2020 Modern QM2

Problem QM2:

1. Consider an electron (a spin ½ particle) in the spin state $|\psi\rangle = \frac{1}{\sqrt{25}}(3|+\rangle + 4|-\rangle)$, where $|+\rangle$ and $|-\rangle$ are the spin up and spin down state, respectively, in the z-direction. The electron passes through an ***x*-direction** Stern-Gerlach analyzer, what is the probability that it will be measured in the $|-\rangle_x$ state?
2. After the electron passes through the Stern-Gerlach analyzer, at time $t = 0$ the particle is measured to be in the the $|-\rangle_x$ state (i.e. for the following problem, the initial state is $|\psi(t=0)\rangle = |-\rangle_x$), also at this moment ($t = 0$), the electron enters a uniform magnetic field in the z direction: $\vec{B} = B_0\hat{z}$. What is the state $|\psi(t)\rangle$ at some time t later (in the S_z basis)?

   Hint: the Hamiltonian can be written as $H \doteq \frac{\hbar\omega_0}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
3. What is the probability of measuring the particle in state $|\phi\rangle = \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle)$ at some time t later?

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1. The eigen state $|-\rangle_x$ in the S_z basis is given by:

$$|-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

By the Born rule the probability of being the in the, $|-\rangle_x$ state is:

$$P = |\langle -|_x \psi\rangle|^2$$
$$= \left|\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \end{pmatrix}\frac{1}{\sqrt{25}}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right|^2$$
$$= \left|\frac{1}{\sqrt{50}}(3-5)\right|^2$$
$$= \frac{1}{50}$$

2. The Schrodinger equation is given:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \quad (1)$$

where:

$$\hat{H} = -\vec{\mu}\cdot\vec{B} \quad (2)$$

Since this doesn't depend on time the solution is:

$$|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\psi(0)\rangle \quad (3)$$

You can easily very it satisfies Eq. (1) and meets the boundary condition. Now, looking at Eq. (2), we note:

$$\vec{\mu} = \gamma \frac{\hbar}{2}\, \vec{\sigma} \tag{4}$$

Where $\gamma = -g\frac{e}{2m}$ and g is the gyromagnetic constant. We'll use $g = 2$, which is correct to leading order (we'll ignore higher order corrections to this) and $\gamma = -\frac{e}{m}$. Thus,

$$\hat{H} = \frac{1}{2}\frac{e}{m}\, \vec{B} \cdot \vec{\sigma} \tag{5}$$

Inserting Eq. (5) into (3):

$$\psi(t) = \; e^{-it\,\frac{e|\vec{B}|}{m}\,\hat{B}\cdot\vec{\sigma}}\;\psi(0)$$

We used $\vec{B} = |\vec{B}|\,\hat{B}$. Letting $\omega = \frac{e}{m}|\vec{B}|$ be the classical cyclotron frequency then:

$$\psi(t) = \; e^{-\frac{i\omega t}{2}\,\hat{B}\cdot\vec{\sigma}}\;\psi(0)$$

We now use that fact that if we have a vector $\vec{A} = a\,\hat{A}$ then:

$$e^{i\,a\,\hat{A}\cdot\vec{\sigma}} = I\cos(a) + i(\hat{A}\cdot\vec{\sigma})\sin(a)$$

Therefore, the general solution to Eq. (1) is:

$$\psi(t) = \left(I\cos\left(\frac{\omega t}{2}\right) - i(\hat{B}\cdot\vec{\sigma})\sin\left(\frac{\omega t}{2}\right)\right)\,\psi(0)$$

Now we insert $\hat{B} = B_0\,\hat{z}$ and $\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$:

$$\psi(t) = \frac{1}{\sqrt{2}}\left(I\cos\left(\frac{\omega t}{2}\right) - i\sigma_z\sin\left(\frac{\omega t}{2}\right)\right)\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\psi(t) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}\cos\left(\frac{\omega t}{2}\right) & 0\\ 0 & \cos\left(\frac{\omega t}{2}\right)\end{pmatrix} + \begin{pmatrix}-i\sin\left(\frac{\omega t}{2}\right) & 0\\ 0 & i\sin\left(\frac{\omega t}{2}\right)\end{pmatrix}\right]\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\left(\frac{\omega t}{2}\right) - \mathrm{isin}\left(\frac{\omega t}{2}\right) & 0\\ 0 & \cos\left(\frac{\omega t}{2}\right) + \mathrm{isin}\left(\frac{\omega t}{2}\right)\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}e^{-i\frac{\omega t}{2}} & 0\\ 0 & e^{+i\frac{\omega t}{2}}\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\boxed{\psi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-\frac{i\omega t}{2}} \\ -e^{\frac{i\omega t}{2}} \end{pmatrix}}$$

Note: alternatively, you can solve the eigenvalues and eigenvectors of Eq. (5), use $|\psi(t)\rangle = \sum_n c_n e^{-\frac{iE_n t}{\hbar}} |n\rangle$ and use the boundary condition to find the coefficients.

3. By the Born rule the probability is given by:

$$
\begin{aligned}
P(t) &= |\langle\phi|\psi(t)\rangle|^2 \\
&= |\frac{1}{\sqrt{2}}(1 \quad -i)\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-\frac{i\omega t}{2}} \\ -e^{\frac{i\omega t}{2}} \end{pmatrix}|^2 \\
&= |\frac{1}{2}\left(e^{-\frac{i\omega t}{2}} + ie^{\frac{i\omega t}{2}}\right)|^2 \\
&= \frac{1}{4}|\left(\cos\left(\frac{\omega t}{2}\right) - i\sin\left(\frac{\omega t}{2}\right) + i\left(\cos\left(\frac{\omega t}{2}\right) + i\sin\left(\frac{\omega t}{2}\right)\right)\right)|^2 \\
&= \frac{1}{4}|\left(\cos\left(\frac{\omega t}{2}\right) - \sin\left(\frac{\omega t}{2}\right) + i\left(\cos\left(\frac{\omega t}{2}\right) - \sin\left(\frac{\omega t}{2}\right)\right)\right)|^2 \\
&= \frac{1}{4}|\cos\left(\frac{\omega t}{2}\right) - \sin\left(\frac{\omega t}{2}\right)|^2\,|1+i|^2 \\
&= \frac{1}{4}\left(\cos^2\left(\frac{\omega t}{2}\right) - 2\sin\left(\frac{\omega t}{2}\right)\cos\left(\frac{\omega t}{2}\right) + \cos^2\left(\frac{\omega t}{2}\right)\right)(1+1) \\
&= \frac{1}{2}\left(\cos^2\left(\frac{\omega t}{2}\right) - 2\sin\left(\frac{\omega t}{2}\right)\cos\left(\frac{\omega t}{2}\right) + \sin^2\left(\frac{\omega t}{2}\right)\right) \\
&= \frac{1}{2}\left(1 - 2\sin\left(\frac{\omega t}{2}\right)\cos\left(\frac{\omega t}{2}\right)\right) \quad (Use\ \sin 2\theta = 2\cos\theta\sin\theta) \\
&= \frac{1}{2}(1 - \sin(\omega t))
\end{aligned}
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