2015 Modern - 2 (QM)

2. (**Required**) A particle of mass m is confined to move freely in a ring of radius R.
(a) Find the energies and the eigenfunctions of the particle.
(b) A perturbation term

$$H' = \begin{cases} V_1, -\alpha < \varphi < 0 \\ V_2, \ 0 < \varphi < \alpha \\ 0, \quad \text{elsewhere} \end{cases}$$

is added to the particle, where ϕ is the azimuthal angle and α is an arbitrary fixed value. Find the first order corrections of the energies for the three lowest energy states.

a) The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{1}{R^2}\frac{d^2}{d\theta^2}\psi(\theta) = E\psi(\theta)$$

$$\frac{d^2}{d\theta^2}\psi(\theta) = -\frac{2mER^2}{\hbar^2}\psi(\theta)$$

Solution:

$$\psi(\theta) = c_1 e^{ik\theta} + c_2 e^{-ik\theta} \quad (1)$$

where $k = \sqrt{\frac{2mER^2}{\hbar^2}}$

We need $\psi(\theta+2\pi) = \psi(\theta)$

So,

$$2\pi k = 2\pi n \qquad \begin{pmatrix} n \\ \text{integer} \end{pmatrix}$$

$$k = n$$

Since n includes positive and negative integers, in Eq. (1) it shows up twice. Therefore,

$$\psi(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \qquad (2)$$

where $\sum_{n=-\infty}^{\infty} |c_n|^2 = 1$ and $e^{in\theta}$ are the eigenfunctions. We could have guessed Eq. (2) by taking a Fourier perspective.

Now,

$$k^2 = n^2$$

$$\frac{2mER^2}{\hbar^2} = n^2$$

$$E = \frac{\hbar^2}{2mR^2} n^2$$

So

$$\boxed{\psi_n(\theta) = e^{in\theta}, \quad E_n = \frac{\hbar^2}{2mR^2} n^2}$$

Note: to normalize $\psi_n(\theta)$

$$1 = \int_0^{2\pi} d\theta \, |C_n e^{in\theta}|^2 = |C_n|^2 \int_0^{2\pi} d\theta = |C_n|^2 2\pi$$

$$C_n = \frac{1}{\sqrt{2\pi}} \Rightarrow \psi_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi}}$$

b)

$\underline{n=0}$

$$E_0^{(1)} = \langle 0 | H' | 0 \rangle$$

$$= \int_0^{2\pi} d\theta \frac{1}{\sqrt{2\pi}} H'(\theta) \frac{1}{\sqrt{2\pi}}$$

Take $\varphi = \theta - \pi$

$$= \frac{1}{2\pi} \left[\int_{-\alpha}^{0} d\theta \, V_1 + \int_0^{\alpha} d\theta \, V_2 \right]$$

$$= \frac{1}{2\pi} \left[\alpha V_1 + V_2 \alpha \right]$$

$\underline{n = \pm 1}$

$$E_{\pm}^{(1)} = \langle \pm 1 \, | \, H' \, | \pm 1 \rangle$$

$$= \int_0^{2\pi} d\theta \, \frac{\left(e^{\mp in\theta}\right)}{\sqrt{2\pi}} H'(\theta) \frac{e^{\pm in\theta}}{\sqrt{2\pi}}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\theta \, H'(\theta)$$

$$\boxed{E_{\pm}^{(1)} = \frac{\cancel{1}\alpha}{2\pi} \left[V_1 + V_2 \right]}$$