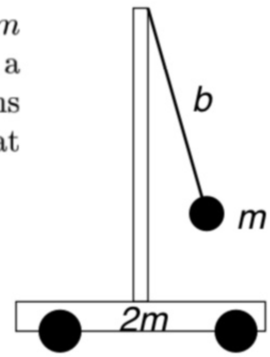
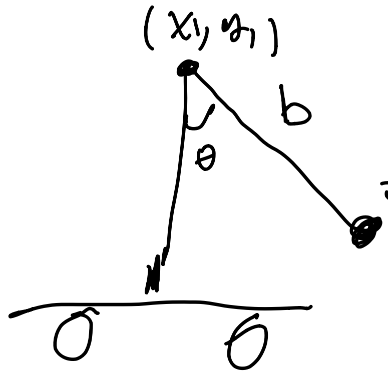
Problem A2: A simple pendulum of length b and mass m is attached to a cart of mass $2m$ that is allowed to roll on a horizontal surface (see attached figure). Find the equations of motion. Assume that the system is frictionless and that the wheels of the cart are massless.





$$(x_2, y_2) = (x_1 + b\cos\theta,\ y_1 - b\sin\theta)$$

$$T = \frac{2m}{2}\dot{x}_1^2 + \frac{1}{2}m\left((\dot{x}_1 - b\cos\theta\,\dot{\theta})^2 + b^2\cos^2\theta\,\dot{\theta}^2\right)$$

$$= \frac{1}{2}(2m)\dot{x}_1^2 + \frac{1}{2}m\left(\dot{x}_1^2 - 2b\cos\theta\,\dot{x}_1\dot{\theta} + \underbrace{b^2\cos^2\theta\,\dot{\theta}^2 + b^2\sin^2\theta\,\dot{\theta}^2}_{b^2\dot{\theta}^2}\right)$$

$$T = \frac{1}{2}(2m)\dot{x}_1^2 + \frac{1}{2}m\left(\dot{x}_1^2 - 2b\cos\theta\,\dot{x}_1\dot{\theta} + b^2\dot{\theta}^2\right) \quad (1)$$

$$V = mg\,y_2 = mg(y_1 - b\sin\theta) \quad (2)$$

$y_1 \to$ const.

Therefore,

$L = T - V$

The equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\partial L}{\partial x_1}$$

However, $\frac{\partial L}{\partial x_1} = 0$. The $p = \frac{\partial L}{\partial \dot{x}_1} = \text{const.}$

$$p = \frac{\partial L}{\partial \dot{x}_1} = 2m\dot{x}_1 + m\dot{x}_1 - mb\cos\theta\,\dot{\theta}$$

$$\boxed{p = 3m\dot{x}_1 - mb\dot{\theta}\cos\theta} \qquad (3)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

$$\frac{d}{dt}\left(-b\cancel{m}\cos\theta\,\dot{x}_1 + \cancel{m}b^2\dot{\theta}\right) = \cancel{m}gb\frac{\partial}{\partial\theta}\sin\theta$$

$$-b\left(-\sin\theta\,\dot{\theta}\dot{x}_1 + \cos\theta\,\ddot{x}_1\right) + b^2\ddot{\theta} = gb\cos\theta$$

$$\boxed{b\sin\theta\,\dot{\theta}\dot{x}_1 - b\cos\theta\,\ddot{x}_1 + b\ddot{\theta} = gb\cos\theta} \qquad (4)$$

Eq. (3) & (4) probably sufficient for full points.

Full sol^n below (not needed)

Note we can also use conservation of energy

$$E = T + V$$

$$E = \frac{1}{2}(2m)\dot{x}_1^2 + \frac{1}{2}m\left(\dot{x}_1^2 - 2b\cos\theta\,\dot{x}_1\dot{\theta} + b^2\dot{\theta}^2\right) - mgb\sin\theta$$

(y_1 arbitrary take $y_1 = 0$)

$$E = \frac{3m}{2}\dot{x}_1^2 + \frac{1}{2}m\left(-2b\cos\theta\,\dot{x}_1\dot{\theta} + b^2\dot{\theta}^2\right) - mgb\sin\theta$$

From (3)

$$\dot{x}_1 = \frac{p + mb\cos\theta\,\dot{\theta}}{3m}$$

$$E = \frac{3m}{2}\left(\frac{p + mb\cos\theta\,\dot{\theta}}{3m}\right)^2 + \frac{m}{2}\left(-2b\cos\theta\,\dot{\theta}\left(\frac{p + mb\cos\theta\,\dot{\theta}}{3m}\right) + b^2\dot{\theta}^2\right)$$

$$- mgb\sin\theta$$

$$E = \frac{1}{6m}\left(p + mb\cos\theta\,\dot{\theta}\right)^2 + \frac{m}{2}\left(-\frac{2bp}{3m}\cos\theta\,\dot{\theta} - \frac{2b^2}{3}\cos^2\theta\,\dot{\theta}^2 + b^2\dot{\theta}^2\right)$$

$$- mgb\sin\theta$$

$$E = \frac{p^2}{6m} + \cancel{\frac{pb\cos\theta\,\dot\theta}{3}} + \frac{m b^2 \cos^2\theta\,\dot\theta^2}{6}$$

$$- \cancel{\frac{bp}{3}\cos\theta\,\dot\theta} - \frac{b^2 m}{3}\cos^2\theta\,\dot\theta^2 + \frac{mb^2}{2}\dot\theta^2 - mgb\sin\theta$$

$$E - \frac{p^2}{6m} = -\frac{m b^2 \cos^2\theta\,\dot\theta^2}{6} + \frac{mb^2}{2}\dot\theta^2 - mgb\sin\theta$$

$$\frac{E}{m} - \frac{p^2}{6m^2} = \left[\frac{1}{2} - \frac{\cos^2\theta}{6}\right] b^2\dot\theta^2 - gb\sin\theta$$

$$\underbrace{\frac{6Em - p^2}{6m^2}}_{A} = \frac{(3-\cos^2\theta)}{6} b^2\dot\theta^2 - gb\sin\theta$$

$$\frac{(3-\cos^2\theta)}{6} b^2\dot\theta^2 = A + gb\sin\theta$$

$$\dot\theta^2 = \frac{6(A + gb\sin\theta)}{(3-\cos^2\theta)b^2}$$

$$\boxed{\dot\theta = \pm\sqrt{\frac{6(A+gb\sin\theta)}{(3-\cos^2\theta)b^2}}} \quad (5)$$

Eq. (3) and (5) are 1^{st} order ODE.

Solving (5) gives (3)