Section A

1. **(Required)** A particle in an infinite square well potential between $x = 0$ and $x = L$ is described initially by a normalized wavefunction that represents a superposition of the ground and first excited states: $\Psi(x, 0) = \frac{1}{\sqrt{2}}[\psi_1(x) + \psi_2(x)]$, where $\psi_1(x)$ and $\psi_2(x)$ are themselves normalized.
   a. What is $\Psi(x, t)$at a later time t?
   b. Show that the average energy is equal to $(E_1 + E_2)/2$, where E_1 and E_2 are energies of the ground and first excited states respectively.
   c. Show that the average position of the particle oscillates with time as $\langle x \rangle = \frac{L}{2} + A\cos(\Omega t)$, where $A = \int x\psi_1^*\psi_2 dx$ and $\Omega = (E_2 - E_1)/\hbar$
   d. Show that $A = -\frac{16L}{9\pi^2}$
   e. Briefly discuss the relevance of the expression for Ω.

a) $\Psi(x,t) = \sum_n c_n e^{\frac{-iE_nt}{\hbar}} \psi_n(x)$

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[e^{-\frac{iE_1t}{\hbar}}\psi_1(x) + e^{\frac{-iE_2t}{\hbar}}\psi_2(x)\right]$$

b)

$$\langle E \rangle = \langle \Psi | \hat{H} | \Psi \rangle$$

$$= \langle \Psi | \hat{H} \left(\frac{1}{\sqrt{2}} e^{-\frac{iE_1t}{\hbar}} |\psi_1\rangle + \frac{1}{\sqrt{2}} e^{\frac{-iE_2t}{\hbar}} |\psi_2\rangle \right)$$

$$= \langle \Psi | \left(\frac{E_1}{\sqrt{2}} e^{\frac{-iE_1t}{\hbar}} |\psi_1\rangle + \frac{E_2}{\sqrt{2}} e^{-\frac{iE_2t}{\hbar}} |\psi_2\rangle \right)$$

$$= \frac{1}{\sqrt{2}}\left(\langle\psi_1| e^{\frac{iE_1t}{\hbar}} + \langle\psi_2| e^{\frac{iE_2t}{\hbar}}\right)\frac{1}{\sqrt{2}}\left(E_1 e^{\frac{-iE_1t}{\hbar}}|\psi_1\rangle + E_2 e^{\frac{-iE_2t}{\hbar}}|\psi_2\rangle\right)$$

$$= \frac{1}{2}\left[E_1\langle\psi_1|\psi_1\rangle + E_2\langle\psi_2|\psi_2\rangle\right]$$

$$\boxed{\langle E\rangle = \frac{1}{2}\left[E_1 + E_2\right]}$$

c) $\langle\hat{x}\rangle = \langle\Psi|\hat{x}|\Psi\rangle$

$$= \frac{1}{\sqrt{2}}\left(\langle\psi_1| e^{\frac{iE_1t}{\hbar}} + \langle\psi_2| e^{\frac{iE_2t}{\hbar}}\right)\hat{x}\,\frac{1}{\sqrt{2}}\left(e^{\frac{-iE_1t}{\hbar}}|\psi_1\rangle + e^{\frac{-iE_2t}{\hbar}}|\psi_2\rangle\right)$$

$$= \frac{1}{2}\Big[\langle\psi_1|\hat{x}|\psi_1\rangle + e^{\frac{i(E_1-E_2)t}{\hbar}}\langle\psi_1|\hat{x}|\psi_2\rangle + e^{\frac{i(E_2-E_1)t}{\hbar}}\langle\psi_2|\hat{x}|\psi_1\rangle$$

$$+ \langle\psi_2|x|\psi_2\rangle\Big]$$

From symmetry $\langle\psi_1|x|\psi_1\rangle = \langle\psi_2|x|\psi_2\rangle = \frac{L}{2}$

Use $\Omega = \frac{E_2 - E_1}{\hbar}$ $\quad A = \langle\psi_1|x|\psi_2\rangle$

$$= \int dx\, \psi_1^*(x)\, x\, \psi_2(x)$$

$$= \frac{1}{2}\left[L + e^{-i\Omega t} A + e^{i\Omega t} A^* \right]$$

Now, for particle in box

$\psi_n(x)$ are real. $A = A^*$

$$\langle E \rangle = \frac{1}{2}\left[L + A\left(e^{-i\Omega t} + e^{i\Omega t}\right)\right]$$

$$= \frac{L}{2} + A \, \frac{e^{-i\Omega t} + e^{i\Omega t}}{2}$$

$$\Rightarrow \boxed{\langle E \rangle = \frac{L}{2} + A\cos\Omega t}$$

d) $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$

$$A = \int_0^L dx \, \psi_1^*(x) \, x \, \psi_2(x)$$

$$= \int_0^L dx \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) x \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right)$$

$$= \frac{2}{L} \int_0^L dx \sin\left(\frac{\pi x}{L}\right) x \sin\left(\frac{\pi x}{2L}\right)$$

let

$$y = \frac{\pi x}{L}$$

$$dx = \frac{L}{\pi} dy$$

$$\frac{2}{\cancel{L}} \int_0^\pi \frac{\cancel{L}}{\pi} dy \sin(y) \left(\frac{L}{\pi} y\right) \sin(2y)$$

$$= \frac{2}{\pi^2} L \int_0^\pi dy\, y \sin(y) \sin(2y)$$

Use $\sin(2y) = 2\sin y \cos y$

$$= \frac{4}{\pi^2} L \int_0^\pi dy\, y \sin^2 y \cos y$$

$$= \frac{4}{\pi^2} L \int_0^\pi y \, d\left(\frac{\sin^3 y}{3}\right)$$

(Integrate by parts)

$$= \frac{4}{\pi^2} L \left(\left[\cancel{y} \frac{\sin^3 y}{3} \right]_0^{\pi} - \frac{1}{3}\int_0^{\pi} dy \, \sin^3 y \right)$$

Now,

$$\int dy \, \sin^3 y = \int dy \, \sin y \, (1 - \cos^2 y) .$$

$$= \int dy \left(\sin y - \sin y \cos^2 y \right)$$

$$= -\cos y + \frac{\cos^3 y}{3}$$

$$A = \frac{4L}{\pi^2} \left(-\frac{1}{3} \left[-\cos y + \frac{\cos^3 y}{3} \right]_0^{\pi} \right)$$

$$= \frac{4L}{3\pi^2} \left(\left(\cos \pi - \frac{\cos^3 \pi}{3} \right) - \left(\cos 0 - \frac{\cos^3 0}{3} \right) \right)$$

$$= \frac{4L}{3\pi^2} \left(-1 + \frac{1}{3} - \left(1 - \frac{1}{3} \right) \right)$$

$$= \frac{4L}{3\pi^2} \left(\frac{-4}{3} \right) \Rightarrow \boxed{A = \frac{-16L}{9\pi^2}}$$