4. A 1-D particle in an infinite square well $(0 \le x \le a)$ has the initial wavefunction $\Psi(x,0) = Ax(a-x)$ inside the well and $\Psi(x,0) = 0$ outside the well.
(a) Find the normalization constant A.
(b) Find the expectation value for the position x.
(c) Find the expectation value for x^2.
(d) Use the results of (a) and (b), find the standard deviation σ_x.
(e) If $\sigma_p = \frac{\hbar\sqrt{10}}{a}$, is the Heisenberg Uncertainty Principle satisfied?

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a) We need:

$$\int_{-\infty}^{\infty} dx\, |\Psi(x,0)|^2 = 1$$

$$|A|^2 \int_0^a dx\, x^2(a-x)^2 = 1$$

$$|A|^2 \int_0^a dx\, (a^2x^2 - 2ax^3 + x^4) = 1$$

$$|A|^2 \left(\frac{a^2x^3}{3} - \frac{2ax^4}{4} + \frac{x^5}{5}\right)\Big|_0^a = 1$$

$$|A|^2 \left(\frac{a^5}{3} - \frac{a^5}{2} + \frac{a^5}{5}\right) = 1$$

$$|A|^2 a^5 \left(\frac{10-15+6}{30}\right) = 1$$

Therefore:

$$\boxed{A = \sqrt{\frac{30}{a^5}}}$$

b)

$$\langle x \rangle = \int_{-\infty}^{\infty} dx \Psi^*(x,0) x \Psi(x,0)$$

$$= |A|^2 \int_0^a dx\, x(a-x)x\, x(a-x)$$

$$= \frac{30}{a^5}\int_0^a dx\, x^3(a-x)^2$$

$$= \frac{30}{a^5}\int_0^a dx\, x^3(a^2 - 2ax + x^2)$$

$$= \frac{30}{a^5}\int_0^a dx\, (a^2x^3 - 2ax^4 + x^5)$$

$$= \frac{30}{a^5}\left(\frac{a^6}{4} - \frac{2a^6}{5} + \frac{a^6}{6}\right)$$
$$= 30\,a\left(\frac{15 - 24 + 10}{60}\right)$$

And we get:

$$\boxed{\langle x\rangle = \frac{a}{2}}$$

c) Similar to the above:

$$\langle x^2\rangle = \int_{-\infty}^{\infty} dx \Psi^*(x,0) x^2 \Psi(x,0)$$
$$= |A|^2 \int_0^a dx\; x(a-x)x^2\; x(a-x)$$
$$= \frac{30}{a^5}\int_0^a dx\; x^4(a-x)^2$$
$$= \frac{30}{a^5}\int_0^a dx\; x^4(a^2 - 2ax + x^2)$$
$$= \frac{30}{a^5}\int_0^a dx\; (a^2x^4 - 2ax^5 + x^6)$$
$$= \frac{30}{a^5}\left(\frac{a^7}{5} - \frac{2a^7}{6} + \frac{a^7}{7}\right)$$
$$= 30\,a^2\left(\frac{1}{5} - \frac{1}{3} + \frac{1}{7}\right)$$
$$= 30\,a^2\left(\frac{21 - 35 + 15}{105}\right)$$

Thus,

$$\boxed{\langle x^2\rangle = \frac{2}{7}a^2}$$

Note: One can show,

$$\int_0^a dx\; x^m(a-x)^2 = \frac{2}{(m+1)(m+2)(m+3)}a^{m+2}$$

So that in general:

$$\langle x^n\rangle = \frac{60}{(n+3)(n+4)(n+5)}a^n$$

d)

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2$$
$$= \left(\frac{2}{7} - \frac{1}{4}\right)a^2$$

$$= \left(\frac{8-7}{28}\right) a^2$$

$$\boxed{\sigma_x = \frac{a}{\sqrt{28}}}$$

e) The uncertainty principle states:

$$\sigma_x \sigma_x \geq \frac{\hbar}{2}$$

So,

$$\sigma_x \sigma_p = \hbar \sqrt{\frac{10}{28}}$$

$$= \hbar \sqrt{\frac{5}{14}}$$

Now

$$\sqrt{\frac{5}{14}} > \frac{1}{2}$$

since $\frac{5}{14} > \frac{1}{4}$ (alternatively, just plug into calculator and verify). Therefore, **yes, it is satisfied**.