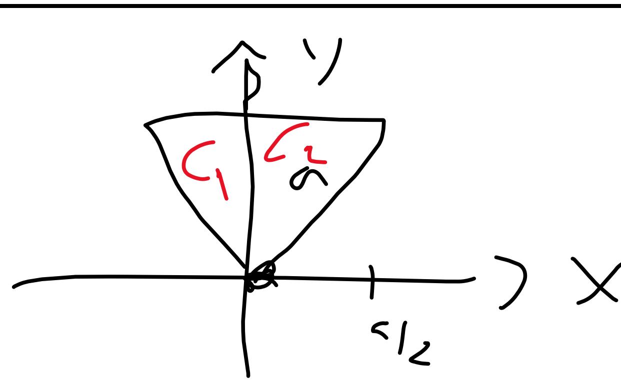
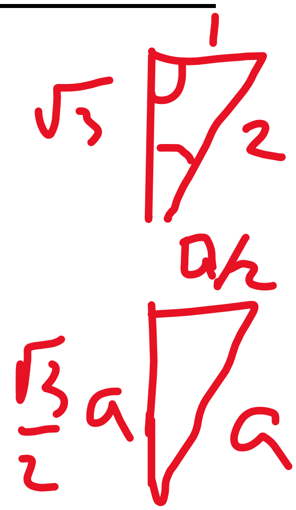
2018 Classical – 3 (CM) - LeonC

3) Consider a thin equilateral triangle of mass m with sides of length a. The triangle is suspended from one apex such that the triangle is constrained to rotate in the plane of the triangle.
   a. Find the moment of inertia about the pivot.
   b. Find the frequency of small oscillations assuming the triangle is subject to gravity.
   c. Flip the triangle over and place the pivot at the center of one side so that the apex hangs down. Do parts (a) and (b) for this configuration.





a) The moment of inertia is given by:

$$I = \int dm\, r^2$$

Assuming uniform distribution:

$$dm = \rho\, dA$$
$$= \frac{m}{\frac{\sqrt{3}a}{2}\frac{a}{2}}\, dA$$

$$\rightarrow\ dm = \frac{4m}{\sqrt{3}a^2}\, dA$$

Thus,

$$I = \frac{4m}{\sqrt{3}a^2}\int dA\, r^2 \qquad (1)$$

Let's evaluate the integral in Eq. (1):

$$\int_{C_1+C_2} dA\, r^2 = \int_{C_1} dA\, r^2 + \int_{C_2} dA\, r^2$$
$$= 2\int_{C_1} dA\, r^2 \quad (From\ symmetry)$$

At a given x we integrate from the line to the top. That is from the line that goes through the origin and has the point $\left(\frac{1}{2}a, \frac{\sqrt{3}}{2}a\right)$: $y = \sqrt{3}x$. The top is $y = \frac{\sqrt{3}}{2}a$. Therefore:

$$= 2\int_0^{\frac{a}{2}} dx \int_{\sqrt{3}ax}^{\frac{\sqrt{3}a}{2}} dy \quad r^2$$
$$= 2\int_0^{\frac{a}{2}} dx \int_{\sqrt{3}x}^{\frac{\sqrt{3}a}{2}} dy \ (x^2 + y^2)$$
$$= 2\int_0^{\frac{a}{2}} dx \left(yx^2 + \frac{y^3}{3}\right)\Big|_{\sqrt{3}x}^{\frac{\sqrt{3}a}{2}}$$
$$= 2\int_0^{\frac{a}{2}} dx \left(\frac{\sqrt{3}a}{2}x^2 + \frac{1}{3}\frac{\sqrt{3}a}{2}\left(\frac{3a^2}{4}\right) - \sqrt{3}x^3 - \frac{1}{3}\sqrt{3}x(3x^2)\right)$$
$$= 2\sqrt{3}\int_0^{\frac{a}{2}} dx \left(\frac{1}{2}ax^2 + \frac{a^3}{8} - x^3 - x^3\right)$$
$$= 2\sqrt{3}\int_0^{\frac{a}{2}} dx \left(-2x^3 + \frac{1}{2}ax^2 + \frac{a^3}{8}\right)$$
$$= 2\sqrt{3}\left(-2\frac{1}{4}\frac{a^4}{16} + \frac{a}{2}\frac{1}{3}\frac{a^3}{8} + \frac{a^3}{8}\frac{a}{2}\right)$$
$$= 2\sqrt{3}a^4\left(-\frac{1}{32} + \frac{1}{48} + \frac{1}{16}\right)$$
$$= 2\sqrt{3}a^4\left(\frac{5}{96}\right)$$
$$= \frac{5}{48}\sqrt{3}\, a^4$$

