2012 Classical – 7 (EM)

7. Write down the equation of motion for a charged particle in superimposed uniform, parallel electric and magnetic fields, both in the z-direction, and solve it, given that the particle starts from the origin with velocity $v\hat{x}$. A screen is placed at $x = a$ where $a << \frac{mv}{|q|B}$. Show that the locus of points of arrival of particles with given m and q, but different speeds v, is approximately a parabola. Hoes does this locus depend on m and q?

Solving for $x(t), y(t), z(t)$

From the Lorentz force:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

For $\boldsymbol{E} = E\,\hat{\boldsymbol{z}}$ and $\boldsymbol{B} = B\,\hat{\boldsymbol{z}}$:

$$m\frac{d^2x}{dt^2} = q\,Bv_y \tag{1}$$

$$m\frac{d^2y}{dt^2} = -q\,Bv_x \tag{2}$$

$$m\frac{d^2z}{dt^2} = qE \tag{3}$$

With boundary conditions:

$$x(0) = y(0) = z(0) = 0$$
$$v_y(0) = v_z(0) = 0$$
$$v_x(0) = v$$

For the z direction the solution to Eq. (3) with boundary conditions is

$$\boxed{z(t) = \frac{qE}{2m}\,t^2} \tag{4}$$

x and y direction

Rewriting Eq. (1) and (2):

$$\frac{dv_x}{dt} = \frac{q\,B}{m}v_y \tag{5}$$

$$\frac{dv_y}{dt} = -\frac{q\,B}{m}v_x \tag{6}$$

Let $\omega = \frac{qB}{m}$. Then:

$$\begin{pmatrix} \frac{dv_x}{dt} \\ \frac{dv_y}{dt} \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} \tag{7}$$

Let A be the matrix in Eq. (7). We can solve Eq. (7) by finding the eigenvalues and eigenvectors of A.

$$|A - \lambda I| = 0$$
$$\lambda^2 + \omega^2 = 0 \rightarrow \lambda = \pm i\omega$$

The eigenvectors are:

$$\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \pm i\omega \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Looking at the first row:

$$v_2 = \pm i \, v_1$$

Let $v_1 = v$ since that's a

Thus, the two eigenvectors are:

$$v_+ = \begin{pmatrix} 1 \\ i \end{pmatrix}, v_- = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

And

$$v(t) = c_1 e^{+i\,\omega t} \begin{pmatrix} 1 \\ i \end{pmatrix} + c_2 e^{-i\omega t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Using $v(0) = \begin{pmatrix} v \\ 0 \end{pmatrix}$

$$v = c_1 + c_2$$
$$0 = ic_1 - ic_2$$

Therefore:

$$c_1 = c_2 = \frac{v}{2}$$

and

$$v(t) = \frac{v}{2} e^{i\,\omega t} \begin{pmatrix} 1 \\ i \end{pmatrix} + \frac{v}{2} e^{-i\omega t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

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$$\rightarrow \quad v_x(t) = v \cos \omega t \tag{8}$$
$$v_x(t) = v \sin \omega t \tag{9}$$

Integrating Eq. (8) and using $x(0) = 0$:

$$\boxed{x(t) = \frac{v}{\omega}\sin\omega t} \tag{10}$$

Similarly for Eq. (9) we get:

$$\boxed{y(t) = \frac{v}{\omega}(1 - \cos\omega t)} \tag{11}$$

Again, $\omega = \frac{qB}{m}$.

Locus of points

From Eq. (8) and (9):

$$x^2 + \left(y - \frac{v}{\omega}\right)^2 = \frac{v^2}{\omega^2}$$

$$\rightarrow \ (y - \frac{v}{\omega})^2 = \frac{v^2}{\omega^2} - a^2 \tag{12}$$

Now, plugging in $x(t) = a$ into Eq. (10):

$$a = \frac{v}{\omega}\sin\omega t$$

$$\frac{\omega a}{v} = \sin\omega t \tag{13}$$

We're given

$$a \ll \frac{mv}{|q|B} = \frac{v}{\omega} \rightarrow \ \frac{\omega a}{v} \ll \ 1$$

Hence, we can take $\sin^{-1}()$ of both sides of Eq. (13).

$$\sin^{-1}\left(\frac{\omega a}{v}\right) = \omega t$$

$$\rightarrow \ t = \frac{1}{\omega}\sin^{-1}\left(\frac{\omega a}{v}\right) \tag{14}$$

By $\frac{\omega a}{v} \ll \ 1$ then to second order approximation:

$$t \ \approx \frac{a}{v} \qquad (from\ \sin^{-1}(\epsilon) \approx \epsilon \ \ for\ |\epsilon| \ll 1$$

From Eq. (4)

$$\frac{2m}{qE}z(t) = t^2 \approx \frac{a^2}{v^2}$$

$$\rightarrow \ a^2 \approx \frac{2mv^2}{qE} z \tag{15}$$

Inserting Eq. (15) into (12):

$$(y - \frac{v}{\omega})^2 \approx \frac{v^2}{\omega^2} - \frac{2mv^2}{qE} z$$

Then:

$$z \approx \ -\frac{qE}{2mv^2} (y - \frac{v}{\omega})^2 + \frac{qE}{2m\omega^2}$$

Inserting $\omega^2 = \frac{qB}{m}$:

$$z \approx \ -\frac{qE}{2mv^2} \left(y - v \sqrt{\frac{m}{qB}} \right)^2 + \frac{E}{2B} = \ -\frac{qE}{2mv^2} \left(y^2 - 2v \sqrt{\frac{m}{qB}} \ y + \frac{mv^2}{qB} \right) + \frac{E}{2B}$$

$$\boxed{z \approx \ -\frac{qE}{2mv^2} y^2 + \frac{E}{v} \sqrt{\frac{q}{mB}} \ y}$$