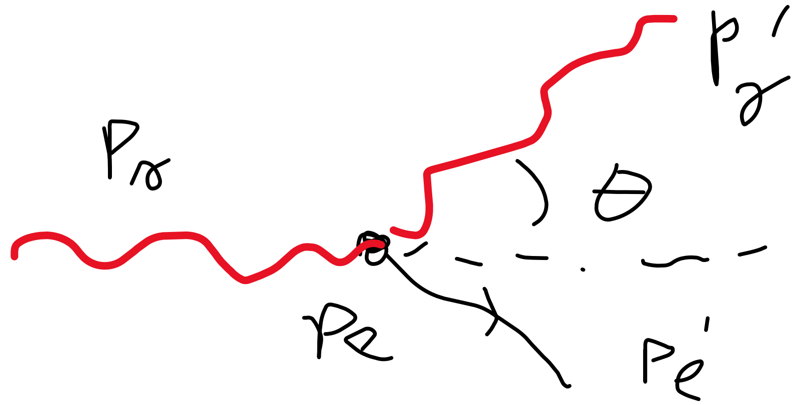
Modern Spring 2006

Problem A1: A highly energetic photon (wavelength λ) collides with an electron at rest. After the collision the angle θ between the incident photon direction and the final direction is measured. Calculate the change in the wavelength, $\Delta\lambda = \lambda' - \lambda$ in terms of θ, where λ' is the scattered photon wavelength. What is particularly remarkable about this result?



The 4-momentum of the incident photon and electron are:

$$p_\gamma^\mu = (E_\gamma, c\boldsymbol{p}_\gamma) = \left(\frac{hc}{\lambda}, c\boldsymbol{p}_\gamma\right)$$
$$p_e^\mu = (m_e c^2\,, \boldsymbol{0})$$

Since the photon is massless $|c\boldsymbol{p}_\gamma| = E_\gamma = \frac{hc}{\lambda}$. The 4-momentum of the outgoing photon and electron are:

$$p'^\mu_\gamma = (E'_\gamma, c\boldsymbol{p}'_\gamma) = \left(\frac{hc}{\lambda'}, c\boldsymbol{p}'_\gamma\right)$$
$$p'^\mu_e = (m, c\boldsymbol{p_e}')$$

As before, $|c\boldsymbol{p}'_\gamma| = E'_\gamma = \frac{hc}{\lambda'}$. By 4-momentum conservation:

$$p_\gamma^\mu + p_e^\mu = p'^\mu_\gamma + p'^\mu_e$$
$$p_\gamma^\mu - p'^\mu_\gamma = p'^\mu_e - p_e^\mu \qquad (1)$$

Square both sides of Eq. (1):

$$p_\gamma^2 - 2p_\gamma \cdot p'_\gamma + {p'_\gamma}^2 = p_e^2 - 2p_e \cdot p'_e + {p'_e}^2$$
$$(Use\ on\ mass\ shell\ condition)$$
$$0 - 2p_\gamma \cdot p'_\gamma + 0 = m_e^2c^4 - 2p_\gamma \cdot p'_\gamma + m_e^2c^4$$
$$-2\left(E_\gamma E'_\gamma - c\boldsymbol{p}_\gamma \cdot c\boldsymbol{p}'_\gamma\right) = 2m_e^2c^4 - 2\left(E_eE'_e - c\boldsymbol{p}_e \cdot c\boldsymbol{p}'_e\right)$$

$$-2\left(E_\gamma E'_\gamma - E_\gamma E'_\gamma \cos\theta\right) = 2m_e^2c^4 - 2(E_eE'_e - 0)$$

Here we used $c\boldsymbol{p}_\gamma \cdot c\boldsymbol{p}'_\gamma = |c\boldsymbol{p}_\gamma||c\boldsymbol{p}'_\gamma|\cos\theta = E_\gamma E'_\gamma \cos\theta$

$$-2E_\gamma E'_\gamma(1-\cos\theta) = 2m_e^2c^4 - 2(m_ec^2E'_e - 0)$$

$$\left(\div -2 \ and \ use \ E = \frac{hc}{\lambda} \ for \ \gamma's\right)$$

$$\frac{hc}{\lambda}\frac{hc}{\lambda'}(1-\cos\theta) = -m_e^2c^4 + m_ec^2E'_e$$

$$\frac{(hc)^2}{\lambda\lambda'}(1-\cos\theta) = -m_e^2c^4 + m_ec^2E'_e \qquad (2)$$

Now, from conservation of energy:

$$m_ec^2 + E_\gamma = E'_e + E'_\gamma$$

$$E'_e = m_ec^2 + E_\gamma - E'_\gamma$$

$$\rightarrow E'_e = m_ec^2 + \frac{hc}{\lambda} - \frac{hc}{\lambda'} \qquad (3)$$

Insert Eq. (3) into (2):

$$\frac{(hc)^2}{\lambda\lambda'}(1-\cos\theta) = -m_e^2c^4 + m_ec^2\left(m_ec^2 + \frac{hc}{\lambda} - \frac{hc}{\lambda'}\right)$$

$$\frac{(hc)^2}{\lambda\lambda'}(1-\cos\theta) = -m_e^2c^4 + m_e^2c^4 + m_ec^2\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right)$$

$$\frac{(hc)^2}{\lambda\lambda'}(1-\cos\theta) = m_ec^2\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right)$$

$$\frac{(hc)^2}{\lambda\lambda'}(1-\cos\theta) = m_ec^3h\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)$$

$$\frac{(hc)^2}{\lambda\lambda'}(1-\cos\theta) = m_ec^3h\left(\frac{\lambda'-\lambda}{\lambda\lambda'}\right)$$

$$h(1-\cos\theta) = m_ec(\lambda'-\lambda)$$

$$\rightarrow \boxed{\Delta\lambda = \frac{h}{m_ec}(1-\cos\theta)}$$

Since $(1-\cos\theta) \geq 0$ the photon always increases its wavelength, hence always loses energy. The maximum energy loss occurs at $\cos\theta = -1$, backward scattering. Finally, $\Delta\lambda \sim \frac{1}{m_e}$ and thus in general the more massive the object the less change in wavelength you'll see in Compton scattering.

The Compton effect shows light cannot be purely explained by waves.