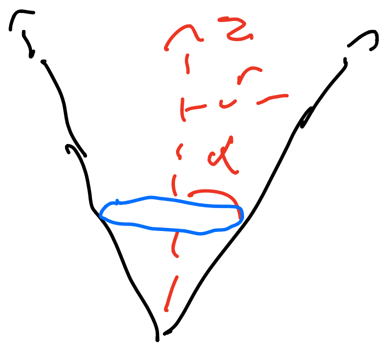
2. A particle of mass m slides without friction on the inside of a cone which has its vertex at the origin, its axis along the z-axis, and whose sides make an angle α with the vertical.

   (a) Choose as independent variables, r, the distance of the particle from the vertex of the cone, and ϕ, the azimuthal angle aroung the axis of the cone, and write down the Lagrangian of the system.

   (b) Derive the Euler-Lagrange equations for the two variables r and ϕ.

   (c) Shown that circular motion inside the cone is possible for any value of r and determine the corresponding angular velocity as a function of r.



$$\frac{r}{z} = \tan\alpha$$

$$z = \frac{r}{\tan\alpha}$$

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2 \right)$$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + \frac{\dot{r}^2}{\tan^2\alpha} \right)$$

Now,

$$1 + \frac{1}{\tan^2\alpha} = 1 + \frac{1}{\frac{\sin^2\alpha}{\cos^2\alpha}} = \frac{\sin^2\alpha + \cos^2\alpha}{\sin^2\alpha}$$

$$= \frac{1}{\sin^2\alpha}$$

$$T = \frac{1}{2} m \left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\theta}^2 \right) \qquad (1)$$

Now,

$$V = mgz$$

$$V = \frac{mgr}{\tan\alpha} \qquad (2)$$

From Eq. (1) and (2)

$$L = \frac{1}{2} m \left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\theta}^2 \right) - \frac{mgr}{\tan\alpha}$$

b) $\frac{\partial L}{\partial \theta} = 0 \Rightarrow p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \text{const}$

$$p_\theta = \frac{1}{\cancel{2}} m r^2 (\cancel{2}\dot{\theta})$$

$$\boxed{p_\theta = m r^2 \dot{\theta}} \qquad (3)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$

$$\cancel{\tfrac{1}{2}}\cancel{m}\,\frac{d}{dt}\left(\frac{\cancel{2}\dot{r}}{\sin^2\alpha}\right) = \frac{\cancel{m}}{\cancel{2}}\dot{\theta}^2(\cancel{2}r) - \frac{\cancel{m}g}{\tan\alpha}$$

$$\frac{\ddot{r}}{\sin^2\alpha} = r\dot{\theta}^2 - \frac{g}{\tan\alpha} \qquad (4)$$

From Eq. (3) $\dot{\theta} = \frac{p_\theta}{mr^2}$. Plugging this into (4),

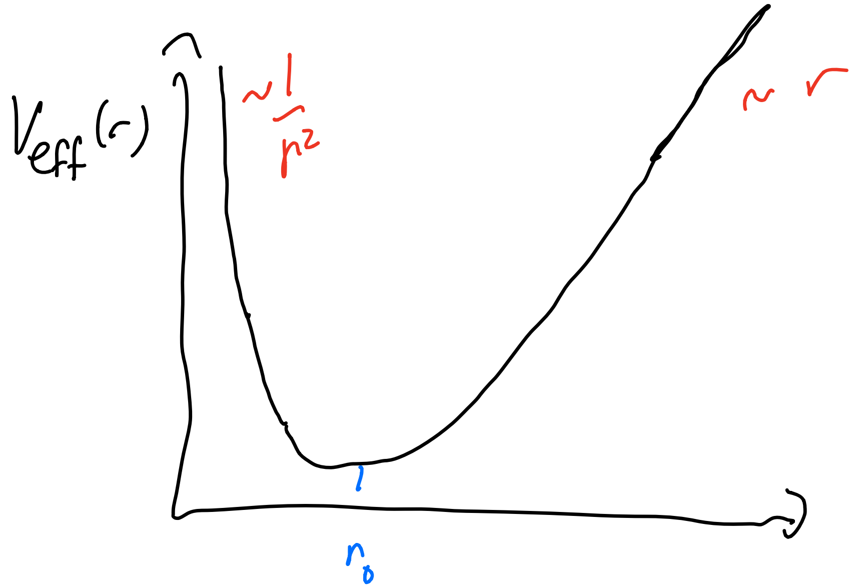
$$\frac{\ddot{r}}{\sin^2\alpha} = \frac{p_\theta^2}{m^2r^3} - \frac{g}{\tan\alpha}$$

$$\boxed{\ddot{r} = \sin^2\alpha\left[\frac{p_\theta^2}{m^2r^3} - \frac{g}{\tan\alpha}\right]}$$

c) $$m\ddot{r} = m\sin^2\alpha\left[\frac{p_\theta^2}{m^2r^3} - \frac{g}{\tan\alpha}\right]$$

$$= -\frac{d}{dr}\left(\sin^2\alpha\left[\frac{p_\theta^2}{2mr^2} + \frac{mgr}{\tan\alpha}\right]\right)$$

$$m\ddot{r} = -\frac{d}{dr}V_{eff}(r)$$



$$\frac{dV_{eff}}{dr} = 0 = m\sin^2\alpha\left[\frac{p_\theta^2}{n^2 r_0^3} - \frac{g}{\tan\alpha}\right]$$

$$\Rightarrow \quad r_0 = \sqrt[3]{\frac{p_\theta^2 \tan\alpha}{n^2 g}}$$

Therefore equilibrium exists. Adjust p_θ to set for any r.

From Eq. (4),

$$\ddot{r} = \sin^2\alpha\left(r\dot{\theta}^2 - \frac{g}{\tan\alpha}\right)$$

Necessary condition for circular orbit is

$$\ddot{r} = 0$$

$$0 = \sin^2\alpha\left(r\dot{\theta}^2 - \frac{g}{\tan\alpha}\right)$$

$$0 = r\dot{\theta}^2 - \frac{g}{\tan\alpha}$$

$$\boxed{\dot{\theta}(r) = \sqrt{\frac{g}{r\tan\alpha}}}$$