Therefore:

$$I = \frac{4m}{\sqrt{3}a^2}\frac{7}{48}\sqrt{3}\, a$$
$$\boxed{I = \frac{5}{12}ma^2}$$

b) We have:

$$I\frac{d^2\theta}{dt^2} = -mg\, r_{cm}\sin\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{mg\, r_{cm}}{I}\sin\theta$$

Using the small angle approximation:

$$\frac{d^2\theta}{dt^2} \approx -\frac{mg\, r_{cm}}{I}\theta$$

$$\omega^2 = \frac{mg\, r_{cm}}{I}$$

Using the result from (a) and $r_{cm} = \frac{a\sqrt{3}}{3}$ (see diagram in c) then

$$\omega^2 = \frac{mg\ \frac{a\sqrt{3}}{3}}{\frac{5}{12}ma^2}$$

$$= \frac{12\ g\sqrt{3}}{5a(3)}$$

$$\rightarrow \boxed{\omega^2 = \frac{4\sqrt{3}}{5}\frac{g}{a}}$$

c) From the parallel axis theorem:

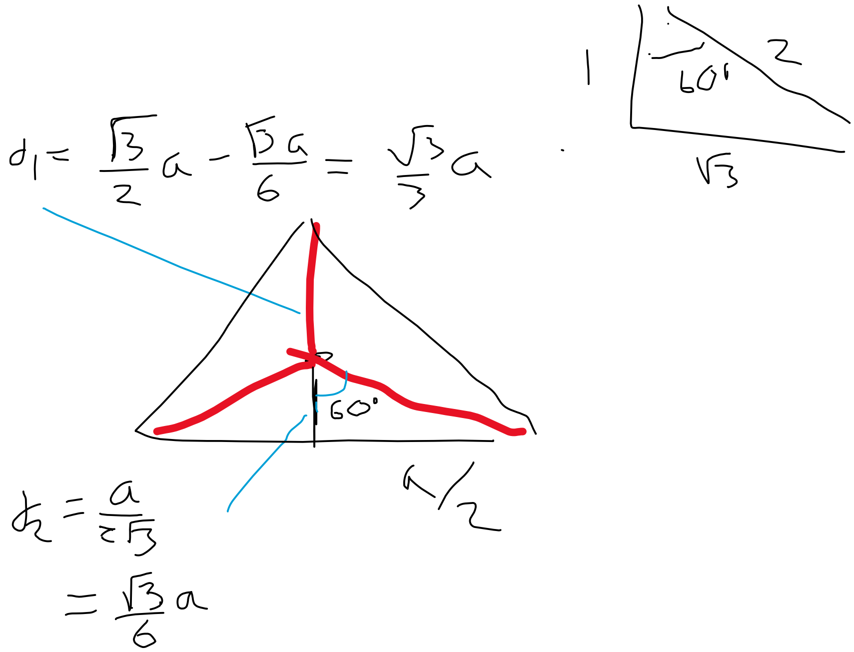
$$I = I_{cm} + md^2 \rightarrow I_{cm} = I - md^2 = I_{cm}$$

If we let I_1 and I_2 be the moment of inertia for the a and c confirmation, respectively. Also, let $d_1\ and\ d_2$ be the distances of the CM from the pivot point, respectively, we get:

$$I_1 - md_1^2 = I_2 - md_2^2$$

$$I_2 = I_1 - md_1^2 + md_2^2$$

Solving:



