Problem A1: In a two-body scattering event, $A + B \to C + D$, it is convenient to introduce Mandelstam variables

$$s = \frac{(p_A + p_B)^2}{c^2}, \qquad t = \frac{(p_A - p_C)^2}{c^2}, \qquad u = \frac{(p_A - p_D)^2}{c^2},$$

where p_A, p_B, p_C and p_D are the four momenta of A, B, C, and D respectively. The squares in the numerators are the relativistically invariant dot product. Let m_A, m_B, m_C and m_D denote the rest masses of A, B, C and D respectively.

a) Show that $s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2$.

b) Show that the center of mass energy of A is $\frac{(s+m_A^2-m_B^2)c^2}{2\sqrt{s}}$.

c) Show that the lab energy of A (B at rest) is $\frac{(s-m_A^2-m_B^2)c^2}{2m_B}$.

d) Show that the total Center of Mass energy is $\sqrt{s}c^2$.

For simplicity let $c = 1$. We have 4-momentum conservation:

$$p_A + p_B = p_C + p_D$$

And the on-mass shell condition:

$$p_i^2 = E_i^2 - |\boldsymbol{p}_i|^2 = m_i^2 \qquad (i = A, B, C, D)$$

This is all we need to solve everything here.

a)

$$s + t + u = (p_A + p_B)^2 + (p_A - p_c)^2 + (p_A - p_D)^2$$
$$= p_A^2 + 2p_A \cdot p_B + p_B^2 + p_A^2 - 2p_A \cdot p_C + p_C^2 + p_A^2 - 2p_A \cdot p_D + p_D^2$$

$$= m_A^2 + 2p_A \cdot p_B + m_B^2 + m_A^2 - 2p_A \cdot p_C + m_C^2 + m_A^2 - 2p_A \cdot p_D + m_D^2 \qquad (from\ on - mass\ shell)$$

$$= 3m_A^2 + +m_B^2 + m_C^2 + m_D^2 + \ 2p_A \cdot p_B - 2p_A \cdot p_C - 2p_A \cdot p_D$$

$$= 3m_A^2 + +m_B^2 + m_C^2 + m_D^2 + \ 2p_A \cdot (p_B - p_C - p_D)$$
$$= 3m_A^2 + +m_B^2 + m_C^2 + m_D^2 + \ 2p_A \cdot (-p_A) \cdot \ (from\ \ 4 - momenutm\ conservation)$$

$$= 3m_A^2 + +m_B^2 + m_C^2 + m_D^2 - \ 2m_A^2 \qquad (from\ on - mass\ shell)$$

$$= m_A^2 + +m_B^2 + m_C^2 \ + m_D^2$$

b) In the CM frame,

$$p_A^\mu = (E_A, \boldsymbol{p})$$
$$p_B^\mu = (E_B, -\boldsymbol{p})$$

Thus,

$$p_A + p_B = (E_A + E_B, \boldsymbol{0})$$

and,

$$s = (p_A + p_B) = (E_A + E_B)^2$$
$$\rightarrow \ E_A + E_B = \sqrt{s} \qquad (1)$$

From the on-mass shell condition:

$$E_A^2 - |\boldsymbol{p}|^2 = m_A^2 \qquad (2)$$
$$E_B^2 - |\boldsymbol{p}|^2 = m_B^2 \qquad (3)$$

Subtract Eq. (2) and (3) to get rid of $|\boldsymbol{p}|^2$:

$$E_A^2 - E_B^2 = m_A^2 - m_B^2 \qquad (now, factorize\ LHS)$$
$$(E_A - E_B)(E_A + E_B) = m_A^2 - \ m_B^2 \qquad (use\ \ Eq.(1))$$
$$(E_A - E_B)\sqrt{s} \ = \ m_A^2 - \ m_B^2$$
$$E_A - E_B \ = \frac{m_A^2 - \ m_B^2}{\sqrt{s}} \qquad (4)$$

Now, get rid of E_B by adding Eq. (2) and (4):

$$2E_A = \sqrt{s} + \frac{m_A^2 - \ m_B^2}{\sqrt{s}}$$
$$2E_A = \frac{s + \ m_A^2 - \ m_B^2}{\sqrt{s}}$$
$$E_A = \frac{s + \ m_A^2 - \ m_B^2}{2\sqrt{s}}$$

c) In the rest frame of B:

$$p_A^\mu = (E_A, \boldsymbol{p_A})$$

$$p_B^\mu = (m_B, \mathbf{0})$$

Now,

$$s = (p_A + p_B)^2$$
$$s = p_A^2 + 2p_A \cdot p_B + p_B^2$$
$$s = m_A^2 + 2(E_A E_B - \boldsymbol{p_A} \cdot \mathbf{0}) + m_B^2$$
$$s = m_A^2 + 2E_A m_B + m_B^2$$
$$s - m_A^2 - m_B^2 = 2E_A m_B$$

Therefore,

$$E_A = \frac{s - m_A^2 - m_B^2}{2m_B}$$

d) Basically, done in a):

$$s = (p_A + p_B) = (E_A + E_B)^2 = E_{CM}^2$$
$$\rightarrow E_{CM} = \sqrt{s}$$