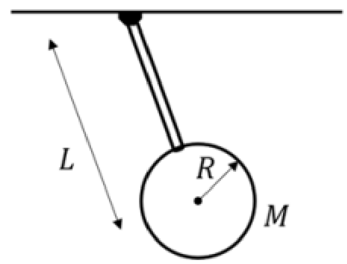
Problem CM3: A physical pendulum consists of a uniform disk of mass M and radius R rigidly attached to a massless rod such that the distance from the pivot to the center of the disk is L.

a) Find the period of small oscillation.
b) For what length of rod will the period be a minimum?



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a) The moment of inertia for the disk about it center-of-mass (CM):

$$
\begin{aligned}
I_{CM} &= \int dm\, r^2 \\
&= \int \rho\, dA\, r^2 \\
&= \int \frac{M}{\pi R^2}\, dA\, r^2 \qquad (since\ disk\ is\ uniform) \\
&= \frac{M}{\pi R^2}\int dA\, r^2 \\
&= \frac{M}{\pi R^2}\int_0^{2\pi} d\theta \int_0^R dr\, r\, r^2 \\
&= \frac{M}{\pi R^2}\, 2\pi \int_0^R dr\, r^3 \\
&= \frac{M}{\pi R^2}\, 2\pi \left(\frac{R^4}{4}\right) \\
&= \frac{1}{2} M R^2
\end{aligned}
$$

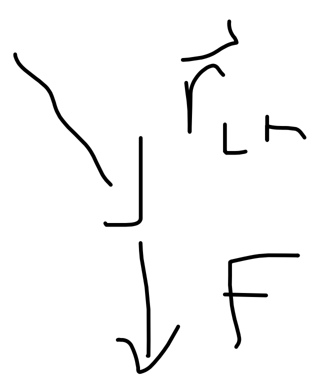
To get the moment of inertia for the pivot point given use the parallel axis theorem:

$$
\begin{aligned}
I &= I_{CM} + ML^2 \\
&= M\left(\frac{1}{2}R^2 + L^2\right)
\end{aligned}
$$

Now, look at the torque:

$$\tau = I\boldsymbol{\alpha} = \boldsymbol{F} \times \boldsymbol{r_{cm}}$$

$$
\begin{aligned}
I\frac{d^2\theta}{dt^2} &= -|F||r_{cm}|\sin\theta \\
\frac{d^2\theta}{dt^2} &= -\frac{|F||r_{cm}|}{I}\sin\theta
\end{aligned}
$$



$$\frac{d^2\theta}{dt^2} = -\frac{Mg|r_{cm}|}{I}\sin\theta$$
$$\frac{d^2\theta}{dt^2} = -\frac{gL}{\frac{1}{2}R^2 + L^2}\sin\theta \quad \left(Use \sin\theta \approx \theta\right)$$
$$\frac{d^2\theta}{dt^2} \approx -\frac{gL}{\frac{1}{2}R^2 + L^2}\theta$$

Then,

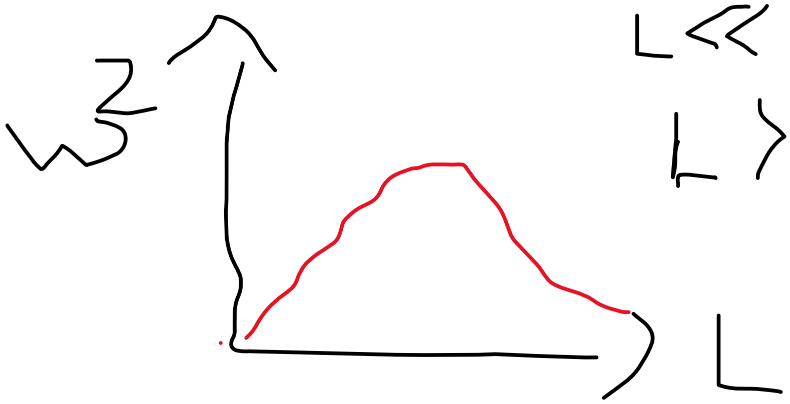
$$\boxed{\omega^2 = \frac{gL}{\frac{1}{2}R^2 + L^2}} \tag{1}$$

Note: when $L \gg R$ this reduces to the formula for a pendulum of a point particle: $\omega^2 = \frac{g}{L}$.

b) Since $\omega = \frac{2\pi}{T}$ minimizing T is the same as maximizing ω, which itself is the same as maximizing ω^2.

$L \ll R$: $\omega^2 \sim L$

$L \gg R$: $\omega^2 \sim \frac{1}{L}$



$$0 = \frac{d\omega^2}{dL} = \frac{g\left(\frac{1}{2}R^2 + L^2\right) - gL(2L)}{\left(\frac{1}{2}R^2 + L^2\right)^2}$$
$$= \frac{g\left(\frac{1}{2}R^2 - L^2\right)}{\left(\frac{1}{2}R^2 + L^2\right)^2}$$
$$\rightarrow \boxed{L = \frac{R}{\sqrt{2}} = \frac{\sqrt{2}}{2}R}$$

Note: Here the pivot point is actually *inside* the disk.