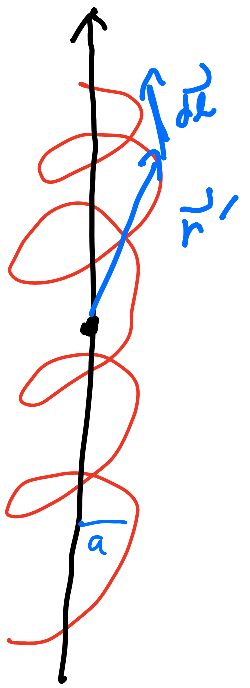
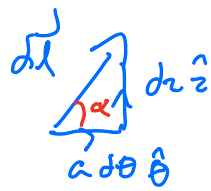
6) A conducting wire is wrapped around a vertical cylinder of radius a to form a helical coil with pitch angle α (i.e. any segment of the wire makes an angle α with the horizontal). The helix has N complete turns. If the wire carries a current I, show that the B field at the center of the helix is,

$$\frac{1}{2}\left(\frac{\mu_0 N I}{a}\right)(1+\pi^2 N^2 \tan^2\alpha)^{-1/2}$$





$$\frac{dz}{a\,d\theta} = \tan\alpha$$

Parametrizing the helix:

$$r(s) = a$$
$$\theta(s) = s$$
$$z(s) = a\tan\alpha\, s$$

where $-\frac{N}{2}(2\pi) \le s \le \frac{N}{2}(2\pi)$

$$\Rightarrow \quad \vec{r}\,' = a\hat{r} + s\hat{\theta} + a\tan\alpha\, s\,\hat{z}$$

$$d\vec{l} = ds\,\hat{\theta} + a\tan\alpha\, ds\,\hat{z}$$

By the Biot-Savart law,

$$\vec{B}(\vec{0}) = \frac{\mu_0}{4\pi}\int_C \frac{I\,d\vec{l}\times\vec{r}\,'}{|\vec{r}\,'|^3} \qquad (1)$$

$$d\vec{l} \times \vec{r}\,' = d\vec{l} \times (a\hat{r} + \vec{l})$$

$$= a\, d\vec{l} \times \hat{r} + \underbrace{d\vec{l} \times \vec{l}}_{=0}$$

$d\vec{l} \perp \vec{l}$

$$d\vec{l} \times \vec{r}\,' = a(ds\,\hat{\theta} + a \tan\alpha\, ds\,\hat{z}) \times \hat{r}$$

$$= a\, ds\, \hat{\theta} \times \hat{r} + a^2 \tan\alpha\, ds\, \hat{z} \times \hat{r}$$

$$d\vec{l} \times \vec{r}\,' = a\, ds\, \hat{z} - a^2 \tan\alpha\, ds\, \hat{\theta} \qquad (2)$$

The magnetic field must point in the $\hat{z}$ direction. Therefore, from (1) and (2)

$$B_z(\vec{0}) = \frac{\mu_0 I}{4\pi} \int_{-\frac{N}{2}(2\pi)}^{\frac{N}{2}(2\pi)} ds \frac{a}{(a^2 + z^2)^{3/2}}$$

Using $z(s) = a \tan\alpha\, s$,

$$B_z(\vec{0}) = \frac{\mu_0 I}{4\pi} \int_{-N\pi}^{N\pi} ds \frac{a}{(\underbrace{a^2 + a^2 \tan^2\alpha\, s^2}_{\text{factor out } a^2})^{3/2}}$$

$$B_z(\vec{0}) = \frac{\mu_0 I}{4\pi a} \int_{-N\pi}^{N\pi} ds \frac{1}{(1 + \tan^2\alpha\, s^2)^{3/2}}$$

Let $y = \tan\alpha\, s$. Then

$$\boxed{B_z(0) = \frac{\mu_0 I}{4\pi a \tan\alpha} \underbrace{\int_{-\tilde{y}}^{\tilde{y}} dy \frac{1}{(1+y^2)^{3/2}}}_{\text{Let } = J}} \quad (3)$$

where $\tilde{y} = \tan\alpha\,(Nh)$.

$$J = \int_{-\tilde{y}}^{\tilde{y}} dy \frac{1}{(1+y^2)^{3/2}}$$

Now let $y = \tan\varphi \Rightarrow dy = \frac{1}{\cos^2\varphi} d\varphi$

$$J = \int_{-\tan^{-1}(\tilde{y})}^{\tan^{-1}(\tilde{y})} \frac{d\varphi}{\cos^2\varphi} \frac{1}{(\underbrace{1+\tan^2\varphi}_{= 1/\cos^2\varphi})^{3/2}}$$

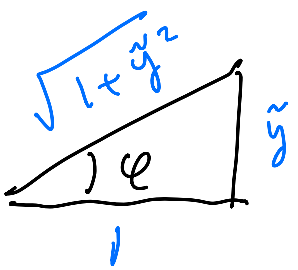
$$= \int_{-\tan^{-1}(\tilde{y})}^{\tan^{-1}(\tilde{y})} d\varphi \frac{1}{\cos^2\varphi} \frac{1}{\left(\frac{1}{\cos^3\varphi}\right)}$$

$$= \int_{-\tan(\tilde{y})}^{\tan^{-1}(\tilde{y})} d\varphi \quad \cos\varphi$$

$$\bar{J} = \sin\varphi \Big|_{-\tan^{-1}(\tilde{y})}^{\tan^{-1}(\tilde{y})}$$

$$= 2\sin\varphi \Big|_{0}^{\tan^{-1}(\tilde{y})}$$ Since $\sin\varphi$ is odd function

Now,



$$\tan\varphi = \frac{\tilde{y}}{1}$$

$$\Rightarrow \sin\varphi = \frac{\tilde{y}}{\sqrt{1+\tilde{y}^2}}$$

So,

$$J = 2\frac{\tilde{y}}{\sqrt{1+\tilde{y}^2}}$$ Use definition of $\tilde{y}$

$$J = 2\frac{N\pi\tan\alpha}{\sqrt{1+N^2\pi^2\tan^2\alpha}} \qquad (4)$$

Plugging (4) into (3)

$$B_z(0) = \frac{\mu_0 I}{\cancel{4}_{2}\pi a \cancel{\tan\alpha}} \frac{\cancel{2} N \cancel{\pi} \cancel{\tan\alpha}}{\sqrt{1 + N^2\pi^2\tan^2\alpha}}$$

Therefore,

$$\boxed{B_z(0) = \frac{\mu_0 N I}{2a} \frac{1}{\sqrt{1 + N^2\pi^2\tan^2\alpha}}}$$