2017 Modern - 3 (QM)

3) A spin-1/2 particle with gyromagnetic ratio γ is at rest in a static magnetic field $B_0\hat{\boldsymbol{k}}$ (along the z direction). Now you turn on a small transverse radio-frequency field $\vec{\boldsymbol{B}}(t) = B_{rf}\cos(\omega t)\,\hat{\boldsymbol{i}}$ (along the x direction) with $B_{rf} \ll B_0$. At $t = 0$, the spin is in the down state $\begin{pmatrix}0\\1\end{pmatrix}$. Using the first-order time-dependent perturbation theory, find the probability of the spin in the up state $\begin{pmatrix}1\\0\end{pmatrix}$ as a function of t.

$\underline{0^{th} \text{ order}}$

$$\hat{H}_0 = -\vec{\mu}\cdot\vec{B}$$

Here: $$\vec{B} = B_0\,\hat{k}$$

$$\vec{\mu} = \gamma\vec{S} = \frac{\gamma\hbar}{2}\vec{\sigma}$$

$$\Rightarrow \quad \hat{H}_0 = -\frac{\gamma\hbar B_0}{2}\,\vec{\sigma}\cdot\hat{k}$$

$$\hat{H}_0 = -\frac{\hbar\omega_0}{2}\sigma_z \qquad (\omega_0 = \gamma B_0)$$

Then

$$i\hbar\frac{\partial}{\partial t}|\psi^{(0)}\rangle = \hat{H}_0\,|\psi^{(0)}\rangle$$

$$i\cancel{\hbar}\begin{pmatrix}\partial\psi_1^{(0)}/\partial t \\ \partial\psi_2^{(0)}/\partial t\end{pmatrix} = -\frac{\cancel{\hbar}\omega_0}{2}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_1^{(0)} \\ \psi_2^{(0)}\end{pmatrix}$$

$$\begin{pmatrix}\partial\psi_1^{(0)}/\partial t \\ \partial\psi_2^{(0)}/\partial t\end{pmatrix} = \frac{i\omega_0}{2}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\psi_1^{(0)} \\ \psi_2^{(0)}\end{pmatrix}$$

$$\partial\psi_1^{(0)}/\partial t = i\omega_0\psi_1^{(0)}$$

$$\partial\psi_2^{(0)}/\partial t = -i\omega_0\psi_2^{(0)}$$

$$\Rightarrow \begin{pmatrix}\psi_1^{(0)}(t) \\ \psi_2^{(0)}(t)\end{pmatrix} = c_1^{(0)}e^{i\omega_0 t}\begin{pmatrix}1 \\ 0\end{pmatrix} + c_2^{(0)}e^{-i\omega_0 t}\begin{pmatrix}0 \\ 1\end{pmatrix} \quad (1)$$

since $\begin{pmatrix}\psi_1(0) \\ \psi_2(0)\end{pmatrix} = \begin{pmatrix}0 \\ 1\end{pmatrix}$

$$\Rightarrow \boxed{\begin{pmatrix}\psi_1(t) \\ \psi_2(t)\end{pmatrix} = e^{-i\omega_0 t}\begin{pmatrix}0 \\ 1\end{pmatrix}}$$

<u>1st order</u>

$$\hat{H} = \hat{H}_0 + H'$$

The solution will be of the (1), except now the coefficients have time dependance.

$$\begin{pmatrix} \psi_1^{(1)} \\ \psi_2^{(1)} \end{pmatrix} = c_1^{(1)}(t)\, e^{\frac{i\omega t}{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2^{(1)}(t)\, e^{-\frac{i\omega t}{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

From time-dependent perturbation theory, we know

$$c_2^{(1)}(t) = \frac{1}{i\hbar} \int_0^t dt' \langle - | H' | i \rangle\, e^{i\left(\frac{E_- - E_i}{\hbar}\right)t}$$

where $|i\rangle$ is the initial state:

Since $|i\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $E_- = \hbar\omega_0/2$

$E_i = E_+ = -\frac{\hbar\omega_0}{2}$

and

$$\hat{H}' = -\frac{\gamma\hbar}{2} \vec{\sigma} \cdot (B_{rf}\, \hat{i}) \cos(\omega t) = -\hbar \frac{\omega'}{2} \cos(\omega t)\, \sigma_x$$

where $\omega' = \gamma B_{rf}$

and $\quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

$$\Rightarrow \quad \langle - | \hat{H}' | + \rangle = \langle - | H' | + \rangle$$

$$= -\hbar\omega' \cos(\omega t) \langle - | \sigma_x | + \rangle$$

$$= -\hbar\omega' \cos(\omega t)$$

Then

$$c_2^{(1)}(t) = \frac{1}{i\cancel{\hbar}} \int_0^t dt' \left(-\cancel{\hbar}\,\omega' \cos\omega t'\right) e^{i \frac{\cancel{\hbar}\omega_0 t'}{\cancel{\hbar}}}$$

$$= -\frac{1}{i}\,\omega' \int_0^t dt' \cos\omega t' \, e^{i\omega_0 t'}$$

(Use $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$$= -\frac{1}{2i}\,\omega' \int_0^t dt' \left[e^{i(\omega+\omega_0)t'} + e^{-i(\omega-\omega_0)t'} \right]$$

$$= \frac{-\omega'}{2i} \left[\frac{e^{i(\omega-\omega_0)t'}}{i(\omega+\omega_0)} - \frac{e^{-i(\omega-\omega_0)t'}}{i(\omega-\omega_0)} \right]_0^t$$

$$= \frac{\omega'}{2}\left[\frac{e^{i\overbrace{(\omega-\omega_0)t}^{\text{Let } = \alpha}} - 1}{\omega+\omega_0} - \frac{e^{-i(\omega-\omega_0)t} - 1}{(\omega-\omega_0)} \right]$$

$$= \frac{\omega'}{2} \frac{(\omega-\omega_0)\left(e^{i\alpha}-1\right) - (\omega+\omega_0)\left(e^{-i\alpha t}-1\right)}{(\omega+\omega_0)(\omega-\omega_0)}$$

$$= \frac{\omega'}{2} \frac{\omega\left(e^{i\alpha} - e^{-i\alpha}\right) - \omega_0\left(e^{i\alpha} + e^{i\alpha}\right) - 2\omega}{(\omega^2-\omega_0^2)}$$

$$\boxed{C_2^{(1)}(t) = \omega'\left[\frac{i\omega \sin\alpha - \omega_0 \cos\alpha - 2\omega}{\omega^2-\omega_0^2}\right]}$$

Then

$$P(t) = \left|C_2^{(1)}(t)\right|^2$$

$$= \frac{\omega'^2}{(\omega^2-\omega_0^2)^2}\left[(-\omega_0\cos\alpha - 2\omega)^2 + \omega^2\sin^2\alpha\right]$$

$$= \frac{\omega'^2}{(\omega^2 - \omega_0^2)^2} \left[\omega_0^2 \cos^2\alpha + 4\omega_0 \omega \cos\alpha + 4\omega^2 + \omega^2 \sin^2\alpha \right]$$