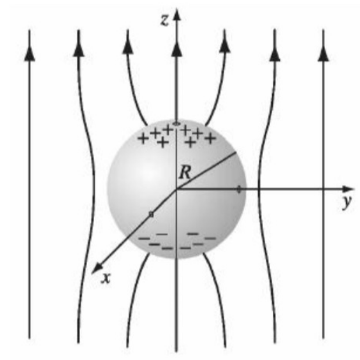
2017 Classical - 5 (EM)

Section B: Electricity and Magnetism

5) An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $\vec{\boldsymbol{E}} = E_0 \hat{\mathbf{z}}$. The field "pushes" positive charge to the "northern" surface of the sphere, and (symmetrically) negative charge to the "southern" surface. This induced charge, in turn, distorts the field in the neighborhood of the sphere (see figure to the right). Determine the **electric potential in the region outside the sphere** ($r > R$) and the **induced surface charge density**.



The solution to Laplace's equation in spherical co-ordinates with azimuthal symmetry is:

$$V(r,\cos\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos\theta) \qquad (1)$$

Since uncharged metal, equipotential at surface of sphere. Choose it $V = 0$. Then, we have two boundary conditions:

$$V(R,\cos\theta) = 0 \qquad \text{BC1}$$

$$r \gg R: \quad V(r,\cos\theta) = -E_0 z = -E_0 r\cos\theta \qquad \text{BC2}$$

Imposition BC1 on Eq. (1):

$$0 = \sum_{\ell=0}^{\infty} \left[A_\ell R^\ell + \frac{B}{R^{\ell+1}} \right] P_\ell(\cos\theta)$$

$$\Rightarrow \quad A_\ell R^\ell + \frac{B_\ell}{R^{\ell+1}} = 0$$

$$\Rightarrow \quad B_\ell = -A_\ell R^{2\ell+1} \qquad (2)$$

Inserting (2) into (1):

$$V(r,\cos\theta) = \sum_{\ell=0}^{\infty} \left[r^\ell - \frac{R^{2\ell+1}}{r^{\ell+1}} \right] A_\ell P_\ell(\cos\theta) \qquad (3)$$

Impose BC2 on Eq. 3. Since $r \gg R$ the second term in the brackets doesn't contribute.

$$-E_0 r\cos\theta = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \qquad (4)$$

Then:
$$A_\ell = \begin{cases} -E_0 & \ell = 1 \\ 0 & \text{otherwise} \end{cases}$$

Two ways to view why. RHS and LHS of (1) are Taylor series in r. Since the LHS has only the $\ell = 1$ term then so must RHS. Also, $P_1(\cos\theta) = \cos\theta$.

The second way is to use the orthogonality of the Legendre polynomials and, again, $P_1(\cos\theta) = \cos\theta$.

Thus,

$$\boxed{V(r, \cos\theta) = -\left[r - \frac{R^3}{r^2}\right] E_0 \cos\theta}$$

Now,

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$$

At the surface, the normal direction is the $\hat{r}$ direction. Therefore,

$$\sigma(\cos\theta) = -\varepsilon_0 \frac{\partial}{\partial r} V(r, \cos\theta)\Big|_{r=R}$$

$$= \varepsilon_0 \left(1 + 2\frac{R^3}{r^3}\right) E_0 \cos\theta \Big|_{r=R}$$

$$\boxed{\sigma(\cos\theta) = 3\varepsilon_0 E_0 \cos\theta}$$

Note: $\sigma > 0 \qquad 0 < \theta < \pi/2$

$\sigma < 0 \qquad \frac{\pi}{2} < \theta < \pi$

as depicted in picture.