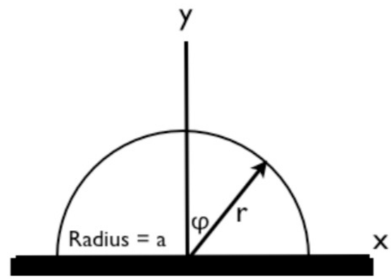
2013 Classical - 1 (CM)

1. Mechanics



Consider a mass that starts at the very top of a hemisphere of radius *a* as shown in the above figure. The mass slides down the hemisphere without friction. Use of the method of Lagrange multipliers is advised to solve this problem.

(a) Write down the Lagrangian in terms of the two generalized coordinates *r* and ϕ. Include the constraint in terms of the Lagrange multiplier λ.

(b) Solve the problem to find the angle ϕ at which the mass falls off.

a) $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - mgr\cos\varphi$

$+ \lambda(r-a)$

b) $\underline{\lambda}$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\lambda}}\right) = \frac{\partial L}{\partial \lambda}$

$0 = r - a \qquad (1)$

$\underline{\varphi}$ $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{\partial L}{\partial \varphi}$

$$\cancel{m} r^2 \ddot{\varphi} = \cancel{m} g r \sin\varphi$$

$$\ddot{\varphi} = \frac{g}{r} \sin\varphi \qquad (2)$$

r

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$

$$m \ddot{r} = m r \dot{\varphi}^2 - m g \cos\varphi + \lambda$$

$$m \ddot{r} = m r \dot{\varphi}^2 - m g \cos\varphi + \lambda \qquad (3)$$

From Eq. (1) $r - a = 0 \Rightarrow \ddot{r} = 0$.
Inserting into Eq. (3)

$$0 = m a \dot{\varphi}^2 - m g \cos\varphi + \lambda$$

$$\lambda = m g \cos\varphi - m a \dot{\varphi}^2 \qquad (4)$$

Use conservation of energy $\left(\frac{\partial L}{\partial t} = 0\right)$.

$$E = T + V$$

$$E = \frac{1}{2} m\left(\dot{r}^2 + r^2 \dot{\varphi}^2\right) + mgr\cos\varphi$$

$$= \frac{1}{2} m a^2 \dot{\varphi}^2 + mga\cos\varphi$$

Note: Normal force does no work since $\perp$ to motion.

Assume starts with $T \approx 0$ (small nudge)

$$E = \cancel{m} g \cancel{a} = \frac{1}{2} \cancel{m} a^{\cancel{2}} \dot{\varphi}^2 + \cancel{m} g \cancel{a} \cos\varphi$$

$$g = \frac{1}{2} a \dot{\varphi}^2 + g\cos\varphi$$

$$g(1 - \cos\varphi) = \frac{1}{2} a \dot{\varphi}^2$$

$$a\dot{\varphi}^2 = 2g(1 - \cos\varphi) \quad (5)$$

Insert into Eq. (4)

$$\lambda = mg\cos\varphi - 2gm(1-\cos\varphi)$$

$$= -2gm + 3mg\cos\varphi$$

$$\lambda = 3gm\left(\cos\varphi - \frac{2}{3}\right)$$

Mass falls off when normal force is zero.

$$\lambda = 0 = 3gm\left(\cos\varphi - \frac{2}{3}\right)$$

$$\Rightarrow \boxed{\varphi = \cos^{-1}\left(\frac{2}{3}\right)}$$