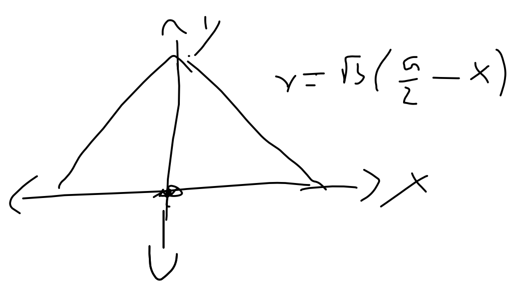
Therefore,

$$I_2 = \frac{5}{12}ma^2 - m\left(\frac{1}{3}\right)a^2 + m\frac{3}{36}a^2$$
$$= \left(\frac{5}{12} - \frac{1}{3} + \frac{1}{12}\right)ma^2$$
$$= \left(\frac{5-4+1}{12}\right)ma^2$$

And:

$$\boxed{I_2 = \frac{1}{6}ma^2}$$

Alternatively:



$$I_2 = \frac{4m}{\sqrt{3}a^2}\int dA\, r^2 \qquad (2)$$

Then the integral is:

$$\int dA\, r^2 = 2\int_0^{\frac{a}{2}} dx \int_0^{\sqrt{3}\left(\frac{a}{2}-x\right)} dy \quad r^2$$
$$= 2\int_0^{\frac{a}{2}} dx \int_0^{\sqrt{3}\left(\frac{a}{2}-x\right)} dy \ \ (x^2+y^2)$$
$$= 2\int_0^{\frac{a}{2}} dx \left(x^2 y + \frac{y^3}{3}\right)\Big|_0^{\sqrt{3}\left(\frac{a}{2}-x\right)}$$
$$= 2\int_0^{\frac{a}{2}} dx \left(x^2\sqrt{3}\left(\frac{a}{2}-x\right) + \frac{1}{3}\sqrt{3}\,(3)\left(\frac{a}{2}-x\right)^3\right)$$
$$= 2\sqrt{3}\int_0^{\frac{a}{2}} dx \left(x^2\left(\frac{a}{2}-x\right) + \left(\frac{a}{2}-x\right)^3\right)$$
$$= 2\sqrt{3}\int_0^{\frac{a}{2}} dx \left(\frac{a}{2}x^2 - x^3 - x^3 + 3x^2\frac{a}{2} - 3x\frac{a^2}{4} + \frac{a^3}{8}\right)$$
$$= 2\sqrt{3}\int_0^{\frac{a}{2}} dx \left(\frac{a}{2}x^2 - x^3 - x^3 + 3x^2\frac{a}{2} - 3x\frac{a^2}{4} + \frac{a^3}{8}\right)$$
$$= 2\sqrt{3}\int_0^{\frac{a}{2}} dx \left(-2x^3 + 2ax^2 - 3\frac{a^2}{4}x + \frac{a^3}{8}\right)$$
$$= 2\sqrt{3}\left(-2\frac{1}{4}x^4 + 2a\frac{1}{3}x^3 - 3\frac{a^2}{8}x^2 + \frac{a^3}{8}\,x\right)\Big|_0^{\frac{a}{2}}$$
$$= 2\sqrt{3}\left(-\frac{1}{2}\frac{a^4}{16} + \frac{2a}{3}\,\frac{a^3}{8} - 3\frac{a^2}{8}\frac{a}{4} + \frac{a^3}{8}\,\frac{a}{2}\right)$$
$$= 2\sqrt{3}a^4\left(-\frac{1}{32} + \frac{1}{12} - \frac{3}{32} + \frac{1}{16}\right)$$
$$= 2\sqrt{3}a^4\frac{1}{48}$$
$$= \frac{\sqrt{3}}{24}a^4$$

Inserting the result into Eq. (2):

$$I_2 = \frac{4m}{\sqrt{3}a^2}\frac{\sqrt{3}}{24}a^4$$
$$= \frac{m}{6}a^2$$

As before.