2007 Spring B1 (EM)

Problem B1: A circular loop of radius R carries a current I.

a) Show that the magnetic field, B, at any point on the axis of the loop points along the axis and derive an expression for the magnetic field strength in terms of R and I, and the distance z from the center of the loop.

b) Now consider two such identical loops parallel to each other with their centers along the same axis separated by a distance b with each carrying current I in the same direction. Choose the origin of the z axis midway between the two loops so that the centers are at $z = -b/2$ and $z = b/2$. Now expand the total magnetic field in powers of z for points near $z = 0$. Show that the lowest order term (zeroth order in z) is

$$B_0 = \mu_0 I R^2 / d^3$$

with $d^2 = R^2 + b^2/4$.

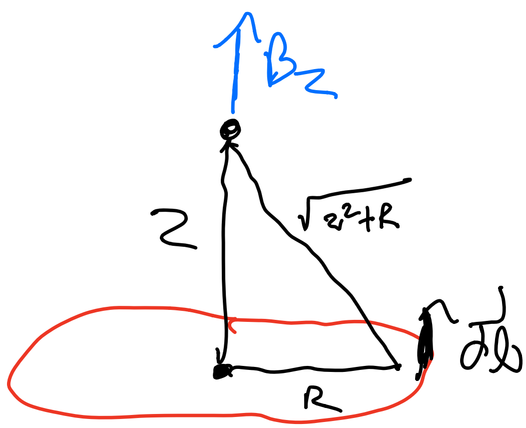
c) Show that the lowest order correction to the expression above has the form

$$B_2 = \alpha B_0 z^2,$$

and evaluate the constant α in terms of R, b, and d.

d) For the case of $b = R$ (Helmholtz coils), show that the lowest order correction to B_0 is of order z^4. You need not evaluate this term.

a)



By cylindrical symmetry $\vec{B}$ points in the $\hat{z}$ direction.

By Biot Savart. $B_z = \frac{\mu_0}{2\pi} \int \frac{I \, d\vec{l} \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^2}$

$$= \frac{\mu_0}{4\pi} I \oint \frac{dl}{\sqrt{z^2+R^2}} \cos\theta$$

$$= \frac{\mu_0}{4\pi} I \oint \frac{dl}{z^2+R^2} \frac{R}{\sqrt{z^2+R^2}}$$

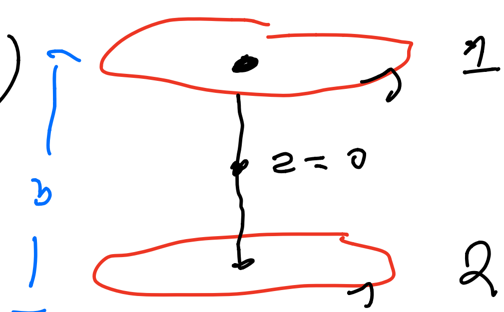
$$= \frac{\mu_0 I}{4\pi} \frac{R}{(z^2+R^2)^{3/2}} \oint dl$$

$$= \frac{\mu_0 I}{4\pi} \frac{R}{(z^2+R^2)^{3/2}} (2\pi R)$$

$$B_z = \frac{\mu_0 I R^2}{2(z^2+R^2)^{3/2}}$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 I R^2}{2(z^2+R^2)^{3/2}} \hat{z}}$$

b)



$$B_z = B_z^1 + B_z^2$$

$$= \frac{\mu_0 I R^2}{2\left((z+\frac{b}{2})^2 + R^2\right)^{3/2}} + \frac{\mu_0 I R^2}{2\left((z-\frac{b}{2})^2 + R^2\right)^{3/2}}$$

$$B_z(z) = \frac{\mu_0 I R^2}{2}\left[\frac{1}{\left[(z+\frac{b}{2})^2+R^2\right]^{3/2}} + \frac{1}{\left[(z-\frac{b}{2})^2+R^2\right]^{3/2}}\right] \quad (1)$$

