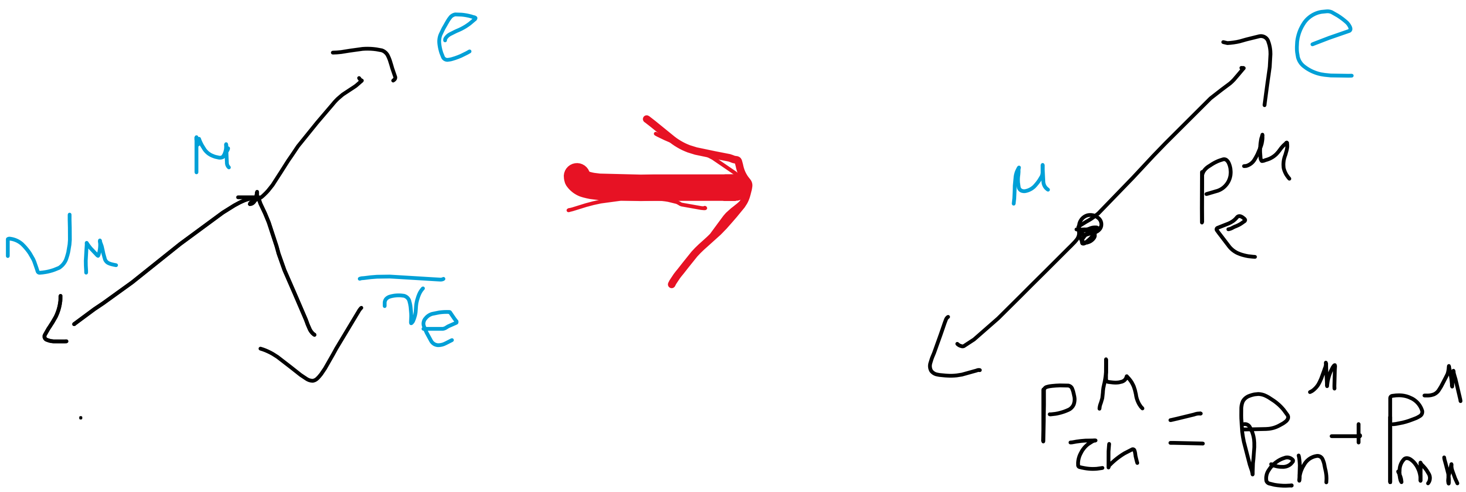
3. (Special Relativity/Modern Physics) **REQUIRED**

When a muon decays at rest,

$$\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$$

why can the electron only have a maximum energy of 53 MeV? The mass of the muon is 106 MeV/c[2] and you can assume that the neutrinos are massless. You can also ignore the mass of the electron.



We’re in the rest frame of the of the muon, thus its 4-momentum is:

$$p_m^\mu = (m_m, \mathbf{0})$$

The electron’s 4-momentum is:

$$p_e^\mu = (E_e, \boldsymbol{p})$$

Let the 4-momentum of the two-neutrino system be:

$$p_{2n}^\mu = p^\mu_{e.n.} + p^\mu{}_{m.n.} = (E_{2n}, -\boldsymbol{p})$$

By conservation of 3-momentum the two-neutrino system needs momentum equal in magnitude and opposite of direction of the electron.

By conservation of energy:

$$m_m = E_e + E_{e.n.} + E_{m.n}$$

$$m_m = E_e + E_{2n} \tag{1}$$

Thus, if we can show that the least energy the two-neutrino system can have is that of the electron, $E_e \leq E_{2n}$ then from Eq. (1) we would show $E_e \leq \frac{m_m}{2}$, as required.

First, due to the on-mass shell condition on the electron:

$E_e^2 - |\boldsymbol{p}|^2 = 0$

$$\rightarrow E_i = |\boldsymbol{p}| \tag{2}$$

Now, the invariant mass squared of the two-neutrino system is:

$p_{2n}^2 = \left(p^{\mu}_{e.n.} + p^{\mu}{}_{m.n.}\right)^2$
$= p_{e.n.}^2 + 2p_{e.n.} \cdot p_{m.n.} + p_{m.n.}^2$ $\quad (use\ on\ mass\ shell\ and\ 4-product\ def.)$
$= 0 + 2(E_{e.n.}E_{m.n.} - |\boldsymbol{p}_{e.n}||\boldsymbol{p}_{m.n}|\cos\theta_{2n}) + 0$ $\quad (use\ E = |\boldsymbol{p}|\ for\ neutrinos)$
$= 2(E_{e.n.}E_{m.n.} - E_{e.n.}E_{m.n.}\cos\theta_{2n})$

$$p_{2n}^2 = 2E_{e.n.}E_{m.n.}(1 - \cos\theta_{2n}) \tag{3}$$

Since energy is positive and $\cos\theta_{2n} \leq 1$

$$p_{2n}^2 \geq 0 \tag{4}$$

We can also express the invariant mass of the two-neutrino system as:

$$p_{2n}^2 = E_{2n}^2 - |\boldsymbol{p}|^2$$

$$E_{2n}^2 = |\boldsymbol{p}|^2 + 2E_{e.n.}E_{m.n.}(1 - \cos\theta_{2n}) \geq |\boldsymbol{p}|^2 \quad (from\ (4))$$

$$E_{2n}^2 \geq |\boldsymbol{p}|^2 \quad (use\ Eq.(2))$$
$$E_{2n}^2 \geq E_e^2$$

$$E_{2n} \geq E_e \tag{5}$$

Therefore, using (5) in Eq. (1)

$$m_m = E_e + E_{2n} \geq E_e + E_e = 2E_2$$

$$\rightarrow \boxed{E_e \leq \frac{m_m}{2} = 53\ MeV}$$