Problem A4: In the standard derivation of the Bohr formula, the nucleus is stationary and, therefore, the implicit assumption is that the mass of the nucleus is infinite. However, the electron actually orbits around the center of mass of the electron-nucleus system. The proper derivation would replace the electron mass with the *reduced mass* of the system given by

$$\mu = \frac{m_e M}{m_e + M},$$

where M is the nuclear mass. Use μ to find the correct expression for the Bohr energy E_n. Show that the relative error in the ground-state energy of hydrogen is given by

$$\frac{\Delta E}{E} = \frac{m_e}{m_p},$$

where ΔE is the difference between the energy assuming a stationary nucleus and the energy for finite-mass nucleus.

For Bohr model, angular momentum is quantized:

$$l = n\hbar$$

$$\Rightarrow \quad n\hbar = mvr \qquad (1)$$

Also, e^- in circular orbit due to Coulomb force,

$$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r^2} \qquad (2)$$

Using (1),

$$\frac{m}{r_n}\left(\frac{n\hbar}{m_e r_n}\right)^2 = \frac{1}{4\pi\varepsilon_0}\frac{Ze^3}{r_n^2}$$

$$\frac{n^2\hbar^2}{m_e r_n^3} = \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_n^2}$$

$$r_n = \frac{4\pi\varepsilon_0 \, n^2\hbar^2}{Ze^2 m_e} \quad (3)$$

Now, looking at energy,

$$E_n = T + V$$

$$= \frac{1}{2}mv^2 - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_n}$$

Using (2)

$$= \frac{1}{2}\left(\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_n^2}\right)r_n - \frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r_n}$$

$$E_n = -\frac{1}{8\pi\varepsilon_0}\frac{Ze^2}{r_n}$$

Inserting (3),

$$E_n = -\frac{1}{8\pi\varepsilon_0}\frac{1}{\frac{4\pi\varepsilon_0 n^2\hbar^2}{Ze^2 m_e}} \times Ze^2$$

$$\boxed{E_n = -\frac{m_e Z^2 e^4}{32\pi^2\varepsilon_0^2\hbar^2}\frac{1}{n^2}} \quad (4)$$

Let $E = E_1$. Also, let E' be Eq. (4) with $m_e \to \mu$

Then,

$$E' = E \frac{M}{m_e}$$

$$E' = E \frac{1}{\cancel{m_e}} \left(\frac{\cancel{m_e} M}{m_e + M} \right)$$

$$= E \left(\frac{M}{m_e + M} \right)$$

Taking $M = m_p$

$$E' = E \left(\frac{m_p}{m_p + m_e} \right) \qquad \frac{1/m_p}{1/m_p}$$

$$E' = E \frac{1}{1 + \frac{m_e}{m_p}}$$

$$E' \left(1 + \frac{m_e}{m_p} \right) = E$$

$$E' + \frac{m_e}{m_p} E' = E$$

$$\frac{m_e}{m_p} E' = E - E'$$

$$\frac{m_e}{m_p} E' = \Delta E$$

$$\Rightarrow \boxed{\frac{\Delta E}{E'} = \frac{m_e}{m_p}}$$

Note using E for stationary nucleus and E' for finite-mass nucleus. Different notation than question.

In general,

$$\frac{\Delta E}{E} = f\left(\frac{m_e}{m_p}\right)$$

$$= c_0 + c_1\left(\frac{m_e}{m_p}\right) + c_2\left(\frac{m_e}{m_p}\right)^2 + \ldots$$

$\Delta E / E \rightarrow$ dimensionless need to be function of dimensionless parameter(s)

$\varepsilon = \frac{m_e}{m_p} \rightarrow$ perturbative parameter

$\frac{\Delta E}{E} = 0$ when $\frac{m_e}{m_p} = 0$

Therefore,

$$\frac{\Delta E}{E} = c_1 \left(\frac{m_e}{m_p}\right) + c_2 \left(\frac{m_e}{m_p}\right)^2 + \dots$$