2012 Classical - 8 (SM)

THERMODYNAMICS/STATISTICAL PHYSICS

8. The dry adiabatic lapse rate:

   (a) Consider a horizontal slab of air whose thickness (height) is dz. If this slab is at rest, the pressure holding it up from below must balance both the pressure from above and the weight of the slab. Use this fact to find an expression for dP/dz, the variation of pressure with altitude, in terms of the density of air, ρ, and acceleration due to gravity, g.

   (b) Use the ideal gas law to write the density of air in terms of pressure, temperature, and the average mass m of the air molecules. Show, then, that the pressure obeys the **barometric equation**

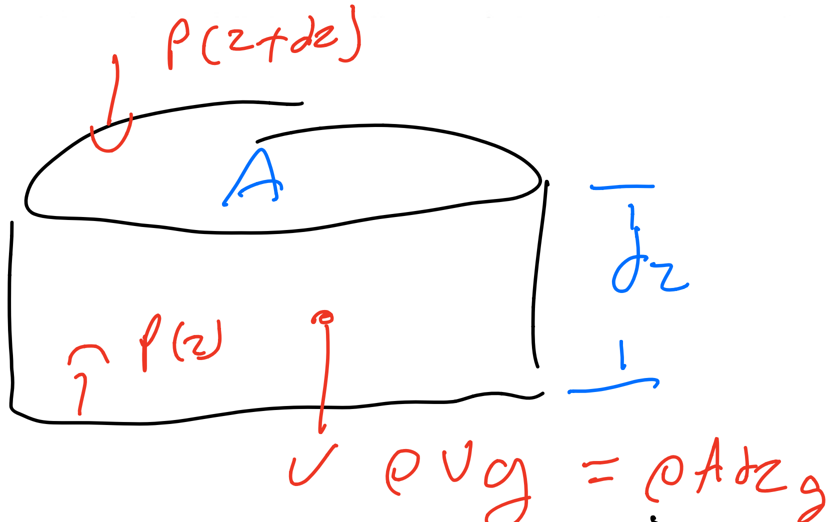
$$\frac{dP}{dz} = -\frac{mg}{kT}P. \tag{2}$$

   (c) Prove that

$$\frac{dT}{dP} = \frac{2}{f+2}\frac{T}{P}. \tag{3}$$

   for an ideal gas with f degrees of freedom expanding adiabatically. You can assume the equipartition theorem.

   (d) Solve for $T(z)$ using (2) and (3).



a) Using equilibrium condition,

$$-AP(z+dz) + AP(z) - \rho A g\, dz$$

$$0 = -A\,dP - \rho A g\, dz$$

$$\boxed{\frac{\partial P}{\partial z} \simeq -\rho g}$$

b)

$$PV = nRT = N k_B T$$

Now, $\rho = \frac{Nm}{V} = m \frac{N}{V}$ (Use gas law)

$$\rho = m \frac{P}{k_B T}$$

$$\boxed{\frac{dP}{dz} = - \frac{mg}{k_B T} P}$$

c) $U = \frac{f}{2} N k_B T$ (Equipartition Theorem)

$$dU = \delta Q - P dV$$

Adiabatic $\quad \delta Q = 0$

$$\frac{f}{2} N k_B \, \partial T = - P \, \partial V$$

Now $\quad P \, \partial V = \partial(PV) - V \partial P$

$$= N k_B \, \partial T - V \partial P$$

So,

$$\frac{f}{2} N k_B \, dT = - \left(N k_B \, \partial T - V \partial P \right)$$

$$\left(\frac{f}{2} + 1 \right) N k_B \, \partial T = V \partial P$$

$$\left(\frac{f+2}{2} \right) \cancel{N k_B} \, \partial T = \frac{\cancel{N k_B} T}{P} \partial P$$

$$\boxed{\frac{\partial T}{\partial P} = \frac{2}{f+2} \frac{T}{P}}$$

d) $$\frac{\partial T}{\partial z} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial z}$$

$$= \left(\frac{2}{f+2}\frac{T}{p}\right)\left(-\frac{mg}{k_B T}p\right)$$

$$\frac{\partial T}{\partial z} = -\frac{2}{f+2}\frac{mg}{k_B}$$

$$\Rightarrow \boxed{T(z) = -\frac{2}{f+2}\frac{mg}{k_B}(z-z_0)+T_0}$$