2009 Classical - C9 (SM)

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Problem C2: An ideal gas goes through an expansion from V_i to V_f. The initial pressure is P_i. Compute the work done by the gas assuming the expansion was at constant pressure. Compute the work done by the gas assuming the expansion was at constant temperature. In which case does the gas do more work?

$$W = \int_{V_i}^{V_f} dV \; P$$

Positive because W done by gas

Constant P

$$W_1 = P_i \int_{V_i}^{V_f} dV = P_i (V_f - V_i)$$

$$= P_i \Delta V$$

Constant T

$$P_i V_i = nRT$$

$$PV = nRT = P_i V_i$$

$$\Rightarrow \quad P = \frac{nRT}{V} = \frac{P_i V_i}{V}$$

$$W_2 = \int_{V_i}^{V_f} dV \; \frac{P_i V_i}{V}$$

$$W_2 = P_i V_i \int_{V_i}^{V_f} \frac{dV}{V}$$

$$= P_i V_i \ln\left(\frac{V_f}{V_i}\right) \qquad V_f = V_i + \Delta V$$

$$= P_i V_i \ln\left(\frac{V_i + \Delta V}{V_i}\right)$$

$$= P_i V_i \ln\left(1 + \frac{\Delta V}{V_i}\right)$$

Now $$\frac{1}{1+x} = 1 - x + x^2 + \dots$$

Integrating from 0 to x

$$\ln(1+x) - \ln(1) = x - \frac{x^2}{2} + \dots$$

$$\Rightarrow \quad \ln(1+x) < x$$

Alternatively, $f(x) = \ln(1+x)$ $\quad f(0) = 0$

$g(x) = x$ $\quad g(0) = 0$

$g'(x) = 1$

$$f'(x) = \frac{1}{1+x} < 1 \quad \text{for} \quad x > 0$$

Therefore, $\quad f(x) < g(x) \quad$ for $x > 0$

$$\Rightarrow \ln(1+x) < x$$

Since gas is expanding $\Delta V > 0$, and

$$W_2 = P_i V_i \ln\left(1 + \frac{\Delta V}{V_i}\right) < P_i V_i \left(\frac{\Delta V}{V_i}\right)$$

$$= P_i \Delta V$$

$$= W_1$$

Therefore, $W_2 < W_1$ and the gas does more work with const. pressure.