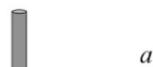


5. The potential at the surface of a hollow sphere with radius R is given by

$$V_o(\theta) = k \cos(3\theta)$$

where k is a constant and θ is the polar angle. Using the solutions to Laplace's equation and the appropriate boundary conditions, determine the electric potential inside and outside of this spherical hollow shell and the surface charge density $\sigma(\theta)$.



The solution to Laplace's equation in spherical coordinates when there is azimuthal symmetry is:

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

Inside

Solution blows up at $r=0$ unless $B_l = 0$

$$\Rightarrow V_{in}(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

Outside

Solution blows up at $r \rightarrow \infty$ unless $A_l = 0$

$$V_{out}(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

$r=R$

Need

$$V_{in}(R, \theta) = V_{out}(R, \theta)$$

$$\sum_{\ell=0}^{\infty} A_{\ell} R^{\ell} P_{\ell}(\cos \theta) = \sum_{\ell=0}^{\infty} \frac{B_{\ell}}{R^{\ell+1}} P_{\ell}(\cos \theta)$$

$$\sum_{\ell=0}^{\infty} \left(A_{\ell} R^{\ell} - \frac{B_{\ell}}{R^{\ell+1}} \right) P_{\ell}(\cos \theta) = 0$$

$$\Rightarrow A_{\ell} R^{\ell} - \frac{B_{\ell}}{R^{\ell+1}} = 0$$

$$B_{\ell} = A_{\ell} R^{2\ell+1}$$

We also need :

$$V_m(R, \cos \theta) = V_0(\theta) = k \cos(3\theta)$$

$$\Rightarrow \sum_{\ell=0}^{\infty} A_{\ell} R^{\ell} P_{\ell}(\cos \theta) = k \cos(3\theta)$$

Now

$$\cos(3\theta) = 4\cos^3 \theta - 3\cos \theta$$

$$\text{Also, } P_0(\cos \theta) = 1$$

$$P_1(\cos \theta) = \cos \theta$$

$$P_2(\cos \theta) = \frac{1}{2}(3\cos^2 \theta - 1)$$

$$P_3(\cos \theta) = \frac{1}{2}(5\cos^3 \theta - 3\cos \theta)$$

Since $\cos(3\theta)$ is odd in 3rd order, need only $P_1(\cos\theta)$ and $P_3(\cos\theta)$.

$$\cos 3\theta = \frac{8}{5} P_3(\cos\theta) - \frac{3}{5} P_1(\cos\theta)$$

$$V(R, \theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \frac{8}{5} k P_3(\cos\theta) - \frac{3k}{5} P_1(\cos\theta)$$

$$\Rightarrow A_3 R^3 = \frac{8}{5} k$$

$$A_3 = \frac{8k}{5R^3}$$

$$B_3 = \frac{8kR^4}{5}$$

$$A_1 R^1 = -\frac{3}{5} k$$

$$A_1 = -\frac{3}{5} \frac{k}{R}$$

$$B_1 = -\frac{3}{5} k R^2$$

$$V_{\text{out}}(r, \theta) = -\frac{3k}{5} \left(\frac{R}{r}\right)^2 P_1(\cos\theta) + \frac{8k}{5} \left(\frac{R}{r}\right)^4 P_3(\cos\theta)$$

$$V_{\text{in}}(r, \theta) = -\frac{3k}{5} \left(\frac{r}{R}\right) P_1(\cos\theta) + \frac{8k}{5} \left(\frac{r}{R}\right)^3 P_3(\cos\theta)$$

$$\sigma = -\epsilon_0 \left[\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r} \right] \Big|_{r=R}$$

$$\frac{\partial V_{\text{out}}}{\partial r} \Big|_{r=R} = -\frac{3k}{5} R^2 \frac{(-2)}{R^3} P_1(\cos\theta) + \frac{8kR^4}{5} \frac{(-4)}{R^5} P_3(\cos\theta)$$

$$= \frac{6k}{5R} P_1(\cos\theta) - \frac{32k}{5R} P_3(\cos\theta)$$

$$\frac{\partial V_{\text{in}}}{\partial r} \Big|_{r=R} = -\frac{3k}{5R} P_1(\cos\theta) + \frac{24k}{5R} P_3(\cos\theta)$$

$$\sigma = -\epsilon_0 \left[\frac{9k}{5R} P_1(\cos\theta) - \frac{56k}{5R} P_3(\cos\theta) \right]$$

$$\sigma = \frac{e_0 k}{5R} [-9P_1(\cos\theta) + 56 P_3(\cos\theta)]$$