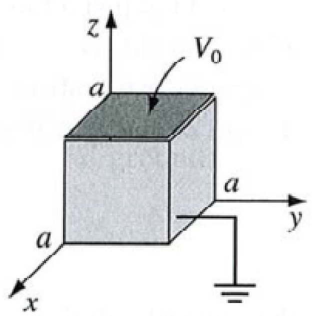
6. A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded (see figure below). The top is made of a separate sheet of metal, insulated from the other sides, and held at constant potential V_0. Determine the electric potential inside the box.



Inside the box:

$$0 \le x \le a, \qquad 0 \le y \le a, \qquad 0 \le z \le a$$

The boundary conditions from the six faces of the cube are:

$$V(0,y,z) = V(a,y,z) = 0 \qquad (BC1\ a\ \&b)$$

$$V(x,0,z) = V(x,a,z) = 0 \qquad (BC2\quad a\ \&b)$$

$$V(x,y,0) = 0,\ \ V(x,y,a) = V_0 \qquad (BC3\quad a\quad \&b)$$

The potential satisfies Laplace's equation:

$$\nabla^2 V(x,y,z) = 0$$

Note that the trivial solution $V(x,y,z) = 0$ is not a solution since it doesn't satisfy BC3 b.

To solve, we'll use the method of separation of variables:

$$V(x,y,z) = X(x)Y(y)Z(z)$$

Then

$$\nabla^2 X(x)Y(y)Z(z) = 0$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) X(x)Y(y)Z(z) = 0$$

$$\frac{\partial^2 X(x)Y(y)Z(z)}{\partial x^2} + \frac{\partial^2 X(x)Y(y)Z(z)}{\partial y^2} + \frac{\partial^2 X(x)Y(y)Z(z)}{\partial z^2} = 0$$

$$Y(y)Z(z)\frac{d^2X(x)}{dx^2} + X(x)Z(z)\frac{d^2Y(y)}{dy^2} + X(x)Y(y)\frac{d^2Z(z)}{dz^2} = 0$$

Divide by $X(x)Y(y)Z(z)$:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0 \tag{1}$$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} \qquad \textit{(moved } Z(z) \textit{ term to other side)}$$

The left-hand side (LHS) depends on x, y while the right-hand side (RHS) on z. Since these are all independent variables, this can only happen if both sides are constant. Therefore:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = const.$$
$$\rightarrow \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = const. - \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}$$

The same logic can be applied here: the LHS depends on x and the RHS on y. Therefore, it must by constant. So, we get:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \lambda_x \tag{2}$$
$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = \lambda_y \tag{3}$$
$$\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = \lambda_z \tag{4}$$

Where the $\lambda's$ are constant. Plugging this into Eq. (1):

$$\lambda_x + \lambda_y + \lambda_z = 0 \tag{5}$$

Thus, the $\lambda's$ cannot have all the same sign. The solutions to Eq. (2)-(4) will be sinusoidal or hyperbolic functions, depending on the sign of the $\lambda's$ (negative in the former, positive in the latter). We'll first solve Eq. (2) subject to BC1:

$$X(0)Y(y)Z(z) = 0 = X(a)Y(y)Z(z)$$

This is true for all y, z in the box. Since the trivial case of $V(x, y, z) = 0$ cannot be a solution then $Y(y)Z(z)$ can zero for all y,z, then this can only hold when:

$$X(0) = 0 = X(a) \tag{6}$$

Since $X(x)$ crosses the x axis twice, it cannot be hyperbolic. The best way to see this is that $X(0) = 0$ implies that $X(x)$ is hyberolic sine. But the $\sinh(a) = 0$ condition cannot be met (prove it). Therefore, λ_x cannot be positive.

If $\lambda_x = 0$ then the solution $X(x)$ is linear. The BC's from Eq. (6) imply the function must be $X(x) = 0$, again not allowed.

Therefore

$$\lambda_x < 0$$

and the solution Eq. (2) is:

$$X(x) = \ a_{x,1}\sin(k_x x) + a_{x,2}\cos(k_x x)$$

Where:

$$-k_x^2 = \lambda_x$$

and $a_{x,1}/a_{x,2}$ are constants. From $X(0) = 0$:

$$X(0) = \ a_{x,1}\sin(k_x 0) + a_{x,2}\cos(k_x 0)$$

$$0 = \ a_{x,2}$$

From $X(a) = 0$:

$$X(a) = 0 = a_{x,1}\sin(k_x a)$$

This can only occur if

$$k_x a = m\pi \qquad\qquad (m \ an \ integer)$$

The logic here is similar to the quantum mechanics problem of a particle in a 1D box.

