5.

a) Using Maxwell's equations, show that the electromagnetic wave equations in charge free and current free space are given by

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

and

$$\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

.

b) Using the results of part a), determine the speed of EM waves. What is the physical significance of this?

c) Write down the real electric and magnetic fields for a monochromatic plane wave of amplitude E_0, frequency ω and phase angle zero that is travelling in the positive x direction and polarized in the z direction. Show that these fields are solutions to the wave equation.

d) Using the electric and magnetic fields from part c), determine the Poynting vector. Explain the physical significance of the Poynting vector.

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad (1)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad (2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad (3)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad (4)$$

Tak $\vec{\nabla}\times$ of Eq. (3)

$$\underbrace{\vec{\nabla} \times (\vec{\nabla} \times \vec{E})}_{\text{use } \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}} = -\frac{\partial}{\partial t}(\underbrace{\vec{\nabla} \times \vec{B}}_{\text{use Eq. (4)}})$$

$$\vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{E}}_{=0 \text{ from } (1)}) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$-\nabla^2\vec{E} = -\mu_0\varepsilon_0\frac{\partial^2 E}{\partial t^2}$$

$$\boxed{\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}} \qquad (5)$$

Similarly, taking $\vec{\nabla}\times$ of Eq. (4),

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \mu_0\varepsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

$$\vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B} = -\mu_0\varepsilon_0\frac{\partial}{\partial t}\left(\frac{\partial\vec{B}}{\partial t}\right)$$

$$\boxed{\nabla^2\vec{B} = \mu_0\varepsilon_0\frac{\partial^2\vec{B}}{\partial t^2}} \qquad (6)$$

b) Insert $f(\vec{r}-\vec{v}t)$ into (5),

$$f''(\vec{r}-\vec{v}t) = \mu_0\varepsilon_0 \, v^2 f''(\vec{r}-\vec{v}t)$$

$$1 = \mu_0 \varepsilon_0 v^2$$

$$\Rightarrow \boxed{v = \frac{1}{\sqrt{\mu_0\varepsilon_0}}} = c$$

c) For convenience, let $\tilde{E}$ & $\tilde{B}$ be complex electric & magnetic fields. Later, we will make everything real.

Let $\tilde{\vec{E}}(\vec{r},t) = \tilde{\vec{E}}_0 \, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\tilde{\vec{B}}(\vec{r},t) = \tilde{\vec{B}}_0 \, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

From (1) $\quad \vec{\nabla}\cdot\tilde{\vec{E}}(r,t) = 0$ $\quad$ For plane wave $\vec{\nabla} \to i\vec{k}$

$$i\vec{k}\cdot\tilde{\vec{E}}_0 \, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \Rightarrow$$

$$\Rightarrow i\vec{k}\cdot\tilde{\vec{E}}_0 = 0$$

Similarly From (2)

$$i\vec{k}\cdot\tilde{\vec{B}}_0 = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\tilde{\vec{B}}(r,t)$$

$$i\vec{k}\times\vec{E}_0\, e^{i(\vec{k}\cdot r-\omega t)} = i\omega\,\vec{B}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\vec{B}_0 = \frac{\vec{k}\times\vec{E}_0}{\omega}$$

$$\vec{B}_0 \perp \vec{E}_0 \perp \vec{k}$$

$$B_0 = \frac{k E_0}{\omega} = \frac{1}{c}E_0 \qquad \left(\frac{k}{\omega}=\frac{1}{c}\right)$$

So, let $\tilde{\vec{E}}_0 = \tilde{E}_0\,\hat{n}$

$$\tilde{\vec{E}}(\vec{r},t) = \tilde{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\,\hat{n}$$

$$\tilde{\vec{B}}(r,t) = -\frac{\vec{k}\times\tilde{E}_0}{\omega}\, e^{i(\vec{k}\cdot r-\omega t)}$$

$$\tilde{\vec{B}}(r,t) = -\frac{\hat{k}\times\tilde{\vec{E}}(\vec{r},t)}{c}$$

Now $\tilde{E}_0 = |\tilde{E}_0| e^{i\delta}$

$$\tilde{\vec{E}}(\vec{r},t) = |\tilde{E}_0| e^{i(\vec{k}\cdot\vec{r} - \omega t + \delta)} \hat{n}$$

$$\vec{E}(\vec{r},t) = \mathrm{Re}(\tilde{\vec{E}}(\vec{r},t))$$

$$\boxed{\vec{E}(\vec{r},t) = E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)\hat{n}}$$

$$\vec{B}(\vec{r},t) = \mathrm{Re}(\tilde{\vec{B}}(\vec{r},t))$$

$$\boxed{\vec{B}(\vec{r},t) = \frac{E_0}{c} \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)(\hat{k} \times \hat{n})}$$

Given: $\delta = 0$ $\quad \hat{n} = \hat{z}$ $\quad \hat{k} = \hat{x}$ $\quad \frac{k}{\omega} = \frac{1}{c}$

$$\vec{E}(\vec{r},t) = E_0 \cos(kx - \omega t)\hat{z}$$

$$\boxed{\vec{E}(\vec{r},t) = E_0 \cos\left(\frac{\omega}{c}(x - ct)\right)\hat{z}}$$

$$\hat{k} \times \hat{n} = \hat{x} \times \hat{z} = -\hat{y}$$

$$\boxed{\vec{B}(\vec{r},t) = -\frac{E_0}{c} \cos\left(\frac{\omega}{c}(x - ct)\right)\hat{y}}$$

c) $\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$

$\hat{z} \times \hat{y} = -\hat{x}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$\vec{S} = -\frac{E_0^2}{c} \cos^2\left(\frac{\omega}{c}(x-ct)\right) (-\hat{x})$$

$$\boxed{\vec{S} = \frac{E_0^2}{\mu_0 c} \cos^2\left(\frac{\omega}{c}(x-ct)\right) \hat{x}}$$

$\vec{S} \rightarrow$ energy flux density

(energy per unit area per time)

Note: energy density,

$$u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)$$

Show $\Rightarrow$ $\vec{S} = cu\,\hat{n}$