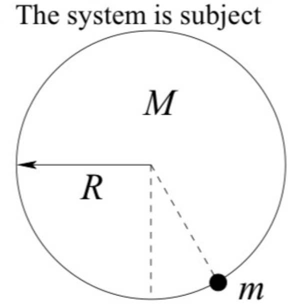
Section A

1. (**Required**) A disk of radius R and mass M is free to rotate about a light axle running through its center. It also has a mass m, imbedded on its rim. The system is subject to gravity.
   a. Write down the Lagrangian for the system.
   b. Find the equation of motion from the Lagrangian.
   c. Find the *stable* equilibrium position for the system. It may be obvious but some proof is needed.
   d. Find the frequency, ω, for small oscillations about the equilibrium position.



a) For the disk

$$I = \int dm\, r^2 \qquad \frac{dm}{dA} = \frac{M}{\pi R^2}$$

$$= \frac{M}{\pi R^2}\int dA\, r^2$$

$$= \frac{M}{\pi R^2}\int_0^{2\pi} d\theta \int_0^R dr\, r\, r^2$$

$$= \frac{M}{\cancel{\pi} R^2}(2\cancel{\pi})\int_0^R dr\, r^3$$

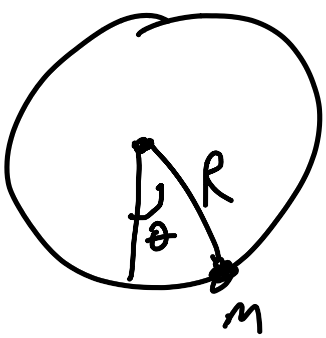
$$= \frac{2M}{R^2}\left(\frac{r^4}{4}\Big|_0^R\right) = \frac{1}{2}MR^2$$

So,

$$T = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m R^2 \dot{\theta}^2$$

$$= \frac{1}{2}\left(\frac{MR^2}{2}\right)\dot{\theta}^2 + \frac{1}{2} m R^2 \dot{\theta}^2$$

$$T = \frac{1}{2} R^2 \left[\frac{M}{2} + m\right] \dot{\theta}^2$$



$$V = mgy$$

$$= mg(-R\cos\theta)$$

$$= -mgR\cos\theta$$

$$\boxed{L = \frac{1}{2} R^2 \left[\frac{M}{2} + m\right] \dot{\theta}^2 + mgR\cos\theta}$$

b)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = \frac{\partial L}{\partial \theta}$$

$$\frac{d}{dt}\left(R^2\left[\frac{M}{2} + m\right]\dot\theta \right) = -mgR\sin\theta$$

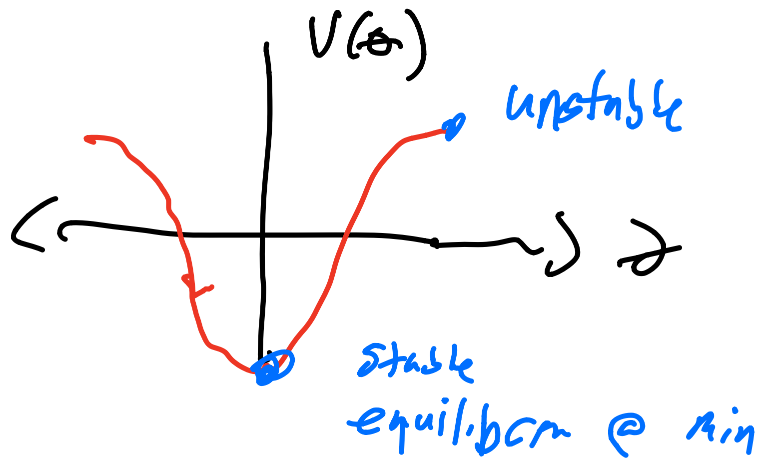
$$R^{\cancel{2}}\left[\frac{M}{2} + m\right]\ddot\theta = -mg\cancel{R}\sin\theta \quad (1)$$

$$\boxed{\ddot\theta = -\frac{mg}{\left[\frac{M}{2}+m\right]R}\sin\theta} \quad (2)$$

Sanity checks:

- units match in Eq. (2)
- If $M=0$ get formula for a pendulum
- Eq. (1) is what you get if you used
  $$\vec\tau = I\vec\alpha = \vec F \times \vec R$$

i) $V(\theta) = -mgR\cos\theta$



$$\frac{\partial V}{\partial \theta} = mgR\sin\theta$$

$$\frac{\partial^2 V}{\partial \theta^2} = mgR\cos\theta$$

Equilibrium at $\frac{\partial V}{\partial \theta} = 0 \Rightarrow \theta = 0, \pi$

$$\left.\frac{\partial^2 V}{\partial \theta^2}\right|_{\theta=0} > 0$$

$$\left.\frac{\partial^2 V}{\partial \theta^2}\right|_{\theta=\pi} < 0 \Rightarrow \boxed{\theta = 0 \text{ stable}}$$

d) Looking at Eq. (2) and using small angle approx:

$$\ddot{\theta} \approx -\frac{mg}{\left(\frac{M}{2}+m\right)R}\theta$$

Equation for simple harmonic motion is

$\ddot{\theta} = -\omega^2 \theta$

$$\Rightarrow \quad \boxed{\omega^2 = \frac{mg}{\left(\frac{M}{2}+m\right)R}}$$