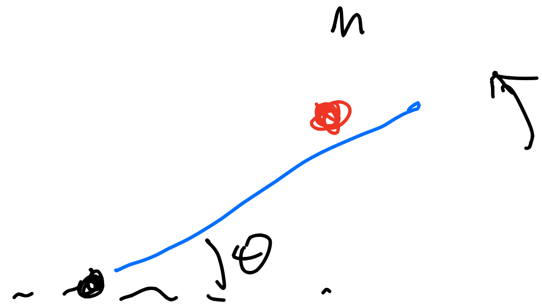
2. A particle of mass m rests on a smooth plane that is initially horizontal. The plane is raised to an inclination angle θ at a constant rate α (so $\theta = 0$ at $t = 0$), causing the particle to move down the plane. Determine the motion of the particle. Hint: It is best to place your origin at the fixed end of the plane and define the distance r from the origin as one generalized coordinate and the angle of inclination θ as the other. Your differential equation for the motion will have a general solution to the homogeneous part (r_h) and a particular solution to the non-homogeneous part (r_p).



$$T = \frac{1}{2} m \left[\dot{r}^2 + r^2 \dot{\theta}^2 \right]$$

$$V = mgr \sin\theta$$

Insert now $\theta = \alpha t$

$$T = \frac{1}{2} m \left[\dot{r}^2 + r^2 \alpha^2 \right]$$

$$V = mgr \sin(\alpha t)$$

$$L = T - V$$

From the E-L equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$

$$\frac{d}{dt}\left(\cancel{m}\dot{r}\right) = \cancel{m} r \alpha^2 - \cancel{m} g \sin(\alpha t)$$

$$\ddot{r} = \alpha^2 r - g \sin(\alpha t)$$

$$\boxed{\ddot{r} - \alpha^2 r = -g \sin(\alpha t)}$$

(1)

Solution of (1) can be separated into:

$$r(t) = r_h(t) + r_p(t)$$

where

$$\ddot{r}_h(t) - \alpha^2 r_h(t) = 0 \qquad (2)$$

and $r_p(t)$ is a particular solution of Eq. (1).

The solution to Eq. (2) is

$$r_h(t) = c_1 \cosh(\alpha t) + \sinh(\alpha t)$$

For particular solution, try

$$r_p(t) = A \sin(\alpha t)$$

Then (1) becomes,

$$-A\alpha^2 \sin(\alpha t) - A\alpha^2 \sin \alpha t = -g \sin \alpha t$$

$$-2A\alpha^2 \sin \alpha t = -g \sin \alpha t$$

$$\Rightarrow \quad A = \frac{g}{2\alpha^2}$$

Therefore, the solution is

$$r(t) = c_1 \sinh(\alpha t) + c_2 \cosh(\alpha t) + \frac{g}{2\alpha^2} \sin(\alpha t)$$

Since started at rest,

$$\dot{r}(0) = 0 = c_1 \alpha \cosh(\alpha 0) + c_2 \alpha \sinh(\alpha \cdot 0) + \frac{g}{2\alpha^{\cancel{2}}} \cancel{\alpha} \cos(\alpha \cdot 0)$$

$$0 = \alpha c_1 + 0 + \frac{g}{2\alpha}$$

$$c_1 = -\frac{g}{2\alpha^2}$$

Let $r(0) = r_0$. Then,

$$r(0) = -\frac{g}{2\alpha^2} \sinh(0) + c_2 \cosh(0) + \frac{g}{2\alpha^2} \sin(0)$$

$$r_0 = 0 + c_2 + 0$$

$$c_2 = r_0$$

Therefore,

$$\boxed{r(t) = -\frac{g}{2\alpha^2} \sinh(\alpha t) + r_0 \cosh(\alpha t) + \frac{g}{2\alpha^2} \sin(\alpha t)}$$