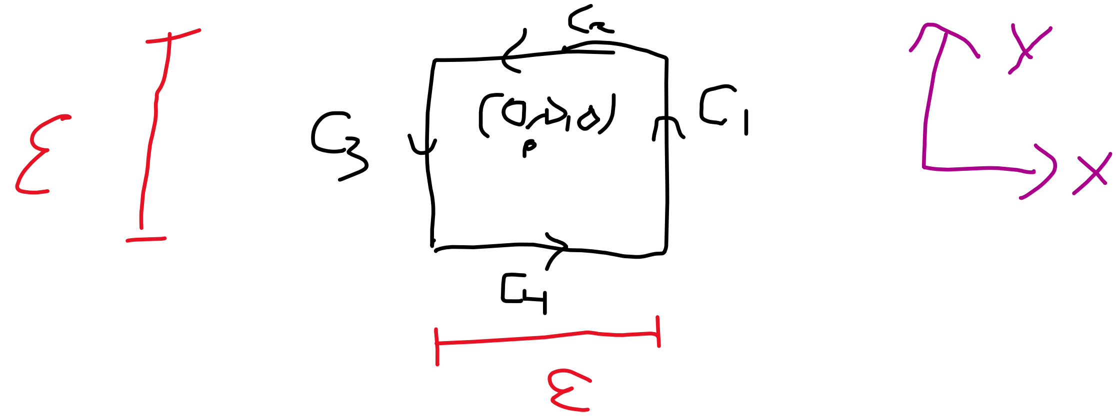
Problem B2: Start from the integral form of Faraday's law of induction,

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t} \int \vec{B} \cdot d\vec{A},$$

and derive its differential form:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B}.$$

Hint: Use infinitesimal square loops in the three different planes.



For this we'll make use of:

$$\int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} dx\, f(x) \approx f(0)\epsilon \tag{1}$$

which follows from the mean value theorem for integrals and continuity. Intuitivaly, it just states an infinitesimal integral around the origin is about the length integrated over times the value of the function at the origin. (We'll treat Eq. (1) as an equality even though it's not. We're okay because corrections are of the order of ϵ^2 and don't contribute in the end when $\epsilon \to 0$).

We'll start in the xy plane with $z = 0$ (see drawing). The magnetic flux is then:

$$\int d\boldsymbol{A} \cdot \boldsymbol{B}(x,y,0) = \int dA\, B_z(x,y,z) = \int_{-\epsilon/2}^{\epsilon/2} dx \int_{-\epsilon/2}^{\epsilon/2} dy\, B_z(x,y,0)$$

Applying Eq. (1) twice:

$$\int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} dx \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} dy\, B_z(x,y,0) = \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} dx\ B_z(x,0,0)\,\epsilon$$
$$= B_z(0,0,z)\,\epsilon^2$$

Thus,

$$-\frac{\partial}{\partial t}\int d\boldsymbol{A}\cdot\boldsymbol{B} = \frac{\partial B_z(0,0,0)}{\partial t}\,\epsilon^2 \qquad (2)$$

Now, let's evaluate:

$$\oint d\boldsymbol{\ell}\cdot\boldsymbol{E}$$

From the drawing:

$$\oint d\boldsymbol{\ell}\cdot\boldsymbol{E} = \left(\int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}\right) d\boldsymbol{\ell}\cdot\boldsymbol{E} \qquad (3)$$

Let's evaluate the first term.

$$\int_{C_1} d\boldsymbol{\ell}\cdot\boldsymbol{E}(x,y,0) = \int_{C_1} d\hat{\boldsymbol{y}}\cdot\boldsymbol{E}(x,y,0)$$
$$= \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} dy\, E_y\left(\frac{\epsilon}{2},y,0\right) \qquad (use\ Eq.1)$$
$$= \epsilon\, E_y\left(\frac{\epsilon}{2},0,0\right)$$

For C_2 the procedure is similar, except now: $d\boldsymbol{\ell} = d\hat{\boldsymbol{x}}$ and we're going in the negative direction

$$\int_{C_2} d\boldsymbol{\ell}\cdot\boldsymbol{E}(x,y,0) = \int_{C_2} d\hat{\boldsymbol{x}}\cdot\boldsymbol{E}(x,y,0)$$
$$= \int_{\frac{\epsilon}{2}}^{-\frac{\epsilon}{2}} dx\, E_x\left(x,\frac{\epsilon}{2},0\right) \qquad (flip\ terms\ in\ intergal)$$
$$= -\int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} dx\, E_x\left(x,\frac{\epsilon}{2},0\right) \qquad (use\ Eq.1)$$
$$= -\epsilon\, E_x\left(0,\frac{\epsilon}{2},0\right)$$