Now expanding about $z=0$

$$B_z(z) = B_z(0) + z\, B'(z)|_{z=0} + \ldots$$

Therefore,

$$B_z(0) = \frac{\mu_0 I R^2}{2}\left[\frac{1}{(b^2/4+R^2)^{3/2}} + \frac{1}{(b^2/4+R^2)^{3/2}}\right]$$

$$\boxed{B_z(0) = \frac{\mu_0 I R^2}{\delta^3}}$$

c) $(z \pm b/2)^2 + R^2$

$$\left(z^2 \pm zb + \frac{b^2}{4} \right) + R^2$$

$$= z^2 \pm bz + d^2$$

Then,

$$\frac{1}{\left((z \pm b/2)^2 + R^2 \right)^{3/2}}$$

$$= \frac{1}{(z^2 \pm bz + d^2)^{3/2}} = \frac{1}{d^3 \left(1 + \underbrace{\frac{\pm bz + z^2}{d^2}} \right)^{3/2}}$$

For

$$(1+\varepsilon)^n \simeq 1 + n\varepsilon + \frac{n(n-1)}{2}\varepsilon^2 + \dots$$

$|\varepsilon| < 1$

Since $|z| < b/2$ $z < d$ ($b < d$ by definition)

Take $\varepsilon = \frac{\pm bz + z^2}{d^2}$

$$= \frac{1}{d^3} \left(1 \right) - \frac{3}{2} \left(\frac{\pm bz + z^2}{d^2} \right)$$

$$+\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{\pm bz+z^2}{d^2}\right)^3$$

$$+ \ldots\ldots)$$

Keeping only 2^{nd} order terms,

$$\frac{1}{d^3}\left[1-\frac{3}{2}\frac{\pm bz+z^2}{d^2}\right.$$

$$\left.+\frac{15}{8}\left(\frac{b^2z^2}{d^4}\right)\right]$$

$$\boxed{\frac{1}{\left((z\pm b/2)^2+R^2\right)^{3/2}} = \frac{1}{d^3}\left[1\mp\frac{3}{2}\frac{bz}{d^2}+\left(-\frac{3}{2d^2}+\frac{15b^2}{8d^4}\right)z^2\right]}$$

And,

$$B_z(z)=\mu_0 I R^2\left[\frac{1}{\left((z+b/2)^2+R^2\right)^{3/2}}+\frac{1}{\left((z-b/2)^2+R^2\right)^{3/2}}\right]$$

$$= \frac{\mu_0 I R^2}{2 d^3}\left[1 - \cancel{\frac{3}{2}\frac{b^2}{d^2}} + \left(\frac{15}{8}\frac{b^2}{d^4} - \frac{3}{2d^2}\right) z^2 \right.$$

$$\left. + 1 + \cancel{\frac{3}{2}\frac{b^2}{d^2}} + \left(\frac{15}{8}\frac{b^2}{d^4} - \frac{3}{2d^2}\right) z^2 \right]$$

$$B_z(z) = \frac{\mu_0 I R^2}{2 d^3}\left[2 + 2\left(\frac{15}{8}\frac{b^2}{d^4} - \frac{3}{2d^2}\right) z^2 \right]$$

$$\boxed{B_z(z) = B_0 + B_0 \left(\frac{15 b^2}{8 d^4} - \frac{3}{2 d^2}\right) z^2}$$

Now $\alpha = \frac{15 b^2}{8 d^4} - \frac{3}{2 d^2} \cdot \frac{4d^2}{4d^2}$

$$\alpha = \frac{15 b^2 - 12 d^2}{8 d^4}$$

$$\alpha = \frac{15 b^2 - 12 (R^2 + b^2/4)}{8 d^4}$$

$$\alpha = \frac{12(b^2 - R^2)}{8 (R^2 + b^2/4)^2}$$

$$\boxed{\alpha = \frac{3}{2}\frac{b^2 - R^2}{(R^2 + b^2/4)^2}}$$

d) From expression for α

$\alpha = 0$ when $b = R$,

Looking at Eq. (1)

$$B_z(-z) = B_z(z)$$

Therefore, the function is even and all odd terms in Taylor expansion, including the third term, are zero.

Thus, the next order correction term would be $O(z^4)$.