Therefore, $k_x = \frac{m\pi}{a}$ and:

$$X(x) = a_{x,1}\sin\left(\frac{m\pi x}{a}\right) \tag{7}$$

This forms a complete set to the possibilities for $X(x)$.

Now, the $Y(y)$ has the analogous conditions to $X(x)$, therefore the solution is the same:

$$Y(y) = a_{y,1}\sin\left(\frac{n\pi y}{a}\right) \tag{8}$$

Here, n is some integer. From Eq. (5) then:

$$\lambda_x + \lambda_y + \lambda_z = 0$$

$$-\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{a}\right)^2 + \lambda_z = 0$$

$$\lambda_z = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2 \geq 0$$

It will be useful to define:

$$k_z^{m,n} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2}$$

Therefore, λ_z is positive and the solution to $Z(z)$ is hyperbolic:

$$Z(z) = a_{z,1}\ \sinh(k_z^{m,n}\, z) + a_{z,2}\cosh(k_z^{m,n}\, z)$$

From the condition $Z(0) = 0$:

$$0 = 0 + a_{z,2} \rightarrow a_{z,2} = 0$$

Therefore,

$$Z(z) = a_{z,1} \sinh(k_z^{m,n} z) \tag{9}$$

Note: the boundary condition $V(a, y, z) = V_0$ cannot be used here because then we would have:

$$X(x)Y(y)Z(a) = V_0$$

Unlike before where the RHS was zero, here it's not. Thus, we cannot use the same logic as before and say something about $Z(a)$.

From Eq. (7), (8) and (9) then a particular solution is:

$$X(x)Y(y)Z(z) = c_{m,n} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sinh(k_z^{m,n} z)$$

Here we have absorbed all the constants into $c_{m,n}$. These form a complete set of solutions. The general solution is thus obtained by summing over all possibilities:

$$V(x, y, z) = \sum_{m,n \in \mathbb{Z}} c_{m,n} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sinh(k_z^{m,n} z)$$

To solve for $c_{m,n}$, we invoke the above boundary condition $V(x, y, a) = V_0$:

$$V_0 = \sum_{m,n \in \mathbb{Z}} c_{m,n} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sinh(k_z^{m,n} a) \tag{10}$$

We can now use the orthogonality of the sine function from Fourier analysis:

$$\int_0^a dx \, \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{l\,\pi x}{a}\right) = \frac{a}{2}\, \delta_{ml}$$

$$\int_0^a dy \, \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{r\,\pi y}{a}\right) = \frac{a}{2}\, \delta_{nr}$$

Therefore, multiplying both sides of Eq. (10) by $\sin\left(\frac{l\,\pi x}{a}\right) \sin\left(\frac{r\,\pi y}{a}\right)$, integrating from $x = 0$ to $x = a$ and invoking orthogonality:

$$V_0 \int_0^a dx \;\; \sin\left(\frac{l\,\pi x}{a}\right) \int_0^a dy \;\; \sin\left(\frac{r\,\pi y}{a}\right) = \sum_{m,n \in \mathbb{Z}} c_{m,n} \frac{a}{2}\, \delta_{ml} \frac{a}{2}\, \delta_{nr} \;\sinh(k_z^{m,n} a)$$

$$V_0 \left(-\frac{a}{l\pi} \cos\left(\frac{l\,\pi x}{a}\right) |_0^a \right) \left(-\frac{a}{y\pi} \cos\left(\frac{r\,\pi y}{a}\right) |_0^a \right) = \frac{a^2}{4}\, c_{l,r} \sinh\left(k_z^{l,r}\, a\right)$$

$$V_0 \frac{a^2}{lr\pi^2} (-\,\cos(l\,\pi) + 1)(-\,\cos(r\,\pi) + 1) = \frac{a^2}{4}\, c_{l,r} \sinh\left(k_z^{l,r}\, a\right)$$

$$c_{l,r} = \frac{1}{\sinh\left(k_z^{l,r}\, a\right)} \frac{4V_0}{lr\pi^2}\left(-\,(-1)^l + 1\right)\left(-\,(-1)^r + 1\right)$$

We see that if l is even then $c_{l,r} = 0$ and similarly for r. Therefore,

$$c_{l,r} = \begin{cases} \dfrac{1}{\sinh\left(k_z^{l,r}\, a\right)} \dfrac{16V_0}{lr\pi^2} & l, r\ \ both\ odd \\ 0 & otherwise \end{cases}$$

Using this and our earlier definition of $k_z^{l,r}$:

$$\boxed{V(x,y,z) = \frac{16V_0}{\pi^2} \sum_{m,n\ \ odd} \frac{1}{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \frac{\sinh\left(z\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2}\right)}{\sinh\left(a\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2}\right)}}$$