2015 Classical - 5 (EM)

B

5. The potential at the surface of a hollow sphere with radius R is given by

$$V_o(\theta) = k\cos(3\theta)$$

where k is a constant and θ is the polar angle. Using the solutions to Laplace's equation and the appropriate boundary conditions, determine the electric potential inside and outside of this spherical hollow shell and the surface charge density $\sigma(\theta)$.

a

The solution to Laplace's equation in spherical co-ordinates when there is Azimuthal symmetry is:

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

Inside

Solution blows up at $r=0$ unless $B_\ell = 0$

$$\Rightarrow \quad V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

Outside

Solution blows up at $r \to \infty$ unless $A_\ell = 0$

$$V_{out}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

$r=R$

Need

$$V_{in}(R,\theta) = V_{out}(R,\theta)$$

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

$$\sum_{l=0}^{\infty} \left(A_l R^l - \frac{B_l}{R^{l+1}} \right) P_l(\cos\theta) = 0$$

$$\Rightarrow \quad A_l R^l - \frac{B_l}{R^{l+1}} = 0$$

$$\boxed{B_l = A_l R^{2l+1}}$$

We also need:

$$V_{in}(R, \cos\theta) = V_0(\theta) = k\cos(3\theta)$$

$$\Rightarrow \quad \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = k \cos(3\theta)$$

Now

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$$

Also,

$$P_0(\cos\theta) = 1$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$

$$P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

Since $\cos(3\theta)$ is odd and 3rd order, need only $P_1(\cos\theta)$ and $P_3(\cos\theta)$.

$$\cos 3\theta = \frac{8}{5} P_3(\cos\theta) - \frac{3}{5} P_1(\cos\theta)$$

$$V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = \frac{8}{5} k\, P_3(\cos\theta) - \frac{3k}{5} P_1(\cos\theta)$$

$$\Rightarrow \quad A_3 R^3 = \frac{8}{5} k$$

$$\boxed{A_3 = \frac{8k}{5R^3}} \qquad \boxed{B_3 = \frac{8kR^4}{5}}$$

$$A_1 R^1 = -\frac{3}{5} K$$

$$\boxed{A_1 = -\frac{3}{5}\frac{k}{R}} \qquad B_1 = \frac{-3kR^2}{5}$$

$$\boxed{\begin{aligned} V_{out}(r,\theta) &= -\frac{3k}{5}\left(\frac{R}{r}\right)^2 P_1(\cos\theta) + \frac{8k}{5}\left(\frac{R}{r}\right)^4 P_3(\cos\theta) \\ V_{in}(r,\theta) &= -\frac{3k}{5}\left(\frac{r}{R}\right) P_1(\cos\theta) + \frac{8k}{5}\left(\frac{r}{R}\right)^3 P_3(\cos\theta) \end{aligned}}$$

$$\sigma = -\varepsilon_0 \left[\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r} \right] \Bigg|_{r=R}$$

$$\frac{\partial V_{out}}{\partial r}\Bigg|_{r=R} = -\frac{3kR^2}{5}\frac{(-2)}{R^3} P_1(\cos\theta) + \frac{8kR^4}{5}\frac{(-4)}{R^5} P_3(\cos\theta)$$

$$= \frac{6k}{5R} P_1(\cos\theta) - \frac{32}{5}\frac{k}{R} P_3(\cos\theta)$$

$$\frac{\partial V_{in}}{\partial r}\Bigg|_{r=R} = -\frac{3k}{5R} P_1(\cos\theta) + \frac{24k}{5R} P_3(\cos\theta)$$

$$\sigma = -\varepsilon_0 \left[\frac{9k}{5R} P_1(\cos\theta) - \frac{56k}{5R} P_3(\cos\theta) \right]$$

$$\boxed{\sigma = \frac{\varepsilon_0 K}{5R}\left[-9P_1(\cos\theta) + 56\,P_3(\cos\theta)\right]}$$