2017 Modern - 5 (QM)

5) A particle with positive energy ($E > 0$) moves in the $+x$ direction in a region where the potential is given by

$$V(x) = 0 \qquad \text{for } x < 0$$
$$= -V_0 \qquad \text{for } x > 0$$

where V_0 is a positive, real constant.

a. Solve the Schrödinger equation for the regions $x < 0$ and $x > 0$.
b. State the appropriate boundary conditions for this problem. Use these boundary conditions to relate the incident wave amplitude to the reflected wave amplitude.
c. Determine the reflection coefficient when $V_0 = 2E$. Your final answer should be in numeric form. Discuss the physical significance of your answer in terms of incident, reflected, and transmitted waves.

a) $\underline{x<0}$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi = E$$

$$\frac{d^2\psi}{dx^2} = -k_1^2\psi$$

$$k_1 = \sqrt{\frac{2mE}{\hbar}}$$

$$\Rightarrow \quad \psi_1(x) = C_1 e^{ik_1x} + C_2 e^{-ik_1x}$$

$\underline{x>0}$ Similarly

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} = -k_2^2 \psi$$

$$k_2 = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

$$\psi_2(x) = d_1 e^{ik_2 x} + d_2 e^{ik_2 x}$$

b) Impose continuity and differentiability at $\underline{x=0}$

$$\psi_1(0) = \psi_2(0)$$

$$\boxed{c_1 + c_2 = d_1 + d_2} \quad (1)$$

$$\psi_1'(0) = \psi_2'(0)$$

$$ik_1 c_1 - ik_1 c_2 = ik_2 d_1 - ik_2 d_2$$

$$k_1 c_1 - k_1 c_2 = k_2 d_1 - k_2 d$$

$$\boxed{k_1 (c_1 - c_2) = k_2 (d_1 - d_2)} \quad (2)$$

Because it's initially moving in +x in direction

$c_1 e^{ik_1 x} \rightarrow$ incident

$c_2 e^{-ik_1 x} \rightarrow$ reflecting

$d_1 e^{ik_2 x} \rightarrow$ transmitted

$d_2 = 0$ (No incoming from $x > 0$)

Therefore, Eq. (1) and (2) become:

$$c_1 + c_2 = d_1 \quad (3)$$

$$k_1 (c_1 - c_2) = k_2 d_1 \quad (4)$$

Taking $k_2 (3) - (4)$

$$k_2 (c_1 + c_2) - k_1 (c_1 - c_2) = 0$$

$$(k_2 - k_1) c_1 + (k_2 + k_1) c_2 = 0$$

$$(k_1 + k_2) c_2 = (k_1 - k_2) c_1$$

$$\frac{c_2}{c_1} = \frac{-k_2 + k_1}{k_1 + k_2}$$

$$= \frac{-\sqrt{\frac{2m(E+V_0)}{\hbar^2}} + \sqrt{\frac{2mE}{\hbar^2}}}{\sqrt{\frac{2m(E+V_0)}{\hbar^2}} + \sqrt{\frac{2mE}{\hbar^2}}}$$

$$\boxed{\frac{c_2}{c_1} = \frac{-\sqrt{E+V_0} + \sqrt{E}}{\sqrt{E+V_0} + \sqrt{E}}}$$

c) $R = \left|\frac{c_2}{c_1}\right|^2 = \left|\frac{\sqrt{3E} - \sqrt{E}}{\sqrt{3E} + \sqrt{E}}\right|^2$

$$= \left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^2 = \frac{3-2\sqrt{3}+1}{3+2\sqrt{3}+1}$$

$$= \frac{4-2\sqrt{3}}{4+2\sqrt{3}} \cdot \frac{4-2\sqrt{3}}{4-2\sqrt{3}}$$

$$= \frac{16-16\sqrt{3}+4(3)}{16+12}$$

$$= \frac{28-16\sqrt{3}}{28}$$

$$\boxed{R = 1 - \frac{4}{7}\sqrt{3}} \simeq$$

$$T = \frac{4}{7}\sqrt{3}$$

Since $E > V(+\infty)$

this is a scattering state

(bound if $E < V(+\infty)$ and $E < V(-\infty)$)