We can the integrals for C_3 and C_4 similarly (being careful with negative signs):

$$\int_{C_3} d\boldsymbol{\ell}\cdot\boldsymbol{E}(x,y,0) = \int_{C_3} (d\hat{\boldsymbol{y}})\cdot\boldsymbol{E}(x,y,0)$$

$$= \int_{\frac{\epsilon}{2}}^{-\frac{\epsilon}{2}} dy\, E_y\left(-\frac{\epsilon}{2}, y, 0\right) \qquad (flip\ integral\ and\ use\ Eq. 1)$$

$$= -\epsilon\, E_y\left(-\frac{\epsilon}{2}, 0{,}0\right)$$

$$\int_{C_4} d\boldsymbol{\ell} \cdot \boldsymbol{E}(x, y, 0) = \int_{C_4} (d\hat{\boldsymbol{x}}) \cdot \boldsymbol{E}(x, y, 0)$$

$$= \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} dx\, E_x\left(x, -\frac{\epsilon}{2}, 0\right) \qquad (use\ Eq. 1)$$

$$= \epsilon\, E_x\left(0, -\frac{\epsilon}{2}, 0\right)$$

Putting this all together:

$$\oint d\boldsymbol{\ell} \cdot \boldsymbol{E} = \epsilon\, E_y\left(\frac{\epsilon}{2}, 0{,}0\right) - \epsilon\, E_x\left(0, \frac{\epsilon}{2}, 0\right) - \epsilon\, E_y\left(-\frac{\epsilon}{2}, 0{,}0\right) + \epsilon\, E_x\left(0, -\frac{\epsilon}{2}, 0\right) \qquad (factor\ \epsilon)$$

$$= \epsilon\left(E_y\left(\frac{\epsilon}{2}, 0{,}0\right) - E_x\left(0, \frac{\epsilon}{2}, 0\right) - E_y\left(-\frac{\epsilon}{2}, 0{,}0\right) + E_x\left(0, -\frac{\epsilon}{2}, 0\right)\right) \qquad (group\ \ E_x\ \&E_y)$$

$$\oint d\boldsymbol{\ell} \cdot \boldsymbol{E} = \epsilon\left(\left[-E_x\left(0, \frac{\epsilon}{2}, 0\right) + E_x\left(0, -\frac{\epsilon}{2}, 0\right)\right] + \left[E_y\left(\frac{\epsilon}{2}, 0{,}0\right) - E_y\left(-\frac{\epsilon}{2}, 0{,}0\right)\right]\right) \qquad (4)$$

Equating Eq. (2) and (4):

$$\oint d\boldsymbol{\ell} \cdot \boldsymbol{E} = -\frac{\partial}{\partial t}\int d\boldsymbol{A} \cdot \boldsymbol{B}$$

$$\epsilon\left(\left[-E_x\left(0, \frac{\epsilon}{2}, 0\right) + E_x\left(0, -\frac{\epsilon}{2}, 0\right)\right] + \left[E_y\left(\frac{\epsilon}{2}, 0{,}0\right) - E_y\left(-\frac{\epsilon}{2}, 0{,}0\right)\right]\right) = -\epsilon^2\, \frac{\partial B_z(0{,}0{,}0)}{\partial t}$$

Divide both sides by ϵ^2

$$\left[\frac{-E_x\left(0, \frac{\epsilon}{2}, 0\right) + E_x\left(0, -\frac{\epsilon}{2}, 0\right)}{\epsilon}\right] + \left[\frac{E_y\left(\frac{\epsilon}{2}, 0{,}0\right) - E_y\left(-\frac{\epsilon}{2}, 0{,}0\right)}{\epsilon}\right] = -\frac{\partial B_z(0{,}0{,}0)}{\partial t} \qquad (5)$$

Let $\epsilon \to 0$ and note:

$$\lim_{\epsilon \to 0} \frac{E_x\left(0, \frac{\epsilon}{2}, 0\right) - E_x\left(0, -\frac{\epsilon}{2}, 0\right)}{\epsilon} = \frac{\partial E_x(0{,}0{,}0)}{\partial y}$$

$$\lim_{\epsilon \to 0} \frac{E_y\left(\frac{\epsilon}{2}, 0,0\right) - E_y\left(-\frac{\epsilon}{2}, 0,0\right)}{\epsilon} = \frac{\partial E_y(0,0,0)}{\partial x}$$

Thus, Eq. (5) becomes

$$-\frac{\partial E_x(0,0,0)}{\partial y} + \frac{\partial E_y(0,0,0)}{\partial x} = -\frac{\partial B_z(0,0,0)}{\partial t} \tag{6}$$

Or:

$$(\nabla \times E)_z(0,0,0) = -\frac{\partial B_z(0,0,0)}{\partial t}$$

We will now use the cyclic symmetries of xyz to get the x and y components.

Going the to the yz plane and doing the exact same thing results in Eq. (6) but with: $x, y , z \to y, z, x$:

$$-\frac{\partial E_y(0,0,0)}{\partial z} + \frac{\partial E_z(0,0,0)}{\partial y} = -\frac{\partial B_x(0,0,0)}{\partial t}$$
$$\to (\nabla \times E)_x(0,0,0) = -\frac{\partial B_x(0,0,0)}{\partial t}$$

Going to the zx plane and doing the same as we did in the xy but with $x, y , z \to z, x, y$:

$$-\frac{\partial E_z(0,0,0)}{\partial x} + \frac{\partial E_x(0,0,0)}{\partial z} = -\frac{\partial B_y(0,0,0)}{\partial t}$$
$$\to (\nabla \times E)_y(0,0,0) = -\frac{\partial B_y(0,0,0)}{\partial t}$$

Therefore,

$$\boldsymbol{\nabla} \times \boldsymbol{E}(0,0,0) = -\frac{\partial \boldsymbol{B}(0,0,0)}{\partial t}$$

Nothing about relied on being at the origin. Therefore, can do at any arbitrary point and

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$