2007 Fall Classical - B1 (EM)

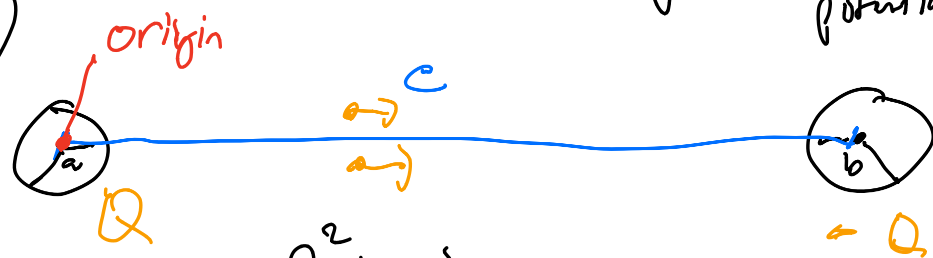
Problem B1: a) Two conducting spheres of radii a and b have their centers a distance c apart. If $c >> a$ and $c >> b$, show that the capacitance of the system is given by

$$C = 4\pi\epsilon_0 \left(\frac{1}{a} + \frac{1}{b} - \frac{2}{c}\right)^{-1}.$$

b) Two infinitely long conducting cylinders of radii a and b are separated by a distance c. Again $c >> a$ and $c >> b$. Find an approximate expression for the capacitance of a length L of this system.

$$C = \frac{Q}{V} = \frac{\text{charge}}{\text{potential}}$$

a)



$$V = -\int_1^2 d\vec{l} \cdot \vec{E}$$

$$= -\int_a^{c-b} dr \left[\frac{Q}{4\pi\varepsilon_0}\frac{1}{r^2} + \frac{Q}{4\pi\varepsilon_0}\frac{1}{(c-r)^2}\right]$$

$$= -\frac{Q}{4\pi\varepsilon_0}\int_a^{c-b} dr \left[\frac{1}{r^2} + \frac{1}{(c-r)^2}\right]$$

$$= -\frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{r} + \frac{1}{c-r}\right]_a^{c-b}$$

$$= -\frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{c-b} + \frac{1}{b} - \left(-\frac{1}{a} + \frac{1}{c-a}\right)\right]$$

$$V = \frac{Q}{4\pi\varepsilon_0}\left[-\frac{1}{c-b} + \frac{1}{b} + \frac{1}{a} - \frac{1}{c-a}\right]$$

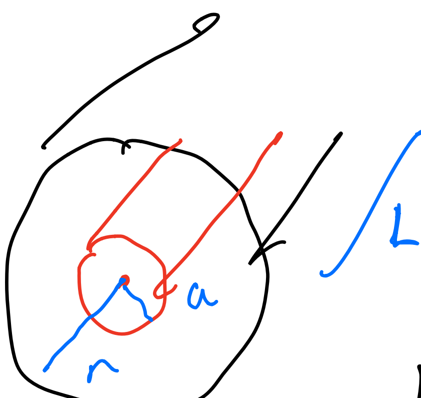
Then,

$$C = \frac{Q}{V} = \frac{4\pi\varepsilon_0}{\frac{1}{a} + \frac{1}{b} - \frac{1}{c-b} - \frac{1}{c-a}}$$

If $c \gg a, b$ then

$$\boxed{C = 4\pi\varepsilon_0\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{c}\right)^{-1}}$$

b)



$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

let $\lambda = Q/L$

$$E(2\pi r \cancel{L}) = \frac{\lambda \cancel{L}}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi\varepsilon_0} \frac{1}{r}$$

$$\Rightarrow \boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0} \frac{1}{r} \hat{r}}$$

Therefore,

$$V = -\int_{r=a}^{r=c-b} d\vec{l} \cdot \vec{E}$$

$$= -\frac{\lambda}{2\pi\varepsilon_0} \int_a^{c-b} dr \left[\frac{1}{r} + \frac{1}{c-r} \right]$$

$$= -\frac{Q/L}{2\pi\varepsilon_0} \Big[\ln(r) - \ln(c-r) \Big]_a^{c-b}$$

$$V = -\frac{Q}{2\pi\varepsilon_0 L} \Big[\ln(c-b) - \ln(b) - \ln(a) + \ln(c-a) \Big]$$

$$V = -\frac{Q}{2\pi\varepsilon_0 L} \left[\ln\left(\frac{c}{b} - 1\right) + \ln\left(\frac{c}{a} - 1\right) \right]$$

$$V = \frac{Q}{2\pi\varepsilon_0 L}\left[\ln\left(\frac{1}{\frac{c}{b}-1}\right) + \ln\left(\frac{1}{\frac{c}{a}-1}\right)\right]$$

$$\Rightarrow \; C = \frac{Q}{V} = 2\pi\varepsilon_0 L \frac{1}{\ln\left(\frac{1}{\frac{c}{b}-1}\right) + \ln\left(\frac{1}{\frac{c}{a}-1}\right)}$$

Now use $c \gg a, b$

$$C \simeq 2\pi\varepsilon_0 L \frac{1}{\ln\left(\frac{1}{\frac{c}{b}}\right) + \ln\left(\frac{1}{\frac{c}{a}}\right)}$$

$$C \simeq \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{ab}{c^2}\right)}$$

Need $C > 0$

since $c \gg a, b$

$\ln\left(\frac{ab}{c^2}\right)$

takes "$-$"

"$\ln\left(\frac{ab}{c^2}\right)$"

$= -\ln\left(\frac{ab}{c^2}\right)$

$$C = 2\pi\varepsilon_0 L$$

$$C = \frac{4\pi\varepsilon_0 L}{\ln\left(\frac{c^2}{ab}\right)}$$

$$= \ln\left(\frac{c^2}{ab}\right)$$