Problem A2: Consider a quantum system with the Hamiltonian

$$H = E_0 \begin{pmatrix} 5 & -2 \\ 1 & 2 \end{pmatrix},$$

where E_0 is a positive real number.

a. What are the eigenvalues of H?
b. What are the eigenvectors (normalized, matrix form) of H?
c. What is the expectation value of H when the system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix} ?$$

Bonus (CL): what's wrong with H?

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a) To find the eigenvalues solve:

$$\det(H - \lambda I) = 0$$

$$\begin{vmatrix} 5E_0 - \lambda & -2E_0 \\ E_0 & 2E_0 - \lambda \end{vmatrix} = 0$$

$$(5E_0 - \lambda)(2E_0 - \lambda) + 2E_0^2 = 0$$

$$10E_0^2 - 5E_0\lambda - 2E_0\lambda + \lambda^2 + 2E_0^2 = 0$$

$$\lambda^2 - 7E_0\lambda + 12E_0^2 = 0$$

$$(\lambda - 3E_0)(\lambda - 4E_0) = 0$$

Therefore,

$$\boxed{\lambda = 3E_0, 4E_0}$$

b) We need to solve for:

$$H\psi = E\psi$$

For both eigenvalues

$\underline{2E_0}$

$$E_0 \begin{pmatrix} 5 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 3E_0 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 3 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

From first row:

$$5\psi_1 - 2\psi_2 = 3\psi_1$$

$$\psi_1 = \psi_2 \rightarrow \frac{\psi_2}{\psi_1} = 1$$

One can easily verify that $H \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Now,

$$\psi_l = A_l \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where A_l is a normalizing constant. Its value is:

$$|\psi_l|^2 = 1$$
$$|A|^2 \left(1^2 + 1^2\right) = 1$$
$$|A|^2\, 2 = 1 \rightarrow A = \frac{1}{\sqrt{2}}$$

$$\boxed{\psi_l = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$\underline{4E_0}$

$$E_0 \begin{pmatrix} 5 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 4E_0 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 4 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

From first row:

$$5\psi_1 - 2\psi_2 = 4\psi_1$$
$$\psi_1 - 2\psi_2 = 0 \rightarrow\rightarrow \quad \frac{\psi_2}{\psi_1} = \frac{1}{2}$$

Again, you can see $H\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 4\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. So,

$$\psi_u = A_u \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Normalizing:

$$|A_u|^2 \left(2^2 + 1^2\right) = 1$$
$$\rightarrow A_u = \frac{1}{\sqrt{5}}$$

$$\boxed{\psi_l = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

c) The expectation value is given by:

$$\langle\psi|H|\psi\rangle = \vec{\psi}^{\dagger} E_0 \begin{pmatrix} 5 & -2 \\ 1 & 2 \end{pmatrix} \vec{\psi}$$
$$= \frac{E_0}{10} \begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$
$$= \frac{E_0}{10} \begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} 13 \\ 5 \end{pmatrix}$$
$$= \frac{E_0}{10} \left(3(13) + 5\right)$$

$$\boxed{\langle\psi|H|\psi\rangle = 4.4E_0}$$

Bonus: As we see, the eigenvectors are NOT orthogonal despite the eigenvalues being different. This is due to the fact that H is not Hermitian. Thus, this Hamiltonian is non-physical.

The Hamiltonian needs to Hermitian for the evolution operator:

$$U(t) = e^{-iHt/\hbar}$$

to be unitarian. This is needed for the probabilities to sum up to 1.