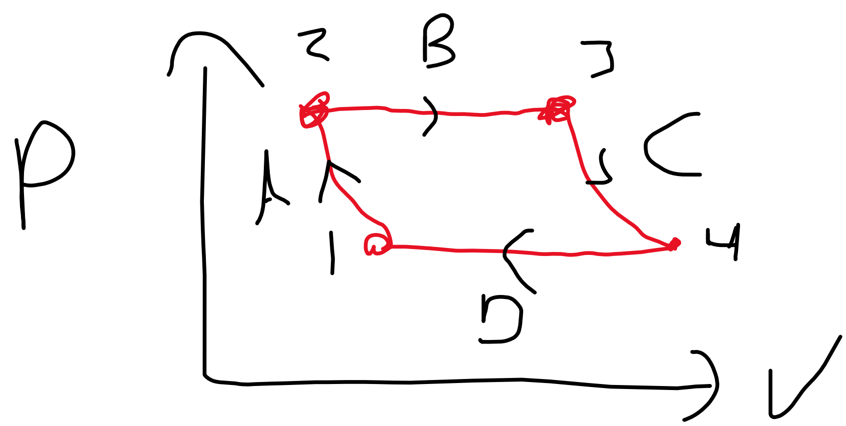
Classical 2014 – 8 (ST)

8. The complete thermodynamic cycle of a heat engine using an ideal gas with constant specific heat capacities consists of four steps. Step A is an *adiabatic* compression from pressure P_1 and volume V_1 to pressure P_2 and volume V_2. Step B is an *isobaric* expansion at pressure P_2 from volume V_2 to volume V_3. Step C is an *adiabatic* expansion from pressure P_2 and volume V_3 to pressure P_1 and volume V_4. Step D is an *isobaric* compression at pressure P_1 from volume V_4 back to the original volume V_1.

a) Make a *PV* diagram for the complete cycle.
b) Show that the ratio of the heat flow out of the engine during step D to the heat flow into the engine during step B is given by
$Q_{out}/Q_{in} = (T_4\text{-}T_1)/(T_3\text{-}T_2)$
c) The efficiency of the engine is defined as the ratio of the work done by the engine to the input heat transfer. Use the ideal gas equation together with the fact that PV^{γ} is constant for an adiabatic process, where γ is the adiabatic index of the gas, to show that the efficiency of the engine is given by
$e = 1 - \left(\frac{P_1}{P_2}\right)^{\alpha}$,
where $\alpha = \frac{(\gamma-1)}{\gamma}$.

"Science owes more to the steam engine than the steam engine owes to science." — Lawrence Joseph Henderson (Attributed)

a)



The following is not necessary, but let's note that from first law of thermodynamics:

$$dU = \delta Q + dW$$

If we have an adiabatic process $\delta Q = 0$ and

$$dU = dW = -PdV$$

For an ideal gas, $dU = C_V dT$ and $PV = nRT$. Thus,

$$C_V dT = -\frac{nRT}{V} dV$$
$$C_V \ln\left(\frac{T}{T_0}\right) = -nR \ln\left(\frac{V}{V_0}\right)$$

$$\frac{T}{T_0} = \left(\frac{V}{V_0}\right)^{-\frac{nR}{C_V}}$$

$$\frac{PV}{P_0 V_0} = \left(\frac{V}{V_0}\right)^{-\frac{nR}{C_V}} \rightarrow PV^{1+\frac{nR}{C_V}} = const.$$

Hence,

$$P \sim \frac{1}{V^\gamma}$$

during A and C. Also, $\gamma = 1 + \frac{nR}{c_V} = 1 + \frac{2}{f}$ (see b).

b) For an isobaric process the work done is:

$$W = -\int_{V_1}^{V_2} P dV = -P\,\Delta V$$

For an ideal gas:

$$U = \frac{1}{2} f nR\, T = C_V T$$

$$\Delta U_B = Q_B - W_B$$
$$C_V\ \Delta T_B = Q_B - P_2(V_3 - V_2)$$
$$C_V\,\Delta T_B = Q_B - nR\,\Delta T_B$$

$$\rightarrow\ Q_B = (C_V + nR)\,\Delta T_B$$

$$Q_B = C_P\ \Delta T_B$$

Similarly:

$$Q_D = C_P\ \Delta T_D$$

Therefore,

$$\frac{Q_{out}}{Q_{in}} = \left|\frac{Q_D}{Q_B}\right| = \left|\frac{\Delta T_D}{\Delta T_B}\right| = \frac{T_4 - T_1}{T_3 - T_2}$$

c) In a cycle the system returns to the same state so:

$$\Delta U = 0$$

Since A and C are adiabatic, all the heat change comes from steps B and D.

$$\Delta U = -Q_{out} + Q_{in} - W$$

$$\rightarrow W = Q_{in} - Q_{out}$$

Therefore, the efficiency is:

$$e = \frac{W}{Q_{in}}$$

$$e = \frac{Q_{in} - Q_{out}}{Q_{in}}$$

$$= 1 - \frac{Q_{out}}{Q_{in}}$$

$$= 1 - \frac{T_4 - T_1}{T_3 - T_2}$$

$$e = 1 - \frac{T_1}{T_2}\left(\frac{\frac{T_4}{T_1} - 1}{\frac{T_3}{T_2} - 1}\right) \qquad (1)$$

Now, for an ideal gas:

$$PV = nRT$$

During an adiabatic process:

$$PV^{\gamma} = const. = C \qquad (Given)$$

Then,

$$V = \left(\frac{C}{P}\right)^{\frac{1}{\gamma}}$$

and:

$$PV = nRT$$

$$P\left(\frac{C}{P}\right)^{\frac{1}{\gamma}} = nRT$$

$$T = \frac{C^{\frac{1}{\gamma}}}{nR}\ P^{1-\frac{1}{\gamma}}$$

$$\rightarrow\ T = \frac{C^{\frac{1}{\gamma}}}{nR}\ P^{\alpha} \qquad (2)$$

Since $P_2 = P_3$ and $P_1 = P_4$:

$$\frac{P_1}{P_2} = \frac{P_4}{P_3} \rightarrow \left(\frac{T_1}{T_2}\right)^{\frac{1}{\alpha}} = \left(\frac{T_4}{T_4}\right)^{\frac{1}{\alpha}}$$

$$\frac{T_1}{T_2} = \frac{T_4}{T_3}$$

$$\rightarrow \frac{T_4}{T_1} = \frac{T_3}{T_2} \tag{3}$$

Insert Eq. (3) into (1):

$$e = 1 - \frac{T_1}{T_2}\left(\frac{\frac{T_3}{T_2} - 1}{\frac{T_3}{T_2} - 1}\right)$$

$$= 1 - \frac{T_1}{T_2} \quad (Use\ Eq.\ (2)$$

$$\boxed{e = 1 - \left(\frac{P_1}{P_2}\right)^{\alpha}}$$